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Boundary layers for compressible Navier–Stokes equations with density-dependent viscosity and cylindrical symmetry

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Abstract

In this paper, we consider the zero shear viscosity limit for the Navier–Stokes equations of compressible flows with density-dependent viscosity coefficient and cylindrical symmetry. The boundary layer effect as the shear viscosity $\mu = \varepsilon\rho^\theta$ goes to zero (in fact, $\varepsilon \to 0$ in this paper, which implies $\mu \to 0$) is studied. We prove that the boundary layer thickness is of the order $O(\varepsilon^\alpha)$, where $0 < \alpha < \frac{1}{2}$ for the constant initial data and $0 < \alpha < \frac{1}{4}$ for the general initial data, which extend the result in Frid and Shelukhin (1999) [4] to the case of density-dependent viscosity coefficient.

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Keywords: Navier–Stokes equations; Density-dependent viscosity; Cylindrical symmetry; Zero shear viscosity limit; Boundary layers; BL-thickness

1. Introduction

In this paper, we shall study the effect of boundary layers as the shear viscosity μ goes to zero (in fact, $\varepsilon \to 0$ in this paper, which implies $\mu \to 0$) for an initial–boundary value problem to the Navier–Stokes equations of compressible flows with density-dependent viscosity coefficient and cylindrical symmetry. We restrict ourselves to the flows between two circular coaxial cylinders and assume that the corresponding solutions depend only on the radial variable x in $\Omega := \{x \mid 0 < a < x < b\}$ and the time variable $t \in [0, T]$. Precisely, under the cylindrical symmetric transformation:

$$\vec{u} = \left(u\frac{x_1}{x} - v\frac{x_2}{x}, u\frac{x_2}{x} + v\frac{x_1}{x}, w\right), \qquad x = \sqrt{x_1^2 + x_2^2}, \qquad u = u(x,t), \qquad v = v(x,t), \qquad w = w(x,t),$$

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we consider the following three-dimensional compressible Navier–Stokes equations

$$\begin{cases} \partial_t\rho + \nabla\cdot(\rho\vec{u}) = 0, \\ \partial_t(\rho\vec{u}) + \nabla\cdot(\rho\vec{u}\otimes\vec{u}) + \nabla p = \operatorname{div}\big(\mu\big(\nabla\vec{u} + \nabla\vec{u}^T\big)\big) + \nabla(\lambda \operatorname{div}\vec{u}). \end{cases}$$

Here ρ is the density, $p = A\rho^\gamma$ is the pressure, where $\gamma > 1$ is the gas constant and A is a positive constant, which will be normalized to be 1; u is the component of the velocity vector $\vec{u}$ along the radial variable x $(x \in \Omega)$, v is the angular component of $\vec{u}$, w is the axial component of $\vec{u}$; λ and μ are the bulk and shear viscosity coefficients, respectively. $\lambda > 0$ is a positive constant and $\mu = \varepsilon\rho^\theta$ with $0 \leqslant \theta \leqslant \gamma$ and ε being a positive constant.

The reduced system is now of the form (see [8] for the case of constant viscosity coefficient):

$$\begin{cases} \rho_t + (\rho u)_x + \dfrac{\rho u}{x} = 0, \\ \rho\left(u_t + uu_x - \dfrac{v^2}{x}\right) + p_x - (2\mu+\lambda)\left(u_x + \dfrac{u}{x}\right)_x - 2\mu_x u_x = 0, \\ \rho\left(v_t + uv_x + \dfrac{uv}{x}\right) - \mu\left(v_x + \dfrac{v}{x}\right)_x - \mu_x v_x + \mu_x\dfrac{v}{x} = 0, \\ \rho(w_t + uw_x) - \mu\left(w_{xx} + \dfrac{w_x}{x}\right) - \mu_x w_x = 0. \end{cases} \tag{1.1}$$

In the domain $Q_T := \Omega \times (0,T)$, we consider the initial–boundary value problem given by (1.1) and

$$(\rho,u,v,w)|_{t=0} = (\rho_0,u_0,v_0,w_0)(x), \quad x \in \Omega := (a,b), \tag{1.2}$$

$$u|_{x=a,b} = 0, \qquad (v,w)|_{x=a} = (v_1,w_1)(t), \qquad (v,w)|_{x=b} = (v_2,w_2)(t). \tag{1.3}$$

The viscosity coefficient is often assumed to be a positive constant. However, it is well known that the viscosity coefficient of the flow is not constant. It is motivated by the physical consideration that in the derivation of the Navier–Stokes equations from the Boltzmann equation through the Chapman–Enskog expansion to the second order, cf. [1, 6], the viscosity coefficient is a function of the temperature. Especially, for isentropic flow, this dependence of the viscosity is translated into the dependence on the density. For more physical background, please refer to [22,23,29] and references therein.

The asymptotic behavior of viscous flows, as the viscosity vanishes, is one of the important topics in the theory of compressible flows, and the problem of small viscosity finds many applications, for example, in the boundary layer theory [14].

Shelukhin studied the zero shear viscosity limit for flows with heat conductivity between two parallel plates in [19,20] and the passage to the limit for a free-boundary problem of describing a joint motion of two compressible fluids with different viscosities, as the shear viscosity of the fluids vanishes in [21].

Frid and Shelukhin in [4] investigated the cylinder isentropic problem (1.1)–(1.3) with constant shear viscosity coefficient μ (i.e. $\theta = 0$). They proved that the problem possessed a unique global strong solution when $\rho_0 \in H^1(\Omega)$, $\inf_\Omega \rho_0 > 0$, $u_0, v_0, w_0 \in H_0^1(\Omega)$, and also investigated the boundary layer effect and proved the existence of a boundary layer thickness (BL-thickness; see Definition 1.1 below) which is close to the value $O(\mu^{\frac{1}{2}})$. For the non-isentropic flows with constant shear viscosity coefficient μ and constant heat conductivity, Frid and Shelukhin in [5] justified the vanishing shear viscosity limit. For the non-isentropic flows with constant shear viscosity coefficient μ and non-constant heat conductivity, Jiang and Zhang in [7] studied the boundary layer effect and convergence rates as the shear viscosity μ goes to zero. Their results showed that the BL-thickness and a convergence rate were of the orders $O(\mu^\alpha)$ with $0 < \alpha < \frac{1}{2}$ and $O(\sqrt{\mu})$, respectively.

But there are no relevant results for the problem with density-dependent viscosity coefficient. The main aim of this paper is to extend Frid and Shelukhin's result in [4] to the case of density-dependent viscosity coefficient. We establish the convergence result as $\varepsilon \to 0$ and also give BL-thickness of the order $O(\varepsilon^\alpha)$, where $0 < \alpha < \frac{1}{2}$ for the constant initial data and $0 < \alpha < \frac{1}{4}$ for the general initial data.

The boundary layers problem has been one of the fundamental and important issues in fluid dynamics. The Prandtl boundary layer (characteristic boundaries) are studied for the linearized case in [26–28] by using asymptotic analysis, while the boundary layer stability in the case of non-characteristic boundaries and one spatial dimension is discussed in [11,16]. For incompressible Navier–Stokes equations, there are a number of papers dedicated to the questions of the boundary layers, see [3,9,12,13,18,24,25] for example. We also mention that the fluids with $\mu = 0$ and $\lambda > 0$ were discussed in [10,15,17].

To formulate the main results of this paper, we need the following notations which will be used.

Notation. From now on, we use the notations $\|(f_1, f_2, \dots, f_n)\|^2 = \|f_1\|^2 + \|f_2\|^2 + \cdots + \|f_n\|^2$ for functions $f_1, f_2, \dots, f_n$ belonging to the same functional space equipped with a norm $\|\cdot\|$. $W^{m,p}(\mathcal{O})$ and $W_0^{m,p}(\mathcal{O})$ ($W^{m,2}(\mathcal{O}) = H^m(\mathcal{O})$, $W_0^{m,2}(\mathcal{O}) = H_0^m(\mathcal{O})$) with $m \in \mathbb{Z}_+$ and $p \geqslant 1$ denote the usual Sobolev spaces defined over $\mathcal{O}$, $W^{0,p}(\mathcal{O}) \equiv L^p(\mathcal{O})$. The symbol $C^k(\mathcal{O})$ (resp. $C^k(\overline{\mathcal{O}})$) with non-negative integer k represents the set of functions having continuous derivatives up to order k in $\mathcal{O}$ (resp. in $\overline{\mathcal{O}}$). For simplicity, we will use the abbreviations:

$$\|\cdot\|_{W^{m,p}} := \|\cdot\|_{W^{m,p}(\Omega)}, \qquad \|\cdot\|_{L^p} := \|\cdot\|_{L^p(\Omega)}.$$

$L^p(I, B)$ (resp. $\|\cdot\|_{L^p(I,B)}$) denotes the space of all strongly measurable, pth-power integrable (essentially bounded if $p = \infty$) functions from I to B (resp. norms), where $I \subset \mathbb{R}$ and B is a Banach space.

Then, our main results can be stated as follows.

Theorem 1.1. *Assume that the initial and boundary data satisfy the conditions*:

$$\begin{aligned}
&(\rho_0, v_0, w_0) \in W^{1,2}(\Omega), \qquad u_0 \in W_0^{1,2} \cap W^{2,2}(\Omega), \qquad \rho_0 > 0,\\
&\|\rho_0^{-1}\|_{C(\bar{\Omega})} < \infty, \qquad \|(v_i, w_i)\|_{C^1([0,T])} < \infty \quad (i = 1, 2),\\
&u_0(a) = u_0(b) = 0, \qquad (v_0, w_0)(a) = (v_1, w_1)(0), \qquad (v_0, w_0)(b) = (v_2, w_2)(0).
\end{aligned}$$

Then there exists a positive constant ε_0, such that for all $\varepsilon \leqslant \varepsilon_0$, we have:

(i) *For any $T > 0$, there exists a unique solution (ρ, u, v, w) of the problem* (1.1)–(1.3) *satisfying*

$$\begin{aligned}
&\rho \in L^\infty(0, T; W^{1,2}), \qquad \rho_t \in L^\infty(0, T; L^2), \qquad \rho > 0,\\
&u \in L^\infty(0, T; W_0^{1,2} \cap W^{2,2}), \qquad u_t \in L^\infty(0, T; L^2) \cap L^2(0, T; W^{1,2}),
\end{aligned}$$

where the norms are all uniform in ε. There also exists a positive constant C independent of ε, such that $|(v, w)(x, t)| \leqslant C$ for all $(x, t) \in [a, b] \times [0, T]$, and

$$\varepsilon^{\frac{1}{2}} \|(v_x, w_x)\|^2_{L^\infty(0,T;L^2)} + \varepsilon^{\frac{3}{2}} \|(v_{xx}, w_{xx})\|^2_{L^2(0,T;L^2)} \leqslant C.$$

(ii) *There exist functions $(\bar{\rho}, \bar{u}, \bar{v}, \bar{w})$ such that as $\varepsilon \to 0$,*

$$\begin{aligned}
&(\rho, u) \to (\bar{\rho}, \bar{u}) \quad \textit{strongly in } C^\alpha(\bar{Q}_T) \textit{ for any } \alpha \in \left(0, \frac{1}{2}\right),\\
&(u_x, v, w) \to (\bar{u}_x, \bar{v}, \bar{w}) \quad \textit{strongly in } L^{p_1}(Q_T) \textit{ for any } p_1 \in [1, \infty),\\
&(\rho_x, \rho_t, u_{xx}, u_t) \rightharpoonup (\bar{\rho}_x, \bar{\rho}_t, \bar{u}_{xx}, \bar{u}_t) \quad \textit{weakly-}* \textit{ in } L^\infty(0, T; L^2),\\
&(\varepsilon v_x^2, \varepsilon w_x^2) \to (0, 0) \quad \textit{strongly in } L^{p_2}(Q_T) \textit{ for any } p_2 \in [1, 2],
\end{aligned}$$

where $Q_T = (a, b) \times (0, T)$.

Furthermore, the limit functions $(\bar{\rho}, \bar{u}, \bar{v}, \bar{w})$ solve the problem (1.1)–(1.3) *with $\varepsilon = 0$ in the following sense*:

$$\int_{Q_T} x\bar{\rho}(\psi_t + \bar{u}\psi_x)\,dx\,dt + \int_{\Omega} x\rho_0\psi(x, 0)\,dx = 0, \tag{1.4}$$

$$\int_{Q_T} x\left[\bar\rho\bar u\phi_t + \left(\bar\rho\bar u^2 + \bar\rho^\gamma - \lambda\bar u_x - \lambda\frac{\bar u}{x}\right)\phi_x\right]dx\,dt$$
$$+ \int_{Q_T}\left(\bar\rho\bar v^2 + \bar\rho^\gamma - \lambda\bar u_x - \lambda\frac{\bar u}{x}\right)\phi\,dx\,dt + \int_\Omega x\rho_0 u_0\phi(x,0)\,dx = 0, \tag{1.5}$$

$$\int_{Q_T} x\bar\rho\bar v\left(\phi_t + \bar u\phi_x - \frac{\bar u}{x}\phi\right)dx\,dt + \int_\Omega x\rho_0 v_0\phi(x,0)\,dx = 0, \tag{1.6}$$

$$\int_{Q_T} x\bar\rho\bar w(\phi_t + \bar u\phi_x)\,dx\,dt + \int_\Omega x\rho_0 w_0\phi(x,0)\,dx = 0, \tag{1.7}$$

for any text functions $\phi, \psi \in C^1(\bar Q_T)$ with $\phi(x,T) = \psi(x,T) = 0$ for all $x \in \Omega$, and $\phi(a,t) = \phi(b,t) = 0$ for all $t \in (0,T)$.

Next, we state the result about the boundary layer effect. At first, let us give the definition of BL-thickness, which mainly comes from [4].

Definition 1.1. A function $\delta(\varepsilon)$ is called a BL-thickness for the problem (1.1)–(1.3) with vanishing shear viscosity μ (in fact, vanishing ε implies vanishing shear viscosity μ), if $\delta(\varepsilon) \downarrow 0$ as $\varepsilon \downarrow 0$, and

$$(\rho, u) \to (\bar\rho, \bar u) \quad \text{strongly in } C^\alpha(\bar Q_T) \text{ for any } \alpha \in \left(0, \frac{1}{2}\right), \text{ as } \varepsilon \to 0, \tag{1.8}$$

$$\lim_{\varepsilon\to 0}\left\|(v - \bar v, w - \bar w)\right\|_{L^\infty(0,T;L^\infty(\Omega_{\delta(\varepsilon)}))} = 0, \tag{1.9}$$

$$\liminf_{\varepsilon\to 0}\left\|(v - \bar v, w - \bar w)\right\|_{L^\infty(0,T;L^\infty(\Omega))} > 0, \tag{1.10}$$

where $\Omega_\delta := \{x \in \Omega \mid a+\delta < x < b-\delta\}$, and (ρ, u, v, w) (resp. $(\bar\rho, \bar u, \bar v, \bar w)$) is the solution to the problem (1.1)–(1.3) (resp. to (1.4)–(1.7)).

Clearly, this definition does not determine the BL-thickness uniquely, since any function $\delta_*(\varepsilon)$ satisfying the inequality $\delta_*(\varepsilon) \geqslant \delta(\varepsilon)$ for small ε is also a BL-thickness.

Theorem 1.2 *(BL-thickness for the constant initial data). Under the conditions of Theorem* 1.1*, we assume in addition that the initial data satisfy $(\rho, u, v, w)|_{t=0} = (\rho_0, 0, 0, 0)$ where the initial density ρ_0 is positive constant. Then the limit problem* (1.4)–(1.7) *has only trivial solution $(\bar\rho, \bar u, \bar v, \bar w) = (\rho_0, 0, 0, 0)$, and any function $\delta(\varepsilon)$, satisfying the conditions $\delta(\varepsilon) \to 0$ and $\varepsilon^{\frac{1}{2}}/\delta(\varepsilon) \to 0$ as $\varepsilon \to 0$, is a BL-thickness such that*

$$(\rho, u) \to (\rho_0, 0) \quad \textit{strongly in } C^\alpha(\bar Q_T) \textit{ for any } \alpha \in \left(0, \frac{1}{2}\right), \textit{ as } \varepsilon \to 0,$$

$$\lim_{\varepsilon\to 0}\left\|(v, w)\right\|_{L^\infty(0,T;L^\infty(\Omega_{\delta(\varepsilon)}))} = 0,$$

and

$$\liminf_{\varepsilon\to 0}\left\|(v, w)\right\|_{L^\infty(0,T;L^\infty(\Omega))} > 0,$$

whenever the boundary conditions v_i and w_i $(i = 1, 2)$ are not identical zero.

