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Quasi-neutral limit for a viscous capillary model of plasma

Limite quasi neutre pour un modèle visqueux capillaire de plasmas

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Abstract

The purpose of this work is to study the quasi-neutral limit of a viscous capillary model of plasma expressed as a so-called Navier–Stokes–Poisson–Korteweg model. The existence of global weak solutions for a given Debye length λ is obtained in a periodic box domain T^3 or a strip domain $\mathrm{T}^2 \times (0,1)$. The convergence when λ goes to zero to solutions to the compressible capillary Navier–Stokes equations, in the torus T^3, turns out to be global in time in energy norm.

Résumé

Le but de ce travail est d'étudier la limite quasi-neutre d'un modèle de plasma que l'on désignera comme le modèle de Navier–Stokes–Poisson–Korteweg. L'existence de solutions globales faibles, pour une longueur de Debye λ fixée, est obtenue dans un domaine périodique T^3 ou un domaine de type bande $\mathrm{T}^2 \times (0,1)$. La convergence quand λ tend vers 0 vers les solutions des équations de Navier–Stokes capillaires compressibles, dans le tore T^3, est alors globale en temps.

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1. Introduction

For the description of physical phenomena in plasmas and semiconductors, hydrodynamical models are widely used, see [9]. In the inviscid situation, the Euler–Poisson system has been extensively studied (see [6,7]). Local in time existence and convergence results in the quasi-neutral limit in one space dimension have been obtained for instance in [4] by using pseudo-differential techniques whenever the hole density depends on the electrostatic potential. They assume that the initial density is bounded and bounded away from zero by a positive constant. Recently in [10], the quasi-neutral limit for solutions of the nonstationary multi-dimensional Euler–Poisson equations has been also justified in a periodic domain T^3 when the hole density for semiconductors or the ion density for plasma is a given function. In particular, when the hole density is constant, the limit electron velocity and electrostatic potential satisfy the classical incompressible Euler equations. Similar upper and lower bounds on the initial density are assumed.

We are interested here in the quasi-neutral limit for the 3D isothermal Navier–Stokes–Poisson system for a nonmagnetic plasma consisting of two species of charged particles: simply charged ions i or protons with positive charge and electrons e with negative charge. We use a fluid description for ions, denoting ρ and u respectively the ions density and velocity. Ions and electrons interact through the electrostatic potential ϕ. Electrons are assumed to be thermalized and follow a nondimensional Maxwell–Boltzmann distribution $\rho_e := \exp\phi$ connecting the scaled electron density ρ_e and potential ϕ. Moreover, surface tension is taken into account through a Korteweg type model of capillarity. Let us finally emphasize that the initial density is not assumed to be bounded from below by a positive constant.

In nondimensional form, the dynamics is described by the following Navier–Stokes–Poisson–Korteweg (NSPK) system with Reynolds number ν^{-1}, Weber number σ and dimensionless Debye length λ

$$\partial_t\rho + \operatorname{div}(\rho u) = 0, \tag{1}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) = -\nabla p(\rho) - \beta^2\rho\nabla\phi + \operatorname{div}\bigl(2\rho\nu D(u) + K\bigr) - \alpha u, \tag{2}$$

$$-\lambda^2\Delta\phi = \rho - \exp\phi, \tag{3}$$

$$K_{ij} = \frac{\sigma}{2}\bigl(\Delta\rho^2 - |\nabla\rho|^2\bigr)\delta_{ij} - \sigma\partial_i\rho\partial_j\rho, \tag{4}$$

where $p(\rho)$ denotes the pressure, $D(u)_{ij} = (\partial_i u_j + \partial_j u_i)/2$ the strain tensor, and K the capillarity tensor. Let us observe that $\operatorname{div} K = \sigma\rho\nabla\Delta\rho$, and that the contribution of capillary effects to energy will be proportional to $\sigma|\nabla\rho|^2/2$ (see [1,2]). The dimensionless coefficient β, defined as the ratio between the thermal velocity of electrons and the ions velocity is assumed of order one. The presence of a positive damping coefficient α has some importance when dealing with the stability of solutions for small densities. It allows us to prove the existence of global weak solutions of the systems (NSK) and (NSPK) as in [2]. When α is allowed to be nonnegative, as in [1], the classical definition of weak solutions has to be slightly changed. We choose test functions for the momentum equations which are somehow supported on the sets of positive ρ. Basically, the idea is to consider test functions of the form $\rho\varphi$ in the momentum equations, where φ is smooth in space and time. Indeed, in the complement of the set of vanishing ρ, the space of regularity of the density allows to recover the usual momentum equations. In other words, this technical point reduces to multiplying the momentum equations by ρ and consider the obtained equation in the sense of distributions.

