# Positive solutions to superlinear second-order divergence type elliptic equations in cone-like domains

# Solutions positives des équations elliptiques de type divergence du second ordre superlinéaires sur des domaines coniques

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## Abstract

We study the problem of the existence and nonexistence of positive solutions to the superlinear second-order divergence type elliptic equation with measurable coefficients $-\nabla \cdot a \cdot \nabla u = u^p$ (∗), $p > 1$, in an unbounded cone-like domain $G \subset \mathbb{R}^N$ ($N \geqslant 3$). We prove that the critical exponent $p^*(a, G) = \inf\{p > 1: (∗)$ *has a positive supersolution at infinity in $G$*} for a nontrivial cone-like domain is always in $(1, \frac{N}{N-2})$ and depends both on the geometry of the domain $G$ and the coefficients $a$ of the equation.

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## Résumé

Nous étudions le problème d'existence ou non existence de solutions positives d'équations elliptiques de type divergence du second-ordre superlinéaires à coefficients mesurables $-\nabla \cdot a \cdot \nabla u = u^p$ (∗), $p > 1$, sur un domaine conique $G \subset \mathbb{R}^N$ ($N \geqslant 3$). Nous prouvons que l'exposant critique $p^*(a, G) = \inf\{p > 1: (∗)$ *a une supersolution positive à l'infini dans $G$*} pour un domaine conique non–trivial est toujours dans $(1, \frac{N}{N-2})$, dépend à la fois de la géométrie du domaine $G$ et des coefficients $a$ de l'équation.

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## 1. Introduction

We study the existence and nonexistence of positive solutions and supersolutions to the superlinear second-order divergence type elliptic equation

$$-\nabla \cdot a \cdot \nabla u = u^p \quad \text{in } G. \tag{1.1}$$

Here $p > 1$, $G \subset \mathbb{R}^N$ ($N \geqslant 3$) is an unbounded domain (i.e., connected open set) and $-\nabla \cdot a \cdot \nabla :=$ $-\sum_{i,j=1}^{N} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial}{\partial x_j})$ is a second order divergence type elliptic expression. We assume throughout the paper that the matrix $a = (a_{ij}(x))_{i,j=1}^{N}$ is symmetric measurable and uniformly elliptic, i.e. there exists an ellipticity constant $\nu = \nu(a) > 0$ such that

$$\nu^{-1}|\xi|^2 \leqslant \sum_{i,j=1}^{N} a_{ij}(x)\xi_i\xi_j \leqslant \nu|\xi|^2, \quad \text{for all } \xi \in \mathbb{R}^N \text{ and almost all } x \in G. \tag{1.2}$$

The qualitative theory of semilinear equations of type (1.1) in unbounded domains of different geometries has been extensively studied because of various applications in mathematical physics and the rich mathematical structure. One of the features of Eq. (1.1) in unbounded domains is the nonexistence of positive solutions for certain values of the exponent $p$. Such nonexistence phenomena have been known at least since the celebrated paper by Gidas and Spruck [14], where it was proved that the equation

$$-\Delta u = u^p \tag{1.3}$$

has no positive classical solutions in $\mathbb{R}^N$ ($N \geqslant 3$) for $1 \leqslant p < \frac{N+2}{N-2}$. Though this results is sharp (for $p \geqslant \frac{N+2}{N-2}$ there are classical positive solutions), the critical exponent $p^* = \frac{N+2}{N-2}$ is highly unstable with respect to any changes in the statement of the problem. In particular, for any $p \in (\frac{N}{N-2}, \frac{N+2}{N-2}]$ one can produce a smooth potential $W(x)$ squeezed between two positive constants such that equation $-\Delta u = W(x)u^p$ has a positive solution in $\mathbb{R}^N$ ([34], see also [11] for more delicate results). If one looks for supersolutions to (1.3) in $\mathbb{R}^N$ or studies (1.3) in exterior domains then the value and the properties of the critical exponent change. The following result is well-known (see, e.g. [4,6]). *If $N \geqslant 3$ and $1 < p \leqslant \frac{N}{N-2}$ then there are no positive supersolutions to (1.3) outside a ball in $\mathbb{R}^N$. The value of the critical exponent $p^* = \frac{N}{N-2}$ is sharp in the sense that (1.1) has (infinitely many) positive solutions outside a ball for any $p > p^*$.* This statement has been extended in different directions by many authors (see, e.g. [3,5,7,8,10,12,17–20,25,32,35,37,38]). In particular, in [17] it was shown that the critical exponent $p^* = \frac{N}{N-2}$ is stable with respect to the change of the Laplacian by a second-order uniformly elliptic divergence type operator with measurable coefficients, perturbed by a potential, for a sufficiently wide class of potentials (see also [18] for equations of type (1.3) in exterior domains in presence of first order terms).

In this paper we develop a new method of studying nonexistence of positive solutions to (1.1) in cone-like domains (as a model example of unbounded domains in $\mathbb{R}^N$ with nontrivial geometry). The method is based upon the maximum principle and asymptotic properties at infinity of the corresponding solutions to the homogeneous linear equation. This approach was first proposed in [17]. In the framework of our method we are able to establish the nonexistence results for (1.1) with measurable coefficients in the cone-like domains without any smoothness of the boundary in the setting of the most general definition of weak supersolutions.

We say that $u$ is a *solution* (*supersolution*) to Eq. (1.1) if $u \in H^1_{\text{loc}}(G)$ and

$$\int_G \nabla u \cdot a \cdot \nabla \varphi \, dx = (\geqslant) \int_G u^p \varphi \, dx \quad \text{for all } 0 \leqslant \varphi \in H^1_c(G),$$

where $H_c^1(G)$ stands for the set of compactly supported elements from $H_{\text{loc}}^1(G)$. By the weak Harnack inequality for supersolutions (see, e.g. [15, Theorem 8.18]) any nontrivial nonnegative supersolution to (1.1) is positive in $G$. We say that Eq. (1.1) has a *solution (supersolution) at infinity* if there exists a closed ball $\bar{B}_\rho$ centered at the origin with radius $\rho > 0$ such that (1.1) has a solution (supersolution) in $G \setminus \bar{B}_\rho$.

We define the *critical exponent* to Eq. (1.1) by

$$p^* = p^*(a, G) = \inf\{p > 1\colon (1.1) \text{ has a positive supersolution at infinity in } G\}.$$

In this paper we study the critical exponent $p^*(a, G)$ in a class of cone-like domains

$$\mathcal{C}_\Omega = \{(r, \omega) \in \mathbb{R}^N\colon \omega \in \Omega,\ r > 0\},$$

where $(r, \omega)$ are the polar coordinates in $\mathbb{R}^N$ and $\Omega \subseteq S^{N-1}$ is a subdomain (a connected open subset) of the unit sphere $S^{N-1}$ in $\mathbb{R}^N$. The following proposition collects some properties of the critical exponent and positive supersolutions to (1.1) on cone-like domains.

