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The strong minimum principle for quasisuperminimizers of non-standard growth

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Abstract

We prove the strong minimum principle for non-negative quasisuperminimizers of the variable exponent Dirichlet energy integral under the assumption that the exponent has modulus of continuity slightly more general than Lipschitz. The proof is based on a new version of the weak Harnack estimate.

Résumé

Nous prouvons le fort principe du minimum pour des quasisuperminimizeurs non-négatifs de problème de Dirichlet de l'exposant variable en supposant que l'exposant a le module de continuité un peu plus général que Lipschitz. La démonstration est fondée sur une nouvelle version de la faible inégalité de Harnack.

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1. Introduction

It is well known that solutions of linear and non-linear PDE as well as (quasi)minimizers of variational integrals often satisfy the strong maximum principle: a bounded non-constant continuous solution u cannot attain its maximum or minimum in a domain. For the minimum and a non-negative solution u the claim follows easily from Harnack's inequality, which states that

$$\sup_B u \leqslant C \inf_B u$$

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for all small balls $B \Subset \Omega$. In fact, Harnack's inequality implies that the set $\{u = 0\}$ is open and the continuity of u guarantees that the set $\{u > 0\}$ is open. If we are only interested in the minimum principle, it is enough to consider supersolutions or quasisuperminimizers and similar reasoning works since non-negative lower semicontinuous representatives satisfy the weak Harnack inequality

$$\left(\fint_{2B} u^h\,dx\right)^{1/h} \leqslant C \inf_B u \tag{1.1}$$

for some exponent $h > 0$ and for all small balls $B \Subset \Omega$. This holds in particular for quasiminimizers and quasisuperminimizers of the p-Dirichlet energy integral

$$\int_\Omega |\nabla u|^p\,dx$$

for $p > 1$; see [4].

In this note we consider quasisuperminimizers of the $p(\cdot)$-Dirichlet energy integral

$$\int_\Omega |\nabla u|^{p(x)}\,dx.$$

Such energies arise for instance in fluid dynamics [23] and image processing [20]. This case is much more complicated and the strong minimum principle is an open question in general. The problems arise from two facts: (weak) Harnack estimates include an additional term; and the homogeneity is missing, that is, the set of K-quasisuperminimizers is not closed under multiplication with a positive number, cf. Remark 3.3. Recall that in [15], P. Harjulehto, T. Kuusi, T. Lukkari, N. Marola, and M. Parviainen proved by De Giorgi's method that non-negative quasiminimizers satisfy Harnack's inequality

$$\sup_{Q_R} u \leqslant C\big(\|u\|_{L^\varepsilon}\big)\Big(\inf_{Q_R} u + R\Big)$$

whenever the cube Q_R with the side-length R is small enough and p is log-Hölder continuous with $1 < p^- \leqslant p^+ < \infty$. Using Moser's iteration this inequality was obtained earlier by Yu. Alkhutov [3] (for $\varepsilon = \infty$) and then by P. Harjulehto, J. Kinnunen and T. Lukkari [14] (for $\varepsilon > 0$ arbitrarily small).

It is known that the constant C cannot be independent of u, see [14, Example 3.10]. It is not known whether the additional R on the right hand side is needed or not; however, all known regularity proofs result in the additional term R. The same additional term R appears also in the proofs of the weak Harnack estimate analogous to (1.1). Notice that the strong minimum principle can be proved without the weak Harnack estimate by potential theoretic tools, see e.g. [19, Theorem 4.1], but this approach requires the homogeneity.

As far as we know, the only proof of the strong maximum principle in the variable exponent case is by a direct method, i.e. by choosing suitable test functions. This result is due to X.-L. Fan, Y.Z. Zhao and Q.-H. Zhang [9] under the assumption that $p \in C^1(\overline{\Omega})$ with $1 < p^- \leqslant p^+ < \infty$. Recently, R. Fortini, D. Mugnai and P. Pucci [10] were able to prove the *weak* maximum principle for subsolutions of more general equations under the assumption that p is only log-Hölder continuous.

In this paper we prove the strong maximum and minimum principle in the variable exponent case using Harnack's inequality. Our results apply to the larger class of quasisubminimizers, rather than subminimizers as in the previous papers. Our proof relies on new versions of the weak Harnack estimate (Theorem 4.6) with more precise control of the error term based on the modulus of continuity of the exponent p. In Section 5, we consider Dini-type continuity conditions on p, including for instance the case

$$\big|p(x) - p(y)\big| \leqslant c|x-y| \log\left(e + \frac{1}{|x-y|}\right)$$

which is slightly weaker than the Lipschitz continuity. In this case we obtain the error term $\exp(-1/(R \log R))$ which is so small that the strong minimum principle can be achieved by an iterative process, see Theorem 5.3. For quasiminimizers we obtain similarly the strong maximum principle.