Motivated by the ideas of [7], we can give some results about the BL-thickness for the general initial data under some additional initial assumptions. The main result is as follows:

Theorem 1.3 *(BL-thickness for the general initial data). Let the assumptions given in Theorem* 1.1 *hold. Assume in addition that $(v_0, w_0) \in H^3(\Omega)$ and $\rho_0 \in H^2(\Omega)$. Then, any function $\delta(\varepsilon)$, satisfying the conditions $\delta(\varepsilon) \to 0$*

and $\varepsilon^{\frac{1}{4}}/\delta(\varepsilon) \to 0$*, as* $\varepsilon \to 0$*, is a BL-thickness such that* (1.8)–(1.10) *hold, whenever the boundary conditions* (v_1, w_1, v_2, w_2) *are not identical* $(\bar{v}(a,t), \bar{w}(a,t), \bar{v}(b,t), \bar{w}(b,t))$.

Remark 1.1. When $\theta = 0$, all the results in Theorems 1.1, 1.2 and 1.3 hold without the restriction $\varepsilon \leqslant \varepsilon_0$ (we can choose the constant ε_0 as in (2.21)), which can be proved by using the ideas in [4,7]. So, in this paper, we only consider the case when $0 < \theta \leqslant \gamma$.

Remark 1.2. The above results also hold for the case when $\gamma = 1$, where $\frac{\rho^\gamma}{\gamma-1}$ in (2.2) is replaced by $(1 - \rho + \rho \ln \rho)$, which is positive for $\rho > 0$. We don't refer this case in the present paper, see [4] for details.

The difficulty of this problem is to obtain the upper and lower bounds of the density ρ, especially for the lower bound. When ε is small enough in some sense, using some new *a priori* estimates for the solution, we can obtain the bounds of the density. The key ideas are using the classical continuity method and the following *a priori* assumption:

$$\sup_{t\in[0,T]} \varepsilon^2 \int_a^b \left(\rho^\theta\right)_x^2 dx \leqslant 1, \quad \text{for any } T > 0.$$

In Lemmas 2.2–2.4, we get some uniform *a priori* estimates (with respect to ε) for the solution. Then, using these estimates and the smallness restriction on ε, we can obtain

$$\sup_{t\in[0,T]} \varepsilon^2 \int_a^b \left(\rho^\theta\right)_x^2 dx \leqslant \frac{1}{2}, \quad \text{for any } T > 0.$$

Using the classical continuity method, we can close this estimate.

Compared to [4,7], there are some other difficulties in our paper. Firstly, we have to seek a new method to estimate the bound of ρ because of the effect of density-dependent shear viscosity μ (see Lemma 2.2). Secondly, in order to obtain the bound of v, we must deal with $\int_a^b \frac{v^{2N}}{x^{2N}}\,dx$ instead of $\int_a^b v^{2N}\,dx$ as in [7], where $N \in \mathbb{N}$. Finally, when we want to get the BL-thickness results, we have to obtain the estimates:

$$\delta \int_{a+\delta}^{b-\delta} \left| \left(\frac{v}{x}\right)_x \right| dx \leqslant C\varepsilon^{\frac{1}{2}} \quad \big(\text{see (3.16) for the constant initial data } (\rho_0, 0, 0, 0)\big), \tag{1.11}$$

or

$$\delta \int_{a+\delta}^{b-\delta} \left| \left(\frac{v - \bar{v}}{x}\right)_x \right| dx \leqslant C\varepsilon^{\frac{1}{4}} \quad \big(\text{see (3.40) for the general initial data}\big), \tag{1.12}$$

instead of estimating in [7]

$$\delta \int_{a+\delta}^{b-\delta} |v_x|\,dx \leqslant C\varepsilon^{\frac{1}{2}}, \tag{1.13}$$

or

$$\delta \int_{a+\delta}^{b-\delta} \left|(v - \bar{v})_x\right| dx \leqslant C\varepsilon^{\frac{1}{4}}. \tag{1.14}$$

In fact, as in [7], when we estimate (1.13) and (1.14) directly, we will encounter the terms like

$$\int_0^t \int_a^b \frac{\mu_x v}{\rho x} z_x \varphi_v'' \xi_\delta(x)\,dx\,d\tau, \qquad z = v_x, \tag{1.15}$$

because of the effect of density-dependent shear viscosity μ. However, we have no method to estimate (1.15).

The paper is organized as follows. We first derive some *a priori* estimates in Section 2. Then, at the beginning of Section 3, we illustrate the proof of Theorem 1.1 concisely, since the process of it is similar to that in [2,4,5,19,20]. Next, in Section 3.1, we apply uniform estimates in Section 2 to prove Theorem 1.2 by adapting and modifying the techniques used in [4,7]. Finally, in Section 3.2, we give some results on the convergence rate (w.r.t. ε) and the BL-thickness for the case of general initial data under certain additional initial conditions and this completes the proof of Theorem 1.3.

We should mention that the methods introduced by Frid and Shelukhin in [4,5] and Jiang and Zhang in [7] will play a crucial role in our proof here.

2. *A priori* estimates

This section is devoted to the derivation of *a priori* estimates uniform in ε for the solution to the problem (1.1)–(1.3). For simplicity, throughout the rest of this paper we shall denote by C the various positive constants dependent on the initial data, γ, λ and T, but independent of ε.

At first, we give the basic energy estimates.

Lemma 2.1. *For any $t \in (0,T)$, we have*

$$\int_a^b x\rho(x,t)\,dx = \int_a^b x\rho_0(x)\,dx > 0, \tag{2.1}$$

and

$$\int_a^b x\left\{\rho\left(u^2+v^2+w^2\right)+\frac{\rho^\gamma}{\gamma-1}\right\}dx + \int_0^t\int_a^b x\left\{(2\mu+\lambda)\left(\frac{u^2}{x^2}+u_x^2\right)+\mu\left(v_x-\frac{v}{x}\right)^2+\mu w_x^2\right\}dx\,d\tau \leqslant C. \tag{2.2}$$

Proof. Firstly, it follows that (2.1) holds from $(1.1)_1$ and boundary conditions (1.3).

Secondly, rewrite (1.1) as

$$\begin{aligned}&\left\{\frac{1}{2}\rho\left(u^2+v^2+w^2\right)x+\frac{\rho^\gamma}{\gamma-1}x\right\}_t+\left\{\frac{1}{2}x\rho u\left(u^2+v^2+w^2\right)+\frac{\gamma}{\gamma-1}\rho^\gamma ux\right\}_x\\&\quad+\left\{(2\mu+\lambda)\left(u_x^2+\frac{u^2}{x^2}\right)+\mu\left(v_x-\frac{v}{x}\right)^2+\mu w_x^2\right\}x\\&=\left\{(2\mu+\lambda)u_xux\right\}_x+\{\mu vv_xx\}_x-\left\{\mu v^2\right\}_x+\{\mu ww_xx\}_x.\end{aligned} \tag{2.3}$$

Integrating (2.3) with respect to (x,t) over $(a,b)\times(0,t)$ and using the boundary conditions $u(a,t)=u(b,t)=0$, we have

$$\begin{aligned}&\int_a^b x\left\{\frac{1}{2}\rho\left(u^2+v^2+w^2\right)+\frac{\rho^\gamma}{\gamma-1}\right\}dx+\int_0^t\int_a^b x\left\{(2\mu+\lambda)\left(u_x^2+\frac{u^2}{x^2}\right)+\mu\left(v_x-\frac{v}{x}\right)^2+\mu w_x^2\right\}dx\,d\tau\\&=\int_a^b x\left\{\frac{1}{2}\rho_0\left(u_0^2+v_0^2+w_0^2\right)+\frac{\rho_0^\gamma}{\gamma-1}\right\}dx+\int_0^t \mu vv_xx|_{x=a}^b\,d\tau-\int_0^t \mu v^2|_{x=a}^b\,d\tau+\int_0^t \mu ww_xx|_{x=a}^b\,d\tau.\end{aligned} \tag{2.4}$$

On the other hand, integrating Eqs. $(1.1)_3$–$(1.1)_4$ over (a,x) and (x,b), respectively, we have by using the boundary conditions (1.3) that

$$\begin{aligned}\mu v v_x x|_{x=a} ={}& \frac{a}{b-a}v_1(t)\int_a^b \mu\left(v_x-\frac{v}{x}\right)dx-\frac{a}{b-a}v_1(t)\frac{d}{dt}\int_a^b\int_a^x \rho v\,dy\,dx\\ &-\frac{2av_1(t)}{b-a}\int_a^b\int_a^x\frac{1}{y}\rho uv\,dy\,dx-\frac{av_1(t)}{b-a}\int_a^b \rho uv\,dx\\ &+\frac{2a}{b-a}v_1(t)\int_a^b\int_a^x \mu\left(v_x-\frac{v}{y}\right)\frac{1}{y}\,dy\,dx+\left(\mu v^2\right)\big|_{x=a},\end{aligned} \tag{2.5}$$

$$\begin{aligned}\mu v v_x x|_{x=b} ={}& \frac{b}{b-a}v_2(t)\int_a^b \mu\left(v_x-\frac{v}{x}\right)dx+\frac{b}{b-a}v_2(t)\frac{d}{dt}\int_a^b\int_x^b \rho v\,dy\,dx\\ &+\frac{2bv_2(t)}{b-a}\int_a^b\int_x^b\frac{1}{y}\rho uv\,dy\,dx-\frac{bv_2(t)}{b-a}\int_a^b \rho uv\,dx\\ &-\frac{2b}{b-a}v_2(t)\int_a^b\int_x^b \mu\left(v_x-\frac{v}{y}\right)\frac{1}{y}\,dy\,dx+\left(\mu v^2\right)\big|_{x=b},\end{aligned} \tag{2.6}$$

$$\begin{aligned}\mu w w_x x|_{x=a} ={}& \frac{1}{b-a}\Bigg\{aw_1(t)\int_a^b \mu w_x\,dx-aw_1(t)\frac{d}{dt}\int_a^b\int_a^x \rho w\,dy\,dx-aw_1(t)\int_a^b\int_a^x\frac{1}{y}\rho uw\,dy\,dx\\ &-aw_1(t)\int_a^b \rho uw\,dx+aw_1(t)\int_a^b\int_a^x\frac{1}{y}\mu w_x\,dy\,dx\Bigg\},\end{aligned} \tag{2.7}$$

and

$$\begin{aligned}\mu w w_x x|_{x=b} ={}& \frac{1}{b-a}\Bigg\{bw_2(t)\int_a^b \mu w_x\,dx+bw_2(t)\frac{d}{dt}\int_a^b\int_x^b \rho w\,dy\,dx+bw_2(t)\int_a^b\int_x^b\frac{1}{y}\rho uw\,dy\,dx\\ &-bw_2(t)\int_a^b \rho uw\,dx-bw_2(t)\int_a^b\int_x^b\frac{1}{y}\mu w_x\,dy\,dx\Bigg\}.\end{aligned} \tag{2.8}$$

Substituting (2.5)–(2.8) into (2.4), we have

$$\begin{aligned}&\int_a^b x\left\{\frac{1}{2}\rho\left(u^2+v^2+w^2\right)+\frac{\rho^\gamma}{\gamma-1}\right\}dx\\ &\quad+\int_0^t\int_a^b x\left\{(2\mu+\lambda)\left(u_x^2+\frac{u^2}{x^2}\right)+\mu\left(v_x-\frac{v}{x}\right)^2+\mu w_x^2\right\}dx\,d\tau\\ &\leqslant C+C\int_0^t\int_a^b x\mu\left(v_1^2(\tau)+v_2^2(\tau)\right)dx\,d\tau+\frac{1}{2}\int_0^t\int_a^b \mu\left(v_x-\frac{v}{x}\right)^2 x\,dx\,d\tau\end{aligned}$$

$$
\begin{aligned}
&+C\int_0^t\int_a^b \rho v^2\,dx\,d\tau + C\int_0^t\int_a^b \rho\big(v_{1t}^2(\tau)+v_{2t}^2(\tau)\big)x\,dx\,d\tau \\
&+\frac{1}{4}\int_a^b \rho\big(u^2+v^2+w^2\big)x\,dx\,d\tau + C\int_a^b \rho\big(v_1^2(t)+v_2^2(t)+w_1^2(t)+w_2^2(t)\big)x\,dx \\
&+C\int_0^t\int_a^b \big(\big|v_1(\tau)\big|+\big|v_2(\tau)\big|\big)\rho\big(u^2+v^2\big)x\,dx\,d\tau + \frac{1}{2}\int_0^t\int_a^b \mu w_x^2 x\,dx\,d\tau \\
&+C\int_0^t\int_a^b x\mu\big(w_1^2(\tau)+w_2^2(\tau)\big)\,dx\,d\tau + C\int_0^t\int_a^b \rho w^2 x\,dx\,d\tau \\
&+C\int_0^t\int_a^b \rho\big(w_{1t}^2(\tau)+w_{2t}^2(\tau)\big)x\,dx\,d\tau + C\int_0^t\int_a^b \big(\big|w_1(\tau)\big|+\big|w_2(\tau)\big|\big)\rho\big(u^2+w^2\big)x\,dx\,d\tau \\
\leqslant{}& C+\frac{1}{2}\int_0^t\int_a^b \mu\left(v_x-\frac{v}{x}\right)^2 x\,dx\,d\tau + \frac{1}{2}\int_0^t\int_a^b \mu w_x^2 x\,dx\,d\tau + \frac{1}{4}\int_a^b \rho\big(u^2+v^2+w^2\big)x\,dx\,d\tau \\
&+C\int_0^t\int_a^b \rho\big(u^2+v^2+w^2\big)x\,dx\,d\tau + C\int_0^t\int_a^b x\rho^\gamma\,dx\,d\tau.
\end{aligned}
$$

Here we have used the Cauchy–Schwartz inequality, the Young inequality and $0<\theta\leqslant\gamma$.

By Gronwall's inequality, we deduce (2.2). This completes the proof of Lemma 2.1. □

In order to estimate the upper and lower bounds of ρ, we suppose the following *a priori* assumption holds:

$$
\sup_{t\in[0,T]} \varepsilon^2\int_a^b \big(\rho^\theta\big)_x^2\,dx \leqslant 1. \tag{2.9}
$$

Lemma 2.2. *Under the assumptions of Theorem* 1.1 *and the a priori assumption* (2.9)*, if $\varepsilon\leqslant\varepsilon_0<1$, then there exists a positive constant C, such that*

$$
C^{-1}\leqslant\rho(x,t)\leqslant C,\quad \textit{for all } (x,t)\in Q_T. \tag{2.10}
$$

Proof. For any fixed point $(x_0,t_0)\in Q_T$, we want to obtain the bound of ρ at a fixed point (x_0,t_0). First, by (2.1) and the assumptions in Theorem 1.1, there exists $x_1=x_1(t_0)\in(a,b)$ such that $(b-a)x_1\rho(x_1,t_0)=\int_a^b x\rho(x,t_0)\,dx=\int_a^b x\rho_0(x)\,dx$, which implies that there exists a positive constant C such that

$$
C^{-1}\leqslant\rho(x_1,t_0)\leqslant C. \tag{2.11}
$$

Let $L(x,t)=\frac{2\varepsilon}{\theta}\rho^\theta+\lambda\ln\rho$. We shall examine the difference $L(x_1,t)-L(x_0,t)$ by considering its evolution along particle trajectories. Define $X_j(t)$ by

$$
\frac{d}{dt}X_j=u(X_j,t),\qquad X_j(t_0)=x_j,\quad j=0,1,
$$

and let $\Delta L(t)=L(X_1(t),t)-L(X_0(t),t)$. We then obtain from (1.1) that

$$\frac{d}{dt}\Delta L(t) = -(2\mu+\lambda)\left(\frac{u}{x}+u_x\right)\Big|_{X_0}^{X_1} = -\int_{X_0}^{X_1}\left(\rho u_t + \rho u u_x - \frac{\rho v^2}{x} + p_x + 2\mu_x\frac{u}{x}\right)dx$$
$$= -\frac{d}{dt}I(t) - \Delta p - \int_{X_0}^{X_1}\frac{\rho}{x}\left(u^2-v^2\right)dx - \int_{X_0}^{X_1}2\mu_x\frac{u}{x}\,dx, \tag{2.12}$$

where $I(t)=\int_{X_0}^{X_1}\rho u\,dx$, and $\Delta p = p(\rho(\cdot,t))|_{X_0}^{X_1}$. Now define $\alpha(t)=\frac{\Delta p(t)}{\Delta L(t)}$. It is easy to see that $\alpha(t)\geqslant 0$ by the representations of L and p.

