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Comparison results and steady states for the Fujita equation with fractional Laplacian

Résultats de comparaison et solutions stationnaires de l'équation de Fujita avec Laplacien fractionnel

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Abstract

We study a semilinear PDE generalizing the Fujita equation whose evolution operator is the sum of a fractional power of the Laplacian and a convex non-linearity. Using the Feynman–Kac representation we prove criteria for asymptotic extinction versus finite time blow up of positive solutions based on comparison with global solutions. For a critical power non-linearity we obtain a two-parameter family of radially symmetric stationary solutions. By extending the method of moving planes to fractional powers of the Laplacian we prove that all positive steady states of the corresponding equation in a finite ball are radially symmetric.

Résumé

Nous étudions une équation de réaction-diffusion semilinéaire (généralisant l'équation de Fujita), dont l'opérateur d'évolution est la somme d'une puissance fractionnelle du Laplacien et d'une non-linéarité convexe. A l'aide de la représentation de Feynman–Kac nous exhibons des critères entrainant l'extinction asymptotique, respectivement l'explosion en temps fini, de solutions positives. Ces critères s'obtiennent en comparant avec des solutions globales. Pour une certaine puissance critique de la non-linéarité nous obtenons une famille paramétrisée de solutions stationnaires à symétrie radiale. Par extension de la méthode de déplacement d'hyperplans à des puissances fractionnelles du Laplacien, nous prouvons que toute solution positive stationnaire de l'équation correspondante dans une boule finie comporte une symétrie radiale.

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1. Introduction

We consider the "generalized Fujita equation"

$$\begin{aligned} &\frac{\partial}{\partial t}u(t,x) = Lu(t,x) + G\big(u(t,x)\big), \quad t \geqslant 0,\ x \in \mathbf{R}^d, \\ &u(0,x) = \varphi(x) \geqslant 0, \end{aligned} \tag{1.1}$$

where $L = \Delta_\alpha$ is the fractional power $-(-\Delta)^{\alpha/2}$ of the Laplacian, $0 < \alpha \leqslant 2$, and $G : \mathbf{R}_+ \to \mathbf{R}_+$ is a convex function satisfying conditions (2.1) and (2.2) below. Solutions will be understood in the mild sense (see e.g. [20]) so that (1.1) makes sense for any non-negative bounded measurable function φ on $\mathbf{R}^d$.

A well known fact is that for any non-trivial initial value φ there exists a number $T_\varphi \in (0, \infty]$ such that (1.1) has a unique solution u on $\mathbf{R}^d \times [0, T_\varphi)$ which is bounded on $\mathbf{R}^d \times [0, T]$ for any $0 < T < T_\varphi$, and if $T_\varphi < \infty$, then $\|u(\cdot, t)\|_{L^\infty(\mathbf{R}^d)} \to \infty$ as $t \uparrow T_\varphi$.

When $T_\varphi = \infty$ we say that u is a global solution, and when $T_\varphi < \infty$ we say that u blows up in finite time or that u is non-global.

The study of blow up properties of (1.1) goes back to the fundamental work of Fujita [11], who studied Eq. (1.1) with $\alpha = 2$ and $G(z) = z^{1+\beta}$, $\beta > 0$. The investigation of (1.1) with a general α was initiated by Sugitani [23], who showed that if $d \leqslant \alpha/\beta$, then for any non-vanishing initial condition the solution blows up in finite time. Using a Feynman–Kac representation for the solutions of semilinear problems of the form (1.1), this conclusion was re-derived in [4] and the corresponding behaviors of equations with time-dependent non-linearities and of various systems of semilinear pde's were studied.

It is known (e.g. [18,19]) that in supercritical dimensions $d > \alpha/\beta$, Eq. (1.1) admits global as well as non-global positive solutions, depending on the "size" of the initial condition. For short we address this parameter constellation as the *global regime.*

In the first part of the present note we prove two comparison criteria in the global regime:

(i) Assume the initial value $\varphi \geqslant 0$ leads to a globally bounded solution. Then any initial value ψ with $0 \leqslant \psi \leqslant (1-\varepsilon)\varphi$, $\varepsilon > 0$, gives rise to a solution converging to zero.
(ii) Assume the initial value $\varphi \geqslant 0$ leads to a solution which is uniformly bounded away from 0 for all $t > 0$ and all x in some ball $\subset \mathbf{R}^d$. Then any initial value ψ with $\psi \geqslant (1+\varepsilon)\varphi$, $\varepsilon > 0$, gives rise to a solution which blows up in finite time.

The essential tool in proving (i) and (ii) is the probabilistic representation of the solution of (1.1) provided by the Feynman–Kac formula, that was obtained in [4] (see (2.4) below).

Natural candidates for the comparison in (i) and (ii) are (time-)stationary solutions of (1.1), i.e. solutions of the "elliptic" equation

$$\Delta_\alpha u(x) + G\big(u(x)\big) = 0, \quad x \in \mathbf{R}^d. \tag{1.2}$$

In the case $\alpha = 2$ and $G(z) = z^{1+\beta}$, it is known that (see [15,8,13,24])

– for $d > 2$, $1 + \beta < (d+2)/(d-2)$, apart from $u \equiv 0$, no bounded non-negative solution of (1.2) exists,

– for $d > 2$, $\beta = (d+2)/(d-2) - 1$, all bounded stationary solutions of (1.2) are given by the family

$$u_{c,A}(x) = \frac{A(d(d-2))^{(d-2)/2}}{(d(d-2) + (A^{2/(d-2)}\|x-c\|)^2)^{(d-2)/2}}, \quad c, x \in \mathbf{R}^d,\ A \in \mathbf{R}_+ \tag{1.3}$$

(note that the two parameters of the family are the symmetry center c of u and its value A at c),

– for $d > 2$, $\beta > (d+2)/(d-2) - 1$, there exists a one-parameter family u_A, $A > 0$ of solutions of (1.2) with the properties: $u_A(x)$ is symmetric around $x = 0$,

$$u_A(0) = A, \quad \|x\|^{2/\beta} u_A\big(\|x\|\big) \to K(d,\beta) \quad \text{as } \|x\| \to \infty, \tag{1.4}$$

where $K(d,\beta)$ is a constant not depending on A.

In the case $\alpha < 2$, much less is known. For $d > \alpha$, $\beta = (d+\alpha)/(d-\alpha) - 1$, we specify in Proposition 3.1 a two-parameter family $u_{c,A}$, $c \in \mathbf{R}^d$, $0 < A < \infty$, of radially symmetric solutions of

$$\Delta_\alpha u(x) + u^{1+\beta}(x) = 0, \quad x \in \mathbf{R}^d \tag{1.5}$$

with the property

$$u_{c,A}(c) = A, \quad \|x\|^{d-\alpha} u_{c,A}\big(\|x\|\big) \to K(d,\alpha,\beta) \quad \text{as } \|x\| \to \infty,$$

where $K(d,\alpha,\beta)$ is a positive constant. A natural conjecture now is that, like in the Laplacian case, for $\beta = (d+\alpha)/(d-\alpha) - 1$ the $u_{c,A}$ constitute *all* the bounded solutions of (1.5), and that for $\beta < (d+\alpha)/(d-\alpha) - 1$ there are no bounded non-zero solutions at all.