Clearly our assumption on the exponent is stronger than the log-Hölder continuity. However, we note that there exist functions with this continuity modulus which are nowhere differentiable; an example is the Tagaki function, cf. [16, Theorem 4]. Hence our condition is substantially weaker than the assumption $p \in C^1(\overline{\Omega})$ used in [9]. Notice also that our main result can be formulated generally, independent of variable exponent spaces: a certain weak Harnack estimate with an additive error implies the strong minimum principle whenever the additional term is sufficiently small.

2. Preliminaries

The results of this section can be found in [6]; most were first proved in [17].

By $Q(x,r)$ we denote an open cube centered at x with sides parallel to the coordinate axes of length $2r$. By $f \approx g$ we mean that there exists a constant $c > 0$ such that $\frac{1}{c} f \leqslant g \leqslant cf$. By $\Omega \subset \mathbb{R}^n$ we always denote a bounded open set. A bounded measurable function $p : \Omega \to [1,\infty)$ is called a *variable exponent*, and we denote for $A \subset \Omega$

$$p_A^+ := \sup_{x\in A} p(x), \qquad p_A^- := \inf_{x\in A} p(x), \qquad p^+ := \sup_{x\in\Omega} p(x), \qquad p^- := \inf_{x\in\Omega} p(x).$$

We define a *modular* by setting

$$\varrho_{p(\cdot)}(f) := \int_\Omega \left|f(x)\right|^{p(x)}\,dx.$$

The *variable exponent Lebesgue space* $L^{p(\cdot)}(\Omega)$ consists of all measurable functions defined on Ω for which the modular is finite. The Luxemburg norm on this space is defined as

$$\|f\|_{p(\cdot)} := \inf\left\{\lambda > 0\colon\ \varrho_{p(\cdot)}\left(\frac{f}{\lambda}\right) \leqslant 1\right\}.$$

Equipped with this norm $L^{p(\cdot)}(\Omega)$ is a Banach space. The variable exponent Lebesgue space is a special case of a Musielak–Orlicz space. For a constant function p it coincides with the standard Lebesgue space.

The *variable exponent Sobolev space* $W^{1,p(\cdot)}(\Omega)$ consists of functions $u \in L^{p(\cdot)}(\Omega)$ whose distributional gradient ∇u belongs to $L^{p(\cdot)}(\Omega)$. The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is a Banach space with the norm

$$\|u\|_{1,p(\cdot)} := \|u\|_{p(\cdot)} + \|\nabla u\|_{p(\cdot)}.$$

If E is a measurable set with a finite measure, and $p \leqslant q$ are variable exponents, then $L^{p(\cdot)}(E)$ embeds continuously into $L^{q(\cdot)}(E)$. In particular this implies that every function $u \in W^{1,p(\cdot)}(\Omega)$ also belongs to $W^{1,p_\Omega^-}(\Omega)$. The variable exponent Hölder inequality takes the form

$$\int_\Omega fg\,dx \leqslant 2\|f\|_{p(\cdot)}\|g\|_{p'(\cdot)},$$

$1/p(x) + 1/p'(x) \equiv 1$.

The variable exponent p is said to be log-*Hölder continuous* if there is a constant $C_{\log}$ such that

$$\left|p(x) - p(y)\right| \leqslant \frac{C_{\log}}{\log(e + 1/|x-y|)}$$

for all $x, y \in \Omega$. The importance of this condition was realized by Diening [5]. A crucial fact is (see [6, p. 101]) that there is a constant $C > 0$ such that

$$|B|^{p_B^-} \approx |B|^{p_B^+} \tag{2.1}$$

if and only if p is log-Hölder continuous (note that we consider only the case of bounded domains). Under the log-Hölder condition smooth functions are dense in variable exponent Sobolev spaces and there is no confusion in defining the Sobolev space with zero boundary values, $W_0^{1,p(\cdot)}(\Omega)$, as the completion of $C_0^\infty(\Omega)$ with respect to the norm $\|u\|_{1,p(\cdot)}$.

We assume throughout this paper that p is log-Hölder continuous and $1 < p^- \leqslant p^+ < \infty$.

3. Regularity of quasisuperminimizers for log-Hölder continuous exponent

We recall first some (essentially known) auxiliary results for quasisuperminimizers. In particular we need the Lebesgue point property for quasisuperminimizers. For more results on PDE with non-standard growth we refer to the papers [1,2,7,11,21,22,24–27] or the survey [13].

Definition 3.1. Let $\Omega \subset \mathbb{R}^n$ be an open set and $K \geqslant 1$. A function $u \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ is a *K-quasiminimizer in Ω*, if

$$\int_{\{v\neq 0\}} |\nabla u|^{p(x)}\,dx \leqslant K \int_{\{v\neq 0\}} \big|\nabla(u+v)\big|^{p(x)}\,dx$$

for all functions $v \in W^{1,p(\cdot)}(\Omega)$ with compact support in Ω. If the inequality holds only for all non-negative or non-positive v, then u is called a *K-quasisuperminimizer* or *K-quasisubminimizer* in Ω, respectively.