**Proposition 1.1.** *Let $\Omega' \subset \Omega$ are subdomains of $S^{N-1}$. Then*

(i) $1 \leqslant p^*(a, \mathcal{C}_{\Omega'}) \leqslant p^*(a, \mathcal{C}_\Omega) \leqslant \frac{N}{N-2}$;
(ii) *If $p > p^*(a, \mathcal{C}_\Omega)$ then (1.1) has a positive supersolution at infinity in $\mathcal{C}_\Omega$;*
(iii) *If $p > p^*(a, \mathcal{C}_\Omega)$ then (1.1) has a positive solution at infinity in $\mathcal{C}_\Omega$.*

**Remark 1.2.** Assertion (i) follows directly from the definition of the critical exponent $p^*(a, G)$ and the fact that $p^*(a, \mathbb{R}^N) = \frac{N}{N-2}$, see [17]. Property (ii) simply means that the critical exponent $p^*(a, G)$ divides the semiaxes $(1, +\infty)$ into the nonexistence zone $(1, p^*)$ and the existence zone $(p^*, +\infty)$. Existence (or nonexistence) of a positive solution at the critical value $p^*$ itself is a separate issue. Property (iii) says that the existence of a positive supersolution at infinity implies the existence of a positive solution at infinity. More precisely, we prove that if (1.1) has a supersolution $u > 0$ in $\mathcal{C}_\Omega^\rho$ then for any $r > \rho$ it has a solution $w > 0$ in $\mathcal{C}_\Omega^r$ such that $w \leqslant u$.

The value of the critical exponent for the equation $-\Delta u = u^p$ in $\mathcal{C}_\Omega$ with $\Omega \subseteq S^{N-1}$ satisfying mild regularity assumptions was first established by Bandle and Levine [4] (see also [3]). They reduce the problem to an ODE by averaging over $\Omega$. The nonexistence of positive solutions without any smoothness assumptions on $\Omega$ has been proved by Berestycki, Capuzzo-Dolcetta and Nirenberg [5] by means of a proper choice of a test function.

Let $\lambda_1 = \lambda_1(\Omega) \geqslant 0$ be the principal eigenvalue of the Dirichlet Laplace–Beltrami operator $-\Delta_\omega$ in $\Omega$. Let $\alpha_- = \alpha_-(\Omega) < 0$ be the negative root of the equation

$$\alpha(\alpha + N - 2) = \lambda_1(\Omega).$$

The result in [4,5] reads as follows.

**Theorem 1.3.** *Let $\Omega \subseteq S^{N-1}$ be a domain. Then $p^*(id, \mathcal{C}_\Omega) = 1 - \frac{2}{\alpha_-}$, and (1.1) has no positive supersolutions at infinity in $\mathcal{C}_\Omega$ in the critical case $p = p^*(id, \mathcal{C}_\Omega)$.*

Applicability of both ODE and test function techniques seems to be limited to the case of radially symmetric matrices $a = a(|x|)$, whereas the method of the present paper is suitable for studying Eq. (1.1) with a general uniformly elliptic measurable matrix $a$. It is extendable as far as the maximum principle is valid and appropriate asymptotic estimates are available (see the proof of Theorem 1.6 below). Advantages of this approach are its transparency and flexibility. As a first demonstration of the method we give a new proof of Theorem 1.3, which has its own virtue being considerably less technical then in [4,5]. As a consequence of Theorem 1.3 we derive the following result, which says that in contrast to the case of exterior domains the value of the critical exponent on a fixed cone-like domain essentially depends on the coefficients of the matrix $a$ of the equation.

**Theorem 1.4.** *Let $\Omega \subset S^{N-1}$ be a domain such that $\lambda_1(\Omega) > 0$. Then for any $p \in (1, \frac{N}{N-2})$ there exists a uniformly elliptic matrix $a_p$ such that $p^*(a_p, \mathcal{C}_\Omega) = p$.*

**Remark 1.5.** The matrix $a_p$ can be constructed in such a way that (1.1) either *has* or *has no* positive supersolutions at infinity in $\mathcal{C}_\Omega$ in the critical case $p = p^*(a_p, \mathcal{C}_\Omega)$, see Remark 4.5 for details.

The main result of the paper asserts that Eq. (1.1) with arbitrary uniformly elliptic measurable matrix $a$ on a "nontrivial" cone-like domain always admits a "nontrivial" critical exponent.

**Theorem 1.6.** *Let $\Omega \subseteq S^{N-1}$ be a domain and $a$ be a uniformly elliptic matrix. Then $p^*(a, \mathcal{C}_\Omega) > 1$. If the interior of $S^{N-1} \setminus \Omega$ is nonempty then $p^*(a, \mathcal{C}_\Omega) < \frac{N}{N-2}$.*

**Remark 1.7.** It is not difficult to see that in the case $N = 2$ Eq. (1.1) has no positive solutions outside a ball for any $p > 1$. However, when $\mathcal{C}_\Omega$ is a "nontrivial" cone-like domain in $\mathbb{R}^2$, that is $S^1 \setminus \Omega \neq \emptyset$, then all the results of the paper remain true with minor modifications of some proofs.

The rest of the paper is organized as follows. In Section 2 we discuss the maximum and comparison principles in a form appropriate for our purposes and study some properties of linear equations in cone-like domains. Proposition 1.1 is proved in Section 3. Section 4 contains the proof of Theorems 1.3 and 1.4. The proof of Theorem 1.6 as well as some further remarks are given in Section 5.

## 2. Background, framework and auxiliary facts

Let $G \subseteq \mathbb{R}^N$ be a domain in $\mathbb{R}^N$. Throughout the paper we assume that $N \geqslant 3$. We write $G' \Subset G$ if $G'$ is a bounded subdomain of $G$ such that $cl\, G' \subset G$. By $\| \cdot \|_p$ we denote the standard norm in the Lebesgue space $L^p$. By $c, c_1, \ldots$ we denote various positive constants whose exact value is irrelevant.

Let $S^{N-1} = \{x \in \mathbb{R}^N \colon |x| = 1\}$ and $\Omega \subseteq S^{N-1}$ be a subdomain of $S^{N-1}$. Here and thereafter, for $0 \leqslant \rho < R \leqslant +\infty$, we denote

$$\mathcal{C}_\Omega^{(\rho, R)} := \big\{ (r, \omega) \in \mathbb{R}^N \colon \omega \in \Omega,\ r \in (\rho, R) \big\}, \qquad \mathcal{C}_\Omega^\rho := \mathcal{C}_\Omega^{(\rho, +\infty)}.$$

Accordingly, $\mathcal{C}_\Omega = \mathcal{C}_\Omega^0$ and $\mathcal{C}_{S^{N-1}} = \mathbb{R}^N \setminus \{0\}$.