Rewrite (2.12) as the forms of a linear ODE

$$\frac{d}{dt}\Delta L(t) + \alpha(t)\Delta L(t) = -\frac{d}{dt}I(t) - \int_{X_0}^{X_1}\frac{\rho}{x}\left(u^2-v^2\right)dx - \int_{X_0}^{X_1}2\mu_x\frac{u}{x}\,dx,$$

whose solution is

$$\Delta L(t) = e^{-\int_0^t\alpha(\tau)\,d\tau}\big(\Delta L(0)+I(0)\big) - I(t)$$
$$+\int_0^t e^{-\int_s^t\alpha(\tau)\,d\tau}\left\{\alpha(s)I(s) - \int_{X_0}^{X_1}\frac{\rho}{x}\left(u^2-v^2\right)dx - 2\int_{X_0}^{X_1}\mu_x\frac{u}{x}\,dx\right\}ds, \tag{2.13}$$

which implies

$$\left|\Delta L(t)\right| \leqslant \left|\Delta L(0)\right| + \left|I(0)\right| + \left|I(t)\right|$$
$$+\int_0^t e^{-\int_s^t\alpha(\tau)\,d\tau}\left\{\alpha(s)\left|I(s)\right| + \left|\int_{X_0}^{X_1}\left(\frac{\rho}{x}\left(u^2-v^2\right)+2\mu_x\frac{u}{x}\right)dx\right|\right\}ds. \tag{2.14}$$

Now we estimate the third term on the right-hand side of (2.14) as follows:

$$\left|I(t)\right| = \left|\int_{X_0}^{X_1}\rho u\,dx\right| \leqslant \int_{X_0}^{X_1}|\rho u|\,dx \leqslant \int_{X_0}^{X_1}\rho u^2\,dx + C\int_{X_0}^{X_1}x\rho\,dx \leqslant C,$$

where we have used Lemma 2.1.

Substituting the above estimate into (2.14), we have

$$\left|\Delta L(t)\right| \leqslant C + C\int_0^t e^{-\int_s^t\alpha(\tau)\,d\tau}\alpha(s)\,ds + \int_0^t\left|\int_{X_0}^{X_1}\left(\frac{\rho}{x}\left(u^2-v^2\right)+2\mu_x\frac{u}{x}\right)dx\right|d\tau. \tag{2.15}$$

By Lemma 2.1 and the *a priori* assumption (2.9), we have

$$\int_0^t e^{-\int_s^t\alpha(\tau)\,d\tau}\alpha(s)\,ds = 1 - e^{-\int_0^t\alpha(\tau)\,d\tau} \leqslant 1,$$

and

$$\int_0^t\left|\int_{X_0}^{X_1}\left(\frac{\rho}{x}\left(u^2-v^2\right)+2\mu_x\frac{u}{x}\right)dx\right|d\tau$$
$$\leqslant C\int_0^t\int_{X_0}^{X_1}\left(\rho u^2+\rho v^2\right)dx\,d\tau + C\int_0^t\int_{X_0}^{X_1}\varepsilon^2\left(\rho^\theta\right)_x^2\,dx\,d\tau + C\int_0^t\int_{X_0}^{X_1}\frac{u^2}{x^2}\,dx\,d\tau \leqslant C.$$

Thus (2.15) shows

$$\left|\Delta L(t_0)\right| = \left|\frac{2\varepsilon}{\theta}\rho^\theta(x_1,t_0) + \lambda\ln\rho(x_1,t_0) - \frac{2\varepsilon}{\theta}\rho^\theta(x_0,t_0) - \lambda\ln\rho(x_0,t_0)\right| \leqslant C,$$

which implies the pointwise bound (2.10) follows from this and (2.11) under the assumption $\varepsilon \leqslant \varepsilon_0 < 1$. □

Lemma 2.3. *There is a positive constant C independent of ε, such that*

$$-C \leqslant v(x,t), w(x,t) \leqslant C \quad \textit{for any } (x,t) \in \bar{Q}_T. \tag{2.16}$$

Proof. Multiplying $(1.1)_4$ by $2Nxw^{2N-1}$ with $N \in \mathbb{N}$, we have

$$\left(x\rho w^{2N}\right)_t + \left(x\rho u w^{2N}\right)_x = 2Nx(\mu w_x)_x w^{2N-1} + 2N\mu w^{2N-1} w_x.$$

Integrating the above equality over $(a,b)\times(0,t)$, we get

$$\int_a^b x\rho w^{2N}\,dx = \int_a^b x\rho_0 w_0^{2N}\,dx - 2N(2N-1)\int_0^t\int_a^b x\mu w_x^2 w^{2N-2}\,dx\,d\tau + 2N\int_0^t \left(x\mu w_x w^{2N-1}\right)\Big|_{x=a.}^{x=b} \tag{2.17}$$

We can use Lemmas 2.1, 2.2, (2.7) and (2.8) to deal with the boundary terms in (2.17), which can be controlled by CN^2C^{2N} through simple calculation. Then we get

$$\int_a^b x\rho w^{2N}\,dx + 2N(2N-1)\int_0^t\int_a^b x\mu w_x^2 w^{2N-2}\,dx\,d\tau \leqslant \int_a^b x\rho_0 w_0^{2N}\,dx + CN^2C^{2N} \leqslant CN^2C^{2N},$$

where C is a positive constant, independent of N, ε. Now, we raise to the power $\frac{1}{2N}$ for both sides of the above inequality and let $N\to\infty$ to deduce that $\|w(t)\|_{L^\infty} \leqslant C$.

Next, we prove $\|v(t)\|_{L^\infty} \leqslant C$. Multiplying $(1.1)_3$ by $2Nx^{-2N}v^{2N-1}$ with $N\in\mathbb{N}$, we have

$$\left(x^{-2N}\rho v^{2N}\right)_t + \left(x^{-2N}\rho u v^{2N}\right)_x + (4N+1)\rho u v^{2N} x^{-2N-1} = 2Nx^{-2N}\left(\mu\left(v_x + \frac{v}{x}\right)\right)_x v^{2N-1} - 4Nx^{-2N}v^{2N-1}\mu_x\frac{v}{x}.$$

Integrating the above equality over $(a,b)\times(0,t)$ and integrating by parts, we get

$$\begin{aligned}\int_a^b x^{-2N}\rho v^{2N}\,dx &= \int_a^b x^{-2N}\rho_0 v_0^{2N}\,dx - (4N+1)\int_0^t\int_a^b x^{-2N-1}\rho u v^{2N}\,dx\,d\tau \\ &\quad - 2N(2N-1)\int_0^t\int_a^b x^{-2N}\mu v^{2N-2}\left(v_x - \frac{4N+1}{4N-2}\frac{v}{x}\right)^2 dx\,d\tau \\ &\quad + \frac{9N}{4N-2}\int_0^t\int_a^b x^{-2N-2}\mu v^{2N}\,dx\,d\tau \\ &\quad + 2N\int_0^t \left\{x^{-2N}v^{2N-1}\mu\left(v_x - \frac{v}{x}\right)\right\}\Big|_{x=a}^{x=b} d\tau.\end{aligned} \tag{2.18}$$

We can use Lemmas 2.1, 2.2, (2.5) and (2.6) to deal with the boundary terms in (2.18), which also can be controlled by CN^2C^{2N}.

By (2.18) and Lemma 2.2, we get

$$\int_a^b \frac{v^{2N}}{x^{2N}}\,dx \leqslant CN^2C^{2N} + CN\left(\int_0^t \left(1+\left\|u(\tau)\right\|_{L^\infty}\right)\int_a^b \frac{v^{2N}}{x^{2N}}\,dx\,d\tau\right)$$
$$\leqslant CN^2C^{2N} + CN\left(\int_0^t \left(1+\left\|u_x(\tau)\right\|_{L^2}\right)\int_a^b \frac{v^{2N}}{x^{2N}}\,dx\,d\tau\right),$$

where we have used the Cauchy–Schwartz inequality and Sobolev's inequality $\|u(t)\|_{L^\infty} \leqslant \|u_x(t)\|_{L^2} \in L^1(0,T)$. Therefore, Gronwall's inequality yields

$$\int_a^b \frac{v^{2N}}{x^{2N}}\,dx \leqslant CN^2C^{2N}\exp\{CN\},$$

where C is independent of N, ε. Now, we raise to the power $\frac{1}{2N}$ for both sides of the above inequality and let $N\to\infty$, then we get $\|v(t)\|_{L^\infty} \leqslant C$.

This proves Lemma 2.3. □

Lemma 2.4. *For any $t \in (0,T)$, the following estimate holds*

$$\int_a^b \rho_x^2(x,t)\,dx \leqslant C. \tag{2.19}$$

Proof. From $(1.1)_1$ and $(1.1)_2$, we have

$$u_x + \frac{u}{x} = -\frac{\rho_t + u\rho_x}{\rho},$$

and

$$\left[\rho\left(\frac{2\mu+\lambda}{\rho^2}\rho_x + u\right)\right]_t + \left[\rho u\left(\frac{2\mu+\lambda}{\rho^2}\rho_x + u\right)\right]_x + p_x + 2\mu_x\frac{u}{x} + \frac{\rho}{x}\left(u^2-v^2\right) = 0.$$

Multiplying the above equation by $(\frac{2\mu+\lambda}{\rho^2}\rho_x + u)$ and integrating the resulting equation over (a,b), we can get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_a^b \rho\left(\frac{2\mu+\lambda}{\rho^2}\rho_x + u\right)^2 dx + \gamma\int_a^b (2\mu+\lambda)\rho^{\gamma-3}\rho_x^2\,dx \\
&\quad = -\gamma\int_a^b \rho^{\gamma-1}\rho_x u\,dx + \frac{1}{2}\int_a^b \frac{\rho u}{x}\left(\frac{2\mu+\lambda}{\rho^2}\rho_x + u\right)^2 dx \\
&\qquad - \int_a^b \frac{\rho}{x}\left(u^2-v^2\right)\left(\frac{2\mu+\lambda}{\rho^2}\rho_x + u\right)dx - 2\varepsilon\int_a^b \left(\rho^\theta\right)_x\frac{u}{x}\left(\frac{2\mu+\lambda}{\rho^2}\rho_x + u\right)dx.
\end{aligned} \tag{2.20}$$

The first three terms on the right-hand side of (2.20) can be estimated by Lemmas 2.1–2.3 and the Cauchy–Schwartz inequality as follows:

$$\left|\gamma\int_a^b \rho^{\gamma-1}\rho_x u\,dx\right| \leqslant \frac{1}{2}\gamma\int_a^b \lambda\rho^{\gamma-3}\rho_x^2\,dx + C\int_a^b \rho u^2\,dx \leqslant \frac{1}{2}\gamma\int_a^b \lambda\rho^{\gamma-3}\rho_x^2\,dx + C,$$

and

$$\left|\frac{1}{2}\int_a^b \frac{\rho u}{x}\left(\frac{2\mu+\lambda}{\rho^2}\rho_x+u\right)^2 dx\right|+\left|\int_a^b \frac{\rho}{x}\left(u^2-v^2\right)\left(\frac{2\mu+\lambda}{\rho^2}\rho_x+u\right)dx\right|$$
$$\leqslant C\|u(t)\|_{L^\infty}\int_a^b \rho\left(\frac{2\mu+\lambda}{\rho^2}\rho_x+u\right)^2 dx+C\int_a^b \rho\left(\frac{2\mu+\lambda}{\rho^2}\rho_x+u\right)^2 dx+C\int_a^b \rho\left(u^2-v^2\right)^2 dx$$
$$\leqslant C\left(1+\int_a^b u_x^2\,dx\right)\int_a^b \rho\left(\frac{2\mu+\lambda}{\rho^2}\rho_x+u\right)^2 dx+C+C\int_a^b u_x^2\,dx,$$

where we have also used Sobolev's inequality $\|u(t)\|_{L^\infty}\leqslant\|u_x(t)\|_{L^2}\in L^1(0,T)$.

And we can bound the last term on the right-hand side of (2.20) by using the *a priori* assumption (2.9):

$$\left|2\varepsilon\int_a^b \left(\rho^\theta\right)_x\frac{u}{x}\left(\frac{2\mu+\lambda}{\rho^2}\rho_x+u\right)dx\right|\leqslant C\int_a^b \rho u^2\left(\frac{2\mu+\lambda}{\rho^2}\rho_x+u\right)^2 dx+C\varepsilon^2\int_a^b \left(\rho^\theta\right)_x^2 dx$$
$$\leqslant C+C\int_a^b u_x^2\,dx\int_a^b \rho\left(\frac{2\mu+\lambda}{\rho^2}\rho_x+u\right)^2 dx.$$

Hence, inserting the above three estimates into (2.20) and using Lemmas 2.1–2.3, and applying Gronwall's inequality, we obtain (2.19).

The proof of Lemma 2.4 is completed. □

Now we prove that the *a priori* assumption (2.9) is closed. From Lemmas 2.2 and 2.4, if we choose

$$C\varepsilon_0^2\leqslant\frac{1}{2},\qquad \varepsilon_0<1, \tag{2.21}$$

then we have

$$\varepsilon^2\int_a^b \left(\rho^\theta\right)_x^2 dx=\theta^2\varepsilon^2\int_a^b \rho^{2\theta-2}\rho_x^2\,dx\leqslant C\varepsilon^2\leqslant C\varepsilon_0^2\leqslant\frac{1}{2},$$

and this closes the *a priori* assumption (2.9).