The pressure function p will be related to the density ρ by a general barotropic constitutive law

$$p = p(\rho), \quad p\in C^1[0,\infty), \quad p(0) = 0. \tag{5}$$

Let us stress that the pressure does not need to be monotonic in ρ; in particular, nuclear plasmas of protons and electrons can be considered (see [5] and [8] for physical motivations).

We are mainly interested in the formal limit $\lambda\to 0$ of the above system, whereas β, σ and ν remain constant. By letting λ go to zero in (3) we get first the limit relation $\phi = \log\rho$, which by using (1), (2) and (4) formally yields the following limit system

$$\partial_t\rho + \operatorname{div}(\rho u) = 0, \tag{6}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u\otimes u) = -\nabla\big(p(\rho) + \beta^2 p_e(\rho)\big) + \operatorname{div}\big(2\rho\nu D(u) + K\big) - \alpha u, \tag{7}$$

$$K_{ij} = \frac{\sigma}{2}\big(\Delta\rho^2 - |\nabla\rho|^2\big)\delta_{ij} - \sigma\partial_i\rho\partial_j\rho, \tag{8}$$

where $p_e(\rho) := \rho$ is the (isothermal) pressure of the electronic gas.

Our aim is to justify the formal convergence of solutions to the Navier–Stokes–Poisson–Korteweg system (1)–(4) to solutions of the limit Navier–Stokes–Korteweg system (6)–(8) also called (NSK). We are only able to prove the convergence on weak solutions defined as in [1] that means multiplying the momentum equations by ρ even if $\alpha > 0$. To our knowledge, this is the first asymptotic result with a possibly vanishing density. As we shall see, the convergence of $(\exp\phi_\lambda)_{\lambda>0}$ to ρ is strong in $L^2((0,T)\times\Omega)$ for all positive T with a rate of convergence at least of order λ.

In the last section, we discuss the difficulties of dealing with boundaries looking at a strip domain $\Omega = \mathrm{T}^2\times(0,1)$ with boundary conditions for the potential on $\mathrm{T}^2\times\{0\}$ and $\mathrm{T}^2\times\{1\}$. A mathematical proof of convergence is far from being given.

2. Weak formulation and main existence results

This section is devoted to the detailed mathematical setting of the problem. The geometry of the domain, the boundary and initial conditions are given as well as the definition of weak solutions considered throughout the following sections. The existence of global weak solutions is derived from an estimate using the particular expression of the viscous tensor in which the dynamic viscosity μ varies as a linear function of the density: $\mu = \rho\nu$. It has been used in [1] in the case of Korteweg type fluid models in two or three dimensions, and in [2] in the study of the viscous shallow water model, which writes as a 2D compressible barotropic Navier–Stokes equations with degenerate viscosity tensor. Damping effects expressed as linear or nonlinear functions of u naturally arise in the derivation of the Shallow Water model from the bottom friction boundary conditions in the underlying free surface 3D Navier–Stokes model. Such friction forces allow to overcome the degeneracy of the momentum equations in which every other term is multiplied by the density ρ. As a result, friction forces are considered in (1)–(4) for the same purpose as in the Shallow Water model.

The three dimensional space domain is assumed to be either a box with periodic boundary conditions T^3, or a strip $\Omega = \mathrm{T}^2\times(0,1)$, which means that periodic boundary conditions are considered in (x,y) variables. On the boundary $\partial\Omega = \mathrm{T}^2\times\{z=0\}\cup\mathrm{T}^2\times\{z=1\}$, we assume that

$$\begin{aligned}&u\cdot n = 0, \qquad (D(u)\cdot n)_{\tan} = 0, \qquad \partial_n\rho = 0, \quad \text{on } \partial\Omega\\ &\phi_{|z=1} = \phi_1, \qquad \phi_{|z=0} = \phi_0,\end{aligned} \tag{9}$$

for some prescribed constant potentials ϕ_1 and ϕ_0. The notation $f_{\tan}$ on the boundary $\partial\Omega$ denotes for any vector field f the tangential part $f_{\tan} = f - (f\cdot n)n$, n denoting the outward normal to $\partial\Omega$. As usual in such nonhomogeneous boundary conditions, we introduce a lifting potential $\tilde\phi$ given by $\tilde\phi(z) = z\phi_1 + (1-z)\phi_0$.

The initial state of the system will be given by the initial density ρ_0 and momentum m_0:

$$\rho_{|t=0} = \rho_0, \qquad \rho u_{|t=0} = m_0, \tag{10}$$

where we agree that $m_0 = 0$ on $\{x\in\Omega \,/\, \rho_0(x) = 0\}$.