The following lemma is needed in Section 5.

Lemma 3.2. *If u is a K-quasisuperminimizer in Ω and $\alpha \in \mathbb{R}$, then* $\min(\alpha, u)$ *is a K-quasisuperminimizer in Ω.*

Proof. Let $v \in W^{1,p(\cdot)}(\Omega)$ be a non-negative function with a compact support. We denote $\Omega' := \{x \in \Omega\colon\ v(x) \neq 0\}$,

$$U := \big\{x \in \Omega'\colon\ u(x) > \alpha\big\}, \qquad V := \big\{x \in \Omega'\colon\ \min\big(u(x), \alpha\big) + v(x) > u(x)\big\},$$

and $\varphi := (\min(u,\alpha) + v - u)_+$. Then $V = \{x \in \Omega'\colon\ \varphi(x) > 0\}$ and Definition 3.1 implies that

$$\int_V |\nabla u|^{p(x)}\,dx \leqslant K \int_V \big|\nabla(u+\varphi)\big|^{p(x)}\,dx = K \int_V \big|\nabla \min(u,\alpha) + \nabla v\big|^{p(x)}\,dx.$$

Since $U \cup V = \Omega'$, we conclude that

$$\begin{aligned}\int_{\Omega'} \big|\nabla \min(u,\alpha)\big|^{p(x)}\,dx &= \int_{V\setminus U} |\nabla u|^{p(x)}\,dx \leqslant \int_V |\nabla u|^{p(x)}\,dx \\ &\leqslant K \int_{\Omega'} \big|\nabla\big(\min(u,\alpha)+v\big)\big|^{p(x)}\,dx. \qquad \Box\end{aligned}$$

Remark 3.3. If u is a K-quasisuperminimizer and $\alpha > 0$, then αu is a quasisuperminimizer with constant $\max(\alpha^{p^+-p^-}, \alpha^{p^--p^+})K$ depending on α. We skip the easy proof of this property since we do not need it in this paper. The problem here is that the quasisuperminimizing constant of αu depends on α.

We recall from [15] the basic weak Harnack estimates for quasisuperminimizers. In [15], the authors study quasiminimizers, but many of their auxiliary results hold in this more general setting.

Fix a point $x_0 \in \Omega$ and denote $Q_r := Q(x_0, r)$. Throughout the rest of the paper we work in a cube $Q := Q_{2R} \Subset \Omega$. We assume that $R \leqslant \frac{1}{2}$, $1 < q < \frac{n}{n-1}$, and choose Q so small that

$$\int_Q |u|^{p(x)}\,dx \leqslant 1 \quad \text{and} \quad \int_Q |\nabla u|^{p(x)}\,dx \leqslant 1. \tag{3.4}$$

Further, we write

$$\varepsilon = 1 - qp^-/\big(p^-\big)^* \quad \text{and} \quad \delta = p^-/p^+ + \varepsilon - 1. \tag{3.5}$$

Note that p^+ may be viewed as p^+_Q since we are only concerned with Q, similarly for p^-.

The following supremum estimate was proved in [15, Theorem 4.14]. Notice in the original proof that only the quasisubminimizing property is needed.

Theorem 3.6. *Let u be a K-quasisubminimizer in Ω and let $s > p^+ - p^-$. Then for every $l \in (0, qp^-)$ and $\varrho < R$,*

$$\operatorname*{ess\,sup}_{Q_\varrho} u \leqslant k_0 + C\left(\frac{1}{(R-\varrho)^n}\int_{Q_R}(u-k_0)_+^l\,dx\right)^{\delta/((\varepsilon-\delta)qp^- + l\delta)}.$$

The constant C depends on l, n, $p(\cdot)$, q, K and the $L^{q's}(Q)$-norm of u.

Theorem 3.6 leads to the following homogenized weak Harnack inequality, see [15, Theorem 5.7].

Theorem 3.7. *Let u be a non-negative quasisuperminimizer in Ω. Then there exist an exponent $h > 0$ and a constant C, both depending on n, $p(\cdot)$, q, K, and the $L^{q's}(Q)$-norm of u, such that*

$$\left(\fint_{Q_R} u^h\,dx\right)^{1/h} \leqslant C\Big(\operatorname*{ess\,inf}_{Q_{R/2}} u + R\Big)$$

for every cube Q_R for which $Q_{10R} \subset \Omega$, $R \leqslant \frac{1}{2}$ and (3.4) *holds.*

We close this preliminary section by pointing out that locally bounded quasisuperminimizers have Lebesgue points everywhere.