Lemma 2.5. *For any $t\in(0,T)$, we have*

$$\|u\|_{L^\infty(0,T;L^\infty(\Omega))}\leqslant C,$$

and

$$\int_a^b u_x^2\,dx+\int_0^t\int_a^b \left(u_t^2+u_{xx}^2\right)dx\,d\tau\leqslant C. \tag{2.22}$$

Proof. Rewrite $(1.1)_2$ as

$$\rho u_t+\rho u u_x-(2\mu+\lambda)u_{xx}=(2\mu+\lambda)\left(\frac{u_x}{x}-\frac{u}{x^2}\right)+\frac{\rho v^2}{x}-p_x+2\mu_x u_x,$$

and

$$\left(\rho u_t-(2\mu+\lambda)u_{xx}\right)^2\rho^{-1}=\rho^{-1}\left\{(2\mu+\lambda)\left(\frac{u_x}{x}-\frac{u}{x^2}\right)-\rho u u_x+\frac{\rho v^2}{x}-\gamma\rho^{\gamma-1}\rho_x+2\mu_x u_x\right\}^2,$$

which implies

$$\rho u_t^2+\frac{(2\mu+\lambda)^2}{\rho}u_{xx}^2-2\big((2\mu+\lambda)u_xu_t\big)_x+\frac{d}{dt}\big[(2\mu+\lambda)u_x^2\big]-2\mu_tu_x^2+4\mu_xu_xu_t$$
$$=\rho^{-1}\left\{(2\mu+\lambda)\left(\frac{u_x}{x}-\frac{u}{x^2}\right)-\rho uu_x+\frac{\rho v^2}{x}-\gamma\rho^{\gamma-1}\rho_x+2\mu_xu_x\right\}^2.$$

By using Lemmas 2.1–2.4 and the Cauchy–Schwartz inequality, we have

$$\begin{aligned}
&\int_0^t\int_a^b\big(u_t^2+u_{xx}^2\big)\,dx+\int_a^b u_x^2\,dx\\
&\quad\leqslant C+C\int_0^t\int_a^b\big\{|\mu_t|u_x^2+|\mu_xu_xu_t|+u^2u_x^2+(2\mu+\lambda)^2\big(u_x^2+u^2\big)+\mu_x^2u_x^2\big\}\,dx\,d\tau\\
&\quad\leqslant C+C\int_0^t\int_a^b\big\{\varepsilon\rho^{\theta-1}|\rho_t|u_x^2+\varepsilon\big|\big(\rho^\theta\big)_xu_xu_t\big|\big\}\,dx\,d\tau\\
&\qquad+C\int_0^t\|u\|_{L^\infty}^2\left(\int_a^b u_x^2\,dx\right)d\tau+C\int_0^t\int_a^b\mu_x^2u_x^2\,dx\,d\tau\\
&\quad\leqslant C+\int_0^t\int_a^b\varepsilon\left|\rho_xu+\rho u_x+\frac{\rho u}{x}\right|u_x^2+\frac12\int_0^t\int_a^b u_t^2\,dx\,d\tau\\
&\qquad+C\varepsilon^2\int_0^t\int_a^b\big(\rho^\theta\big)_x^2u_x^2\,dx\,d\tau+C\int_0^t\|u_x\|_{L^2}^2\left(\int_a^b u_x^2\,dx\right)d\tau\\
&\quad\leqslant C+C\int_0^t\int_a^b\varepsilon|\rho_xu|u_x^2\,dx\,d\tau+C\int_0^t\int_a^b\varepsilon\big|u_x^3\big|\,dx\,d\tau+C\int_0^t\|u\|_{L^\infty}\int_a^b\varepsilon u_x^2\,dx\,d\tau\\
&\qquad+\frac12\int_0^t\int_a^b u_t^2\,dx\,d\tau+C\varepsilon^2\int_0^t\int_a^b\big(\rho^\theta\big)_x^2u_x^2\,dx\,d\tau+C\int_0^t\|u_x\|_{L^2}^2\left(\int_a^b u_x^2\,dx\right)d\tau\\
&\quad=:C+\sum_{i=1}^6 I_i.
\end{aligned}\tag{2.23}$$

Now we estimate I_i $(i=1,2,3,5,6)$ as follows

$$\begin{aligned}
I_1&\leqslant C\int_0^t\int_a^b\rho_x^2u_x^2\,dx\,d\tau+C\varepsilon^2\int_0^t\int_a^b u^2u_x^2\,dx\,d\tau\\
&\leqslant C\int_0^t\|u_x\|_{L^\infty}^2\int_a^b\rho_x^2\,dx\,d\tau+C\varepsilon^2\int_0^t\int_a^b u^2u_x^2\,dx\,d\tau\\
&\leqslant C\int_0^t\|u_x\|_{L^2}\|u_{xx}\|_{L^2}\,d\tau+C\varepsilon^2\int_0^t\int_a^b u^2u_x^2\,dx\,d\tau
\end{aligned}$$

$$\leqslant \frac{1}{4}\int_0^t\int_a^b u_{xx}^2\,dx\,d\tau + C\int_0^t\int_a^b u_x^2\,dx\,d\tau + C\varepsilon^2\int_0^t \|u_x\|_{L^2}^2\left(\int_a^b u_x^2\,dx\right)d\tau,$$

$$I_2 \leqslant C\varepsilon\int_0^t \|u_x\|_{L^\infty}\|u_x\|_{L^2}^2 \leqslant C\varepsilon\int_0^t \|u_x\|_{L^2}^{\frac{1}{2}}\|u_{xx}\|_{L^2}^{\frac{1}{2}}\|u_x\|_{L^2}^2\,dx$$

$$\leqslant C\varepsilon\int_0^t \left(\|u_x\|_{L^2}+\|u_{xx}\|_{L^2}\right)\|u_x\|_{L^2}^2$$

$$\leqslant \frac{1}{4}\int_0^t \|u_{xx}\|_{L^2}^2\,d\tau + C\varepsilon^2\int_0^t \|u_x\|_{L^2}^4\,d\tau + C\varepsilon\int_0^t \|u_x\|_{L^2}^3\,d\tau$$

$$\leqslant \frac{1}{4}\int_0^t \|u_{xx}\|_{L^2}^2\,d\tau + C\varepsilon^2\int_0^t \|u_x\|_{L^2}^2\|u_x\|_{L^2}^2\,d\tau + C\varepsilon\int_0^t \|u_x\|_{L^2}\|u_x\|_{L^2}^2\,d\tau,$$

$$I_3+I_6 \leqslant C\int_0^t \left(1+\|u_x\|_{L^2}^2\right)\left(\int_a^b u_x^2\,dx\right)d\tau,$$

and

$$I_5 \leqslant C\varepsilon^2\int_0^t \|u_x\|_{L^\infty}^2\int_a^b \left(\rho^\theta\right)_x^2\,dx\,d\tau \leqslant \frac{1}{4}\int_0^t\int_a^b u_{xx}^2\,dx\,d\tau + C\int_0^t\int_a^b u_x^2\,dx\,d\tau.$$

Here we have used Sobolev's inequality.

Substituting the above estimates into (2.23), we have

$$\int_0^t\int_a^b \left(u_t^2+u_{xx}^2\right)dx\,d\tau + \int_a^b u_x^2\,dx \leqslant C + C\int_0^t \left(1+\|u_x\|_{L^2}^2\right)\left(\int_a^b u_x^2\,dx\right)d\tau,$$

which implies (2.22) by Gronwall's inequality. Lemma 2.1 and (2.22) give the bound of $\|u\|_{L^\infty(0,T;L^\infty)} \leqslant C$. The proof is completed. □

Lemma 2.6. *There exists a positive constant C independent of ε, such that*

$$\varepsilon^{\frac{1}{2}}\int_a^b \left(v_x^2+w_x^2\right)dx + \varepsilon^{\frac{3}{2}}\int_0^t\int_a^b \left(v_{xx}^2+w_{xx}^2\right)dx\,d\tau \leqslant C.$$

Proof. From $(1.1)_3$ and Lemma 2.2, we have

$$\mu v_{xx}\rho^{-1} - v_t = uv_x + \frac{uv}{x} - \mu\rho^{-1}\left(\frac{v_x}{x}-\frac{v}{x^2}\right) - \rho^{-1}\mu_x v_x + \rho^{-1}\mu_x\frac{v}{x}.$$

Multiplying the above equation by μv_{xx} and integrating the resulting equation over $(a,b)\times(0,t)$, we get

$$\frac{1}{2}\int_a^b \mu v_x^2\,dx + \int_0^t\int_a^b \mu^2\rho^{-1}v_{xx}^2\,dx\,d\tau$$

$$= \frac{1}{2}\int_a^b \mu_0 v_{0x}^2\,dx - \frac{1}{2}\int_0^t\int_a^b \mu u_x v_x^2\,dx\,d\tau + \int_0^t\int_a^b \mu v_x\left(\frac{uv}{x^2} - \frac{u_x v + uv_x}{x}\right)dx\,d\tau$$

$$
\begin{aligned}
&-\int_0^t\int_a^b \mu_x v_x \frac{uv}{x}\,dx\,d\tau-\int_0^t\int_a^b \mu\rho^{-1}\mu v_{xx}\left(\frac{v_x}{x}-\frac{v}{x^2}\right)dx\,d\tau\\
&-\int_0^t\int_a^b \mu\rho^{-1}v_{xx}\mu_x v_x\,dx\,d\tau+\int_0^t\int_a^b \mu\rho^{-1}v_{xx}\mu_x\frac{v}{x}\,dx\,d\tau\\
&-\frac{\theta}{2}\int_0^t\int_a^b \mu u_x v_x^2\,dx\,d\tau-\frac{\theta}{2}\int_0^t\int_a^b \mu v_x^2\frac{u}{x}\,dx\,d\tau+\int_0^t(\mu v_x v_t)|_{x=a}^{x=b}\,d\tau\\
&+\int_0^t\int_a^b \frac{\mu_x v_x}{x}uv\,dx\,d\tau-\int_0^t\int_a^b \rho^{-1}\mu\mu_x v_x v_{xx}\,dx\,d\tau-\int_0^t\int_a^b \rho^{-1}\mu\frac{\mu_x}{x}v_x^2\,dx\,d\tau\\
&+\int_0^t\int_a^b \rho^{-1}\mu v\frac{\mu_x v_x}{x^2}\,dx\,d\tau+\int_0^t\int_a^b \rho^{-1}\mu_x^2 v_x\frac{v}{x}\,dx\,d\tau-\int_0^t\int_a^b \rho^{-1}\mu_x^2 v_x^2\,dx\,d\tau,
\end{aligned}
$$

where we have used integration by parts and boundary conditions (1.3).

The above equation implies

$$
\begin{aligned}
&\varepsilon\int_a^b v_x^2\,dx+\varepsilon^2\int_0^t\int_a^b v_{xx}^2\,dx\,d\tau\\
&\leqslant C\varepsilon+C\varepsilon\int_0^t\int_a^b |u_x|v_x^2\,dx\,d\tau+C\varepsilon\int_0^t\int_a^b |v_x|\big(1+|u_x|+|v_x|\big)\,dx\,d\tau\\
&\quad+C\varepsilon\int_0^t\int_a^b |\rho_x v_x|\,dx\,d\tau+C\varepsilon^2\int_0^t\int_a^b |v_{xx}|\big(1+|v_x|\big)\,dx\,d\tau\\
&\quad+C\varepsilon^2\int_0^t\int_a^b |v_{xx}\rho_x v_x|\,dx\,d\tau+C\varepsilon^2\int_0^t\int_a^b |\rho_x v_{xx}|\,dx\,d\tau\\
&\quad+C\varepsilon\int_0^t\big|v_x v_t|_{x=a}^{x=b}\big|\,d\tau+C\varepsilon^2\int_0^t\int_a^b |\rho_x|v_x^2\,dx\,d\tau+C\varepsilon^2\int_0^t\int_a^b |\rho_x v_x|\,dx\,d\tau\\
&\quad+C\varepsilon^2\int_0^t\int_a^b \rho_x^2|v_x|\,dx\,d\tau+C\varepsilon^2\int_0^t\int_a^b \rho_x^2 v_x^2\,dx\,d\tau\\
&=: C\varepsilon+\sum_{i=1}^{11}J_i,
\end{aligned}
\tag{2.24}
$$

where we have used Lemmas 2.1–2.5.

Before we estimate the terms on the right-hand side of (2.24), we give the following useful estimates (see (2.10) in [7])

$$
\big|v_x(x,t)\big|\leqslant C\big(1+\|v_x(t)\|_{L^2}^{\frac{1}{2}}\|v_{xx}(t)\|_{L^2}^{\frac{1}{2}}\big),\quad \text{for all } x\in[a,b],
\tag{2.25}
$$

and

$$
\big|w_x(x,t)\big|\leqslant C\big(1+\|w_x(t)\|_{L^2}^{\frac{1}{2}}\|w_{xx}(t)\|_{L^2}^{\frac{1}{2}}\big),\quad \text{for all } x\in[a,b].
\tag{2.26}
$$

Next, we estimate the terms on the right-hand side of (2.24) by virtue of Lemmas 2.1–2.5, (2.25) and (2.26) as follows: Firstly, we have

$$J_1 \leqslant C\varepsilon \int_0^t \left(\|u_x\|_{L^\infty}^2 + 1\right) \int_a^b v_x^2 \,dx\,d\tau, \tag{2.27}$$

$$J_2 + J_3 \leqslant C\varepsilon \int_0^t\int_a^b \left(1 + v_x^2 + u_x^2 + \rho_x^2\right) dx\,d\tau \leqslant C\varepsilon + C\varepsilon \int_0^t\int_a^b v_x^2\,dx\,d\tau, \tag{2.28}$$

$$J_4 \leqslant C\varepsilon^2 + \frac{1}{8}\varepsilon^2 \int_0^t\int_a^b v_{xx}^2\,dx\,d\tau + C\varepsilon^2 \int_0^t\int_a^b v_x^2\,dx\,d\tau, \tag{2.29}$$

$$\begin{aligned} J_5 &\leqslant \frac{1}{16}\varepsilon^2 \int_0^t\int_a^b v_{xx}^2\,dx\,d\tau + C\varepsilon^2 \int_0^t\int_a^b \rho_x^2 v_x^2\,dx\,d\tau \\ &\leqslant \frac{1}{16}\varepsilon^2 \int_0^t\int_a^b v_{xx}^2\,dx\,d\tau + C\varepsilon^2 \int_0^t \left\|v_x(\tau)\right\|_{L^\infty([a,b])}^2 \int_a^b \rho_x^2\,dx\,d\tau \\ &\leqslant \frac{1}{16}\varepsilon^2 \int_0^t\int_a^b v_{xx}^2\,dx\,d\tau + C\varepsilon^2 \int_0^t \left\|v_x(\tau)\right\|_{L^2([a,b])} \left\|v_{xx}(\tau)\right\|_{L^2([a,b])}\,d\tau + C\varepsilon^2 \\ &\leqslant \frac{1}{8}\varepsilon^2 \int_0^t\int_a^b v_{xx}^2\,dx\,d\tau + C\varepsilon^2 \int_0^t\int_a^b v_x^2\,dx\,d\tau + C\varepsilon^2, \end{aligned} \tag{2.30}$$

and

$$J_6 \leqslant \frac{1}{8}\varepsilon^2 \int_0^t\int_a^b v_{xx}^2\,dx\,d\tau + C\varepsilon^2 \int_0^t\int_a^b \rho_x^2\,dx\,d\tau \leqslant \frac{1}{8}\varepsilon^2 \int_0^t\int_a^b v_{xx}^2\,dx\,d\tau + C\varepsilon^2. \tag{2.31}$$

Secondly, the boundary term J_7 can be estimated as

$$J_7 \leqslant C\varepsilon^{\frac{1}{2}} + \frac{1}{8}\varepsilon^2 \int_0^t\int_a^b v_{xx}^2\,dx\,d\tau + C\varepsilon \int_0^t\int_a^b v_x^2\,dx\,d\tau, \tag{2.32}$$

where we have used Young's inequality and (2.25).

Finally, we can estimate J_8–J_{11} as follows:

$$J_8 \leqslant C\varepsilon^2 \int_0^t\int_a^b v_x^2\,dx\,d\tau + C\varepsilon^2 \int_0^t\int_a^b \rho_x^2 v_x^2\,dx\,d\tau \leqslant \frac{1}{8}\varepsilon^2 \int_0^t\int_a^b v_{xx}^2\,dx\,d\tau + C\varepsilon^2 \int_0^t\int_a^b v_x^2\,dx\,d\tau,$$

$$J_9 \leqslant C\varepsilon^2 + C\varepsilon^2 \int_0^t\int_a^b v_x^2\,dx\,d\tau,$$

$$J_{10} \leqslant C\varepsilon^2 \int_0^t\int_a^b \rho_x^2\,dx\,d\tau + C\varepsilon^2 \int_0^t\int_a^b \rho_x^2 v_x^2\,dx\,d\tau \leqslant C\varepsilon^2 + \frac{1}{8}\varepsilon^2 \int_0^t\int_a^b v_{xx}^2\,dx\,d\tau + C\varepsilon^2 \int_0^t\int_a^b v_x^2\,dx\,d\tau,$$

and

$$J_{11} \leqslant C\varepsilon^2 \int_0^t \|v_x\|_{L^\infty}^2 \, d\tau \leqslant \frac{1}{8}\varepsilon^2 \int_0^t \int_a^b v_{xx}^2 \, dx \, d\tau + C\varepsilon^2 \int_0^t \int_a^b v_x^2 \, dx \, d\tau.$$

Substituting the above estimates J_1–J_{11} into (2.24), we get for $\varepsilon \leqslant \varepsilon_0$ that

$$\varepsilon \int_a^b v_x^2 \, dx + \varepsilon^2 \int_0^t \int_a^b v_{xx}^2 \, dx \, d\tau \leqslant C \int_0^t \left(1 + \|u_x\|_{L^2}^2 + \|u_{xx}\|_{L^2}^2\right) \int_a^b \varepsilon v_x^2 \, dx \, d\tau + C\varepsilon^{\frac{1}{2}}.$$

Therefore, we obtain the desired estimate for the angular velocity v by using Lemma 2.5 and Gronwall's inequality. By the similar method, we can also obtain the estimate for the axial velocity w.