*Maximum and comparison principles.* Consider the linear equation

$$-\nabla \cdot a \cdot \nabla u - V u = f \quad \text{in } G, \tag{2.1}$$

where $f \in H^{-1}_{\text{loc}}(G)$ and $0 \leqslant V \in L^1_{\text{loc}}(G)$ is a *form-bounded potential*, that is

$$\int_G V u^2 \, dx \leqslant (1 - \epsilon) \int_G \nabla u \cdot a \cdot \nabla u \, dx \quad \text{for all } 0 \leqslant u \in H^1_c(G) \tag{2.2}$$

with some $\epsilon \in (0, 1)$. A solution (supersolution) to (2.1) is a function $u \in H^1_{\text{loc}}(G)$ such that

$$\int_G \nabla u \cdot a \cdot \nabla \varphi \, dx - \int_G V u \varphi \, dx = (\geqslant)\langle f, \varphi \rangle \quad \text{for all } 0 \leqslant \varphi \in H^1_c(G),$$

where $\langle \cdot, \cdot \rangle$ denotes the duality between $H^{-1}_{\text{loc}}(G)$ and $H^1_c(G)$. If $u \geqslant 0$ is a supersolution to

$$-\nabla \cdot a \cdot \nabla u - V u = 0 \quad \text{in } G, \tag{2.3}$$

then $u$ is a supersolution to $-\nabla \cdot a \cdot \nabla u = 0$ in $G$. Therefore $u$ satisfies on any subdomain $G' \Subset G$ the *weak Harnack inequality*

$$\inf_{G'} u \geqslant \frac{C_W}{\mathrm{mes}(G')} \int\limits_{G'} u \, dx,$$

where $C_W = C_W(G, G') > 0$. In particular, every nontrivial supersolution $u \geqslant 0$ to (2.3) is strictly positive, that is $u > 0$ in $G$.

We define the space $D_0^1(G)$ as the completion of $C_c^\infty(G)$ with respect to the norm $\|u\|_{D_0^1(G)} := \|\nabla u\|_2$. The space $D_0^1(G)$ is a Hilbert and Dirichlet space, with the dual $D^{-1}(G)$, see, e.g. [13]. This implies, amongst other things, that $D_0^1(G)$ is invariant under the standard truncations, e.g. $v \in D_0^1(G)$ implies that $v^+ = v \vee 0 \in D_0^1(G)$, $v^- = -(v \wedge 0) \in D_0^1(G)$. By the Sobolev inequality $D_0^1(G) \subset L^{\frac{2N}{N-2}}(G)$. The Hardy inequality

$$\int\limits_{\mathbb{R}^N} |\nabla u|^2 \, dx \geqslant \frac{(N-2)^2}{4} \int\limits_{\mathbb{R}^N} \frac{|u|^2}{|x|^2} \, dx \quad \text{for all } u \in H_c^1(\mathbb{R}^N), \tag{2.4}$$

implies that $D_0^1(G) \subset L^2(G, |x|^{-2} \, dx)$. Since the matrix $a$ is uniformly elliptic and the potential $V$ is form bounded, the quadratic form

$$Q(u) := \int\limits_G \nabla u \cdot a \cdot \nabla u \, dx - \int\limits_G V u^2 \, dx$$

defines an equivalent norm $\sqrt{Q(u)}$ on $D_0^1(G)$. The following lemma is a standard consequence of the Lax–Milgram Theorem.

**Lemma 2.1.** *Let $f \in D^{-1}(G)$. Then the problem*

$$-\nabla \cdot a \cdot \nabla v - V v = f, \quad v \in D_0^1(G),$$

*has a unique solution.*

The following two lemmas provide the maximum and comparison principles for Eq. (2.1), in a form suitable for our framework. We give the full proofs for completeness, though the arguments are mostly standard.

**Lemma 2.2** (Weak Maximum Principle). *Let $v \in H_{\mathrm{loc}}^1(G)$ be a supersolution to Eq. (2.3) such that $v^- \in D_0^1(G)$. Then $v \geqslant 0$ in $G$.*

**Proof.** Let $(\varphi_n) \subset C_c^\infty(G)$ be a sequence such that $\|\nabla(v^- - \varphi_n)\|_2^2 \to 0$. For every $n \in \mathbb{N}$, set $v_n := 0 \vee \varphi_n \wedge v^-$. Since $0 \leqslant v_n \leqslant v^- \in D_0^1(G)$ and

$$\int\limits_G \left|\nabla(v^- - v_n)\right|^2 dx = \int\limits_{\{0 \leqslant \varphi_n \leqslant v^-\}} \left|\nabla(v^- - \varphi_n)\right|^2 dx + \int\limits_{\{\varphi_n \leqslant 0\}} |\nabla v^-|^2 \, dx$$

$$\leqslant \int\limits_G \left|\nabla(v^- - \varphi_n)\right|^2 dx + \int\limits_{\{\varphi_n \leqslant 0\}} |\nabla v^-|^2 \, dx \to 0,$$

by the Lebesgue dominated convergence, we conclude that $\|\nabla(v^- - v_n)\|_2^2 \to 0$ (cf. [13, Lemma 2.3.4]). Taking $(v_n)$ as a sequence of test functions we obtain

$$0 \leqslant \int_G \nabla v \cdot a \cdot \nabla v_n \, dx - \int_G V v v_n \, dx$$

$$= - \int_G \nabla v^- \cdot a \cdot \nabla v_n \, dx + \int_G V v^- v_n \, dx \to - \int_G \nabla v^- \cdot a \cdot \nabla v^- \, dx + \int_G V |v^-|^2 \, dx \leqslant 0.$$

Thus we conclude that $v^- = 0$. $\quad\square$

**Lemma 2.3** (Weak Comparison Principle). *Let* $0 \leqslant u \in H^1_{\mathrm{loc}}(G)$, $v \in D^1_0(G)$ *and*

$$-\nabla \cdot a \cdot \nabla(u - v) - V(u - v) \geqslant 0 \quad \text{in } G.$$

*Then* $u \geqslant v$ *in* $G$.

**Remark 2.4.** Note that the assertion of Lemma 2.3 follows from Lemma 2.2 if one assumes in addition that $u \in H^1(G)$.

**Proof.** Let $(G_n)_{n \in \mathbb{N}}$ be an exhaustion of $G$, i.e. an increasing sequence of bounded smooth domains such that $G_n \Subset G_{n+1} \Subset G$ and $\bigcup_{n \in \mathbb{N}} G_n = G$. Let $v \in D^1_0(G)$. Let $f \in D^{-1}(G)$ be defined by duality as

$$f := -\nabla \cdot a \cdot \nabla v - V v.$$

Let $v_n \in D^1_0(G_n)$ be the unique weak solution to the linear problem

$$-\nabla \cdot a \cdot \nabla v_n - V v_n = f, \quad v_n \in D^1_0(G_n).$$

Then

$$-\nabla \cdot a \cdot \nabla(u - v_n) - V(u - v_n) \geqslant 0 \quad \text{in } G_n,$$

with

$$u - v_n \in H^1(G_n), \qquad 0 \leqslant (u - v_n)^- \leqslant v_n^+ \in D^1_0(G_n).$$

Therefore $(u - v_n)^- \in D^1_0(G_n)$. By Lemma 2.2 we conclude that $(u - v_n)^- = 0$, that is $v_n \leqslant u$. Let $\bar{v}_n \in D^1_0(G)$ be defined as $\bar{v}_n = v_n$ on $G_n$, $\bar{v}_n = 0$ on $G \setminus G_n$. To complete the proof of the lemma it suffices to show that $\bar{v}_n \to v$ in $D^1_0(G)$. Indeed,

$$Q(\bar{v}_n) = \int_G \nabla \bar{v}_n \cdot a \cdot \nabla \bar{v}_n - \int_G V |\bar{v}_n|^2 = \langle f, v_n \rangle \leqslant c \|f\|_{D^{-1}(G)} \|\bar{v}_n\|_{D^1_0(G)},$$

where $\langle \cdot, \cdot \rangle$ stands for the duality between $D^1_0(G)$ and $D^{-1}(G)$. Hence the sequence $(\bar{v}_n)$ is bounded in $D^1_0(G)$. Thus we can extract a subsequence, which we still denote by $(\bar{v}_n)$, that converges weakly to $v_* \in D^1_0(G)$. Now let $\varphi \in H^1_c(G)$. Then for all $n \in \mathbb{N}$ large enough, we have that $\mathrm{Supp}(\varphi) \subset G_n$ and

$$\int_G \nabla \bar{v}_n \cdot a \cdot \nabla \varphi - \int_G V \bar{v}_n \varphi = \int_{G_n} \nabla v_n \cdot a \cdot \nabla \varphi - \int_{G_n} V v_n \varphi = \langle f, \varphi \rangle.$$