We now give a precise formulation of our results. Formally multiplying (2) by u, integrating by parts, making use of the continuity equation (1) and of the Poisson equation (3) yields the energy inequality

$$\frac{\mathrm{d}E(t)}{\mathrm{d}t} + \int_\Omega \big(\alpha|u|^2 + 2\rho\nu D(u):D(u)\big)(t,x)\,\mathrm{d}x \leqslant 0, \tag{11}$$

where

$$E(t) = \int_\Omega \left(\frac{1}{2}\rho|u|^2 + P(\rho) + \frac{\sigma}{2}|\nabla\rho|^2 + \frac{\lambda^2\beta^2}{2}\big|\nabla(\phi - \tilde{\phi})\big|^2 + \beta^2 F(\phi - \tilde{\phi})\exp\tilde{\phi} + \beta^2\rho\tilde{\phi}\right)(t,x)\,\mathrm{d}x, \tag{12}$$

where

$$P(\rho) = \rho\int_1^\rho \frac{p(s)}{s^2}\,\mathrm{d}s \quad \text{and} \quad F(\psi) = \int_1^\psi \tau\,\mathrm{e}^\tau\,\mathrm{d}\tau = (\psi - 1)\exp\psi. \tag{13}$$

In particular, as soon as the initial data satisfy $E(0) < +\infty$, one has the global estimate $\sup_{t\geqslant 0} E(t) \leqslant E(0)$, which provides global *a priori* bounds for solutions of the above system. We are now able to state the definition of weak solutions

Definition 2.1. Let $\alpha \geqslant 0$ and ρ_0, m_0 such that $E(0) < +\infty$. We shall say that (ρ, u) is a global weak solution of (NSKP) if for all $T > 0$,

- $P(\rho)$ and $F(\phi)$ belong to $L^\infty(0,T;L^1(\Omega))$; $\sqrt{\rho}u$, $\nabla\rho$, and $\nabla\phi$ belong to $L^\infty(0,T;L^2(\Omega))$; finally, u and $\sqrt{\rho}D(u)$ belong to $L^2(0,T;L^2(\Omega))$,
- (1) holds in the sense of distributions,
- for all $v \in (C^\infty((0,T)\times\Omega))^3$, compactly supported in $[0,T)\times\Omega$, one has

$$\begin{aligned}&\int_\Omega m_0\cdot\rho_0 v_0\,\mathrm{d}x + \int_0^T\int_\Omega \big(\rho^2 u\cdot\partial_t v + \rho u\otimes\rho u : D(v) - \rho^2 u\cdot v\,\mathrm{div}\,u \\ &\quad - \alpha\rho u\cdot v + \Xi(\rho)\,\mathrm{div}\,v - \beta^2\rho^2\nabla\phi\cdot v - 2\nu\rho D(u):\rho D(v) \\ &\quad - \nu\rho D(u): v\otimes\nabla\rho - \sigma\rho^2\Delta\rho\,\mathrm{div}\,v - 2\sigma\rho(v\cdot\nabla\rho)\Delta\rho\big)\,\mathrm{d}x\,\mathrm{d}t = 0,\end{aligned} \tag{14}$$

  where

$$\Xi(s) = \int_1^s \tau P'(\tau)\,\mathrm{d}\tau,$$

- finally, for all $\psi \in C^\infty((0,T)\times\Omega)$ compactly supported in $(0,T)\times\Omega$, one has

$$\int_0^T\int_\Omega \big(\lambda^2\nabla\phi\cdot\nabla\psi + \psi\exp\phi - \rho\psi\big)\,\mathrm{d}x\,\mathrm{d}t = 0. \tag{15}$$

Similarly, energy–based analysis can be achieved for the limit system (NSK). Indeed, the energy of (NSK) is obtained by multiplying Eq. (7) by u and integrating by parts. Then, the energy estimate (11) still holds, $E(t)$ being replaced by $\overline{E}(t)$, where

$$\overline{E}(t) = \int_\Omega \left(\frac{1}{2}\rho|u|^2 + P(\rho) + \beta^2\rho\log\rho + \frac{\sigma}{2}|\nabla\rho|^2\right)(t,x)\,\mathrm{d}x. \tag{16}$$

Weak solutions can be defined similarly as in Definition 2.1 where the conditions involving the potential ϕ are ignored.