Lemma 3.8. *Let u be a quasisuperminimizer in Ω. Then*

$$u^*(x) := \lim_{r\to 0}\operatorname*{ess\,inf}_{B(x,r)} u$$

defines a lower semicontinuous representative of u. Moreover, if u is essentially locally upper bounded, then u^* has Lebesgue points everywhere in Ω.

Proof. The proof follows by imitating the standard argument. Notice first that quasisuperminimizers are essentially bounded from below by Theorem 3.6. Hence we may follow the proof of [14, Theorem 4.1] and conclude that u^* is a lower semicontinuous representative of u.

Assume that u is essentially bounded from above. Then by the proof of [14, Theorem 4.1],

$$u^*(x) = \lim_{r\to 0}\fint_{B(x,r)} u\,dy,$$

and the estimate

$$\fint_{B(x,r)}\big|u^*(y) - u^*(x)\big|\,dy \leqslant \fint_{B(x,r)}\big|u^*(y) - m_r\big|\,dy + \big|m_r - u^*(x)\big|$$

implies that

$$\lim_{r\to 0}\fint_{B(x,r)}\big|u^*(y) - u^*(x)\big|\,dy = 0.$$

Here we denoted $m_r := \operatorname{ess\,inf}_{B(x,r)} u$. □

4. Regularity of quasisuperminimizers

This section includes a new version of the weak Harnack estimate with better and more precise control of the error term based on the modulus of continuity of the exponent p. If p is only log-Hölder continuous, then we regain previous results of [15].

In addition to the assumptions $1 < p^- \leqslant p^+ < \infty$, we assume throughout this section that

$$\big|p(x) - p(y)\big| \leqslant \omega\big(|x-y|\big) \leqslant \frac{c}{\log(1+\frac{1}{t})} \tag{4.1}$$

for all $x, y \in \Omega$ and some $c > 0$, where $\omega : [0, \infty) \to [0, \infty)$ is a modulus of continuity, i.e. a continuous increasing function with $\omega(0) = 0$. Thus, in particular, p is log-Hölder continuous.

We start by homogenizing the inequality of Theorem 3.6; compared to the results in [15], the additional error term R is replaced by $e^{-1/\omega(R)}$.

Lemma 4.2. *Let ω be the modulus of continuity of p and let u be a K-quasisubminimizer in Ω. Then for every $s > p^+ - p^-$ and for every $l \in (0, qp^-)$ and $\varrho < R$, we have*

$$\operatorname*{ess\,sup}_{Q_\varrho} u \leqslant k_0 + e^{-1/\omega(R)} + \left(\frac{C}{(R-\varrho)^n} \int_{Q_R} (u-k_0)_+^l \, dx \right)^{1/l}.$$

The constant C depends on l, n, $p(\cdot)$, q, K and the $L^{q's}(Q)$-norm of u.

Proof. Let

$$\alpha := \frac{\delta}{(\varepsilon - \delta)qp^- + l\delta},$$

where $\delta, \varepsilon > 0$ are defined in (3.5). We obtain by Theorem 3.6 and Young's inequality that

$$\begin{aligned} \operatorname*{ess\,sup}_{Q_\varrho} u &\leqslant k_0 + C\left(\frac{\theta}{\theta(R-\varrho)^n} \int_{Q_R} (u-k_0)_+^l \, dx \right)^{\alpha} \\ &\leqslant k_0 + C\left(\frac{\theta}{(R-\varrho)^n} \int_{Q_R} (u-k_0)_+^l \, dx \right)^{\frac{1}{l}} + \theta^{-\frac{\alpha}{1-\alpha l}}. \end{aligned}$$

By the definition of α, δ and ε, we find that

$$\frac{1}{1-\alpha l} > \frac{l\delta}{(\varepsilon-\delta)qp^-} = \frac{p^+ l\delta}{(p^+ - p^-)qp^-} \geqslant \frac{C}{p^+ - p^-} \geqslant \frac{C}{\omega(R)}.$$

The claim follows by choosing $\theta = e^{\frac{1-\alpha l}{\alpha\omega(R)}}$, since then $\theta^{-\frac{\alpha}{1-\alpha l}} = e^{-1/\omega(R)}$ and the previous estimate $\frac{1}{1-\alpha l} \geqslant \frac{C}{\omega(R)}$ implies that $\theta \leqslant C$. □

Next we prove the weak Harnack inequality for non-negative quasisuperminimizers. We proceed as in DiBenedetto and Trudinger [4]; cf. [15] for the variable exponent modification.

In what follows, we denote

$$D(k, r) := \{ x \in Q_r : u(x) < k \}.$$

The proof of the next lemma is similar to that of [15, Lemma 5.1], and is hence omitted.