By the weak continuity we conclude that

$$\int_G \nabla v_* \cdot a \cdot \nabla \varphi - \int_G V v_* \varphi = \langle f, \varphi \rangle.$$

Therefore $v_* \in D^1_0(G)$ satisfies

$$-\nabla \cdot a \cdot \nabla v - V v = f, \quad v \in D^1_0(\mathcal{C}_\Omega).$$

Hence $v_* = v$. Furthermore,

$$Q(\bar{v}_n - v) = \langle f, \bar{v}_n \rangle - 2\langle f, v \rangle + \langle f, v \rangle.$$

Since $\langle f, \bar{v}_n \rangle \to \langle f, v \rangle$ it follows that $\bar{v}_n \to v$ in $D_0^1(G)$. □

*Minimal positive solution in cone like domains.* Let $\Omega \subseteq S^{N-1}$ be a domain. Consider the equation

$$-\nabla \cdot a \cdot \nabla u - V u = 0 \quad \text{in } \mathcal{C}_\Omega, \tag{2.5}$$

where $0 \leqslant V \in L^1_{\text{loc}}(\mathcal{C}_\Omega^\rho)$ is a form-bounded potential. We say that $v > 0$ is a *minimal positive solution* to (2.5) in $\mathcal{C}_\Omega^\rho$ if $v$ is a solution to (2.5) in $\mathcal{C}_\Omega^\rho$ and for any positive supersolution $u > 0$ to (2.5) in $\mathcal{C}_\Omega^r$ with $r \in (0, \rho)$ there exists $c > 0$ such that

$$u \geqslant c v_\psi \quad \text{in } \mathcal{C}_\Omega^\rho.$$

Note, that such definition of a minimal positive solution, adopted for the framework of cone-like domains, is slightly different from the notion of the minimal positive solution at infinity introduced by Agmon [1] (see also [22–24,28,29]).

Below we construct a minimal positive solution to (2.5) in $\mathcal{C}_\Omega^\rho$. Let $0 \leqslant \psi \in C_c^\infty(\Omega)$ and $\theta_\rho \in C^\infty[\rho, +\infty)$ be such that $\theta_\rho(\rho) = 1$, $0 \leqslant \theta_\rho \leqslant 1$ and $\theta_\rho = 0$ for $r \geqslant \rho + \epsilon$ with some $\epsilon > 0$. Thus $f_\psi := \nabla \cdot a \cdot \nabla(\psi \theta_\rho) \in D^{-1}(\mathcal{C}_\Omega^\rho)$. Let $w_\psi$ be the unique solution to the problem

$$-\nabla \cdot a \cdot \nabla w - V w = f_\psi, \quad w \in D_0^1(\mathcal{C}_\Omega^\rho), \tag{2.6}$$

which is given by Lemma 2.1. Set $v_\psi := w_\psi + \psi \theta_\rho$. Then $v_\psi$ is the solution to the problem

$$-\nabla \cdot a \cdot \nabla v - V v = 0, \quad v - \psi \theta_\rho \in D_0^1(\mathcal{C}_\Omega^\rho). \tag{2.7}$$

By the weak Harnack inequality $v_\psi > 0$ in $\mathcal{C}_\Omega^\rho$. Notice that $v_\psi$ actually does not depend on the particular choice of the function $\theta_\rho$ (this easily follows, e.g., from Lemma 2.2).

**Lemma 2.5.** *$v_\psi$ is a minimal positive solution to Eq. (2.5) in $\mathcal{C}_\Omega^\rho$.*

**Proof.** Choose $\Omega' \Subset \Omega$ such that $\text{Supp}(\psi) \Subset \Omega'$. Let $\epsilon > 0$ be such that $\theta_\rho = 0$ for all $r \geqslant \rho + \epsilon$. Let $u > 0$ be a positive supersolution to (2.5) in $\mathcal{C}_\Omega^r$ with $r \in (0, \rho)$. By the weak Harnack inequality there exists $m = m(\Omega', \epsilon) > 0$ such that

$$u > m \quad \text{in } \mathcal{C}_{\Omega'}^{(\rho, \rho+\epsilon)}.$$

Choose $c > 0$ such that $c\psi < m$. Then $u - c\psi\theta_\rho \geqslant 0$ in $\mathcal{C}_\Omega^\rho$, $cw_\psi \in D_0^1(\mathcal{C}_\Omega^\rho)$ and

$$(-\nabla \cdot a \cdot \nabla - V)\big((u - c\psi\theta_\rho) - cw_\psi\big) = (-\nabla \cdot a \cdot \nabla - V)u \geqslant 0 \quad \text{in } \mathcal{C}_\Omega^\rho.$$

By Lemma 2.3 we conclude that $u - c\psi\theta_\rho \geqslant cw_\psi$, that is $u \geqslant cv_\psi$ in $\mathcal{C}_\Omega^\rho$. □

**Remark 2.6.** Let $\Gamma_a(x, y)$ be the positive minimal Green function to the equation $-\nabla \cdot a \cdot \nabla u = 0$ in $\mathbb{R}^N$. Then for any domain $\Omega \subseteq S^{N-1}$ the function $\Gamma_a(x, 0)$ is a positive solution to

$$-\nabla \cdot a \cdot \nabla u = 0 \quad \text{in } \mathcal{C}_\Omega. \tag{2.8}$$

By Lemma 2.3 and the classical estimate [21] we conclude that any minimal positive solution $v_\psi$ to (2.8) in $\mathcal{C}_\Omega^\rho$ obeys the upper bound

$$v_\psi \leqslant c_1 \Gamma_a(x, 0) \leqslant c_2 |x|^{2-N} \quad \text{in } \mathcal{C}_\Omega^\rho. \tag{2.9}$$

*Nonexistence Lemma.*     The next lemma (compare [17], [30, p. 156]) is the key tool in our proofs of nonexistence of positive solutions to nonlinear equation (1.1).

**Lemma 2.7** (Nonexistence Lemma). *Let $0 \leqslant V \in L^1_{\text{loc}}(\mathcal{C}^\rho_\Omega)$ satisfy*

$$|x|^2 V(x) \to \infty \quad \text{as } x \in \mathcal{C}^\rho_{\Omega'} \text{ and } |x| \to \infty \tag{2.10}$$

*for a subdomain $\Omega' \subseteq \Omega$. Then the equation*

$$-\nabla \cdot a \cdot \nabla u - Vu = 0 \quad \text{in } \mathcal{C}^\rho_\Omega \tag{2.11}$$

*has no nontrivial nonnegative supersolutions.*

The proof is based upon the following well-known result (see, e.g., [1, Theorem 3.3]).