The global existence of weak solutions for the limit system (NSK) in the sense of Definition 2.1 was obtained in [1] without damping ($\alpha = 0$). When the damping α is positive, the velocity u makes sense by itself independently of the density ρ since u belongs to $L^2(0,T;(L^2(\Omega))^3)$, so that existence of global weak solutions can be obtained in the classical sense of weak solutions (without multiplying the momentum equation by the density as it is in 2.1) (see [2] for details). The main idea introduced in [2] and [1] is to look for a Lyapunov functional that provides additional *a priori* bounds. More precisely, such a functional is formally obtained by multiplying the momentum equations (7) by $\nabla\log\rho$ and deriving an evolution equation on $\nabla\sqrt{\rho}$ from the mass conservation equation (6). The particular structure of the viscosity tensor – which is degenerate with respect to the density – allows to get the following estimate

$$\begin{aligned}&\int_\Omega \big(8\nu p'(\rho)|\nabla\sqrt{\rho}|^2+2\nu\sigma|\nabla^2\rho|^2\big)\,\mathrm{d}x+\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\left(-2\nu\alpha\log\rho+\frac12|u+2\nu\nabla\log\rho|^2\right)\mathrm{d}x\\ &\quad=\frac12\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\rho|u|^2\,\mathrm{d}x+\int_\Omega 2\nu\rho\nabla u:{}^t\nabla u\,\mathrm{d}x\leqslant\int_\Omega 2\nu\rho D(u):D(u)\,\mathrm{d}x,\end{aligned}\tag{17}$$

which gives parabolic estimates on the density $\rho\in L^2(0,T;H^2(\Omega))$ by using the energy estimate (11) and allows to pass to the limit in the quadratic terms in $\nabla\rho$ when proving the existence of solutions.

Theorem 2.2. *Assume that the initial data (ρ_0,m_0) are taken in such a way that $E(0)<+\infty$ and that the initial density ρ_0 satisfies*

$$\nabla\sqrt{\rho_0}\in L^2(\Omega),\quad \textit{and}\quad -\log_-\rho_0\in L^1(\Omega).\tag{18}$$

Then, there exists two global weak solutions to (NSK) in the sense 2.1 such that in addition $\rho\in L^2(0,T;H^2(\Omega))$, $\nabla\sqrt{\rho}\in(L^\infty(0,T;L^2(\Omega)))^3$.

Let us recall that such an extra *a priori* bound holds because the boundary $\partial\Omega$ has no curvature. If the boundary has some nonzero curvature, some boundary integrals do not disappear in the calculation when multiplying the momentum equation by $\nabla\rho/\rho$ and we are not able to control them. For more details on difficulties associated with boundary conditions, we refer to [1] and [2].

In the case of (NSKP) model, the procedure to derive the Lyapunov functional can be easily adapted. As a matter of fact, the presence of the extra term $-\beta^2\rho\nabla\phi$ in the right hand side of the momentum equations generates on the left hand side of (17)

$$\int_\Omega \rho\nabla\phi\cdot\frac{\nabla\rho}{\rho}\,\mathrm{d}x\tag{19}$$

which for given parameter λ is estimated by

$$\big\|\nabla\phi(t,\cdot)\big\|_{(L^2(\Omega))^3}\big\|\nabla\rho(t,\cdot)\big\|_{(L^2(\Omega))^3},$$

uniformly controlled in time in view of energy estimates (11). The compactness of the product $\rho\nabla\phi$ is a consequence of the strong compactness of ρ in $L^2((0,T)\times\Omega)$ and of the weak $(L^2((0,T)\times\Omega))^3$ compactness of $\nabla\phi$. It follows that global existence of weak solutions for a given λ is an easy corollary of the results of [1,2].

Theorem 2.3. *Assume that the initial data (ρ_0,m_0) are taken in such a way that $E(0)<+\infty$ and that the initial density ρ_0 satisfies*

$$\nabla\sqrt{\rho_0}\in(L^2(\Omega))^3,\quad \textit{and}\quad -\log_-\rho_0\in L^1(\Omega).\tag{20}$$

Then, there exists two global weak solutions to (NSPK) in the sense 2.1 such that in addition $\rho\in L^2(0,T;H^2(\Omega))$, $\nabla\sqrt{\rho}\in(L^\infty(0,T;L^2(\Omega)))^3$.

Let us recall that $\log_-$ is defined on $\mathbb{R}_*^+$ by $\log_-(s) = \log\min(1, s)$.

We now wish to study the behavior of weak solutions to (NSPK) when the renormalized Debye length λ tends to zero. The problem is the control of the term $\rho\nabla\phi$ when ρ tends to 0. We are able now to prove the convergence of solutions in the sense 2.1 of (NSPK) to weak solutions of (NSK). Definition 2.1 is a technical mathematical restriction on the set where the density vanishes, the physical meaning being fully preserved: as a matter of fact, the relevance of the compressible fluid equations in the close neighborhood of $\rho^{-1}(\{0\})$ is questionable.

3. Convergence in the quasineutral limit in a torus

We now consider a sequence of global weak solutions $(\rho_\lambda, u_\lambda, \phi_\lambda)$ of (NSPK) in the sense of Definition 2.1 and intend to prove a convergence result in suitable energy norms to global weak solutions of (NSK).