As a first step to answer these questions, in Section 4 we make use of the so called "method of moving planes", which is well known in the Laplacian case, to show symmetry of positive solutions in a ball. Essential tools (like Hopf's boundary lemma) can be carried over to the α-Laplacian case. With this method, we were able to show (see Theorem 4.1) that the following equation has only radially symmetric non-negative solutions:

$$\Delta_\alpha u(x) + F\big(u(x)\big) = 0, \quad x \in B,$$
$$u(x) = 0, \quad x \in \mathbf{R}^d \setminus B.$$

Here B is an open ball centered around 0 and $F : \mathbf{R}_+ \to \mathbf{R}_+$ is non-decreasing. We conjecture that an analogous statement is valid for $F(z) = z^p$ with $p = (d+\alpha)/(d-\alpha)$ and $\mathbf{R}^d$ instead of B. In Section 4.3 we describe several problems we think one would have to overcome for carrying over the moving planes method to the unbounded space setting in this case.

2. Two comparison criteria

In this section we assume that the function G in Eq. (1.1) satisfies the conditions

$$\lim_{z \to 0+} \frac{G(z)}{z^{1+\beta}} = c \in (0, \infty) \tag{2.1}$$

and

$$\int_\theta^\infty \frac{\mathrm{d}z}{G(z)} < \infty \tag{2.2}$$

for certain positive numbers β and θ.

Lemma 2.1. *Let G be a convex function satisfying* (2.1)*, and $\varepsilon > 0$. For any $M > 0$ there exists $\varepsilon' > 0$ such that*

$$\frac{G((1+\varepsilon)z)}{(1+\varepsilon)z} > (1+\varepsilon')\frac{G(z)}{z} \quad \text{for } 0 < z \leqslant M.$$

Proof. By considering G/c instead of G we can assume that $c = 1$. Given $\tilde{\varepsilon} > 0$ there exists $\delta > 0$ such that

$$(1-\tilde{\varepsilon})z^\beta < \frac{G(z)}{z} < (1+\tilde{\varepsilon})z^\beta$$

for $z \in (0, \delta)$. Take $\tilde{\varepsilon} < ((1+\varepsilon)^\beta - 1)/((1+\varepsilon)^\beta + 1)$. Then for $z < \delta/(1+\varepsilon) := x_0$,

$$\frac{G((1+\varepsilon)z)}{(1+\varepsilon)z} > (1-\tilde{\varepsilon})(1+\varepsilon)^\beta z^\beta > \frac{(1-\tilde{\varepsilon})(1+\varepsilon)^\beta}{(1+\tilde{\varepsilon})}\frac{G(z)}{z} = (1+c')\frac{G(z)}{z}$$

where $c' > 0$. Since $z \mapsto G(z)/z$ is continuous and strictly increasing in $(0, \infty)$ it follows that $\inf_{z\in[x_0,M]} (\frac{G((1+\varepsilon)z)}{(1+\varepsilon)z} / \frac{G(z)}{z}) > 1 + c''$ with $c'' > 0$. Taking $\varepsilon' = c' \wedge c''$ yields the assertion. □

Let us observe that if v is a globally bounded solution of (1.1) we necessarily have, for all x,

$$P_t v(0, x) \to 0 \quad \text{as } t \to \infty,$$

where (P_t) is the semigroup with generator L. Indeed, from the integral form of (1.1)

$$\begin{aligned} v(t,x) &= P_t v(0,x) + \int_0^t P_{t-s} G\big(v(s,x)\big)\,\mathrm{d}s \\ &\geqslant P_t v(0,x) + \int_0^t P_{t-s} G\big(P_s v(0,x)\big)\,\mathrm{d}s \\ &\geqslant P_t v(0,x) + \int_0^t G\big(P_t v(0,x)\big)\,\mathrm{d}s \\ &\geqslant t G\big(P_t v(0,x)\big), \end{aligned}$$

where we used in the first inequality that $P_t v(0,\cdot) \leqslant v(t,\cdot)$, and Jensen's inequality after the second line. It follows from the global boundedness of v that

$$\lim_{t\to\infty} P_t v(0,x) \leqslant \lim_{t\to\infty} G^{-1}(\text{Const.}\, t^{-1}) = 0. \tag{2.3}$$

Proposition 2.1. *Let G be a convex, increasing function satisfying* (2.1) *and* (2.2)*. Assume the initial value $\varphi \geqslant 0$ leads to a globally bounded solution of* (1.1)*. Then any initial value ψ with $0 \leqslant \psi \leqslant (1-\varepsilon)\varphi$, $\varepsilon > 0$, gives rise to a solution converging uniformly to zero.*

Proof. Recall that the Feynman–Kac representation of solutions of (1.1) is given by (see [4])

$$u(t,x) = \int_{\mathbf{R}^d} u(0,y)\, p_t(y,x) \mathbf{E}_y \left[\exp \int_0^t \frac{G(u(s,X_s))}{u(s,X_s)}\,\mathrm{d}s \,\middle|\, X_t = x \right] \mathrm{d}y, \tag{2.4}$$

where (X_t) is the Lévy process with generator L, and $p_t(x,y)$, $t > 0$, $x, y \in \mathbf{R}^d$, are its transition densities.

Suppose that v is a globally bounded solution of (1.1) and that $0 \leqslant u(0,\cdot) \leqslant (1-\varepsilon)v(0,\cdot)$ where $0 < \varepsilon < 1$. As (1.1) preserves ordering we have $u(t,x) \leqslant v(t,x)$ for all $t \geqslant 0$ and $x \in \mathbf{R}^d$, which together with (2.4) improves to

$$u(t,x) \leqslant \int_{\mathbf{R}^d} (1-\varepsilon)v(0,y)\,p_t(y,x)\mathbf{E}_y\left[\exp\int_0^t \frac{G(v(s,X_s))}{v(s,X_s)}\,\mathrm{d}s \,\middle|\, X_t = x\right]\mathrm{d}y = (1-\varepsilon)v(t,x)$$

uniformly in t and x. Inserting this bound again into the Feynman–Kac representation of u yields

$$u(t,x) \leqslant \int_{\mathbf{R}^d} u(0,y)\,p_t(y,x)\mathbf{E}_y\left[\exp\int_0^t \frac{G((1-\varepsilon)v(s,X_s))}{(1-\varepsilon)v(s,X_s)}\,\mathrm{d}s \,\middle|\, X_t = x\right]\mathrm{d}y.$$