Lemma 4.3. *Let ω be the modulus of continuity of p and let u be a non-negative K-quasisuperminimizer in Ω. Then there exists a constant $\gamma_0 \in (0, 1)$, depending on n, $p(\cdot)$, q, K, and the $L^{q's}(Q)$-norm of u, such that if*

$$|D(\theta, R)| \leqslant \gamma_0 |Q_R|$$

for some $\theta > 0$, then

$$\operatorname*{ess\,inf}_{Q_{R/2}} u + e^{-1/\omega(R)} \geqslant \frac{\theta}{2}.$$

Next we generalize [15, Lemma 5.2].

Lemma 4.4. *Let ω be the modulus of continuity of p and let u be a non-negative K-quasisuperminimizer in Ω. Then for every $\gamma \in (0, 1)$ there exists a constant $\mu > 0$, depending on γ, n, $p(\cdot)$, q, K, and the $L^{q's}(Q)$-norm of u, such that if*

$$|D(\theta, R)| \leqslant \gamma |Q_R|$$

for some $\theta > 0$, then

$$\operatorname*{ess\,inf}_{Q_{R/2}} u + e^{-1/\omega(R)} \geqslant \mu\theta.$$

Proof. Let i_0 be a positive integer to be fixed later. Let us first assume that $\theta > 2^{i_0} e^{-1/\omega(R)}$. For $e^{-1/\omega(R)} < h < k < \theta$ we set

$$v := \begin{cases} 0, & \text{if } u \geqslant k, \\ k-u, & \text{if } h < u < k, \\ k-h, & \text{if } u \leqslant h. \end{cases}$$

Then $v \in W^{1,p(\cdot)}_{\mathrm{loc}}(\Omega)$ and $\nabla v = -\nabla u \chi_{\{h<u<k\}}$ a.e. in Ω. Clearly, $v = 0$ in $Q_R \setminus D(k,R)$, and since $|D(k,R)| \leqslant |D(\theta,R)| \leqslant \gamma |Q_R|$ we obtain $|Q_R \setminus D(k,R)| \geqslant (1-\gamma)|Q_R|$. Hence we may apply Sobolev's inequality

$$\biggl(\int_{Q_R} v^{n'}\,dx\biggr)^{1/n'} \leqslant C \int_{\Delta} |\nabla v|\,dx,$$

where $\Delta := D(k,R) \setminus D(h,R)$ and C depends on γ and n. We have

$$(k-h)\bigl|D(h,R)\bigr| = \int_{D(h,R)} v\,dx \leqslant \bigl|D(h,R)\bigr|^{1/n} \biggl(\int_{Q_R} v^{n'}\,dx\biggr)^{1/n'},$$

from which it follows by Hölder's inequality and the assumption $\|\nabla u\|_{L^{p(\cdot)}(Q_R)} \leqslant 1$ that

$$\begin{aligned}(k-h)\bigl|D(h,R)\bigr|^{1/n'} &\leqslant C \int_{\Delta} |\nabla v|\,dx \\ &\leqslant C\|1\|_{L^{p'(\cdot)}(\Delta)} \|\nabla v\|_{L^{p(\cdot)}(\Delta)} \\ &\leqslant C|\Delta|^{1-1/p^-} \biggl(\int_{D(k,R)} |\nabla v|^{p(x)}\,dx\biggr)^{1/p^+}.\end{aligned}$$

The Caccioppoli estimate [15, Lemma 3.4] implies that

$$\begin{aligned}\int_{D(k,R)} |\nabla v|^{p(x)}\,dx &\leqslant \int_{D(k,R)} |\nabla u|^{p(x)}\,dx \leqslant C \int_{D(k,2R)} \biggl(\frac{(k-u)_+}{R}\biggr)^{p(x)} dx \\ &\leqslant CR^{-p^+} \int_{D(k,2R)} k^{p(x)}\,dx \leqslant CR^{-p^+} \int_{D(k,2R)} \biggl(\frac{k}{e^{-1/\omega(R)}} \cdot e^{-1/\omega(R)}\biggr)^{p(x)} dx \\ &\leqslant CR^{-p^+} \biggl(\frac{k}{e^{-1/\omega(R)}}\biggr)^{p^+} \int_{D(k,2R)} e^{-p(x)/\omega(R)}\,dx \\ &\leqslant CR^{-p^+} \biggl(\frac{k}{e^{-1/\omega(R)}}\biggr)^{p^+} e^{-p^-/\omega(R)} R^n \leqslant Ck^{p^+} R^{n-p^+}.\end{aligned}$$

Here the last inequality follows from the assumption (4.1).