The proof of Lemma 2.6 is completed. □

By virtue of Lemma 2.6, we have the following useful result.

Corollary 2.1. *There exists a positive constant C, independent of ε, such that*

$$\varepsilon^2 \int_0^t \int_a^b \left(v_x^4 + w_x^4\right) dx \, d\tau \leqslant C\varepsilon^{\frac{1}{2}}.$$

Proof. By Sobolev's inequality, (2.25), (2.26) and Lemma 2.6, we have

$$\begin{aligned} \varepsilon^2 \int_0^t \int_a^b \left(v_x^4 + w_x^4\right) dx \, d\tau &\leqslant C\varepsilon^{\frac{3}{2}} \int_0^t \left\|(v_x, w_x)(\tau)\right\|_{L^\infty}^2 \int_a^b \varepsilon^{\frac{1}{2}}\left(v_x^2 + w_x^2\right) dx \, d\tau \\ &\leqslant C\varepsilon^{\frac{3}{2}} \int_0^t \left\|(v_x, w_x)(\tau)\right\|_{L^2} \left\|(v_{xx}, w_{xx})(\tau)\right\|_{L^2} d\tau + C\varepsilon^{\frac{3}{2}} \\ &\leqslant C\varepsilon^{\frac{1}{2}}. \end{aligned}$$

This proves Corollary 2.1. □

Lemma 2.7. *We have*

$$\|u_x\|_{L^\infty(0,T;L^\infty)} \leqslant C,$$

and

$$\sup_{t \in [0,T]} \int_a^b \left(u_t^2 + u_{xx}^2\right) dx + \int_0^T \int_a^b u_{xt}^2 \, dx \, d\tau \leqslant C.$$

Proof. From Lemma 2.5 and Sobolev's inequality, we have

$$\|u\|_{L^\infty(0,T;L^\infty)} + \|u_x\|_{L^2(0,T;L^\infty)} \leqslant C. \tag{2.33}$$

By $(1.1)_1$, Lemmas 2.2, 2.4 and 2.5, we get

$$\int_a^b \rho_t^2 \, dx \leqslant C \int_a^b \left(\rho^2 u_x^2 + \rho_x^2 u^2 + \rho^2 u^2\right) dx \leqslant C. \tag{2.34}$$

Differentiating $(1.1)_2$ with respect to t and using $(1.1)_1$ and $(1.1)_3$, we obtain

$$\begin{aligned}
&\rho u_{tt}+\rho u u_{tx}-(2\mu+\lambda)u_{xxt}\\
&\quad=(2\mu+\lambda)\left(\frac{u_{xt}}{x}-\frac{u_t}{x^2}\right)+(u_t+uu_x)\left((\rho u)_x+\frac{\rho u}{x}\right)-\rho u_t u_x+2\mu_t\left(u_x+\frac{u}{x}\right)_x-p_{xt}\\
&\qquad+\frac{1}{x}\left\{-\left(\rho u v^2\right)_x-\frac{3\rho u v^2}{x}+2\mu\left(v_x+\frac{v}{x}\right)_x v+2\mu_x v_x v-\frac{2}{x}v^2\mu_x\right\}+(2\mu_x u_x)_t.
\end{aligned}$$

Multiplying the above equality by u_t and integrating the resulting equation over (a,b), we deduce

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_a^b \rho u_t^2\,dx+\int_a^b(2\mu+\lambda)u_{xt}^2\,dx\\
&\quad=\int_a^b(2\mu+\lambda)u_t\left(\frac{u_{xt}}{x}-\frac{u_t}{x^2}\right)dx-\int_a^b\left(u_t^2+uu_xu_t\right)_x\rho u\,dx\\
&\qquad+\int_a^b\frac{1}{x}\rho u\left(\frac{1}{2}u_t^2+uu_tu_x\right)dx-\int_a^b\rho u_x u_t^2\,dx+\int_a^b\left(\rho^\gamma\right)_t u_{xt}\,dx-3\int_a^b\frac{\rho u v^2}{x^2}u_t\,dx\\
&\qquad-2\int_a^b\mu\left(v_x+\frac{v}{x}\right)\left(-\frac{vu_t}{x^2}+\frac{v_xu_t}{x}+\frac{vu_{tx}}{x}\right)dx-4\int_a^b\mu_x\frac{v^2u_t}{x^2}\,dx\\
&\qquad+\int_a^b\rho u v^2\left(\frac{u_{xt}}{x}-\frac{u_t}{x^2}\right)dx-2\int_a^b\mu_{xt}u_t\frac{u}{x}\,dx-2\int_a^b\mu_t u_{tx}\left(u_x+\frac{u}{x}\right)dx\\
&\quad=:\sum_{i=1}^{11}R_i. \qquad (2.35)
\end{aligned}$$

Next, we estimate each term on the right-hand side of (2.35) by applying the previous lemmas and (2.34) as follows:

$$R_1=\int_a^b(2\mu+\lambda)u_t\left(\frac{u_{xt}}{x}-\frac{u_t}{x^2}\right)dx\leqslant C\int_a^b\left(|u_tu_{xt}|+u_t^2\right)dx\leqslant\frac{1}{7}\int_a^b\lambda u_{xt}^2\,dx+C\int_a^b\rho u_t^2\,dx,$$

$$\begin{aligned}
R_2&=-\int_a^b\rho u\left(2u_tu_{xt}+u_x^2u_t+uu_tu_{xx}+uu_xu_{xt}\right)dx\\
&\leqslant\frac{1}{7}\int_a^b\lambda u_{xt}^2\,dx+C\int_a^b\left(\rho u_t^2+u_x^2\right)dx+C\int_a^b\left(u_x^4+u_{xx}^2\right)dx\\
&\leqslant\frac{1}{7}\int_a^b\lambda u_{xt}^2\,dx+C\int_a^b\rho u_t^2\,dx+C\left(1+\left\|u_x(t)\right\|_{L^\infty}^2+\left\|u_{xx}(t)\right\|_{L^2}^2\right),
\end{aligned}$$

$$R_3+R_4\leqslant C\left(1+\int_a^b\rho u_t^2\,dx+\left\|u_x(t)\right\|_{L^\infty}\int_a^b\rho u_t^2\,dx\right)\leqslant C+C\left(1+\left\|u_x(t)\right\|_{L^\infty}\right)\int_a^b\rho u_t^2\,dx,$$

$$R_5\leqslant\frac{1}{7}\int_a^b\lambda u_{xt}^2\,dx+C\int_a^b\rho_t^2\,dx\leqslant C+\frac{1}{7}\int_a^b\lambda u_{xt}^2\,dx,$$

$$R_6 + R_9 \leqslant C + \frac{1}{7}\int_a^b \lambda u_{xt}^2\,dx + C\int_a^b \rho u_t^2\,dx,$$

$$\begin{aligned}
R_7 &\leqslant C\varepsilon^2 + \frac{1}{7}\int_a^b \lambda u_{xt}^2\,dx + C\int_a^b \varepsilon^2\big(v_x^2 + v_x^4\big)\,dx + C\int_a^b \rho u_t^2\,dx \\
&\leqslant \frac{1}{7}\int_a^b \lambda u_{xt}^2\,dx + C\int_a^b \rho u_t^2\,dx + C\big(\varepsilon^{\frac{3}{2}} + \varepsilon^{\frac{3}{2}}\big\|v_x(t)\big\|_{L^\infty}^2\big) \\
&\leqslant \frac{1}{7}\int_a^b \lambda u_{xt}^2\,dx + C\int_a^b \rho u_t^2\,dx + C\big\{\varepsilon^{\frac{3}{2}} + \varepsilon^{\frac{3}{2}}\big(\big\|v_x(t)\big\|_{L^2}^2 + \big\|v_{xx}(t)\big\|_{L^2}^2\big)\big\} \\
&\leqslant \frac{1}{7}\int_a^b \lambda u_{xt}^2\,dx + C\int_a^b \rho u_t^2\,dx + C\big(\varepsilon + \varepsilon^{\frac{3}{2}}\big\|v_{xx}(t)\big\|_{L^2}^2\big),
\end{aligned}$$

$$R_8 = -4\theta\varepsilon\int_a^b \rho^{\theta-1}\rho_x\frac{v^2}{x^2}u_t\,dx \leqslant C\int_a^b \rho u_t^2\,dx + C\varepsilon^2\int_a^b \rho_x^2\,dx \leqslant C\varepsilon^2 + C\int_a^b \rho u_t^2\,dx,$$

and

$$\begin{aligned}
R_{10} + R_{11} &= 2\int_a^b \mu_t\left(-u_{tx}u_x + u_t\frac{u_x}{x} - u_t\frac{u}{x^2}\right)dx \\
&\leqslant C\varepsilon\int_a^b \rho^{\theta-1}|\rho_t|\big(|u_t u_x| + |u_t| + |u_{tx}u_x|\big)\,dx \\
&\leqslant \frac{1}{7}\int_a^b \lambda u_{tx}^2\,dx + C\varepsilon^2\int_a^b \rho_t^2\,dx + C\int_a^b \rho u_t^2 u_x^2\,dx + C\int_a^b \rho u_t^2\,dx + C\varepsilon^2\int_a^b \rho_t^2 u_x^2\,dx \\
&\leqslant \frac{1}{7}\int_a^b \lambda u_{tx}^2\,dx + C\varepsilon^2 + C\big(1 + \big\|u_x(t)\big\|_{L^2}^2 + \big\|u_{xx}(t)\big\|_{L^2}^2\big)\int_a^b \rho u_t^2\,dx \\
&\quad + C\varepsilon^2\big(\big\|u_x(t)\big\|_{L^2}^2 + \big\|u_{xx}(t)\big\|_{L^2}^2\big).
\end{aligned}$$

Substituting the estimates of R_i $(i = 1, \dots, 11)$ into (2.35), we get from Lemmas 2.1, 2.2, 2.5, 2.6, and Gronwall's inequality that

$$\sup_{t\in[0,T]}\int_a^b u_t^2\,dx + \int_0^T\int_a^b u_{xt}^2\,dx\,d\tau \leqslant C.$$

This together with Eq. $(1.1)_2$ and previous lemmas give the bound of $\|u_{xx}\|_{L^\infty(0,T;L^2)}$. The proof of Lemma 2.7 is completed. □

3. Proof of the main results

In this section, we will give the proof of Theorems 1.1–1.3. By virtue of uniform *a priori* estimates given in Section 2, the existence and uniqueness of the global strong solution to the problem (1.1)–(1.3) with any fixed $0 < \varepsilon \leqslant \varepsilon_0$ can be proved in a standard way as that in [19]. Following a process similar to that in [4] (also cf. [2,5,20]), one can also justify the zero shear viscosity limit (in this paper, $\varepsilon \to 0$ implies $\mu \to 0$). Moreover, we can easily get the convergence result (ii) in Theorem 1.1 by Lemmas 2.1–2.7 and Corollary 2.1.

Next, we prove two results about BL-thickness, which corresponds to the constant initial data (see Theorem 1.2) and the general initial data (see Theorem 1.3) respectively.

3.1. BL-thickness for the constant initial data

As in [4,7], to make the proof of the existence of a BL-thickness simpler and the idea clearer, we restrict ourselves to the following constant initial data:

$$\rho|_{t=0} = \rho_0 = \text{const.} > 0, \qquad (u, v, w)|_{t=0} = 0. \tag{3.1}$$

First, we have the following proposition.

Proposition 3.1. *Let the assumptions given in Theorem* 1.2 *and* (3.1) *hold. Then the limit problem* (1.4)–(1.7) *has only the trivial solution* $(\bar\rho, \bar u, \bar v, \bar w) = (\rho_0, 0, 0, 0)$.

The proof of this proposition is the same as that in [4], so we omit it here for brevity.

Assume (3.1) holds. Then by virtue of Proposition 3.1, to prove Theorem 1.2, it suffices to show that

$$\int_a^b \big(|v| + |w|\big)\,dx \leqslant C\varepsilon^{\frac12}, \qquad \delta\int_{a+\delta}^{b-\delta}\left(\left|\left(\frac{v}{x}\right)_x\right| + |w_x|\right)dx \leqslant C\varepsilon^{\frac12}, \tag{3.2}$$

for the solution (ρ, u, v, w) to the problem (1.1)–(1.3) satisfying the condition (3.1), where $\delta = \delta(\varepsilon)$ satisfies the conditions given in Theorem 1.2. In fact, if (3.2) hold, then we have by $W^{1,1}([a+\delta, b-\delta]) \hookrightarrow L^\infty([a+\delta, b-\delta])$

$$\begin{aligned}\|v\|_{L^\infty([a+\delta,b-\delta])} &\leqslant C\left\|\frac{v}{x}\right\|_{L^\infty([a+\delta,b-\delta])} \leqslant C\int_{a+\delta}^{b-\delta}\left|\frac{v}{x}\right|dx + C\int_{a+\delta}^{b-\delta}\left|\left(\frac{v}{x}\right)_x\right|dx \\ &\leqslant C\left(\varepsilon^{\frac12} + \frac{\varepsilon^{\frac12}}{\delta}\right) \to 0, \quad \text{as } \varepsilon \to 0,\end{aligned}$$

and

$$\|w\|_{L^\infty([a+\delta,b-\delta])} \leqslant \int_{a+\delta}^{b-\delta}|w|\,dx + \int_{a+\delta}^{b-\delta}|w_x| \leqslant C\left(\varepsilon^{\frac12} + \frac{\varepsilon^{\frac12}}{\delta}\right) \to 0, \quad \text{as } \varepsilon \to 0.$$

On the other hand, it is easy to see

$$\liminf_{\varepsilon\to0}\big\|(v, w)\big\|_{L^\infty(0,T,L^\infty(\Omega))} > 0, \quad \text{provided } (v_i, w_i)(t) \neq 0 \ \big(i \in \{1, 2\}\big) \text{ for } t > 0.$$

Thus we can conclude that $\delta(\varepsilon)$ is a BL-thickness satisfying the conditions in Theorem 1.2.