The multiplication of the momentum equations by ρ in Definition 2.2 of weak solutions allows to control the nonlinear products in regions where ρ_λ is close to 0, which in fact correspond to regions where ϕ_λ tends to $-\infty$. In previous works related to the asymptotic analysis of the inviscid Euler–Poisson equations, this kind of problem is avoided since the convergence is only local in time and the initial density is assumed to be bounded from below by a positive constant.

Here, we prove the following result

Theorem 3.1. *Let $(\rho_\lambda, u_\lambda, \phi_\lambda)_{\lambda>0}$ be a sequence of global weak solutions of (NSPK) in the sense of Definition* 2.1*, uniformly bounded in energy norm ($E_\lambda(0) \leqslant C$, $\|\nabla\sqrt{\rho_0}\|_{(L^2(\Omega))^3} \leqslant C$ and $\|\log_-\rho_0\|_{L^1} \leqslant C$ for some positive constant C) with ρ_λ in $L^2(0,T;H^2(\Omega))$ and $\sqrt{\rho}_\lambda$ in $L^\infty(0,T;L^2(\Omega))$ uniformly with respect to λ. There exists a subsequence $(\rho_\lambda, u_\lambda, \phi_\lambda)$ and a global weak solution (ρ, u) of (NSK) in the sense of Definition* 2.1 *such that*

$$\rho_\lambda - \exp\phi_\lambda \to 0 \quad \text{in } L^2\big((0,T)\times\Omega\big), \tag{21}$$

$$\rho_\lambda^2\nabla\phi_\lambda \to \rho\nabla\rho \quad \text{in } \big(\mathcal{D}'\big((0,T)\times\Omega\big)\big)^3, \tag{22}$$

$$\rho_\lambda \to \rho \quad \text{in } L^2\big(0,T;H^s(\Omega)\big) \quad \text{for } s<2, \tag{23}$$

$$\rho_\lambda u_\lambda \to \rho u \quad \text{in } \big(L^2\big(0,T;L^2(\Omega)\big)\big)^3, \tag{24}$$

$$\sqrt{\rho_\lambda}u_\lambda \to \sqrt{\rho}u \quad \text{in } \big(L^\infty\big(0,T;L^2(\Omega)\big)\big)^3 \text{ weak } *, \tag{25}$$

$$\sqrt{\rho_\lambda}\nabla u_\lambda \to \sqrt{\rho}\nabla u \quad \text{in } \big(L^2\big(0,T;L^2(\Omega)\big)\big)^9 \text{ weak.} \tag{26}$$

Proof. Uniform bounds are provided by estimate (11) as soon as the initial energy $E_\lambda(0)$ is uniformly bounded in λ. All the convergences except the first two ones have been proved in [1] and are preserved in the present analysis since they derive from uniform estimates with respect to λ. The difficulty in [1] was to prove the strong convergence of $\rho_\lambda u_\lambda$ in $(L^2(0,T;L^2(\Omega)))^3$. Here the novelty is to show that we are able to pass to the limit in the quantity $\rho_\lambda^2\nabla\phi_\lambda$.

A natural idea is to look for additional bounds by considering the estimates obtained by multiplying the momentum equations by $\nabla\log\rho_\lambda$. Integrating by parts, the term involving the electrostatic force writes as

$$\int_\Omega \nabla\phi_\lambda\cdot\nabla\rho_\lambda\,\mathrm{d}x.$$

Thus, using the equation satisfied by the electrostatic potential, we get

$$\int_\Omega \nabla\phi_\lambda\cdot\nabla\rho_\lambda\,\mathrm{d}x = \lambda^2\int_\Omega|\Delta\phi_\lambda|^2\,\mathrm{d}x + \int_\Omega|\nabla\phi_\lambda|^2\exp\phi_\lambda\,\mathrm{d}x = \int_\Omega\left|\frac{\rho_\lambda - \exp\phi_\lambda}{\lambda}\right|^2\mathrm{d}x + 4\int_\Omega\big|\nabla\exp(\phi_\lambda/2)\big|^2\,\mathrm{d}x.$$

It follows that $(\rho_\lambda - \exp\phi_\lambda)/\lambda$ is uniformly bounded in $L^2(0,T;L^2(\Omega))$ and that $\nabla\exp(\phi_\lambda/2)$ is uniformly bounded in $(L^2(0,T;L^2(\Omega)))^3$. The uniform bounds on ρ_λ and u_λ allows us to pass to the limit in all the nonlinear products, as in [1], except in the terms associated with electrostatic forces. In particular, we shall use that ρ_λ converges strongly to some limit ρ in $L^2(0,T;H^s(\Omega))$ for all $s<2$. The nonlinear term in the momentum equations that remains to study is $\rho_\lambda^2\nabla\phi_\lambda$. The goal is to prove that this product converges to $\rho\nabla\rho$ in $(\mathcal{D}'((0,T)\times\Omega))^3$ as λ goes to 0.