Combining the above inequalities we deduce that

$$\begin{aligned}\biggl(\frac{k-h}{k}\biggr)^{(p^-)'} \bigl|D(h,R)\bigr|^{\frac{(p^-)'}{n'}} &\leqslant CR^{\frac{p^-}{p^+}\frac{n-p^+}{p^- -1}} \bigl|D(k,R) \setminus D(h,R)\bigr| \\ &\leqslant CR^{\frac{n-p^-}{p^- -1}} \bigl|D(k,R) \setminus D(h,R)\bigr|.\end{aligned}$$

Here the log-Hölder continuity is used in the last step. The end of the proof is analogical to the proof of [15, Lemma 5.2]; the only difference is that we now consider the cases $\theta > 2^{i_0}e^{-1/\omega(R)}$ and $\theta \leqslant 2^{i_0}e^{-1/\omega(R)}$. (The variable i_0 is fixed in the course of this part of the proof.) □

For the next step we recall the covering theorem due to Krylov and Safonov, see [18]. For the proof, we refer to, e.g., the monograph by Giusti [12].

Lemma 4.5. *Let $E \subset Q_R \subset \mathbb{R}^n$ be a measurable set and let $0 < \delta < 1$. Moreover, let*

$$E_\delta := \bigcup_{x \in Q(y,R),\, 0<\varrho<R} \bigl\{Q(x,3\varrho) \cap Q(y,R):\ \bigl|Q(x,3\varrho) \cap E\bigr| \geqslant \delta\bigl|Q(y,\varrho)\bigr|\bigr\}.$$

Then either $|E| \geqslant \delta|Q(y,R)|$, in which case $E_\delta = Q(y,R)$, or

$$|E_\delta| \geqslant \frac{1}{\delta}|E|.$$

Theorem 4.6. *Let ω be the modulus of continuity of p and let u be a non-negative K-quasisuperminimizer in Ω. Then there exist an exponent $h > 0$ and a constant C, both depending on n, $p(\cdot)$, q, K, and the $L^{q's}(Q)$-norm of u, such that*

$$\biggl(\fint_{Q_R} u^h\,dx\biggr)^{1/h} \leqslant C\Bigl(\operatorname*{ess\,inf}_{Q_{R/2}} u + e^{-1/\omega(R)}\Bigr)$$

for every cube Q_R for which $Q_{10R} \subset \Omega$, $R \leqslant \frac{1}{2}$ and (3.4) *holds.*

Proof. The proof follows closely the reasoning in [15, Theorem 5.7]. However, since the proof of [15, Theorem 5.7] contains a minor mistake, we give the main steps of the proof here.

In order to apply the formula

$$\int_{Q_R} \bigl(u + e^{-1/\omega(R)}\bigr)^h\,dx = h\int_0^\infty t^{h-1}\bigl|A_t^0\bigr|\,dt$$

we estimate the measure of the set

$$A_t^0 := \bigl\{x \in Q_R:\ u(x) + e^{-1/\omega(R)} > t\bigr\}.$$

As in [15], we denote

$$A_t^i := \bigl\{x \in Q_R:\ u(x) + e^{-1/\omega(R)} > t\mu^i\bigr\}, \quad i = 1, 2, \ldots,$$

and conclude that

$$\bigl|\bigl\{x \in Q(z,6\varrho):\ u + e^{-1/\omega(R)} < t\mu^i\bigr\}\bigr| \leqslant \biggl(1 - \frac{\delta}{6^n}\biggr)|Q_{6\varrho}| = \gamma|Q_{6\varrho}|.$$

Hence Lemma 4.4 yields

$$\operatorname*{ess\,inf}_{Q(z,3\varrho)} u + 2e^{-1/\omega(R)} \geqslant t\mu^{i+1},$$

which implies that

$$\operatorname*{ess\,inf}_{Q(z,3\varrho)} u + e^{-1/\omega(R)} \geqslant \frac{t}{2}\mu^{i+1}.$$

In other words,

$$\bigl(A_t^i\bigr)_\delta \subset A_{t/2}^{i+1}.$$

(Here [15] claims that $(A_t^i)_\delta \subset A_t^{i+1}$.)

Therefore Lemma 4.5 implies that

$$\operatorname*{ess\,inf}_{Q_{R/2}} u + e^{-1/\omega(R)} \geqslant t2^{-j}\mu^{j+1}$$

for the smallest integer j satisfying

$$j \geqslant \frac{1}{\log\delta}\log\frac{|A_t^0|}{|Q_R|},$$

and we obtain the estimate

$$\left|A_t^0\right| \leqslant C|Q_R|\Bigl(\operatorname*{ess\,inf}_{Q_{R/2}} u + e^{-1/\omega(R)}\Bigr)^a t^{-a},$$

where $a := \frac{\log\delta}{\log\mu - \log 2} > 0$. For $0 < h < a$, we conclude the claim as in [15]. □

5. Strong minimum principle

As the main result of this paper we show that the weak Harnack estimate of Theorem 4.6 yields the strong minimum principle under the assumption that p has modulus of continuity Φ satisfying the Dini-type condition

$$\int_0^1 \frac{\Phi^{-1}(t)}{t^2}\,dt = \infty. \tag{5.1}$$

We assume here that $\Phi : [0,\infty) \to [0,\infty)$ is a continuous strictly increasing function with $\Phi(0) = 0$. The condition (5.1) holds e.g. if

$$\Phi(t) = |t|\log\biggl(e + \frac{1}{|t|}\biggr). \tag{5.2}$$

Clearly the assumption (5.1) is stronger than the log-Hölder continuity, but weaker than Lipschitz continuity.