Next, we turn to prove (3.2) under the condition (3.1). As in [4,7], let $\varphi_\nu(v) = \sqrt{v^2 + \nu^2}$. Multiplying $(1.1)_3$ by $x\varphi_\nu'(v)$ and integrating the resulting equation over $(a, b) \times (0, t)$, we get

$$\begin{aligned}\int_a^b x\rho\varphi_\nu(v)\,dx &= \int_a^b x\rho_0\nu\,dx - \int_0^t\int_a^b \rho uv\varphi_\nu'(v)\,dx\,d\tau - \int_0^t\int_a^b \mu\varphi_\nu'(v)\left(v_x + \frac{v}{x}\right)dx\,d\tau \\ &\quad - \int_0^t\int_a^b x\mu_x\varphi_\nu'(v)\left(v_x + \frac{v}{x}\right)dx\,d\tau - \int_0^t\int_a^b x\mu\varphi_\nu''(v)\left(v_x + \frac{v}{x}\right)v_x\,dx\,d\tau\end{aligned}$$

$$+\int_0^t x\mu\left(v_x+\frac{v}{x}\right)\varphi_\nu'(v)\,d\tau\Big|_{x=a}^{x=b}+\int_0^t\int_a^b x\mu_x v_x\varphi_\nu'(v)\,dx\,d\tau$$
$$-\int_0^t\int_a^b \mu_x v\varphi_\nu'(v)\,dx\,d\tau. \tag{3.3}$$

Now we estimate the terms on the right-hand side of (3.3) as follows:

Firstly, letting $\nu\to 0$, we get by Lemmas 2.2–2.6 and the fact that $|\varphi_\nu'(v)|=\frac{|v|}{\sqrt{v^2+\nu^2}}\leqslant 1$

$$-\lim_{\nu\to 0}\int_0^t\int_a^b \rho u v\varphi_\nu'(v)\,dx\,d\tau\leqslant C\int_0^t\int_a^b |v|\,dx\,d\tau,$$

$$-\lim_{\nu\to 0}\int_0^t\int_a^b \mu\varphi_\nu'(v)\left(v_x+\frac{v}{x}\right)dx\,d\tau\leqslant C\varepsilon\int_0^t\int_a^b\big(|v_x|+|v|\big)\,dx\,d\tau$$
$$\leqslant C\varepsilon\left(1+\int_0^t\left(\int_a^b v_x^2\,dx\right)^{\frac12}d\tau\right)\leqslant C\varepsilon^{\frac34},$$

$$-\lim_{\nu\to 0}\int_0^t\int_a^b x\mu_x\varphi_\nu'(v)\left(v_x+\frac{v}{x}\right)dx\,d\tau\leqslant C\varepsilon\int_0^t\int_a^b\left|\rho_x\left(v_x+\frac{v}{x}\right)\right|dx\,d\tau$$
$$\leqslant C\varepsilon\left(\int_0^t\int_a^b\rho_x^2\,dx\,d\tau+\int_0^t\int_a^b\big(v_x^2+v^2\big)\,dx\,d\tau\right)$$
$$\leqslant C\varepsilon^{\frac12},$$

$$\lim_{\nu\to 0}\int_0^t\int_a^b x\mu_x v_x\varphi_\nu'(v)\,dx\,d\tau\leqslant C\varepsilon\int_0^t\int_a^b\big(\rho_x^2+v_x^2\big)\,dx\,d\tau\leqslant C\varepsilon^{\frac12},$$

and

$$-\lim_{\nu\to 0}\int_0^t\int_a^b \mu_x v\varphi_\nu'(v)\,dx\,d\tau\leqslant C\varepsilon\int_0^t\int_a^b|\rho_x|\,dx\,d\tau\leqslant C\varepsilon.$$

Secondly, by virtue of the fact that $\varphi_\nu''(v)=\frac{\nu^2}{(v^2+\nu^2)^{\frac32}}\geqslant 0$ and $0\leqslant v^2\varphi_\nu''(v)\leqslant\nu$, we can get

$$-\int_0^t\int_a^b x\mu\varphi_\nu''(v)\left(v_x+\frac{v}{x}\right)v_x\,dx\,d\tau$$
$$=\int_0^t\int_a^b\mu\frac{v^2}{4x}\varphi_\nu''(v)\,dx\,d\tau-\int_0^t\int_a^b x\mu\left(v_x+\frac{v}{2x}\right)^2\varphi_\nu''(v)\,dx\,d\tau$$
$$\leqslant\int_0^t\int_a^b\mu\frac{v^2}{4x}\varphi_\nu''(v)\,dx\,d\tau\leqslant\nu\int_0^t\int_a^b\frac{\mu}{4x}\,dx\,d\tau\to 0,\quad\text{as }\nu\to 0.$$

Finally, for the boundary term in (3.3), we can use (2.25), Lemma 2.6 and the Hölder inequality with $\frac{1}{2}+\frac{1}{4}+\frac{1}{4}=1$, to get

$$\lim_{\nu\to 0}\int_0^t x\mu\left(v_x+\frac{v}{x}\right)\varphi_\nu'(v)\,d\tau\Big|_{x=a}^{x=b}$$
$$\leqslant C\varepsilon\left(1+\int_0^t \left\|v_x(\tau)\right\|_{L^2}^{\frac{1}{2}}\left\|v_{xx}(\tau)\right\|_{L^2}^{\frac{1}{2}}\,d\tau\right)$$
$$\leqslant C\varepsilon+C\varepsilon^{\frac{1}{2}}\left(\varepsilon^{\frac{1}{2}}\int_0^t\left\|v_x(\tau)\right\|_{L^2}^2\,d\tau\right)^{\frac{1}{4}}\left(\varepsilon^{\frac{3}{2}}\int_0^t\left\|v_{xx}(\tau)\right\|_{L^2}^2\,d\tau\right)^{\frac{1}{4}}\leqslant C\varepsilon^{\frac{1}{2}}.$$

Therefore, recalling $\varphi_\nu(v)\to|v|$ as $\nu\to 0$, substituting these estimates into (3.3), using Lemma 2.2 and Gronwall's inequality, we get

$$\int_a^b\left|v(x,t)\right|dx\leqslant C\varepsilon^{\frac{1}{2}}\quad\text{for any } t\in(0,T).$$

In the same way, we can also prove an analogous estimate for w. Thus, we obtain the first inequality in (3.2).

To show the second inequality in (3.2), denoting $z:=v_x-\frac{v}{x}$, and dividing $(1.1)_3$ by ρ, and then differentiating it with respect to x, we deduce

$$z_t+(uz)_x+\left(\frac{uv}{x}\right)_x+\left(\frac{\mu v}{\rho x^2}\right)_x-\left(\frac{\mu}{\rho}z_x\right)_x-\left(\frac{\mu z}{\rho x}\right)_x$$
$$-\left(\frac{1}{\rho}\mu_x z\right)_x+\frac{1}{x}\left\{\frac{\mu v_x\rho_x}{\rho^2}+2\frac{\mu v_x}{\rho x}-\frac{\mu v}{\rho x^2}-\frac{\mu_x v}{\rho x}-\frac{2uv}{x}+u_x v\right\}=0. \tag{3.4}$$

Introduce the function $\xi_\delta(x)$ as follows (cf. [4,7]):

$$\xi_\delta(x)=\begin{cases}x-a, & \text{if } a\leqslant x\leqslant a+\delta,\\ \delta, & \text{if } a+\delta\leqslant x\leqslant b-\delta,\\ b-x, & \text{if } b-\delta\leqslant x\leqslant b.\end{cases}$$

Then, multiplying (3.4) by $\xi_\delta(x)\varphi_\nu'(z)$ and integrating by parts, we obtain

$$\int_a^b\xi_\delta(x)\varphi_\nu(z)\,dx=\nu\int_a^b\xi_\delta(x)\,dx+\int_0^t\int_a^b uzz_x\varphi_\nu''(z)\xi_\delta(x)\,dx\,d\tau+\int_0^t\int_a^b uz\varphi_\nu'(z)\xi_\delta'(x)\,dx\,d\tau$$
$$-\int_0^t\int_a^b\frac{u}{x}z\varphi_\nu'(z)\xi_\delta(x)\,dx\,d\tau-\int_0^t\int_a^b\frac{u_x}{x}v\varphi_\nu'(z)\xi_\delta(x)\,dx\,d\tau$$
$$-\int_0^t\int_a^b\frac{\mu}{\rho}z_x^2\varphi_\nu''(z)\xi_\delta(x)\,dx\,d\tau-\int_0^t\int_a^b\frac{\mu z_x}{\rho}\varphi_\nu'(z)\xi_\delta'(x)\,dx\,d\tau$$
$$-\int_0^t\int_a^b\frac{\mu z}{\rho x}z_x\varphi_\nu''(z)\xi_\delta(x)\,dx\,d\tau-\int_0^t\int_a^b\frac{\mu z}{\rho x}\varphi_\nu'(z)\xi_\delta'(x)\,dx\,d\tau$$
$$+\int_0^t\int_a^b\frac{\mu v}{\rho x^2}z_x\varphi_\nu''(z)\xi_\delta(x)\,dx\,d\tau+\int_0^t\int_a^b\frac{\mu v}{\rho x^2}\varphi_\nu'(z)\xi_\delta'(x)\,dx\,d\tau$$

$$
\begin{aligned}
&- \int_0^t\int_a^b \frac{\mu_x z}{\rho} z_x \varphi_\nu''(z)\xi_\delta(x)\,dx\,d\tau - \int_0^t\int_a^b \frac{\mu_x z}{\rho}\varphi_\nu'(z)\xi_\delta'(x)\,dx\,d\tau \\
&- \int_0^t\int_a^b \frac{1}{x}\left\{\frac{\mu v_x \rho_x}{\rho^2} + 2\frac{\mu v_x}{\rho x} - \frac{\mu v}{\rho x^2} - \frac{\mu_x v}{\rho x} - \frac{2uv}{x} + u_x v\right\}\xi_\delta(x)\varphi_\nu'(z)\,dx\,d\tau \\
&= \nu\int_a^b \xi_\delta(x)\,dx + \sum_{i=1}^{13} K_i. \qquad (3.5)
\end{aligned}
$$

Now we estimate the terms K_i ($i = 1, 2, \ldots, 13$) on the right-hand side of (3.5) as follows: Firstly, by virtue of the fact that $\varphi_\nu''(z) \geqslant 0$ and $0 \leqslant z^2\varphi_\nu'' \leqslant \nu$, we find that

$$
\begin{aligned}
K_1 + K_5 + K_7 + K_{11} &\leqslant -\frac{1}{2}\int_0^t\int_a^b \frac{\mu}{\rho} z_x^2 \varphi_\nu''(z)\xi_\delta(x)\,dx\,d\tau + C\int_0^t\int_a^b \frac{1}{\mu} z^2\varphi_\nu''(z)\xi_\delta(x)\,dx\,d\tau \\
&\quad + C\int_0^t\int_a^b \mu z^2\varphi_\nu''(z)\xi_\delta(x)\,dx\,d\tau + C\int_0^t\int_a^b \frac{\mu_x^2}{\mu} z^2\varphi_\nu''(z)\xi_\delta(x)\,dx\,d\tau \\
&\leqslant C\nu\int_0^t\int_a^b \frac{1}{\mu}\xi_\delta(x)\,dx\,d\tau + C\nu\int_0^t\int_a^b \mu\xi_\delta(x)\,dx\,d\tau \\
&\quad + C\nu\int_0^t\int_a^b \varepsilon\rho_x^2\,dx\,d\tau \to 0, \quad \text{as } \nu \to 0. \qquad (3.6)
\end{aligned}
$$

Here we have used Lemmas 2.2, 2.4 and 2.5.

Secondly, by the definition of $\xi_\delta(x)$, we can estimate K_2 as follows:

$$
\begin{aligned}
K_2 &= \int_0^t\int_a^{a+\delta} uz\varphi_\nu'(z)\,dx\,d\tau - \int_0^t\int_{b-\delta}^b uz\varphi_\nu'(z)\,dx\,d\tau \leqslant C\int_0^t\int_a^{a+\delta} \xi_\delta(x)|z|\,dx\,d\tau + C\int_0^t\int_{b-\delta}^b \xi_\delta(x)|z|\,dx\,d\tau \\
&\leqslant C\int_0^t\int_a^b \xi_\delta(x)|z|\,dx\,d\tau. \qquad (3.7)
\end{aligned}
$$

Here we have used the following inequalities:

$$
\begin{cases}
\left|u(x,t)\right| \leqslant \int_a^x \left|u_y(y,t)\right|dy \leqslant C(x-a) \leqslant C\xi_\delta(x), & \text{for any } x \in [a, a+\delta], \\
\left|u(x,t)\right| \leqslant \int_x^b \left|u_y(y,t)\right|dy \leqslant C(b-x) \leqslant C\xi_\delta(x), & \text{for any } x \in [b-\delta, b].
\end{cases} \qquad (3.8)
$$

For K_3 and K_4, by Lemmas 2.5, 2.7 and the first assertion of (3.2) and $W^{1,1}([a,b]) \hookrightarrow L^\infty([a,b])$, we have by letting $\nu \to 0$ that

$$
\lim_{\nu\to 0}(K_3 + K_4) \leqslant C\int_0^t\int_a^b \xi_\delta(x)|z|\,dx\,d\tau + \int_0^t \|u_x\|_{L^\infty}\int_a^b |v|\,dx\,d\tau
$$

$$\leqslant C\int_0^t\int_a^b \xi_\delta(x)|z|\,dx\,d\tau + C\varepsilon^{\frac{1}{2}}\int_0^t \|u_x\|_{L^\infty}\,d\tau$$
$$\leqslant C\int_0^t\int_a^b \xi_\delta(x)|z|\,dx\,d\tau + C\varepsilon^{\frac{1}{2}}\int_0^t\int_a^b \big(|u_x|+|u_{xx}|\big)\,dx\,d\tau$$
$$\leqslant C\int_0^t\int_a^b \xi_\delta(x)|z|\,dx\,d\tau + C\varepsilon^{\frac{1}{2}}. \tag{3.9}$$

For K_6, recalling the definition of $\xi_\delta(x)$, we see that

$$\begin{aligned}
K_6 &= -\int_0^t\int_a^b \frac{\mu}{\rho}\big(\varphi_\nu(z)\big)_x \xi_\delta'(x)\,dx\,d\tau \\
&= -\int_0^t\int_a^{a+\delta} \frac{\mu}{\rho}\big(\varphi_\nu(z)\big)_x\,dx\,d\tau + \int_0^t\int_{b-\delta}^{b} \frac{\mu}{\rho}\big(\varphi_\nu(z)\big)_x\,dx\,d\tau \\
&= \int_0^t \left\{ \left(\frac{\mu}{\rho}\varphi_\nu(z)\right)\Big|_{x=b-\delta}^{x=b} - \left(\frac{\mu}{\rho}\varphi_\nu(z)\right)\Big|_{x=a}^{x=a+\delta} \right\} d\tau + \int_0^t\int_a^b \left(\frac{\mu}{\rho}\right)_x \varphi_\nu(z)\xi_\delta'(x)\,dx\,d\tau \\
&\leqslant \int_0^t \left(\frac{\mu\varphi_\nu(z)}{\rho}\Big|_{x=b} + \frac{\mu\varphi_\nu(z)}{\rho}\Big|_{x=a} \right) d\tau - \int_0^t\int_a^b \frac{\mu\rho_x}{\rho^2}\varphi_\nu(z)\xi_\delta'(x)\,dx\,d\tau + \int_0^t\int_a^b \frac{\mu_x}{\rho}\varphi_\nu(z)\xi_\delta'(x)\,dx\,d\tau \\
&=: K_6^1 + K_6^2 + K_6^3.
\end{aligned}$$

From (2.25), Lemmas 2.2–2.4 and 2.6, K_6^1, K_6^2 and K_6^3 can be bounded as follows:

$$\begin{aligned}
\lim_{\nu\to 0} K_6^1 &\leqslant C\varepsilon\left(1 + \int_0^t \big\|v_x(\tau)\big\|_{L^2}^{\frac{1}{2}} \big\|v_{xx}(\tau)\big\|_{L^2}^{\frac{1}{2}}\,d\tau\right) \\
&\leqslant C\varepsilon + C\varepsilon^{\frac{1}{2}}\left(\varepsilon^{\frac{3}{2}}\int_0^t \big\|v_{xx}(\tau)\big\|_{L^2}^2\,d\tau\right)^{\frac{1}{4}}\left(\varepsilon^{\frac{1}{2}}\int_0^t \big\|v_x(\tau)\big\|_{L^2}^2\,d\tau\right)^{\frac{1}{4}} \\
&\leqslant C\varepsilon^{\frac{1}{2}}, \\
\lim_{\nu\to 0} K_6^2 &\leqslant C\varepsilon\int_0^t \big\|\rho_x(\tau)\big\|_{L^2}\left\|\left(v_x - \frac{v}{x}\right)(\tau)\right\|_{L^2} d\tau \\
&\leqslant C\varepsilon^{\frac{3}{4}}\left(\varepsilon^{\frac{1}{2}}\int_0^t \big\|v_x(\tau)\big\|_{L^2}^2\,d\tau\right)^{\frac{1}{2}} + C\varepsilon \leqslant C\varepsilon^{\frac{3}{4}},
\end{aligned}$$

and

$$\lim_{\nu\to 0} K_6^3 \leqslant C\varepsilon\int_0^t\int_a^b |\rho_x|\left|v_x - \frac{v}{x}\right|dx\,d\tau \leqslant C\varepsilon^{\frac{3}{4}}.$$