Putting $z(t,x) := (1-\varepsilon)v(t,x)$ in the above inequality and using Lemma 2.1 (with ε in Lemma 2.1 substituted by $\tilde{\varepsilon} := \varepsilon/(1-\varepsilon)$) we get

$$\begin{aligned} u(t,x) &\leqslant \int_{\mathbf{R}^d} u(0,y)\,p_t(y,x)\mathbf{E}_y\left[\exp\int_0^t \frac{1}{1+\varepsilon'}\frac{G((1+\tilde{\varepsilon})z(s,X_s))}{(1+\tilde{\varepsilon})z(s,X_s)}\,\mathrm{d}s \,\middle|\, X_t = x\right]\mathrm{d}y \\ &= \int_{\mathbf{R}^d} u(0,y)\,p_t(y,x)\mathbf{E}_y\big[\mathrm{e}^{(1-\varepsilon'')A_t} \mid X_t = x\big]\,\mathrm{d}y, \end{aligned}$$

where $\varepsilon' > 0$ is given by Lemma 2.1, $\varepsilon'' := \varepsilon'/(1+\varepsilon')$ and

$$A_t := \int_0^t \frac{G(v(s,X_s))}{v(s,X_s)}\,\mathrm{d}s. \tag{2.5}$$

Thus,

$$\begin{aligned} u(t,x) &\leqslant \int_{\mathbf{R}^d} u(0,y)\,p_t(y,x)\mathbf{E}_y\big[\mathrm{e}^{(1-\varepsilon'')A_t}1\big(\mathrm{e}^{A_t} < \big(P_t v(0,x)\big)^{-1/2}\big) \mid X_t = x\big]\,\mathrm{d}y \\ &\quad + \int_{\mathbf{R}^d} u(0,y)\,p_t(y,x)\mathbf{E}_y\big[\mathrm{e}^{(1-\varepsilon'')A_t}1\big(\mathrm{e}^{A_t} \geqslant \big(P_t v(0,x)\big)^{-1/2}\big) \mid X_t = x\big]\,\mathrm{d}y. \end{aligned}$$

Since $\mathrm{e}^{A_t} \geqslant (P_t v(0,x))^{-1/2}$ implies $\mathrm{e}^{(1-\varepsilon'')A_t} \leqslant \mathrm{e}^{A_t}(P_t v(0,x))^{\varepsilon''/2}$, we obtain

$$\begin{aligned} u(t,x) &\leqslant \int_{\mathbf{R}^d} u(0,y)\,p_t(y,x)\big(P_t v(0,x)\big)^{-1/2} + \big(P_t v(0,x)\big)^{\varepsilon''/2}\int_{\mathbf{R}^d} u(0,y)\,p_t(y,x)\mathbf{E}_y[\mathrm{e}^{A_t} \mid X_t = x]\,\mathrm{d}y \\ &\leqslant (1-\varepsilon)\big(\big(P_t v(0,x)\big)^{1/2} + v(t,x)\big(P_t v(0,x)\big)^{\varepsilon''/2}\big) \end{aligned}$$

which tends to 0 uniformly as $t \to \infty$ due to (2.3). □

Proposition 2.2. *Let G be a convex, increasing function satisfying* (2.1) *and* (2.2). *Assume the initial value $\varphi = v(0,\cdot) \geqslant 0$ leads to a globally bounded solution of* (1.1) *which for some open ball $B \subset \mathbf{R}^d$ and some $\kappa > 0$ obeys*

$$\inf_{x\in B} v(t,x) \geqslant \kappa \quad \text{for all sufficiently large } t > 0. \tag{2.6}$$

Then for any $\varepsilon > 0$, the initial condition $(1+\varepsilon)\varphi$ leads to blow-up in finite time.

Proof. By the Feynman–Kac formula,

$$v(t,x) = \int_{\mathbf{R}^d} v(0,y)p_t(y,x)\mathbf{E}_y[\mathrm{e}^{A_t} \mid X_t = x]\,\mathrm{d}y,$$

where A_t is given by (2.5). If $K > 0$ then

$$\int_{\mathbf{R}^d} v(0,y)p_t(y,x)\mathbf{E}_y[\mathrm{e}^{A_t}; A_t \leqslant K \mid X_t = x]\,\mathrm{d}y \leqslant \mathrm{e}^K \mathbf{E}_x v(0,X_t) \to 0$$

as $t \to \infty$ uniformly in x due to (2.3). Therefore, for all $K > 0$ there exists $T_0 = T_0(K,\gamma) > 0$ such that for $t > T_0$

$$\frac{v(t,x)}{\int_{\mathbf{R}^d} v(0,y)p_t(y,x)\mathbf{E}_y[\mathrm{e}^{A_t}; A_t \geqslant K \mid X_t = x]\,\mathrm{d}y} \leqslant 2. \tag{2.7}$$

Without loss of generality we can assume that $\inf_{x\in B_1(0)} v(t,x) \geqslant \kappa$ for all t large enough, where $B_r(x)$ denotes the ball in $\mathbf{R}^d$ of radius r centered at x. Arguing as above we check via the Feynman–Kac representation that for all $t \geqslant 0$ and $x \in \mathbf{R}^d$, $u(t,x) \geqslant (1+\varepsilon)v(t,x)$. Plugging this again into the Feynman–Kac representation for u yields

$$\begin{aligned} u(t,x) &\geqslant (1+\varepsilon)\int_{\mathbf{R}^d} v(0,y)p_t(y,x)\mathbf{E}_y\left[\exp\int_0^t \frac{G((1+\varepsilon)v(s,X_s))}{(1+\varepsilon)v(s,X_s)}\,\mathrm{d}s \,\middle|\, X_t = x\right]\mathrm{d}y \\ &\geqslant (1+\varepsilon)\int_{\mathbf{R}^d} v(0,y)p_t(y,x)\mathbf{E}_y[\mathrm{e}^{(1+\varepsilon')A_t} \mid X_t = x]\,\mathrm{d}y \end{aligned}$$

for some $\varepsilon' > 0$ by Lemma 2.1. Using this and (2.7) we obtain for given $K > 0$ and t big enough that

$$u(t,x) \geqslant (1+\varepsilon)\mathrm{e}^{K\varepsilon'}\int_{\mathbf{R}^d} v(0,y)p_t(y,x)\mathbf{E}_y[\mathrm{e}^{A_t}; A_t \geqslant K \mid X_t = x]\,\mathrm{d}y \geqslant (1+\varepsilon)\mathrm{e}^{K\varepsilon'}\frac{v(t,x)}{2}.$$

Hence, for any $K > 0$ we find $\inf_{x\in B_1(0)} u(t,x) \geqslant \kappa(1+\varepsilon)\mathrm{e}^{K\varepsilon'}/2$ for all sufficiently large t. As is well known (see e.g. [17]), this inequality together with (2.2) are sufficient for finite-time blow-up of u. □

The kind of threshold phenomenon described by the following corollary is well known for the classical Laplacian, see e.g. [9, Theorems 1.1] or [14, Theorem 1.14] for related results.