Recall that u^* stands for the lower semicontinuous representative of the quasisuperminimizer u, see Lemma 3.8.

Theorem 5.3 *(The strong minimum principle). Let $\Omega \subset \mathbb{R}^n$ be a bounded domain and let $p : \Omega \to [1,\infty)$ with $1 < p^- \leqslant p^+ < \infty$ have modulus of continuity satisfying* (5.1). *Let u be a non-negative quasisuperminimizer in Ω. Then either $u^* > 0$ in Ω or $u^* \equiv 0$ in Ω.*

Proof. Throughout the proof, we use the notation u for u^*. In particular, u is lower semicontinuous. Suppose that $u(x_0) = 0$ for some $x_0 \in \Omega$ and choose a cube Q centered at x_0 such that $10Q \subset \Omega$ and the inequalities

$$\int_Q |u|^{p(x)}\,dx \leqslant 1 \quad\text{and}\quad \int_Q |\nabla u|^{p(x)}\,dx \leqslant 1$$

hold. It is enough to show that $u = 0$ in a neighborhood of x_0. In fact, then $\{x \in \Omega\colon u(x) = 0\}$ and $\{x \in \Omega\colon u(x) > 0\}$ are both open and so one of them must be empty, as claimed. For the proof, we consider a chain of cubes joining x_0 and $x \neq x_0$ and iterate the weak Harnack estimate of Theorem 4.6.

Fix for the moment an integer $k \geqslant 3$. Let $(r_i)_{i=0}^k$ be a non-decreasing sequence. We choose points x_i, $i = 1,\dots,k$, in Q such that $|x_{i-1} - x_i| = r_{i-1}$. For each $i = 0,\dots,k$, let Q_i be a cube with the center x_i and the side-length $2r_i$. Then

$$|Q_i \cap Q_{i-1}| \geqslant c|Q_{i-1}| \tag{5.4}$$

for all $i = 1,\dots,k$. We may assume that numbers r_i are so small that $Q_i \subset Q$ for every $i = 0,\dots,k$.

By Theorem 4.6,

$$\biggl(\fint_{Q_i} u^h\,dx\biggr)^{1/h} \leqslant C\Bigl(\inf_{Q_i} u + e^{-\frac{1}{\Phi(r_i)}}\Bigr)$$

for all $i = 0, \dots, k$. Since Lemma 3.2 allows us to consider the truncated function $\min(u, 1)$, we may assume that the constant C_0 (i.e. the $L^{q's}(Q_i)$-norm of u) is independent on the cube Q_i. By inequality (5.4),

$$\inf_{Q_i} u \leqslant \inf_{Q_i \cap Q_{i-1}} u \leqslant \biggl(\fint_{Q_i \cap Q_{i-1}} u^h\,dx \biggr)^{1/h} \leqslant C \biggl(\fint_{Q_{i-1}} u^h\,dx \biggr)^{1/h}.$$

Hence

$$\biggl(\fint_{Q_i} u^h\,dx \biggr)^{1/h} \leqslant C_0 \biggl[\biggl(\fint_{Q_{i-1}} u^h\,dx \biggr)^{1/h} + e^{-\frac{1}{\Phi(r_i)}} \biggr].$$

Iterating this inequality and using the estimate

$$\biggl(\fint_{Q_0} u^h\,dx \biggr)^{1/h} \leqslant C_0 e^{-\frac{1}{\Phi(r_0)}}$$

(which holds since $u(x_0) = 0$), we find that

$$\biggl(\fint_{Q_j} u^h\,dx \biggr)^{1/h} \leqslant \sum_{i=0}^{k} C_0^{k-i+1} e^{-\frac{1}{\Phi(r_i)}} \tag{5.5}$$

for $j = 0, \dots, k$.