Therefore, for $\varepsilon \leqslant \varepsilon_0$, we have

$$\overline{\lim_{\nu\to 0}} K_6 \leqslant \lim_{\nu\to 0}\big(K_6^1 + K_6^2 + K_6^3\big) \leqslant C\varepsilon^{\frac{1}{2}}. \tag{3.10}$$

Similarly, we can get

$$\lim_{\nu\to 0}(K_8+K_{10})\leqslant C\varepsilon\left(1+\int_0^t\int_a^b |v_x|\,dx\,d\tau\right)\leqslant C\varepsilon^{\frac{3}{4}}, \tag{3.11}$$

and

$$\begin{aligned}\lim_{\nu\to 0}K_{12}&\leqslant C\varepsilon\int_0^t\int_a^b|\rho_x||z|\,dx\,d\tau\leqslant C\varepsilon\int_0^t\int_a^b\rho_x^2\,dx\,d\tau+C\varepsilon\int_0^t\int_a^b\left(v_x-\frac{v}{x}\right)^2dx\,d\tau\\&\leqslant C\varepsilon^{\frac{1}{2}}.\end{aligned} \tag{3.12}$$

Furthermore, we have by integrating by parts that

$$\begin{aligned}K_9&=\int_0^t\int_a^b\frac{\mu v}{\rho x^2}\big(\varphi_\nu'(z)\big)_x\xi_\delta(x)\,dx\,d\tau\\&=\int_0^t\int_a^b\frac{\mu v}{\rho x^2}\left(-\xi_\delta'(x)+\frac{\xi_\delta(x)}{x}+\frac{\rho_x}{\rho}\xi_\delta(x)\right)\varphi_\nu'(z)\,dx\,d\tau-\int_0^t\int_a^b\mu\frac{\xi_\delta(x)}{\rho x^2}z\varphi_\nu'(z)\,dx\,d\tau\\&\quad-\int_0^t\int_a^b\frac{\mu_x v}{\rho x^2}\varphi_\nu'(z)\xi_\delta(x)\,dx\,d\tau\\&\leqslant\int_0^t\int_a^b\frac{\mu v}{\rho x^2}\left(-\xi_\delta'(x)+\frac{\xi_\delta(x)}{x}+\frac{\rho_x}{\rho}\xi_\delta(x)\right)\varphi_\nu'(z)\,dx\,d\tau-\int_0^t\int_a^b\frac{\mu_x v}{\rho x^2}\varphi_\nu'(z)\xi_\delta(x)\,dx\,d\tau,\end{aligned}$$

which implies by Lemmas 2.2–2.4

$$\overline{\lim_{\nu\to 0}}K_9\leqslant C\varepsilon\left(1+\int_0^t\int_a^b|\rho_x|\,dx\,d\tau\right)\leqslant C\varepsilon. \tag{3.13}$$

Finally, by Lemmas 2.2–2.4, 2.6, and the first assertion in (3.2), we have

$$\begin{aligned}\lim_{\nu\to 0}K_{13}&=-\int_0^t\int_a^b\frac{1}{x}\left\{\frac{\mu v_x\rho_x}{\rho^2}+2\frac{\mu v_x}{\rho x}-\frac{\mu v}{\rho x^2}-\frac{\mu_x v}{\rho x}-\frac{2uv}{x}+u_x v\right\}\xi_\delta(x)\varphi_\nu'(z)\,dx\,d\tau\\&\leqslant C\varepsilon^{\frac{1}{2}}.\end{aligned} \tag{3.14}$$

Substituting (3.6), (3.7) and (3.9)–(3.14) into (3.5) and sending $\nu\to 0$, we find that

$$\int_a^b\xi_\delta(x)|z|\,dx\leqslant C\int_0^t\int_a^b\xi_\delta(x)|z|\,dx\,d\tau+C\varepsilon^{\frac{1}{2}}, \tag{3.15}$$

which implies by Gronwall's inequality

$$\delta\int_{a+\delta}^{b-\delta}\left|\left(\frac{v}{x}\right)_x\right|x\,dx\leqslant\int_a^b\xi_\delta(x)\left|\left(\frac{v}{x}\right)_x\right|x\,dx=\int_a^b\xi_\delta(x)\left|v_x-\frac{v}{x}\right|dx\leqslant C\varepsilon^{\frac{1}{2}}. \tag{3.16}$$

Thus, the second assertion in (3.2) holds for the angular velocity v.

To establish an analogous estimate for w, we argue in a similar manner. Dividing $(1.1)_4$ by ρ, then differentiating the resulting equation with respect to x and denoting $z_1 := w_x$, we have

$$z_{1t} + (uz_1)_x - \left(\frac{\mu z_{1x}}{\rho}\right)_x - \left(\frac{\mu z_1}{\rho x}\right)_x - \left(\frac{\mu_x z_1}{\rho}\right)_x = 0. \tag{3.17}$$

Multiplying the above equation by $\xi_\delta(x)\varphi'_\nu(z_1)$ and integrating the resulting equation over $(a,b)\times(0,t)$, we get

$$\begin{aligned}\int_a^b \xi_\delta(x)\varphi_\nu(z_1)\,dx &= \nu\int_a^b \xi_\delta(x)\,dx + \int_0^t\int_a^b uz_1z_{1x}\varphi''_\nu(z_1)\xi_\delta(x)\,dx\,d\tau + \int_0^t\int_a^b uz_1\varphi'_\nu(z_1)\xi'_\delta(x)\,dx\,d\tau \\ &\quad - \int_0^t\int_a^b \frac{\mu}{\rho}z_{1x}^2\varphi''_\nu(z_1)\xi_\delta(x)\,dx\,d\tau - \int_0^t\int_a^b \frac{\mu z_{1x}}{\rho}\varphi'_\nu(z_1)\xi'_\delta(x)\,dx\,d\tau \\ &\quad - \int_0^t\int_a^b \frac{\mu z_1}{x\rho}z_{1x}\varphi''_\nu(z_1)\xi_\delta(x)\,dx\,d\tau - \int_0^t\int_a^b \frac{\mu z_1}{x\rho}\varphi'_\nu(z_1)\xi'_\delta(x)\,dx\,d\tau \\ &\quad - \int_0^t\int_a^b \frac{\mu_x z_1}{\rho}z_{1x}\varphi''_\nu(z_1)\xi_\delta(x)\,dx\,d\tau - \int_0^t\int_a^b \frac{\mu_x z_1}{\rho}\varphi'_\nu(z_1)\xi'_\delta(x)\,dx\,d\tau \\ &= \nu\int_a^b \xi_\delta(x)\,dx + \sum_{i=1}^8 L_i, \end{aligned} \tag{3.18}$$

where the terms L_1, L_3, L_5, L_7 can be bounded in the way similar to that in the derivation of (3.6) as follows:

$$\overline{\lim_{\nu\to 0}}(L_1+L_3+L_5+L_7) \leqslant 0.$$

In addition, using the similar arguments as that in estimating K_2 (see (3.7)), L_2 can be estimated by

$$\lim_{\nu\to 0} L_2 \leqslant C\int_0^t\int_a^b \xi_\delta(x)|z_1|\,dx\,d\tau,$$

and L_4 and L_6 can be dealt with by the similar method as that in K_6 and K_8 respectively, which are simpler, and consequently,

$$\lim_{\nu\to 0}(L_4+L_6) \leqslant C\varepsilon^{\frac{1}{2}}.$$

Furthermore, L_8 coincides with K_{12}, yields

$$\lim_{\nu\to 0} L_8 \leqslant C\varepsilon\int_0^t\int_a^b |\rho_x||w_x|\,dx\,d\tau \leqslant C\varepsilon^{\frac{1}{2}}.$$

Substituting the estimates of L_i $(i=1,\dots,8)$ into (3.18), and sending $\nu\to 0$, we find that

$$\int_a^b \xi_\delta(x)|z_1|\,dx \leqslant C\int_0^t\int_a^b \xi_\delta(x)|z_1|\,dx\,d\tau + C\varepsilon^{\frac{1}{2}},$$

which implies by Gronwall's inequality,

$$\delta\int_{a+\delta}^{b-\delta} |w_x|\,dx \leqslant \int_a^b \xi_\delta(x)|z_1|\,dx \leqslant C\varepsilon^{\frac{1}{2}}. \tag{3.19}$$

(3.16) and (3.19) show the second assertion in (3.2) holds.

This completes the proof of Theorem 1.2 under the case that $\bar\rho = \rho_0 = \text{const.} > 0$, $\bar u = \bar v = \bar w = 0$.

3.2. BL-thickness for the general initial data

In this subsection, motivated by the ideas of [7], which deals with constant viscosity, we can also give some further conclusions on convergence rates of the vanishing shear viscosity (in fact, in this paper, $\varepsilon \to 0$ implies $\mu \to 0$) and the BL-thickness for the general initial data under some additional initial assumptions. Let us first recall from (1.4)–(1.7) that $(\bar\rho, \bar u, \bar v, \bar w)$ solves the following initial–boundary value problem in the sense of distribution:

$$\bar\rho_t + (\bar\rho\bar u)_x + \frac{\bar\rho\bar u}{x} = 0, \tag{3.20}$$

$$\bar\rho\left(\bar u_t + \bar u\bar u_x - \frac{\bar v^2}{x}\right) + \bar p_x - \lambda\left(\bar u_x + \frac{\bar u}{x}\right)_x = 0, \tag{3.21}$$

$$\bar\rho\left(\bar v_t + \bar u\bar v_x + \frac{\bar u\bar v}{x}\right) = 0, \tag{3.22}$$

$$\bar\rho(\bar w_t + \bar u\bar w_x) = 0, \tag{3.23}$$

supplemented with the initial and boundary conditions

$$(\bar\rho, \bar u, \bar v, \bar w)|_{t=0} = (\rho_0, u_0, v_0, w_0)(x), \quad x \in \Omega := (a, b), \tag{3.24}$$

$$\bar u|_{x=a} = \bar u|_{x=b} = 0. \tag{3.25}$$

By the similar arguments as those in Section 2 and [7], we have the following uniform estimates for the solution $(\bar\rho, \bar u, \bar v, \bar w)$ of the limit problem (3.20)–(3.25), provided that the initial–boundary conditions stated in Theorem 1.1 are satisfied.

Lemma 3.1. *Under the assumptions of Theorem* 1.1*, we have*

$$\big\|(\bar u, \bar v, \bar w, \bar u_x)\big\|_{L^\infty} \leqslant C, \qquad C^{-1} \leqslant \bar\rho \leqslant C, \qquad \int_a^b \bar\rho_x^2\,dx \leqslant C,$$

$$\int_a^b \bar u_x^2\,dx + \int_0^t\int_a^b \big(\bar u_t^2 + \bar u_{xx}^2\big)\,dx\,d\tau \leqslant C,$$

$$\int_a^b \bar u_t^2\,dx + \int_a^b \bar u_{xx}^2\,dx + \int_0^t\int_a^b \bar u_{xt}^2\,dx\,d\tau \leqslant C,$$

and

$$\int_a^b \big(\bar v_x^2 + \bar w_x^2\big)\,dx \leqslant C.$$

Furthermore, assume in addition that $(v_0, w_0) \in H^3(\Omega)$ *and* $\rho_0 \in H^2(\Omega)$*, we have*

$$\int_a^b \bar\rho_{xx}^2\,dx \leqslant C, \qquad \int_0^t\int_a^b \bar u_{xxx}^2\,dx\,d\tau \leqslant C,$$

$$\int_a^b \big(|\bar v_{xx}|^2 + |\bar w_{xx}|^2\big)\,dx \leqslant C,$$

and

$$\int_a^b \big(|\bar v_{xxx}|^2 + |\bar w_{xxx}|^2\big)\,dx \leqslant C.$$

Similar to that in [7], we deduce from $(1.1)_1$ and (3.20) that

$$(\rho-\bar\rho)_t+u(\rho-\bar\rho)_x+(u-\bar u)\bar\rho_x+u_x(\rho-\bar\rho)+\bar\rho(u-\bar u)_x+\frac{u(\rho-\bar\rho)}{x}+\frac{\bar\rho(u-\bar u)}{x}=0.$$

Multiplying the above equation by $(\rho-\bar\rho)$ and integrating by parts over $(a,b)\times(0,t)$, we have

$$\int_a^b|\rho-\bar\rho|^2\,dx\leqslant C\int_0^t\int_a^b\left|(u-\bar u)_x\right|^2dx\,d\tau, \tag{3.26}$$

where we have used Sobolev's inequality and Gronwall's inequality.

Similarly, we have by $(1.1)_2$, $(1.1)_3$, $(1.1)_4$, (3.21), (3.22) and (3.23) that

$$(v-\bar v)_t+u(v-\bar v)_x+(u-\bar u)\bar v_x+\frac{u(v-\bar v)}{x}+\frac{\bar v(u-\bar u)}{x}=\frac{\mu}{\rho}\left(v_x+\frac{v}{x}\right)_x+\frac{\mu_x}{\rho}v_x-\frac{\mu_x}{\rho}\frac{v}{x},$$
$$(w-\bar w)_t+u(w-\bar w)_x+(u-\bar u)\bar w_x=\frac{\mu}{\rho}\left(w_{xx}+\frac{w_x}{x}\right)+\frac{\mu_x}{\rho}w_x,$$

and

$$(u-\bar u)_t+u(u-\bar u)_x+\bar u_x(u-\bar u)-\frac{(v+\bar v)(v-\bar v)}{x}+\rho^{-1}\left(\rho^\gamma-\bar\rho^\gamma\right)_x+\left(\rho^{-1}-\bar\rho^{-1}\right)\left(\bar\rho^\gamma\right)_x$$
$$-\frac{\lambda}{\rho}\left((u-\bar u)_x+\frac{u-\bar u}{x}\right)_x+\frac{\lambda(\rho-\bar\rho)}{\rho\bar\rho}\left(\bar u_x+\frac{\bar u}{x}\right)_x-\frac{2\mu}{\rho}\left(u_x+\frac{u}{x}\right)_x-\frac{2}{\rho}\mu_x u_x=0.$$

By using the method similar to that in [7], we can easily get

$$\int_a^b\left(|v-\bar v|^2+|w-\bar w|^2\right)dx\leqslant C\left(\varepsilon^{\frac12}+\int_0^t\int_a^b\left|(u-\bar u)_x\right|^2dx\,d\tau\right), \tag{3.27}$$

and

$$\frac{d}{dt}\int_a^b|u-\bar u|^2\,dx+\int_a^b\left|(u-\bar u)_x\right|^2dx$$
$$\leqslant C\left(\varepsilon^2+\int_a^b\left|(\rho-\bar\rho,u-\bar u,v-\bar v)\right|^2dx+\int_a^b|u-\bar u|^2\left(\rho_x^2+\bar\rho_x^2+\bar u_{xx}^2\right)dx\right). \tag{3.28}$$

Here we have used Lemma 3.1 and the estimates in Section 2.

On the other hand, we get from Lemmas 2.4 and 3.1 that

$$C\int_a^b|u-\bar u|^2\left(\rho_x^2+\bar\rho_x^2+\bar u_{xx}^2\right)dx\leqslant C\|u-\bar u\|_{L^\infty}^2\leqslant C\left\|(u-\bar u)\right\|_{L^2}^2+\frac12\left\|(u-\bar u)_x\right\|_{L^2}^2.$$

Then, we obtain from (3.26)–(3.28) that

$$\frac{d}{dt}\int_a^b|u-\bar u|^2\,dx+\int_a^b\left|(u-\bar u)_x\right|^2dx$$
$$\leqslant C\left(\varepsilon^{\frac12}+\int_a^b|u-\bar u|^2\,dx+\int_0^t\int_a^b\left|(u-\bar u)_x\right|^2dx\,d\tau\right). \tag{3.29}$$

Summing (3.26), (3.27) and (3.29) up, we get the following result on convergence rates of the vanishing ε (in this paper, vanishing ε implies vanishing shear viscosity μ).