Step 1: *convergence for* ρ_λ *close to* 0. Let $\varepsilon>0$ and G a nondecreasing function on $\mathbb{R}$ between 0 and 1 such that $G(s)=1$ if $s>1$ and 0 if $s<1/2$. Denoting $H_\varepsilon(s)=G(s/\varepsilon)$, we have

$$\bigl(1-H_\varepsilon(\rho_\lambda)\bigr)\rho_\lambda^2\nabla\phi_\lambda=\bigl(1-H_\varepsilon(\rho_\lambda)\bigr)\rho_\lambda\frac{\rho_\lambda-\exp\phi_\lambda}{\lambda}\lambda\nabla\phi_\lambda+\bigl(1-H_\varepsilon(\rho_\lambda)\bigr)\rho_\lambda\nabla\exp\phi_\lambda. \tag{27}$$

The first term of the right hand side of (27) is bounded in $(L^1((0,T)\times\Omega))^3$ uniformly by $c\varepsilon$, by estimating the three factors of the product respectively in $L^\infty((0,T)\times\Omega)$, $L^2((0,T)\times\Omega)$ and $L^2((0,T)\times\Omega)$. The second term of (27) may be written as

$$\nabla\Bigl(\bigl(1-H_\varepsilon(\rho_\lambda)\bigr)\rho_\lambda\exp\phi_\lambda\Bigr)-\exp\phi_\lambda\bigl(1-H_\varepsilon(\rho_\lambda)-H_\varepsilon'(\rho_\lambda)\rho_\lambda\bigr)\nabla\rho_\lambda. \tag{28}$$

We recall that $\rho_\lambda-\exp\phi_\lambda$ tends strongly to zero in $L^2((0,T)\times\Omega)$ as λ goes to 0, so that the first term of (28) is the gradient of a function uniformly controlled in $L^2((0,T)\times\Omega)$ by $c\varepsilon$. The second term of (28) may be written under the form

$$(\rho_\lambda-\exp\phi_\lambda)\bigl(1-H_\varepsilon(\rho_\lambda)-H_\varepsilon'(\rho_\lambda)\rho_\lambda\bigr)\nabla\rho_\lambda-\rho_\lambda\bigl(1-H_\varepsilon(\rho_\lambda)+H_\varepsilon'(\rho_\lambda)\rho_\lambda\bigr)\nabla\rho_\lambda. \tag{29}$$

The first term of (29) is estimated by $C\lambda$ in $L^1((0,T)\times\Omega)$, therefore controlled by ε for λ small enough (the first factor is bounded by $c\lambda$ in $L^2((0,T)\times\Omega)$, the second one in $L^\infty((0,T)\times\Omega)$ since $|\rho_\lambda H'(\rho_\lambda)|<c$, and the last term is bounded in $L^2((0,T)\times\Omega)$ recalling that ρ_λ is uniformly bounded in $L^\infty(0,T;H^1(\Omega))$. The second part of (29) if bounded by $C\varepsilon$ in $L^2((0,T)\times\Omega)$ in view of similar and in fact simpler arguments.

As a result, we succeed in controlling in $(\mathcal{D}'((0,T)\times\Omega))^3$ by $c\varepsilon$ (for λ small enough) the term

$$\bigl(1-H_\varepsilon(\rho_\lambda)\bigr)\rho_\lambda^2\nabla\phi_\lambda.$$

Step 2: *convergence for* ρ_λ *away from* 0. Given $\varepsilon>0$ and λ small enough, it remains to study the convergence of the term

$$H_\varepsilon(\rho_\lambda)\rho_\lambda^2\nabla\phi_\lambda,$$

that we decompose, introducing the quantity $Z_\lambda=(\rho_\lambda-\exp\phi_\lambda)/\lambda$, by

$$H_\varepsilon(\rho_\lambda)\rho_\lambda^2\nabla\phi_\lambda=H(\rho_\lambda)\rho_\lambda\nabla\exp\phi_\lambda+H_\varepsilon(\rho_\lambda)\rho_\lambda Z_\lambda\lambda\nabla\phi_\lambda. \tag{30}$$

(a) *Study of the first part* $H_\varepsilon(\rho_\lambda)\rho_\lambda\nabla\exp\phi_\lambda$ *of* (30).