The right hand side of the previous inequality can be estimated by

$$\int_{-1}^{k} C_0^{k-t+1} e^{-\frac{1}{\Phi(r(t))}}\,dt,$$

where the function r is chosen so that $r(t) \geqslant r_i$ when $t \in [i-1, i]$. Let us choose

$$r(t) := \Phi^{-1}\biggl(\frac{1}{a(k-t+1) + l(k)} \biggr) \quad \text{and} \quad r_i := r(i);$$

here $a > \log C_0$ is fixed and $l(k)$ will be specified later. Since r is increasing, it is easy to verify that $r_i \leqslant r_{i+1}$. With this choice of r,

$$\int_{-1}^{k} C_0^{k-t+1} e^{-\frac{1}{\Phi(r(t))}}\,dt = \int_{-1}^{k} e^{-\beta(k-t+1) - l(k)}\,dt \leqslant \frac{1}{\beta} e^{-l(k)},$$

where $\beta := a - \log C_0 > 0$. Hence we conclude from (5.5) that

$$\biggl(\fint_{Q_j} u^h\,dx \biggr)^{1/h} \leqslant C e^{-l(k)} \tag{5.6}$$

for $j = 0, \dots, k$.

Let us now show that we can choose an unbounded function $l(k)$ such that the extent $|x - x_0|$ of the chain $(Q_i)_{i=1}^k$ does not tend to 0 when k grows to infinity. By changing variables we see that this length is at least

$$\sum_{i=2}^{k} r_i \geqslant \int_{1}^{k} r(t)\,dt = \int_{1}^{k} \Phi^{-1}\biggl(\frac{1}{a(k-t+1) + l(k)} \biggr) dt = \frac{1}{a} \int_{1/(ak+l(k))}^{1/(a+l(k))} \frac{\Phi^{-1}(z)}{z^2}\,dz.$$

Since

$$\int_{1/(ak+l(k))}^{1/(a+l(k))} \frac{\Phi^{-1}(z)}{z^2}\,dz \geqslant \int_{1/(ak)}^{1/(a+l(k))} \frac{\Phi^{-1}(z)}{z^2}\,dz$$

whenever $l(k) < a(k-1)$, we may choose for every $k \geqslant k_0$ a number $l(k)$ such that the right hand side of the previous inequality is at least 1 and $l(k) \to \infty$ as $k \to \infty$. In fact, by (5.1), we may choose k_0 such that

$$\int_{1/ak}^{1/a} \frac{\Phi^{-1}(z)}{z^2}\,dz > 1$$

for all $k > k_0$. Hence for all $k > k_0$ there is $m(k)$, $1/ak < m(k) < 1/a$, such that

$$\int_{1/ak}^{m(k)} \frac{\Phi^{-1}(z)}{z^2}\,dz = 1.$$

By condition (5.1), $\lim_{k\to\infty} m(k) = 0$. Now, it is enough to set $l(k) := 1/m(k) - a$. Hence we conclude that the length of the chain is at least 1.

Let then $x \in B(x_0, \delta)$, where $\delta < 1$ is so small that the assumptions given in the first part of the proof are fulfilled. For each $k \geqslant k_0$, we can choose a chain $(Q_i)_{i=0}^k$, as above, such that one of the cubes contains the point x. This is possible since the length of the chain is at least 1. Denote the cube in the chain containing x by Q^k. Note that the size of Q^k tends to zero as $k \to \infty$. By Lemma 3.8, u has a Lebesgue point at x, and therefore (5.6) yields

$$u(x) = \lim_{k\to\infty} \fint_{Q^k} u\,dx \leqslant \lim_{k\to\infty} \fint_{Q^k} u^h\,dx \leqslant C \lim_{k\to\infty} e^{-hl(k)} = 0;$$

in the first inequality we used again that $u \leqslant 1$, by truncation, and assumed without loss of generality that $h \in (0,1)$. This completes the proof. □

Note that quasiminimizers are continuous [8]. Using the previous theorem for $u - \inf u$ and $\sup u - u$ we obtain the following corollary.

Corollary 5.7 *(Strong maximum principle). Under the assumptions of Theorem* 5.3 *a bounded, non-constant quasiminimizer cannot attain its infimum or its supremum in a domain.*

Remark 5.8. Under certain circumstances it is possible to reformulate condition (5.1). Suppose that $\Psi(t) := \frac{\Phi(t)}{t}$ is of log-type, i.e.

$$\Psi\big(t^2\big) \approx \Psi(t)$$

for all $t > 0$. Then for some $t_0 > 0$ we have

$$\Phi(t)\Phi^{-1}(t) \approx t^2$$

for $t \in [0, t_0]$. In fact, the required condition is equivalent to

$$\Psi\big(\Phi(s)\big) \approx \Psi(s),$$

where $t := \Phi(s)$. Since $s^2 \leqslant \Phi(s) \leqslant \sqrt{s}$ on some interval $[0, s_0]$, the latter is clear. Hence the integrals

$$\int_0^1 \frac{\Phi^{-1}(t)}{t^2}\,dt \quad \text{and} \quad \int_0^1 \frac{dt}{\Phi(t)}$$

diverge simultaneously, i.e. the Dini-type condition (5.1) is equivalent to

$$\int_0^1 \frac{dt}{\Phi(t)} = \infty$$

whenever Ψ is of log-type. The details of the proofs of these claims are left to the reader.

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