Corollary 2.1. *Let G be a convex, increasing function satisfying* (2.1) *and* (2.2)*, and $\varphi \geqslant 0$ a non-trivial positive bounded solution of*

$$L\varphi(x) + G\big(\varphi(x)\big) = 0.$$

(a) *For each $\varepsilon > 0$ the solution u of* (1.1) *with initial value $u(0,x) = (1+\varepsilon)\varphi(x)$, $x \in \mathbf{R}^d$, blows-up in finite time.*

(b) *For each $\varepsilon \in (0,1)$ the solution u of* (1.1) *with initial value $u(0,x) = (1-\varepsilon)\varphi(x)$, $x \in \mathbf{R}^d$ converges uniformly to 0 as $t \to \infty$.*

Proof. This is immediate from Propositions 2.1 and 2.2. □

3. A class of radially symmetric stationary solutions

We now set out to specify a family of positive stationary solutions of (1.1) in the particular case of $G(z) = z^p$, where $p = (d+\alpha)/(d-\alpha)$. Before doing this, we still consider the case of a general p, and note that the "elliptic" equation

$$\Delta_\alpha u(x) + u^p(x) = 0, \quad x \in \mathbf{R}^d \tag{3.1}$$

can be rewritten in integral form as

$$u(x) = \int_0^\infty \mathbf{E}_x\big[u^p(X_t)\big]\,\mathrm{d}t, \quad x \in \mathbf{R}^d, \tag{3.2}$$

where (X_t) denotes the (symmetric) α-stable process in $\mathbf{R}^d$. Hence, for $d \leqslant \alpha$, due to recurrence of (X_t), the only non-negative solutions of (3.1) are $u \equiv 0$ and $u \equiv \infty$. Therefore, we henceforth assume that $d > \alpha$, in which case (3.2) rewrites as (see [5, p. 264])

$$u(x) = \int_{\mathbf{R}^d} \frac{\mathcal{A}(d,\alpha)u^p(y)}{\|y-x\|^{d-\alpha}}\,\mathrm{d}y, \quad x \in \mathbf{R}^d, \tag{3.3}$$

where $\mathcal{A}(d,\alpha) := \Gamma(\frac{1}{2}(d-\alpha))/[\Gamma(\frac{1}{2}\alpha)2^\alpha \pi^{d/2}]$.

Proposition 3.1. *If $p = (d+\alpha)/(d-\alpha)$, then for any $A \in (0,\infty)$ and $c \in \mathbf{R}^d$ the function*

$$u_{c,A}(x) = \frac{A}{[1 + (A^{2/(d-\alpha)}2^{-1}(\Gamma(\frac{d+\alpha}{2})/\,\Gamma(\frac{d-\alpha}{2}))^{-1/\alpha}\|x-c\|)^2]^{(d-\alpha)/2}}$$

solves (3.1).

Proof. Without loss of generality we assume that c is the origin. Due to (3.3) it suffices to show that

$$u_{0,A}(x) = \int_{\mathbf{R}^d} \frac{\mathcal{A}(d,\alpha)u_{0,A}^p(y)}{\|y-x\|^{d-\alpha}}\,\mathrm{d}y, \quad x \in \mathbf{R}^d. \tag{3.4}$$

Let us write $a := A^{-2/(d-\alpha)}2(\Gamma(\frac{d+\alpha}{2})/\,\Gamma(\frac{d-\alpha}{2}))^{1/\alpha}$. We first note that

$$u_{0,A}(x) = \frac{A}{(1 + 4\pi^2\|\frac{x}{2\pi a}\|^2)^{(d-\alpha)/2}} = A\widehat{B_{d-\alpha}}\left(\frac{x}{2\pi a}\right)$$

(see [10, p. 155]), where for any $f \in L^1(\mathbf{R}^d)$, $\hat{f}(x) := \int_{\mathbf{R}^d} \mathrm{e}^{-2\pi i y\cdot x} f(y)\,\mathrm{d}y$ is the Fourier transform of f, and for any complex w with $\mathrm{Re}(w) > 0$

$$B_w(x) := \frac{1}{\Gamma(\frac{w}{2})(4\pi)^{d/2}} \int_0^\infty r^{(w-d)/2-1}\mathrm{e}^{-r-\|x\|^2/4r}\,\mathrm{d}r.$$

Hence

$$\widehat{u_{0,A}}(x) = A\left[\widehat{B_{d-\alpha}}\left(\frac{\cdot}{2\pi a}\right)\right]\widehat{\ }(x) = A(2\pi a)^d\,\widehat{\widehat{B_{d-\alpha}}}(2\pi a x) = A(2\pi a)^d B_{d-\alpha}(-2\pi a x).$$

Notice that

$$B_w(x) = \frac{2^{(d-w)/2+1}}{\Gamma(\frac{w}{2})(4\pi)^{d/2}}\|x\|^{(w-d)/2}K_{(d-w)/2}\big(\|x\|\big), \quad x \in \mathbf{R}^d, \tag{3.5}$$

where for any complex ν, K_ν is the Macdonald's function [25, §6·22], also known as modified Bessel function of the second kind, which is given by

$$K_\nu(z) = \frac{1}{2}\left(\frac{1}{2}z\right)^\nu \int_0^\infty r^{-\nu-1}\mathrm{e}^{-r-z^2/4r}\,\mathrm{d}r, \quad \mathrm{Re}(z^2) > 0.$$

It follows that

$$\widehat{u_{0,A}}(x) = Aa^{d-\alpha/2}\pi^{(d-\alpha)/2}\frac{2}{\Gamma(\frac{d-\alpha}{2})}\|x\|^{-\alpha/2}K_{\alpha/2}\big(\|2\pi a x\|\big). \tag{3.6}$$

To compute the Fourier transform of the other side of (3.4) we use the convolution theorem, (3.5) and

$$\big[\mathcal{A}(d,\alpha)\|\cdot\|^{-(d-\alpha)}\big]\widehat{\ }(x) = \big(2\pi\|x\|\big)^{-\alpha}, \quad x \in \mathbf{R}^d,\ 0 < \mathrm{Re}(\alpha) < d,$$

(e.g. [10, p. 154]) to obtain

$$\begin{aligned}
\left[\int_{\mathbf{R}^d}\frac{\mathcal{A}(d,\alpha)u_{0,A}^p(y)}{\|y-\cdot\|^{d-\alpha}}\,\mathrm{d}y\right]\widehat{\ }(x) &= (2\pi)^{-\alpha}\|x\|^{-\alpha}A^p\left[\frac{1}{(1+4\pi^2\|\frac{\cdot}{2\pi a}\|^2)^{(d+\alpha)/2}}\right]\widehat{\ }(x)\\
&= (2\pi)^{-\alpha}\|x\|^{-\alpha}A^p\left[\widehat{B_{d+\alpha}}\left(\frac{\cdot}{2\pi a}\right)\right]\widehat{\ }(x)\\
&= (2\pi)^{-\alpha}\|x\|^{-\alpha}A^p(2\pi a)^d B_{d+\alpha}(-2\pi a x)\\
&= 2^{-\alpha}\pi^{(d-\alpha)/2}\|x\|^{-\alpha/2}A^p a^{d+\alpha/2}\frac{2}{\Gamma(\frac{d+\alpha}{2})}K_{-\alpha/2}\big(\|2\pi a x\|\big).
\end{aligned}$$