Theorem 3.1. *Under the assumptions of Theorem* 1.1*, we have*

$$\int_a^b \left|(\rho-\bar\rho, u-\bar u, v-\bar v, w-\bar w)\right|^2 dx + \int_0^t\int_a^b \left|(u-\bar u)_x\right|^2 dx\, d\tau \leqslant C\varepsilon^{\frac12},$$

where C is a positive constant independent of ε.

Now, we can discuss the result on BL-thickness for the general initial data. Before we give the proof of Theorem 1.3, we give some useful estimates.

Lemma 3.2. *Under the assumptions of Theorem* 1.3*, we have*

$$\varepsilon^{\frac12}\int_a^b \rho_{xx}^2 \leqslant C. \tag{3.30}$$

Proof. From $(1.1)_1$ and $(1.1)_2$, we have

$$(2\mu+\lambda)\rho_{xt} = -2(2\mu+\lambda)\rho_x u_x - (2\mu+\lambda)\rho_{xx}u - (2\mu+\lambda)\frac{\rho_x u}{x} - \rho^2(u_t+uu_x) + \frac{\rho^2 v^2}{x} + 2\rho\mu_x u_x - \rho p_x. \tag{3.31}$$

Differentiating (3.31) with respect to x, then multiplying the resulting equation by ρ_{xx}, we have

$$\begin{aligned}
&\rho_{xx}\left\{(2\mu+\lambda)\rho_{xxt} + 2\mu_x\rho_{xt}\right\}\\
&\quad= \bigg\{-4\mu_x\rho_x u_x - 3(2\mu+\lambda)\rho_{xx}u_x - 2(2\mu+\lambda)\rho_x u_{xx}\\
&\qquad - 2\mu_x\rho_{xx}u - (2\mu+\lambda)\rho_{xxx}u - 2\mu_x\frac{\rho_x u}{x} + (2\mu+\lambda)\frac{\rho_x u}{x^2} - (2\mu+\lambda)\frac{\rho_{xx}u}{x}\\
&\qquad - (2\mu+\lambda)\frac{\rho_x u_x}{x} - 2\rho\rho_x(u_t+uu_x) - \rho^2 u_{tx} - \rho^2 u_x^2 - \rho^2 uu_{xx} - \frac{\rho^2 v^2}{x^2} + \frac{2\rho\rho_x v^2}{x}\\
&\qquad + \frac{2\rho^2 vv_x}{x} + 2\rho_x\mu_x u_x + 2\rho\mu_{xx}u_x + 2\rho\mu_x u_{xx} - (\rho p_x)_x\bigg\}\rho_{xx}.
\end{aligned} \tag{3.32}$$

Integrating (3.32) over (a,b), we have by using integration by parts and the previous lemmas that

$$\begin{aligned}
&\frac12\frac{d}{dt}\int_a^b (2\mu+\lambda)\rho_{xx}^2\, dx\\
&\quad= \int_a^b \bigg(\mu_t\rho_{xx}^2 - 2\mu_x\rho_{xx}\rho_{xt} - 2\mu_x\rho_{xx}^2 u + \mu_x\rho_{xx}^2 u - 4\mu_x\rho_x u_x\rho_{xx} - 2\mu_x\frac{\rho_x u\rho_{xx}}{x}\\
&\qquad + 2\rho_x\mu_x u_x\rho_{xx} + 2\rho\mu_{xx}u_x\rho_{xx} + 2\rho\mu_x u_{xx}\rho_{xx}\bigg)dx\\
&\qquad + \int_a^b \left(-\frac52(2\mu+\lambda)\rho_{xx}u_x - 2(2\mu+\lambda)\rho_x u_{xx}\right)\rho_{xx}\, dx\\
&\qquad + \int_a^b \left((2\mu+\lambda)\frac{\rho_x u}{x^2} - (2\mu+\lambda)\frac{\rho_{xx}u}{x} - (2\mu+\lambda)\frac{\rho_x u_x}{x}\right)\rho_{xx}\, dx
\end{aligned}$$

$$+\int_a^b\left(-2\rho\rho_x(u_t+uu_x)-\rho^2u_{tx}-\rho^2u_x^2-\rho^2uu_{xx}-\frac{\rho^2v^2}{x^2}+\frac{2\rho\rho_xv^2}{x}+\frac{2vv_x\rho^2}{x}-(\rho p_x)_x\right)\rho_{xx}\,dx$$

$$\leqslant C\int_a^b\big(1+|\mu_x\rho_x\rho_{xx}|+\rho_{xx}^2+|u_{xx}\rho_x\rho_{xx}|+|\rho_xu_t\rho_{xx}|+|u_{tx}\rho_{xx}|+|u_{xx}\rho_{xx}|$$
$$+|v_x\rho_{xx}|+|\mu_{xx}\rho_{xx}|+|\mu_xu_{xx}\rho_{xx}|+\rho_x^2|\rho_{xx}|+|\rho_x\rho_{xx}|\big)\,dx$$
$$\leqslant C\int_a^b\big(1+\rho_{xx}^2+|u_{xx}\rho_x\rho_{xx}|+|\rho_xu_t\rho_{xx}|+|u_{tx}\rho_{xx}|+|u_{xx}\rho_{xx}|$$
$$+|v_x\rho_{xx}|+\rho_x^2|\rho_{xx}|+|\rho_x\rho_{xx}|\big)\,dx. \tag{3.33}$$

Here we have used the representation $\mu_{xx}=\varepsilon\theta(\theta-1)\rho^{\theta-2}\rho_x^2+\varepsilon\theta\rho^{\theta-1}\rho_{xx}$, the uniform estimates for (ρ,u,v,w,u_x) and the following equality

$$\int_a^b(2\mu+\lambda)\rho_{xxx}\rho_{xx}u\,dx=-\frac{1}{2}\int_a^b(2\mu+\lambda)\rho_{xx}^2u_x\,dx-\int_a^b\mu_x\rho_{xx}^2u\,dx.$$

Noticing that by Sobolev's inequality, we have

$$\int_a^b\big(\rho_x^2+|\rho_x||u_{xx}|+|\rho_x||u_t|\big)|\rho_{xx}|\,dx\leqslant C\left(\int_a^b\rho_{xx}^2+\|\rho_x\|_{L^\infty}^2\int_a^b\big(\rho_x^2+u_{xx}^2+u_t^2\big)\,dx\right)$$
$$\leqslant C\left(1+\int_a^b\rho_{xx}^2\,dx\right), \tag{3.34}$$

and

$$\int_a^b\big(|u_{tx}|+|u_{xx}|+|v_x|\big)|\rho_{xx}|\,dx\leqslant C\left(\int_a^b\rho_{xx}^2\,dx+\int_a^b\big(u_{tx}^2+u_{xx}^2+v_x^2\big)\,dx\right)$$
$$\leqslant C\left(1+\int_a^b u_{tx}^2\,dx+\int_a^b\rho_{xx}^2\,dx+\int_a^b v_x^2\,dx\right). \tag{3.35}$$

Substituting (3.34) and (3.35) into (3.33) and using the Cauchy–Schwartz inequality, we have

$$\frac{d}{dt}\int_a^b\rho_{xx}^2\,dx\leqslant C+C\int_a^b\rho_{xx}^2\,dx+C\int_a^b u_{tx}^2\,dx+C\int_a^b v_x^2\,dx,$$

which implies

$$\frac{d}{dt}\int_a^b\varepsilon^{\frac{1}{2}}\rho_{xx}^2\,dx\leqslant C+C\int_a^b\varepsilon^{\frac{1}{2}}\rho_{xx}^2\,dx+C\int_a^b u_{tx}^2\,dx+C\int_a^b\varepsilon^{\frac{1}{2}}v_x^2\,dx.$$

By Gronwall's inequality, Lemmas 2.6 and 2.7, we deduce (3.30) and the proof of Lemma 3.2 is completed. □

Next, we give the proof of Theorem 1.3.

From $(1.1)_3$ and (3.22), we have

$$v_{xt}+(uv_x)_x+\left(\frac{uv}{x}\right)_x-\left(\frac{\mu}{\rho}v_{xx}\right)_x-\left(\frac{\mu v_x}{\rho x}\right)_x+\left(\frac{\mu v}{\rho x^2}\right)_x-\left(\frac{\mu_x v_x}{\rho}\right)_x+\left(\frac{v\mu_x}{\rho x}\right)_x=0,$$
$$\bar v_{xt}+(\bar u\bar v_x)_x+\left(\frac{\bar u\bar v}{x}\right)_x=0.$$

Denoting $z_2:=(v-\bar v)_x-\frac{v-\bar v}{x}$, we get

$$\begin{aligned}
&z_{2t}+(uz_2)_x+\left(\frac{u(v-\bar v)}{x}\right)_x+\left(\frac{\mu(v-\bar v)}{\rho x^2}\right)_x-\left(\frac{\mu}{\rho}z_{2x}\right)_x\\
&\quad-\left(\frac{\mu z_2}{\rho x}\right)_x-\left(\frac{1}{\rho}\mu_x z_2\right)_x+\frac{1}{x}\left\{\frac{\mu v_x\rho_x}{\rho^2}+2\frac{\mu v_x}{\rho x}-\frac{\mu v}{\rho x^2}-\frac{\mu_x v}{\rho x}\right\}\\
&\quad=\bar v_{xxx}\frac{\mu}{\rho}+\bar v_{xx}\left(\frac{\mu_x}{\rho}-\frac{\mu\rho_x}{\rho^2}\right)-\bar v_x\frac{\mu}{\rho x^2}-\bar v\left(\frac{\mu_x}{\rho x^2}-\frac{\mu\rho_x}{\rho^2x^2}-2\frac{\mu}{\rho x^3}\right)+\bar v_{xx}\frac{\mu_x}{\rho}\\
&\qquad+\bar v_x\left(\frac{\mu_{xx}}{\rho}-\frac{\mu_x\rho_x}{\rho^2}\right)+\frac{2}{x^2}\left\{u(v-\bar v)+\bar v(u-\bar u)\right\}-\frac{1}{x}\left\{(u_x-\bar u_x)v+\bar u_x(v-\bar v)\right\}\\
&\qquad-(u-\bar u)_x\bar v_x-(u-\bar u)\bar v_{xx}-\left(\frac{\bar v_x}{\rho x}-\frac{\bar v\rho_x}{\rho^2x}-\frac{\bar v}{\rho x^2}\right)\mu_x-\mu_{xx}\frac{\bar v}{\rho x}:=\hat R.
\end{aligned}\tag{3.36}$$

Multiplying (3.36) by $\xi_\delta(x)\varphi_\nu'(z_2)$ and integrating the resulting equation over $(a,b)\times(0,t)$, we have

$$\begin{aligned}
\int_a^b\xi_\delta(x)\varphi_\nu(z_2)\,dx&=\nu\int_a^b\xi_\delta(x)\,dx+\int_0^t\int_a^b uz_2z_{2x}\varphi_\nu''(z_2)\xi_\delta(x)\,dx\,d\tau+\int_0^t\int_a^b uz_2\varphi_\nu'(z_2)\xi_\delta'(x)\,dx\,d\tau\\
&\quad-\int_0^t\int_a^b\frac{u}{x}z_2\varphi_\nu'(z_2)\xi_\delta(x)\,dx\,d\tau-\int_0^t\int_a^b\frac{u_x}{x}(v-\bar v)\varphi_\nu'(z_2)\xi_\delta(x)\,dx\,d\tau\\
&\quad-\int_0^t\int_a^b\frac{\mu}{\rho}z_{2x}^2\varphi_\nu''(z_2)\xi_\delta(x)\,dx\,d\tau-\int_0^t\int_a^b\frac{\mu z_{2x}}{\rho}\varphi_\nu'(z_2)\xi_\delta'(x)\,dx\,d\tau\\
&\quad-\int_0^t\int_a^b\frac{\mu z_2}{x\rho}z_{2x}\varphi_\nu''(z_2)\xi_\delta(x)\,dx\,d\tau-\int_0^t\int_a^b\frac{\mu z_2}{x\rho}\varphi_\nu'(z_2)\xi_\delta'(x)\,dx\,d\tau\\
&\quad+\int_0^t\int_a^b\frac{\mu(v-\bar v)}{\rho x^2}z_{2x}\varphi_\nu''(z_2)\xi_\delta(x)\,dx\,d\tau+\int_0^t\int_a^b\frac{\mu(v-\bar v)}{\rho x^2}\varphi_\nu'(z_2)\xi_\delta'(x)\,dx\,d\tau\\
&\quad-\int_0^t\int_a^b\frac{\mu_x z_2}{\rho}z_{2x}\varphi_\nu''(z_2)\xi_\delta(x)\,dx\,d\tau-\int_0^t\int_a^b\frac{\mu_x z_2}{\rho}\varphi_\nu'(z_2)\xi_\delta'(x)\,dx\,d\tau\\
&\quad-\int_0^t\int_a^b\frac{1}{x}\left\{\frac{\mu v_x\rho_x}{\rho^2}+2\frac{\mu v_x}{\rho x}-\frac{\mu v}{\rho x^2}-\frac{\mu_x v}{\rho x}\right\}\xi_\delta(x)\varphi_\nu'(z_2)\,dx\,d\tau\\
&\quad+\int_0^t\int_a^b\hat R\xi_\delta(x)\varphi_\nu'(z_2)\,dx\,d\tau.
\end{aligned}\tag{3.37}$$

We find the last term on the right-hand side of (3.37) can be bounded by using Theorem 3.1, Lemmas 3.1 and 3.2 as follows:

$$\left|\int_0^t\int_a^b \hat{R}\xi_\delta(x)\varphi_\nu'(z_2)\,dx\,d\tau\right| \leqslant C\left(\varepsilon^{\frac{3}{4}}+\int_0^t\int_a^b\left(|u-\bar{u}|+|v-\bar{v}|+\left|(u-\bar{u})_x\right|\right)dx\,d\tau\right)$$
$$\leqslant C\left(\varepsilon^{\frac{3}{4}}+\left(\int_0^t\int_a^b\left(|u-\bar{u}|^2+|v-\bar{v}|^2+\left|(u-\bar{u})_x\right|^2\right)dx\,d\tau\right)^{\frac{1}{2}}\right)$$
$$\leqslant C\varepsilon^{\frac{1}{4}}. \tag{3.38}$$

The other terms on the right-hand side of (3.37) can be dealt with by employing arguments similar to those used in the derivation of (3.16), then we have

$$\int_a^b \xi_\delta(x)|z_2|\,dx \leqslant C\int_0^t\int_a^b \xi_\delta(x)|z_2|\,dx\,d\tau + C\varepsilon^{\frac{1}{4}}, \tag{3.39}$$

where we have also used Theorem 3.1 to get that

$$\int_a^b |v-\bar{v}|\,dx \leqslant C\left(\int_a^b |v-\bar{v}|^2\,dx\right)^{\frac{1}{2}} \leqslant C\varepsilon^{\frac{1}{4}}.$$

Thus, we conclude from (3.39) that

$$\delta\int_{a+\delta}^{b-\delta}\left|\left(\frac{v-\bar{v}}{x}\right)_x\right|x\,dx \leqslant \int_a^b \xi_\delta(x)\left|\left(\frac{v-\bar{v}}{x}\right)_x\right|x\,dx = \int_a^b \xi_\delta(x)|z_2|\,dx \leqslant C\varepsilon^{\frac{1}{4}}. \tag{3.40}$$

An analogous estimate also holds for $(w-\bar{w})_x$ by using similar arguments, which is simpler (where we let $z_3 := (w-\bar{w})_x$). Therefore, we have proved Theorem 1.3 on the BL-thickness for the general initial data.

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