**Lemma 2.8.** *Let $G \subset \mathbb{R}^N$ be a bounded domain and $\lambda_1 = \lambda_1(G) > 0$ be the principal Dirichlet eigenvalue of $-\nabla \cdot a \cdot \nabla$ in $G$. If $\mu > \lambda_1$ then the equation*

$$-\nabla \cdot a \cdot \nabla u = \mu u \quad \text{in } G \tag{2.12}$$

*has no positive supersolutions.*

**Proof of Lemma 2.7.** Let $\lambda_1(\mathcal{C}^{(\rho,2\rho)}_\Omega) > 0$ be the principal Dirichlet eigenvalue of $-\nabla \cdot a \cdot \nabla$ on $\mathcal{C}^{(\rho,2\rho)}_\Omega$. Rescaling the equation $-\nabla \cdot a \cdot \nabla v = \lambda v$ from $\mathcal{C}^{(\rho,2\rho)}_\Omega$ to $\mathcal{C}^{(1,2)}_\Omega$ one sees that

$$\frac{c^{-1}}{\rho^2} \lambda_1(\mathcal{C}^{(1,2)}_\Omega) \leqslant \lambda_1(\mathcal{C}^{(\rho,2\rho)}_\Omega) \leqslant \frac{c}{\rho^2} \lambda_1(\mathcal{C}^{(1,2)}_\Omega),$$

where $c = c(a) > 0$ depends on the ellipticity constant of the matrix $a$ and does not depend on $\rho > 0$.

Let $u \geqslant 0$ be a supersolution to (2.11). Then (2.10) implies that for some $R \gg 1$ one can find $\mu > 0$ such that $V(x) \geqslant \mu \geqslant c\lambda_1(\mathcal{C}^{(1,2)}_{\Omega'})R^{-2}$ in $\mathcal{C}^{(R,2R)}_{\Omega'}$. Hence $u$ is a supersolution to

$$-\nabla \cdot a \cdot \nabla u = \mu u \quad \text{in } \mathcal{C}^{(R,2R)}_{\Omega'}$$

with $\mu > \lambda_1(\mathcal{C}^{(R,2R)}_{\Omega'})$. By Lemma 2.8 we conclude that $u = 0$ in $\mathcal{C}^{(R,2R)}_{\Omega'}$. Therefore by the weak Harnack inequality $u = 0$ in $\mathcal{C}^\rho_\Omega$.   □

## 3. Proof of Proposition 1.1

Property (i) is obvious. We need to prove (ii) and (iii).

(ii) Let $p_0 \geqslant p^*(a, \mathcal{C}^\rho_\Omega)$ be such that Eq. (1.1) with exponent $p_0$ has a positive supersolution $u > 0$ in $\mathcal{C}^\rho_\Omega$. Let $p > p_0$ and $\alpha = \frac{p-1}{p_0-1} > 1$. Set $v := u^{1/\alpha}$. By the weak Harnack inequality $u > 0$ in $\mathcal{C}^\rho_\Omega$. Hence $u^{-s} \in L^\infty_{\text{loc}}(\mathcal{C}^\rho_\Omega)$ for any $s > 0$. Therefore $\nabla v = \alpha^{-1} u^{1/\alpha-1} \nabla u \in L^2_{\text{loc}}(\mathcal{C}^\rho_\Omega)$, that is $v \in H^1_{\text{loc}}(\mathcal{C}^\rho_\Omega)$.

Let $0 \leqslant \varphi \in C^\infty_c(\mathcal{C}^\rho_\Omega)$. Then

$$\int_{\mathcal{C}^\rho_\Omega} \nabla v^\alpha \cdot a \cdot \nabla \varphi \, dx = \alpha \int_{\mathcal{C}^\rho_\Omega} v^{\alpha-1} \nabla v \cdot a \cdot \nabla \varphi \, dx$$

$$= \alpha \int_{\mathcal{C}^\rho_\Omega} \nabla v \cdot a \cdot \nabla(v^{\alpha-1}\varphi) \, dx - \alpha(\alpha-1) \int_{\mathcal{C}^\rho_\Omega} \nabla v \cdot a \cdot \nabla v (v^{\alpha-2}\varphi) \, dx$$

$$\leqslant \alpha \int\limits_{\mathcal{C}_\Omega^\rho} \nabla v \cdot a \cdot \nabla(v^{\alpha-1}\varphi)\, dx.$$

Notice, that $v^{\alpha-1} = u^{1-1/\alpha} \in H^1_{\mathrm{loc}}(\mathcal{C}_\Omega^\rho)$ by the same argument as above. Therefore $v^{\alpha-1}\varphi \in H^1_c(\mathcal{C}_\Omega^\rho)$. We shall prove that the set

$$\mathcal{K}_v = \big\{ v^{\alpha-1}\varphi,\ 0 \leqslant \varphi \in H^1_c(\mathcal{C}_\Omega^\rho) \big\}$$

is dense in the cone of nonnegative functions in $H^1_c(\mathcal{C}_\Omega^\rho)$. Indeed, let $0 \leqslant \psi \in H^1_c(\mathcal{C}_\Omega^\rho)$. Let $\psi_n \in C^\infty_0(\mathcal{C}_\Omega^\rho)$ be an approximating sequence such that $\|\nabla(\psi_n - \psi)\|_2 \to 0$. Set $\varphi_n = v^{1-\alpha}\psi_n^+$. It is clear that $0 \leqslant \varphi_n \in \mathcal{K}_v \subset H^1_c(\mathcal{C}_\Omega^\rho)$ and $\|\nabla(v^{\alpha-1}\varphi_n - \psi)\|_2 \to 0$.

Since $v^\alpha = u$ and $u > 0$ is a supersolution of (1.1), we obtain that

$$\alpha \int\limits_{\mathcal{C}_\Omega^\rho} \nabla v \cdot a \cdot \nabla(v^{\alpha-1}\varphi)\, dx \geqslant \int\limits_{\mathcal{C}_\Omega^\rho} v^{\alpha p_0}\varphi\, dx = \int\limits_{\mathcal{C}_\Omega^\rho} v^p (v^{\alpha-1}\varphi)\, dx$$

for any $0 \leqslant \varphi \in H^1_c(\mathcal{C}_\Omega^\rho)$. Thus $\alpha^{1/(1-p)} v$ is a supersolution to Eq. (1.1) in $\mathcal{C}_\Omega^\rho$ with exponent $p > p_0$.

(iii) The existence of a (bounded) positive solution to Eq. (1.1) with $p > \frac{N}{N-2}$ in $B_r^c$ for any $r > 0$ has been proved in [17]. We shall consider the case $p \leqslant \frac{N}{N-2}$.