Since $K_\nu = K_{-\nu}$ [1, Formula 9.6.6], by comparing the RHS of the last equality with that of (3.6) we see that they are equal for the value of a stated at the beginning of the proof. The result follows from uniqueness of Fourier transforms. □

Remarks. 1. Recall [7,16] that for $0 < \alpha \leqslant 2$ and $d > \alpha$ the Kelvin transform of u is defined by

$$v(x) := \frac{1}{\|x\|^{d-\alpha}}u\left(\frac{x}{\|x\|^2}\right), \quad x \in \mathbf{R}^d,\ x \neq 0. \tag{3.7}$$

A simple calculation shows that for any fixed $c \in \mathbf{R}^d$ the family of solutions $\{u_{c,A}\}_{A\geqslant 0}$ rendered by Proposition 3.1 is invariant under the Kelvin transform with center c

$$v_c(x) := \frac{1}{\|x-c\|^{d-\alpha}}u_{c,A}\left(c + \frac{x-c}{\|x-c\|^2}\right).$$

2. Moreover, if u is *any* given regular positive solution of (3.1) (where $0 < \alpha \leqslant 2$, $p > 1$ and $d > \alpha$) and $x \neq 0$, then its Kelvin transform $v(x)$ satisfies

$$\Delta_\alpha v(x) + \frac{v^p(x)}{\|x\|^{(d+\alpha)-p(d-\alpha)}} = 0. \tag{3.8}$$

Indeed, let G_α denote the Green's operator corresponding to Δ_α, and $x \neq 0$. Then,

$$\begin{aligned}
-G_\alpha\left(-\frac{v^p(x)}{\|x\|^{(d+\alpha)-p(d-\alpha)}}\right) &= \int \frac{\mathcal{A}(d,\alpha)}{\|x-y\|^{d-\alpha}}\cdot\frac{1}{\|y\|^{(d+\alpha)-p(d-\alpha)}}\cdot\frac{1}{\|y\|^{(d-\alpha)p}}\cdot u^p\left(\frac{y}{\|y\|^2}\right)\mathrm{d}y\\
&= \mathcal{A}(d,\alpha)\int \frac{1}{\|x-\frac{z}{\|z\|^2}\|^{d-\alpha}}\cdot\frac{1}{\|z\|^{d-\alpha}}\cdot u^p(z)\,\mathrm{d}z
\end{aligned}$$

$$= \mathcal{A}(d,\alpha)\int \frac{1}{\| x\cdot\|z\| - \frac{z}{\|z\|}\|^{d-\alpha}}\cdot u^p(z)\,\mathrm{d}z$$

$$= \mathcal{A}(d,\alpha)\int_{\mathbf{R}^d} \frac{u^p(z)\,\mathrm{d}z}{\| \frac{x}{\|x\|} - \|x\|\cdot z\|^{d-\alpha}}$$

$$= \frac{1}{\|x\|^{d-\alpha}} u\left(\frac{x}{\|x\|^2}\right) = v(x),$$

where we used the elementary identity $\|x\cdot\|z\| - \frac{z}{\|z\|}\| = \|\frac{x}{\|x\|} - \|x\|\cdot z\|$ in the fourth equality.

3. Proceeding as in the proof of Proposition 3.1 one can verify, again assuming $d > \alpha$ and $p = (d+\alpha)/(d-\alpha)$, that a singular explicit solution to (3.1) is given by

$$u_{\mathrm{sing}}(x) = \left[2^\alpha\left(\Gamma\left(\frac{d+\alpha}{4}\right)\Big/\Gamma\left(\frac{d-\alpha}{4}\right)\right)^2\right]^{1/(p-1)}\cdot\frac{1}{\|x\|^{(d-\alpha)/2}},\quad x\neq 0,$$

and that u_{sing} is a fixed point of the Kelvin transform (3.7).

4. Rotational symmetry of solutions in a ball

Let $F:\mathbf{R}_+\to\mathbf{R}_+$ be non-decreasing, not identically constant, and $\alpha\in(0,2)$. Let $u:\mathbf{R}^d\to\mathbf{R}_+$ be a non-negative bounded solution to

$$\Delta_\alpha u + F(u) = 0,\quad x\in B_1(0),\quad u\equiv 0\quad \text{in } B_1(0)^c, \tag{4.1}$$

i.e., for all x in the unit ball we have

$$u(x) = \int_0^\infty \mathbf{E}_x\big[F(X_t); \sup_{u\leqslant t}|X_u| < 1\big]\,\mathrm{d}t = \int_{B_1(0)} G_\alpha(x,y)F\big(u(y)\big)\,\mathrm{d}y, \tag{4.2}$$

where X is the symmetric α-stable process and $G_\alpha(x,y)$ is the corresponding Green's function for the unit ball.

It is well known that (4.1) possesses non-trivial positive solutions if F is of the form $F(u) = Ku$, where $K>0$ is a constant [12]. Existence of positive solutions to (4.1) for non-linear reaction terms is a delicate question whose answer depends rather sensitively on F. For example in the case of the classical Laplacian ($\alpha=2$) in $d\geqslant 3$ and a non-linearity of power type $F(u)=u^{1+\beta}$ it is well known that there are non-trivial solutions to (4.1) if and only if $1+\beta < (d+2)/(d-2)$, see e.g. [21].

It is tempting to conjecture that this phenomenon extends from $\alpha=2$ to $\alpha\in(0,2]$, the Laplacian being replaced by Δ_α, and the threshold for $1+\beta$ being $(d+\alpha)/(d-\alpha)$. Note that this complements our conjecture given at the end of Section 1: Whereas in the case of $\mathbf{R}^d$ non-trivial solutions should exist iff $1+\beta\geqslant(d+\alpha)/(d-\alpha)$, in the case of bounded balls the condition for existence of non-trivial solutions should be $1+\beta<(d+\alpha)/(d-\alpha)$.