This term can be rewritten under the form

$$H_\varepsilon(\rho_\lambda)\rho_\lambda\nabla\exp\phi_\lambda=\nabla\bigl(H(\rho_\lambda)\rho_\lambda\exp\phi_\lambda\bigr)-\exp\phi_\lambda\bigl(\rho_\lambda H_\varepsilon'(\rho_\lambda)+H_\varepsilon(\rho_\lambda)\bigr)\nabla\rho_\lambda.$$

Since the first term is the gradient of the product of two factors converging strongly in $L^2((0,T)\times\Omega)$ and that the second term is the product of two factors converging strongly in $L^2((0,T)\times\Omega)$, we get the convergence

$$H_\varepsilon(\rho_\lambda)\rho_\lambda\nabla\exp\phi_\lambda\to H_\varepsilon(\rho)\rho\nabla\rho\quad\text{in }\bigl(\mathcal{D}'\bigl((0,T)\times\Omega\bigr)\bigr)^3\text{ as }\lambda\to0.$$

Notice also that $(1-H_\varepsilon(\rho))\rho\nabla\rho$ is smaller than ε in $(L^2((0,T)\times\Omega))^3$ uniformly in λ.

(b) *Study of the second part* $H_\varepsilon(\rho_\lambda)\rho_\lambda Z_\lambda\lambda\nabla\phi_\lambda$ *of* (30).

The goal is to prove that this term is small in $\mathcal{D}'((0,T)\times\Omega)$. It can be rewritten as

$$H_\varepsilon(\rho_\lambda)\rho_\lambda Z_\lambda\lambda\nabla\phi_\lambda=K_M(Z_\lambda)Z_\lambda H_\varepsilon(\rho_\lambda)\rho_\lambda\lambda\nabla\phi_\lambda+\bigl(1-K_M(Z_\lambda)\bigr)H_\varepsilon(\rho_\lambda)\rho_\lambda Z_\lambda\lambda\nabla\phi_\lambda, \tag{31}$$

where $M>1$ and $K_M(s)=1-G(s/M)$ is an additional truncation function.

(b1) Study of the term $K_M(Z_\lambda)H_\varepsilon(\rho_\lambda)\rho_\lambda Z_\lambda\lambda\nabla\phi_\lambda$.
In order to control the first term of (31), we write

$$K_M(Z_\lambda)Z_\lambda H_\varepsilon(\rho_\lambda)\rho_\lambda\lambda\nabla\phi_\lambda = K_M(Z_\lambda)Z_\lambda H_\varepsilon(\rho_\lambda)\rho_\lambda 1_{\rho_\lambda\geqslant R}\lambda\nabla\phi_\lambda + K_M(Z_\lambda)Z_\lambda H_\varepsilon(\rho_\lambda)\rho_\lambda 1_{\rho_\lambda< R}\lambda\nabla\phi_\lambda. \tag{32}$$

The first part converges to zero when R tends to infinity. Indeed, $K_M(Z_\lambda)Z_\lambda$ is bounded in $L^2((0,T)\times\Omega)$ uniformly in M and λ, $\lambda\nabla\phi_\lambda$ is bounded in $(L^2(0,T;H^1(\Omega)))^3$ and therefore in $(L^2(0,T;L^6(\Omega)))^3$. It remains to estimate $\rho_\lambda 1_{\rho_\lambda\geqslant R}$ in $L^\infty(0,T;L^3(\Omega))$ as follows

$$\left\|\rho_\lambda(t,\cdot)1_{\rho_\lambda(t,\cdot)\geqslant R}\right\|_{L^3(\Omega)} \leqslant \frac{1}{R}\left\|\rho_\lambda(t,\cdot)\right\|^2_{L^6(\Omega)} \leqslant \frac{C}{R}\|\rho_\lambda\|^2_{L^\infty(0,T;H^1(\Omega))},$$

so that for large enough R, the first term of (32) is smaller than ε uniformly in M and λ in $L^1((0,T)\times\Omega)$. We now have to control the second part of (32). R being now given, the factors involving ρ_λ can be estimated in $L^\infty((0,T)\times\Omega)$ norm. Since $\lambda\nabla\phi_\lambda$ is uniformly bounded in $(L^2((0,T)\times\Omega))^3$ and in $(L^2(0,T;L^6(\Omega)))^3$, it is also bounded in $(L^4(0,T;L^3(\Omega)))^3$. Then it suffices to estimate $K_M(Z_\lambda)Z_\lambda$ in $L^{4/3}(0,T;L^{3/2}(\Omega))$ and choose suitable constant M. We use the fact that for all $f\in L^2((0,T)\times\Omega)$ such that $f>M$ with M large enough

$$\|f\|_{L^r((0,T)\times\Omega)} \leqslant CM^{(r-2)/r}\|f\|^{2/r}_{L^2((0,T)\times\Omega)}$$

for all $r<2$. This gives

$$\left\|K_M(Z_\lambda)Z_\lambda\right\|_{L^{4/3}(0,T;L^{3/2}(\Omega))} \leqslant CM^{-1/3}.$$

Thus the second term is smaller than ε in $L^1((0,T)\times\Omega)$ for sufficiently large M, uniformly in λ.