Let $u > 0$ be a supersolution to (1.1) with exponent $p \leqslant N/(N-2)$ in $\mathcal{C}_\Omega^\rho$. Fix $\psi \in C^\infty_c(\Omega)$ and $r > \rho$. Let $v_\psi > 0$ be a minimal positive solution in to $-\nabla \cdot a \cdot \nabla v = 0$ in $\mathcal{C}_\Omega^r$, as constructed in (2.6), (2.7). Then $u \geqslant c v_\psi$ in $\mathcal{C}_\Omega^r$ by Lemma 2.3. Without loss of generality we assume that $c = 1$. Thus $v_\psi > 0$ is a subsolution to (1.1) in $\mathcal{C}_\Omega^r$ and $v_\psi \leqslant u$ in $\mathcal{C}_\Omega^r$. We are going to show that (1.1) has a positive solution $w$ in $\mathcal{C}_\Omega^r$ such that $v_\psi \leqslant w \leqslant u$ in $\mathcal{C}_\Omega^r$.

Let $(G_n)_{n \in \mathbb{N}}$ be an exhaustion of $\mathcal{C}_\Omega^r$. Consider the boundary value problem

$$\begin{cases} -\nabla \cdot a \cdot \nabla w = w^p & \text{in } G_n, \\ w = v_\psi & \text{on } \partial G_n. \end{cases} \tag{3.1}$$

Since $G_n \Subset \mathcal{C}_\Omega^r$ is a smooth bounded domain and $v_\psi \in C^{0,\gamma}_{\mathrm{loc}}(\mathcal{C}_\Omega^r)$, the problem (3.1) is well-posed. Clearly, $v_\psi \leqslant u$ is still a pair of sub and supersolutions for (3.1). Notice that we do not assume that $u \in H^1(G_n)$ is bounded. However, since $p \leqslant \frac{N}{N-2} < \frac{N+2}{N-2}$, one can use an $H^1$-version of sub and supersolution method, see e.g. [9, Theorem 2.2]. Thus there exists a weak solution $w_n \in H^1(G_n)$ of (3.1) such that $v_\psi \leqslant w_n \leqslant u$ in $G_n$.

Consider a sequence $(w_n)_{n>1}$ in $G_1$. Choose a function $\theta \in C^\infty_c(G_2)$ such that $0 \leqslant \theta \leqslant 1$ and $\theta = 1$ on $G_1$. Using $\theta^2 w_n \in H^1_c(G_2)$ as a test function we obtain

$$\int\limits_{G_2} w_n^{p+1}\theta^2 = \int\limits_{G_2} \theta^2 \nabla w_n \cdot a \cdot \nabla w_n\, dy + 2\int\limits_{G_2} \theta w_n \nabla w_n \cdot a \cdot \nabla\theta\, dy.$$

Thus, by standard computations

$$\frac{1}{2}\int\limits_{G_2} \theta^2 \nabla w_n \cdot a \cdot \nabla w_n\, dx \leqslant 2\int\limits_{G_2} w_n^2 \nabla\theta \cdot a \cdot \nabla\theta\, dx + \int\limits_{G_2} \theta^2 w_n^{p+1}\, dx$$

$$\leqslant 2c_1 \big\| (\nabla\theta)^2 \big\|_\infty \int\limits_{G_2} u^2\, dx + \int\limits_{G_2} u^{p+1}\, dx.$$

We conclude that $(w_n)$ is bounded in $H^1(G_1)$. By the construction $v_\psi \leqslant w_n \leqslant u \in H^1(G_1)$ for all $n \in \mathbb{N}$. Therefore $(w_n)$ has a subsequence, denoted by $(w_{n_1(k)})_{k \in \mathbb{N}}$, which converges to a function $w^{(1)} \in H^1(G_1)$ weakly in $H^1(G_1)$, strongly in $L^2(G_1)$ and almost everywhere in $G_1$. Hence it is clear that $w^{(1)}$ is a solution to (1.1) in $G_1$ and $v_\psi \leqslant w^{(1)} \leqslant u$.

Now we proceed by the standard diagonal argument (see, e.g., [26, Theorem 2.10]). At the second step, consider a sequence $(w_{n_1(k)})_{k\in\mathbb{N}}$ in $G_2$ (assuming that $n_1(1) > 2$). In the same way as above we obtain a subsequence $(w_{n_2(k)})_{k\in\mathbb{N}}$ that converges to a function $w^{(2)} \in H^1(G_2)$, which is a solution to (1.1) in $G_2$. Moreover, $v_\psi \leqslant w^{(2)} \leqslant u$ in $G_2$ and $w^{(2)} = w^{(1)}$ in $G_1$. Continuing this process, for each fixed $m > 2$ we construct a subsequence $(w_{n_m(k)})_{k\in\mathbb{N}}$ (with $n_m(1) > m$) that converges weakly to $w^{(m)} \in H^1(G_m)$ which is a solution to (1.1) in $G_m$ and such that $v_\psi \leqslant w^{(m)} \leqslant u$ in $G_m$, $w^{(m)} = w^{(m-1)}$ in $G_{m-1}$.

By the diagonal process $(w_{n_m(m)})_{m\in\mathbb{N}}$ is a subsequence of $(w_{n_m(k)})_{k\in\mathbb{N}}$ for every $m \in \mathbb{N}$. Thus for each fixed $k \in \mathbb{N}$ the sequence $(w_{n_m(m)})$ converges weakly to $w^{(k)}$ in $H^1(G_k)$. Let $w_*$ be the weak limit of $(w_{n_m(m)})$ in $H^1_{\text{loc}}(\mathcal{C}^r_\Omega)$. Then $w_*$ is a solution of (1.1) in $\mathcal{C}^r_\Omega$ such that $v_\psi \leqslant w_* \leqslant u$ in $\mathcal{C}^r_\Omega$. $\quad\square$

**Remark 3.1.** The constructed solution $w_*$ is actually locally Hölder continuous. Indeed, since $p \leqslant \frac{N}{N-2} < \frac{N+2}{N-2}$ we conclude by the Brezis–Kato estimate (see, e.g. [33, Lemma B.3]) that $w_* \in L^s_{\text{loc}}(\mathcal{C}^r_\Omega)$ for any $s < \infty$. Then $-\nabla \cdot a \cdot \nabla w_* = w_*^p \in L^s_{\text{loc}}(\mathcal{C}^r_\Omega)$ for any $s < \infty$. Hence the standard elliptic estimates imply that $w_* \in C^{0,\gamma}_{\text{loc}}(\mathcal{C}^r_\Omega)$.

## 4. Proof of Theorems 1.3 and 1.4

In this section we study positive supersolutions at infinity to the model equation

$$-\Delta u = u^p \quad \text{in } \mathcal{C}_\Omega, \tag{4.1}$$

where $p > 1$ and $\Omega$ is a subdomain of $S^{N-1}$. Recall, that $\lambda_1$ denotes the principal eigenvalue of the Dirichlet Laplace–Beltrami operator $-\Delta_\omega$ in $\Omega$ and $\alpha_-$ stands for the negative root of the equation $\alpha(\alpha + N - 2) = \lambda_1$.

Existence of positive supersolutions to (4.1) with $p > p^*(id, \mathcal{C}_\Omega) = 1 - 2/\alpha_-$ can be easily verified. Namely, by direct computations one can find supersolutions of the form $u = cr^{2/(1-p)}\phi$, where $\phi > 0$ is the principal eigenfunction of $-\Delta_\omega$ on $\Omega$ (see also [4,3] for a direct proof of the existence of positive solutions). We are going to prove absence of positive supersolutions to (4.1) in $\mathcal{C}^\rho_\Omega$ for $p \in (1, 1 - 2/\alpha_-)$. Notice that if $u > 0$ is a solution to (4.1) in $\mathcal{C}^\rho_\Omega$ then, by the scaling properties of the Laplacian, $\rho^{\frac{2}{p-1}}u(\rho x)$ is a solution to (4.1) in $\mathcal{C}^1_\Omega$. So in what follows we fix $\rho = 1$.