In our setting, proving existence of a non-trivial positive solution to (4.1) constitutes a challenging project for future research. In the present paper we will not treat this question, but rather prove

Theorem 4.1. *Any non-negative solution to* (4.1) *is symmetric about the origin.*

Our approach is based on the *method of moving planes*, a device that goes back to Alexandrov [2] and has by now a venerable history in the study of symmetries of solutions of pde's, see also [22,13] and [3]. The idea is as follows: Choose any direction in $\mathbf{R}^d$, without loss of generality the x_1-direction, and show that u is mirror

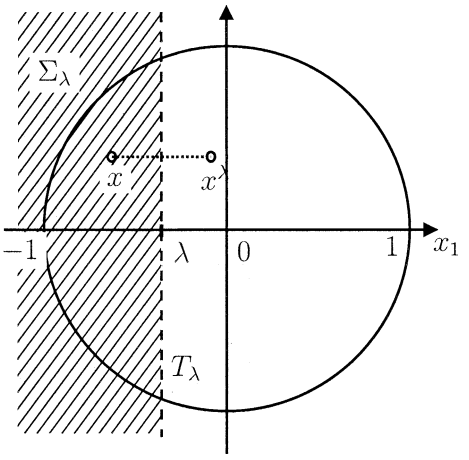

Fig. 1.

symmetric with respect to the hyperplane through the origin with this given direction as a normal vector. In order to achieve this let us define for $\lambda \in (-1, 1)$

$$T_\lambda := \{x \in \mathbf{R}^d\colon\ x_1 = \lambda\}, \quad \Sigma_\lambda := \{x \in \mathbf{R}^d\colon\ x_1 < \lambda\}$$

and for $x \in \mathbf{R}^d$ let $x^\lambda := (2\lambda - x_1, x_2, \ldots, x_d)$ be the image under reflection along T_λ; see Fig. 1.

Define the set Λ by

$$\Lambda := \left\{\lambda \in (-1, 0)\colon\ u(x^\lambda) \geqslant u(x)\ \forall x \in \Sigma_\lambda,\ \frac{\partial}{\partial x_1} u(x) > 0\ \forall x \in T_\lambda \cap B_1(0)\right\}. \tag{4.3}$$

Observe that by the minimum principle we have $u(x^\lambda) > u(x)$ for $x \in \Sigma_\lambda \cap B_1(0)$ and $\lambda \in \Lambda$, i.e., $\lambda \in \Lambda$ means that reflection along T_λ (strictly) increases the value of u. In Section 4.2 we prove that

$$\sup \Lambda = 0 \tag{4.4}$$

so that by continuity $u(-x_1, x_2, \ldots, x_d) \geqslant u(x_1, x_2, \ldots, x_d)$ whenever $x_1 \leqslant 0$. By considering $\lambda > 0$ and working in the opposite direction we can then conclude the reversed inequality and hence obtain the desired symmetry.

4.1. Some preparatory lemmas

Lemma 4.1. *A bounded solution u of* (4.2) *satisfies $u \in C(\mathbf{R}^d) \cap C^\infty(B_1(0))$.*

Proof. Using the explicit form of the Green's kernel of the ball (see e.g. [7, formula (2.3)]) one easily checks that interchange of integration and differentiation is justified. □

Lemma 4.2 (Minimum principle). *Let $D \subset \mathbf{R}^d$ be a bounded domain. Suppose $w : \mathbf{R}^d \to \mathbf{R}_+$ is continuous, $\Delta_\alpha w \leqslant 0$ on D, and satisfies $w \equiv 0$ on D^c. Then either $w \equiv 0$ or $w > 0$ on D.*

Proof. Let $D_\varepsilon := \{x \in D\colon\ w(x) > \varepsilon\}$. By continuity D_ε is open. Assume that $D_\varepsilon \neq \emptyset$ for some $\varepsilon > 0$. Let (X_t) be the α-stable process, and $\tau := \inf\{s\colon\ X_s \notin D\}$ the hitting time of D^c. Then $M_t := w(X_{t\wedge\tau}) - w(X_0) -$

$\int_0^{t\wedge\tau} \Delta_\alpha w(X_s)\,\mathrm{d}s$ is a $\mathbf{P}_x$-martingale for each $x \in D$. Let furthermore $\tau' := \inf\{s\colon\ X_s \in D_\varepsilon\}$. For each $x \in D$ we have $\mathbf{E}_x M_{\tau'} = 0$, or

$$w(x) = \mathbf{E}_x\big[w(X_{\tau'\wedge\tau})\big] + \mathbf{E}_x\left[\int_0^{\tau'\wedge\tau} (-\Delta_\alpha w)(X_s)\,\mathrm{d}s\right] \geqslant \mathbf{E}_x\big[w(X_{\tau'\wedge\tau})\big] \geqslant \varepsilon \mathbf{P}_x(\tau' < \tau) > 0$$

because (X_t) hits any open subset of D with positive probability before exiting from D. □

We will have occasion to consider the behavior of u at the boundary of the ball. In this respect, the following lemma is helpful:

Lemma 4.3 (Hopf's α-stable boundary lemma). *Let $D \subset \mathbf{R}^d$ be open, $w : \mathbf{R}^d \to \mathbf{R}_+$ continuous with $w \equiv 0$ on D^c, $\Delta_\alpha w \leqslant 0$ on D, w not identically zero. Let $x_0 \in \partial D$ satisfy an interior sphere condition, i.e. there exists a ball $B_\delta(x_1) \subset D$ with $\overline{B_\delta(x_1)} \cap D^c = \{x_0\}$, and let ν be an outward pointing unit vector at x_0. Then*

$$\frac{\partial}{\partial \nu} w(x_0) < 0$$

(*in fact,* $\lim_{\varepsilon \searrow 0}(w(x_0) - w(x_0 - \varepsilon\nu))/\varepsilon = -\infty$).

Proof. Because of the interior sphere condition at x_0 we can find a ball $B_\delta(x_1) \subset D$ such that $\overline{B_\delta(x_1)} \cap D^c = \{x_0\}$ and also $\widetilde{B} \subset D$, where $\widetilde{B}$ is the "left half" of a spherical shell around x_1 with interior radius δ and exterior radius $\delta' > \delta$; see Fig. 2. Observe that $w > 0$ in D by Lemma 4.2, in particular $\inf_{\widetilde{B}} w > 0$ because $\widetilde{B}$ is compact and w continuous. Let $\tau := \inf\{t\colon\ X_t \notin B_\delta(x_1)\}$. Then

$$w(x) = \mathbf{E}_x\left[w(X_\tau) - \int_0^\tau (\Delta_\alpha w)(X_t)\,\mathrm{d}t\right] \geqslant \mathbf{E}_x\, w(X_\tau)$$

for $x \in B_\delta(x_1)$. Take $x = x_0 - \varepsilon\nu$ with $\varepsilon > 0$ small enough, and denote by γ the angle between $\overline{x_1 x_0}$ and $\overline{x x_0}$. Then $\gamma \in (-\pi/2, \pi/2)$ because ν is an outward pointing vector, hence $\cos\gamma > 0$. The cosine theorem gives $|x - x_1|^2 = \delta^2 + \varepsilon^2 - 2\delta\varepsilon\cos\gamma$. Using the explicit form of the Poisson kernel for the complement of a ball (see e.g. [7, formula (2.2)] or [6, Theorem A] and rescale) we can estimate

$$w(x) \geqslant \mathbf{E}_x\, w(X_\tau) \geqslant \int_{\widetilde{B}} P(x, y) w(y)\,\mathrm{d}y$$