(b2) Study of the term $H_\varepsilon(\rho_\lambda)\rho_\lambda(1-K_M(Z_\lambda))Z_\lambda\lambda\nabla\phi_\lambda$.
The constant M is now fixed. We recall that if $|Z_\lambda|<M$, i.e.

$$|\rho_\lambda - \exp\phi_\lambda| < M\lambda,$$

then for λ small enough

$$|\rho_\lambda - \exp\phi_\lambda| < \varepsilon/2.$$

If in addition $\rho_\lambda\geqslant\varepsilon$, then

$$\exp\phi_\lambda > \varepsilon/2.$$

Thus $H_\varepsilon(\rho_\lambda)(1-K_M(Z_\lambda))Z_\lambda\exp(-\phi_\lambda/2)$ is bounded in $L^\infty((0,T)\times\Omega)$ by $M\sqrt{2/\varepsilon}$ and we can write

$$H_\varepsilon(\rho_\lambda)\rho_\lambda\big(1-K_M(Z_\lambda)\big)Z_\lambda\lambda\nabla\phi_\lambda = 2H_\varepsilon(\rho_\lambda)\rho_\lambda\big(1-K_M(Z_\lambda)\big)Z_\lambda\exp(-\phi_\lambda/2)\lambda\nabla\exp(\phi_\lambda/2).$$

Then since ρ_λ is bounded in $L^2((0,T)\times\Omega)$, $\nabla\exp(\phi_\lambda/2)$ is bounded in $(L^2((0,T)\times\Omega))^3$ and $(1-K_M(Z_\lambda))Z_\lambda\exp(-\phi_\lambda/2)$ is bounded in $L^\infty((0,T)\times\Omega)$, the second part of the right hand side of (31) is smaller than $C\lambda$ in $L^1((0,T)\times\Omega)$.

This concludes the proof of the asymptotic analysis when $\lambda\to 0$. □

4. Some remarks for the strip domain case

In the case of a strip domain of the form $\mathrm{T}^2\times(0,1)$ with boundary conditions on the top and bottom as specified in (9), global existence results have been derived in Theorem 2.3 for fixed parameter λ.

The question of the quasineutral limit $\lambda \to 0$ in the presence of boundary data has been addressed in [3] in the stationary case for a given density sequence bounded away from zero uniformly in λ. In that case, the convergence is obtained by introducing a boundary layer profile of typical size λ in the neighborhood of the boundary.

In the present case when $\Omega = \mathrm{T}^2 \times (0,1)$, the idea would be to adapt the proof related the periodic framework and to introduce boundary layers as in [3]. Starting from initial data uniformly bounded in energy space such that the bounds (20) are also uniform in λ, additional bounds seem to be necessary to prove the convergence of $\rho_\lambda \nabla\phi_\lambda$ to $\nabla\rho$ in the sense of distributions, ρ being a weak limit of ρ_λ. Multiplying in a similar way the momentum equations by $\nabla \log \rho_\lambda$ yields the identity

$$\int_\Omega \rho_\lambda \nabla\phi_\lambda \cdot \frac{\nabla\rho_\lambda}{\rho_\lambda}\,\mathrm{d}x = \int_{\partial\Omega} \rho_\lambda \partial_n \phi_\lambda \,\mathrm{d}\sigma - \int_\Omega \rho_\lambda \Delta\phi_\lambda \,\mathrm{d}x = \int_{\partial\Omega} \rho_\lambda \partial_n \phi_\lambda \,\mathrm{d}\sigma + \int_\Omega \left(\lambda^2 |\Delta\phi_\lambda|^2 + |\nabla\phi_\lambda|^2 \exp\phi_\lambda\right)\mathrm{d}x.$$

Unfortunately, very little is known about the extra boundary term: the trace of the density ρ_λ is uniformly estimated in $L^\infty(0,T;H^{1/2}(\partial\Omega))$, but we are not able to control uniformly the normal derivative of the potential ϕ_λ in $L^1(0,T;H^{-1/2}(\partial\Omega))$, even assuming that the extra bounds in the periodic case hold.

An other way to try to control the extra term coming from $\rho_\lambda\nabla\phi_\lambda$ consists in integrating by parts in the opposite sense to get

$$\int_\Omega \nabla\phi_\lambda \cdot \nabla\rho_\lambda \,\mathrm{d}x = -\int_\Omega \phi_\lambda \Delta\rho_\lambda \,\mathrm{d}x.$$

This would give the additional bounds if ϕ_λ was bounded uniformly in $L^2(0,T;L^2(\Omega))$. But no information is available on ϕ_λ in regions where ϕ_λ tends to $-\infty$. In other words, our proof fails because of the lack of information on sets where ρ_λ is close to 0.

As a consequence, the justification of the asymptotic behavior in this case therefore seems to be a very challenging mathematical problem.

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