*Minimal solution estimate.*   Here we derive the sharp asymptotic at infinity of the minimal solutions to the equation

$$-\Delta u - \frac{V(\omega)}{|x|^2}u = 0 \quad \text{in } \mathcal{C}_\Omega \tag{4.2}$$

with $V \in L^\infty(\Omega)$. Let $-\Delta_\omega$ be the Dirichlet Laplace–Beltrami operator in $L^2(\Omega)$ and $0 \leqslant V \in L^\infty(\Omega)$. Let $(\tilde{\lambda}_k)_{k\in\mathbb{N}}$ be the sequence of Dirichlet eigenvalues of $-\Delta_\omega - V$, such that $\tilde{\lambda}_1 < \tilde{\lambda}_2 \leqslant \tilde{\lambda}_3 \leqslant \cdots$. By $(\tilde{\phi}_k)_{k\in\mathbb{N}}$ we denote the corresponding orthonormal basis of eigenfunctions in $L^2(\Omega)$, with $\tilde{\phi}_1 > 0$.

From now on we assume that $\tilde{\lambda}_1 > -(N-2)^2/4$. Then the roots of the quadratic equation $\alpha(\alpha + N - 2) = \tilde{\lambda}_k$ are real for each $k \in \mathbb{N}$. By $\tilde{\alpha}_k$ we denote the smallest root of the equation, i.e.,

$$\tilde{\alpha}_k := -\frac{N-2}{2} - \sqrt{\frac{(N-2)^2}{4} + \tilde{\lambda}_k}.$$

Notice that since $\tilde{\lambda}_1 > -(N-2)^2/4$ it follows from the Hardy inequality (2.4) that the potential $V(\omega)|x|^{-2}$ is form bounded.

**Lemma 4.1.** *Let* $\psi \in C_c^\infty(\Omega)$. *Then*

$$v_\psi(x) = \sum_{k=1}^\infty \psi_k r^{\tilde{\alpha}_k} \tilde{\phi}_k(\omega), \quad \text{where } \psi_k = \int_\Omega \psi(\omega) \tilde{\phi}_k(\omega) \, d\omega, \tag{4.3}$$

*is a minimal positive solution to Eq. (4.2) in* $\mathcal{C}_\Omega^1$.

**Proof.** Set $v_k(x) := r^{\tilde{\alpha}_k} \tilde{\phi}_k(\omega)$. Then a direct computation gives that

$$-\Delta v_k - \frac{V(\omega)}{|x|^2} v_k = 0 \quad \text{in } \mathcal{C}_\Omega^1.$$

Recall that $\nabla = \nu \frac{\partial}{\partial r} + \frac{1}{r} \nabla_\omega$, where $\nu = \frac{x}{|x|} \in \mathbb{R}^N$. Since

$$\int_\Omega |\nabla_\omega \tilde{\phi}_k|^2 \, d\omega - \int_\Omega V(\omega) |\tilde{\phi}_k|^2 \, d\omega = \tilde{\lambda}_k,$$

we obtain

$$\epsilon \|\nabla v_k\|_{L^2}^2 \leqslant \int_{\mathcal{C}_\Omega^\rho} \left( |\nabla v_k|^2 - \frac{V(\omega)}{|x|^2} |v_k|^2 \right) dx$$

$$= \int_1^\infty \int_\Omega \left( \left| \frac{\partial}{\partial r} r^{\tilde{\alpha}_k} \tilde{\phi}_k(\omega) \right|^2 + \frac{|r^{\tilde{\alpha}_k} \nabla_\omega \tilde{\phi}_k(\omega)|^2}{r^2} - \frac{V(\omega) |r^{\tilde{\alpha}_k} \tilde{\phi}_k|^2}{r^2} \right) r^{N-1} \, d\omega \, dr$$

$$= \int_1^\infty r^{2\tilde{\alpha}_k + N - 3} (\tilde{\alpha}_k^2 + \tilde{\lambda}_k) \, dr = \frac{\tilde{\alpha}_k^2 + \tilde{\lambda}_k}{2 - N - 2\tilde{\alpha}_k} = -\tilde{\alpha}_k,$$

where $\epsilon > 0$ is the constant in (2.2). Now it is straightforward that $v_k - \tilde{\phi}_k \theta_1 \in D_0^1(\mathcal{C}_\Omega^1)$, so $v_k$ solves the problem

$$-\Delta v - \frac{V(\omega)}{|x|^2} v = 0, \quad v - \tilde{\phi}_k \theta_1 \in D_0^1(\mathcal{C}_\Omega^1).$$

Hence we have

$$\epsilon \|\nabla v_\psi\|_2^2 \leqslant \int_{\mathcal{C}_\Omega^1} \left( |\nabla v_\psi|^2 - \frac{V(\omega)}{|x|^2} |v_\psi|^2 \right) dx = \sum_{k=1}^\infty \psi_k^2 (-\tilde{\alpha}_k)$$

$$\leqslant \frac{N-2}{2} \|\psi\|_2^2 + \|\psi\|_2 \left( \int_\Omega \left( |\nabla_\omega \psi|^2 - V(\omega)\psi^2 + \left( \frac{N-2}{2} \right)^2 \psi^2 \right) d\omega \right)^{\frac{1}{2}}.$$

Hence $v_\psi - \psi \theta_1 \in D_0^1(\mathcal{C}_\Omega^1)$, so $v_\psi$ solves the problem

$$-\Delta v - \frac{V(\omega)}{|x|^2} v = 0, \quad v - \psi \theta_1 \in D_0^1(\mathcal{C}_\Omega^1).$$

By the uniqueness we conclude that $v_\psi$ defined by (4.3) coincides with the minimal solution $v_\psi$ as constructed in (2.6), (2.7). $\quad\square$

**Lemma 4.2.** *Let $v_\psi > 0$ be a minimal solution (4.3) to Eq. (4.2) in $\mathcal{C}_\Omega^1$. Then for any $\Omega' \Subset \Omega$ and $\rho > 1$ there exists $c = c(\Omega', \rho) > 0$ such that*

$$v_\psi(x) \geqslant cr^{\tilde{\alpha}_1} \quad in \ \mathcal{C}_{\Omega'}^\rho. \tag{4.4}$$

**Proof.** By (4.3) one can represent $v_\psi$ as $v_\psi(x) = \psi_1 r^{\tilde{\alpha}_1} \tilde{\phi}_1(\omega) + w(x)$, where

$$w(x) = \sum_{k=2}^\infty \tilde{\psi}_k r^{\tilde{\alpha}_k} \tilde{\phi}_k(\omega).$$