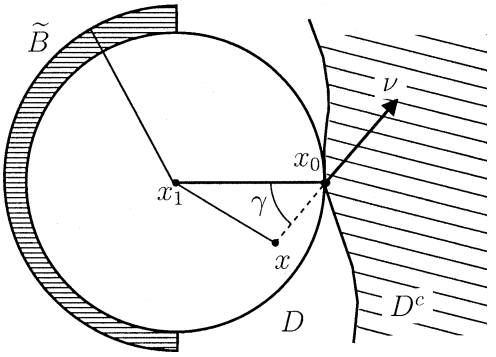

Fig. 2.

$$\geqslant (\inf_{\widetilde{B}} w) C_{\alpha,d} \int_{\widetilde{B}} \left(\frac{\delta^2 - |x - x_1|^2}{|y - x_1|^2 - \delta^2} \right)^{\alpha/2} |x - y|^{-d} \, \mathrm{d}y$$

$$\geqslant C \left(\delta^2 - |x - x_1|^2 \right)^{\alpha/2} \geqslant C' \varepsilon^{\alpha/2}.$$

Thus we see that $\limsup_{\varepsilon \searrow 0} (w(x_0) - w(x_0 - \varepsilon \nu))/\varepsilon = -\infty$. □

Lemma 4.4. *Let $w : \mathbf{R}^d \to \mathbf{R}$ be continuous, bounded with $w \geqslant 0$ on Σ_0, $w(x^0) = -w(x)$. Let $x_* \in T_0$ be such that there exists a $\delta > 0$ with $\Delta_\alpha w \leqslant 0$ on $B_\delta(x_*) \cap \Sigma_0$. Then either*

$$w \equiv 0, \quad \text{or} \quad w > 0 \text{ on } B_\delta(x_*) \cap \Sigma_0 \text{ and } \frac{\partial}{\partial x_1} w(x_*) < 0.$$

Proof. Assume $w \not\equiv 0$. Let $x \in B_\delta(x_*) \cap \Sigma_0$. If $w(x) = 0$ we would have (see e.g. [7])

$$\Delta_\alpha w(x) = c_{\alpha,d} \mathrm{PV} \int_{\mathbf{R}^d} \frac{w(x + y) - 0}{|y|^{d+\alpha}} \, \mathrm{d}y > 0$$

by the non-triviality and symmetry of w, in contradiction to the assumption.

To show that the derivative is non-zero choose $\delta' > 0$ such that $\sup_{\Sigma_0 \setminus B_{\delta'}(x_*)} w > 0$. Let (X_t) be the α-stable process, $\tau := \inf\{t\colon X_t \notin B_{\delta'}(x_*)\}$. Define $v(x) := \mathbf{E}_x w(X_\tau)$, $\tilde{v}(x) := \mathbf{E}_x \int_0^\tau (-\Delta_\alpha w)(X_s) \, \mathrm{d}s$. Observe that

$$\Delta_\alpha v = 0 \quad \text{in } B_{\delta'}(x_*) \quad \text{and} \quad v = w \quad \text{in } \mathbf{R}^d \setminus B_{\delta'}(x_*),$$
$$\Delta_\alpha \tilde{v} = \Delta_\alpha w \quad \text{in } B_{\delta'}(x_*) \quad \text{and} \quad \tilde{v} = 0 \quad \text{in } \mathbf{R}^d \setminus B_{\delta'}(x_*).$$

Uniqueness of the Dirichlet problem for Δ_α in $B_{\delta'}(x_*)$ thus gives $w = v + \tilde{v}$. We have

$$v(x) = \int_{|y - x_*| > \delta'} P(x, y) w(y) \, \mathrm{d}y$$

for $x \in B_{\delta'}(x_*)$, where the Poisson kernel is given by (see e.g. [7, formula (2.2)])

$$P(x, y) = C_{\alpha,d} \left[\frac{\delta'^2 - |x - x_*|^2}{|y - x_*|^2 - \delta'^2} \right]^{\alpha/2} |x - y|^{-d}, \quad x \in B_{\delta'}(x_*), \ y \notin B_{\delta'}(x_*).$$

One checks that $\frac{\partial}{\partial x_1} P(x_*, y) < 0$ for $y \in \Sigma_0$ and $\frac{\partial}{\partial x_1} P(x_*, y) > 0$ for $y \in -\Sigma_0$. The interchange of integration and differentiation is justified because w is bounded, so we can compute

$$\frac{\partial}{\partial x_1} v(x_*) = \int_{|y - x_*| > \delta'} \frac{\partial}{\partial x_1} P(x_*, y) w(y) \, \mathrm{d}y < 0.$$

Furthermore, for $x \in B_{\delta'}(x_*)$

$$\tilde{v}(x) = \int_{B_{\delta'}(x_*)} G(x, y)(-\Delta_\alpha w)(y) \, \mathrm{d}y,$$

where the Green kernel for $B_{\delta'}(x_*)$ is given by (see e.g. [7, formula (2.3)] or [6] and consider the obvious scaling properties of α-stable processes)

$$G(x, y) = c_{\alpha,d} \left(|x - y| \right)^{\alpha - d} \int_0^{w_{\delta'}(x,y)} \frac{r^{\alpha/2 - 1}}{(r + 1)^{d/2}} \, \mathrm{d}r, \tag{4.5}$$

where $w_{\delta'}(x,y) = (\delta'^2 - |x-x_*|^2)(\delta'^2 - |y-x_*|^2)/|x-y|^2$. Inspection shows that for $x, y \in B_{\delta'}(x_*) \cap \Sigma_0$ we have $G(x,y) \geqslant G(x,y^0)$, hence $\tilde{v} \geqslant 0$ in $B_{\delta'}(x_*) \cap \Sigma_0$ and by symmetry $\tilde{v} \leqslant 0$ in $B_{\delta'}(x_*) \cap (-\Sigma_0)$. We conclude that $(\partial/\partial x_1)\tilde{v}(x_*) \leqslant 0$ and thus

$$\frac{\partial}{\partial x_1} w(x_*) \leqslant \frac{\partial}{\partial x_1} v(x_*) < 0. \quad \square$$

4.2. Proof of (4.4)

We proceed in three steps and show that

(1) $\Lambda \supset (-1, -1+\varepsilon)$ for some small $\varepsilon > 0$,
(2) Λ is open. In particular for $\lambda \in \Lambda$ there exists $\varepsilon > 0$ such that $[\lambda, \lambda+\varepsilon) \subset \Lambda$.
(3) From (1) and (2) we conclude that $\Lambda = (-1, \lambda_{\max})$. We finally show that $\lambda_{\max} = 0$.

Step 1. Obviously $(-1, 0, \ldots, 0)$ is an outward pointing direction for each $x \in \partial B_1(0) \cap \Sigma_{-1/2}$. By the boundary lemma and the fact that $(\partial/\partial x_1)u$ is continuous in $B_1(0)$ there is an open neighborhood D of $\partial B_1(0) \cap \Sigma_{-1/2}$ such that $(\partial/\partial x_1)u > 0$ on $D \cap B_1(0)$. Choose $\varepsilon > 0$ so small that $\Sigma_{-1+\varepsilon} \cap B_1(0) \subset D$. Then $-1 + \varepsilon/2 \in \Lambda$.

Step 2. We argue by contradiction. Assume there was $\lambda_* \in \Lambda$ and also a sequence $(\lambda_n) \subset (-1, 0) \setminus \Lambda$ with $\lambda_n \searrow \lambda_*$. From the definition of Λ, possibly passing to a suitable subsequence (which we again would denote by (λ_n)) we can always arrive at one of the following possibilities:

(a) there exists a sequence $(x_n) \subset B_1(0)$, $x_n \in \Sigma_{\lambda_n}$, with $x_n \to x_* \in \overline{B_1(0)}$ and $u(x_n) \geqslant u(x_n^{\lambda_n})$ for all n, or
(b) there exists a sequence $(x_n) \subset B_1(0)$, $x_n \in T_{\lambda_n}$, with $x_n \to x_* \in \overline{B_1(0)}$ and $(\partial/\partial x_1)u(x_n) \leqslant 0$ for all n.

Assume (a) was true. We cannot have $x_* \in \Sigma_{\lambda_*}$ because u is continuous and $u(x^{\lambda_*}) > u(x)$ for $x \in \Sigma_{\lambda_*}$ by the above remark. Hence $x_* \in T_{\lambda_*} \cap \overline{B_1(0)}$. But then we have $(\partial/\partial x_1)u(x_*) = \lim_{n\to\infty}(u(x_n^{\lambda_n}) - u(x_n))/(2\mathrm{d}(x_n, T_{\lambda_n})) \leqslant 0$. By Hopf's boundary lemma, this forces x_* to be away from $\partial B_1(0)$, but then we obtain a contradiction to $\lambda_* \in \Lambda$.

If (b) was true we would again find a point $x_* \in T_{\lambda_*} \cap \overline{B_1(0)}$ with $(\partial/\partial x_1)u(x_*) \leqslant 0$ and arrive at a contradiction.

Step 3. From the preceding steps we know that $\Lambda = (-1, \lambda_{\max})$. If $u(x^{\lambda_{\max}}) \equiv u(x)$ for $x \in \Sigma_{\lambda_{\max}}$ then we have found a symmetry center. As u is continuous, $u = 0$ on $\partial B_1(0)$ and strictly positive inside $B_1(0)$ this can only be true for $\lambda_{\max} = 0$. Indeed, if $\lambda_{\max} < 0$ then by continuity we would have $u(x) \leqslant u(x^{\lambda_{\max}})$ for $x \in \Sigma_{\lambda_{\max}}$, but with $u(x) \not\equiv u(x^{\lambda_{\max}})$. Define $w(x) := u(x^{\lambda_{\max}}) - u(x)$. Observe that w is continuous and bounded, non-negative in $\Sigma_{\lambda_{\max}}$ and $w(x^{\lambda_{\max}}) = -w(x)$. For $x \in \Sigma_{\lambda_{\max}} \cap B_1(0)$ we have

$$\Delta_\alpha w(x) = (\Delta_\alpha)u(x^{\lambda_{\max}}) - (\Delta_\alpha)u(x) = -\big[F\big(u(x^{\lambda_{\max}})\big) - F\big(u(x)\big)\big] \leqslant 0,$$

and we infer from Lemma 4.4 that $(\partial/\partial x_1)w(x) < 0$ for all $x \in T_{\lambda_{\max}} \cap B_1(0)$. In conclusion, $\lambda_{\max} < 0$ implies $\lambda_{\max} \in \Lambda$ which by step 2 forces $\sup \Lambda > \lambda_{\max}$. This is a contradiction.

4.3. Remarks and open questions

1. The approach developed above invites to try to use the moving planes method also for the corresponding problem on $\mathbf{R}^d$ in order to prove the analogue of Gidas and Spruck's theorem [15] in our setting, namely that any solution u of (3.1) with $p = (d+\alpha)/(d-\alpha)$ must be symmetric about some point.

In fact, steps 2 and 3 from Section 4.2 can be carried over easily to the infinite space setting: Once we have, for a given normal direction, a "good" hyperplane T_λ, we can push it along until we hit a symmetry center. Here, "good" means that λ lies in the analogue of (4.3), with $\mathbf{R}^d$ instead of the unit ball. Unfortunately, we have not been able to

implement step 1 in the $\mathbf{R}^d$ setting. In order to get the moving planes method started, one would have to show that for λ sufficiently negative we have $u(x^\lambda) > u(x)$ for all $x \in \Sigma_\lambda$. We conjecture that this is the case. Indeed, using the tools developed in Section 4.1 it is possible to adapt several steps of the proof of Chen and Li, [8] to the case of general α. There is however a technical difficulty which we were not able to overcome, and which we briefly explain below.

Via the Kelvin transformation and (3.8) we may assume that u decays like $c\|x\|^{\alpha-d}$ at infinity. Put $w_\lambda(x) := u(x^\lambda) - u(x)$. Then w_λ is anti-symmetric with respect to reflection at T_λ and satisfies $\Delta_\alpha w_\lambda(x) + c(x)w_\lambda(x) = 0$ on $\mathbf{R}^d$, where $c(x) = p\xi(x)^{p-1}$ with some $\xi(x) \in [u(x) \wedge u_\lambda(x), u(x) \vee u_\lambda(x)]$. Following [8] the task would be to prove that for $\lambda \ll 0$, w_λ does not become negative on Σ_λ. More precisely we would like to show that for $\lambda \ll 0$, a situation where the restriction of w_λ to Σ_λ has a strictly negative minimum leads to a contradiction. However, the non-locality of Δ_α together with the fact that we actually do not have a boundary value problem for w_λ on Σ_λ, but a more complicated restriction in form of an anti-symmetry, seem to invalidate the arguments of the crucial Lemma 2.1 in [8] in the case $\alpha < 2$: A minimum value of the restriction of w_λ to Σ_λ at $x_* \in \Sigma_\lambda$ need not be a global minimum of w_λ, and thus $\Delta_\alpha w_\lambda(x_*) \leqslant 0$ need not lead to a contradiction, at least not without further arguments.

2. It is an open question whether Theorem 4.1 holds if Δ_α is replaced by the generator of a Lévy process with rotationally symmetric increment distribution. We conjecture that the answer is in the affirmative.

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