Notice that $w(x)$ satisfies

$$-\Delta w - \frac{V(\omega)}{|x|^2} w = 0 \quad in \ \mathcal{C}_\Omega^1.$$

Thus by the standard elliptic estimate (see, e.g. [15, Theorem 8.17]) for any $\Omega' \Subset \Omega$ and $\rho > \frac{4}{3}$ one has

$$\sup_{\mathcal{C}_{\Omega'}^{(\rho, 2\rho)}} |w|^2 \leqslant c\rho^{-N} \int_{\mathcal{C}_\Omega^{(\frac{3\rho}{4}, \frac{9\rho}{8})}} |w|^2 \, dx,$$

where the constant $c > 0$ does not depend on $\rho$. Therefore

$$\sup_{\mathcal{C}_{\Omega'}^{(\frac{3\rho}{4}, \frac{9\rho}{8})}} |w|^2 \leqslant c\rho^{-N} \int_{\frac{3\rho}{4}}^{\frac{9\rho}{8}} r^{N-1} \int_\Omega |w|^2 \, d\omega \, dr = c\rho^{-N} \int_{\frac{3\rho}{4}}^{\frac{9\rho}{8}} r^{N-1} \sum_{k=2}^\infty \psi_k^2 r^{2\tilde{\alpha}_k} \, dr$$

$$\leqslant c \int_{\frac{3\rho}{4}}^{\frac{9\rho}{8}} r^{2\tilde{\alpha}_2 - 1} \, dr \|\psi - \psi_1 \tilde{\phi}_1\|_2^2 = c_1 \rho^{2\tilde{\alpha}_2}.$$

So we conclude that

$$v_\psi(x) \geqslant \psi_1 r^{\tilde{\alpha}_1} \phi_1(\omega) - cr^{\tilde{\alpha}_2} \quad in \ \mathcal{C}_{\Omega'}^\rho.$$

Since $\tilde{\alpha}_2 < \tilde{\alpha}_1 < 0$ this implies (4.4).   $\square$

**Remark 4.3.** Related estimates were obtained by Murata [22, pp. 608–612] for the cone with a Lipschitz cross-section $\Omega \subset S^{N-1}$. Notice that if $\Omega$ is a Lipschitz domain (or, more generally, a domain which satisfies the boundary Harnack principle), then the boundary Harnack principle allows one to prove that the function $v_1 = r^{\tilde{\alpha}_1} \tilde{\phi}_1$ is a minimal positive solution to (4.2) in $\mathcal{C}_\Omega^1$. The use of compactly supported function $\psi$ (and hence, of full series expansion in (4.3)) in the construction of the minimal positive solution $v_\psi$ is required for comparison on cones with general nonsmooth cross-sections $\Omega$.

**Proof of Theorem 1.3.** We distinguish the subcritical and critical cases.

*Subcritical case $1 < p < 1 - 2/\alpha_-$.* Assume that $u > 0$ is a supersolution to (4.1) in $\mathcal{C}_\Omega^r$ for some $r \in (0, 1)$. Then $u > 0$ is a supersolution to

$$-\Delta u = 0 \quad in \ \mathcal{C}_\Omega^r. \tag{4.5}$$

By Lemma 2.5 we conclude that $u > cv_\psi$ in $\mathcal{C}_\Omega^1$, where $v_\psi > 0$ is a minimal positive solution (4.3) to Eq. (4.5) in $\mathcal{C}_\Omega^1$. Then by Lemma 4.2 for a subdomain $\Omega' \Subset \Omega$ one has

$$v_\psi \geqslant c|x|^{\alpha_-} \quad in \ \mathcal{C}_{\Omega'}^1. \tag{4.6}$$

So $u > 0$ is a supersolution to

$$-\Delta u - Wu = 0 \quad \text{in } \mathcal{C}^1_\Omega, \tag{4.7}$$

where $W(x) := u^{p-1}(x)$ satisfies

$$W(x) \geqslant c^{p-1}|x|^{\alpha_-(p-1)} \quad \text{in } \mathcal{C}^1_{\Omega'},$$

with $\alpha_-(p-1) > -2$. Now Lemma 2.7 leads to a contradiction.

*Critical case* $p = 1 - 2/\alpha_-$. Let $u > 0$ be a supersolution to (4.1) in $\mathcal{C}^r_\Omega$ with the critical exponent $p^* = 1 - 2/\alpha_-$. Then arguing as in the previous case we conclude that $u$ is a supersolution to (4.7) with $W(x) := u^{p^*-1}(x)$ satisfying

$$W(x) \geqslant \frac{c^{p^*-1}}{|x|^2} \quad \text{in } \mathcal{C}^1_{\Omega'}$$

on a subdomain $\Omega' \Subset \Omega$. Let $\chi_{\Omega'}(\omega)$ be the characteristic function of $\Omega'$. Then $u$ is a supersolution in $\mathcal{C}^1_\Omega$ to the equation

$$-\Delta v - \frac{\epsilon \chi_{\Omega'}(\omega)}{|x|^2} v = 0 \quad \text{in } \mathcal{C}_\Omega, \tag{4.8}$$

for any $\epsilon \in [0, c^{p^*-1}]$. By the variational characterization of the principal Dirichlet eigenvalue one can fix $\epsilon > 0$ small enough in such a way that $\tilde{\lambda}_1 = \tilde{\lambda}_1(-\Delta_\omega - \epsilon \chi_{\Omega'}, \Omega) > -(N-2)^2/4$. Let $w_\psi$ be a minimal positive solution (4.3) to Eq. (4.8) in $\mathcal{C}^1_\Omega$ with such fixed $\epsilon$. Applying Lemma 4.2 to (4.8) we conclude that for a subdomain $\Omega'' \Subset \Omega$ one has

$$u \geqslant c_1 w_\psi \geqslant c_2 |x|^{\tilde{\alpha}_1} \quad \text{in } \mathcal{C}^1_{\Omega''},$$

where $\tilde{\alpha}_1 > \alpha_-$. So $u > 0$ is a supersolution to

$$-\Delta u - Wu = 0 \quad \text{in } \mathcal{C}^1_\Omega, \tag{4.9}$$

where $W(x) := u^{p^*-1}(x)$ satisfies

$$W(x) \geqslant c^{p^*-1}|x|^{\tilde{\alpha}_1(p^*-1)} \quad \text{in } \mathcal{C}^1_{\Omega''}$$

with $\tilde{\alpha}_1(p^*-1) > -2$. This contradicts to Lemma 2.7. □

**Remark 4.4.** Strictly speaking, in the above proof the subcritical case $1 < p < 1 - 2/\alpha_-$ is redundant, due to Proposition 1.1(ii).

Let $\Omega \subset S^{N-1}$ be a domain such that $\lambda_1 = \lambda_1(\Omega) > 0$. Define the operator $L_d$ by

$$L_d = -\frac{\partial^2}{\partial r^2} - \frac{N-1}{r}\frac{\partial}{\partial r} - \frac{d(r)}{\lambda_1}\frac{1}{r^2}\Delta_\omega, \tag{4.10}$$

where $d(r)$ is measurable and squeezed between two positive constants. Then $L_d$ is a divergence type uniformly elliptic operator $-\nabla \cdot a_d \cdot \nabla$ (see, e.g., [36]).

**Proof of Theorem 1.4.** Consider the operator $L_d$ where $d(r) \equiv \alpha(\alpha + N - 2)$ with $\alpha < 2 - N$. Following the lines of the proof of Theorem 1.3 we conclude that $p^*(a_d, \mathcal{C}_\Omega) = 1 - 2/\alpha$. Clearly for any given $p \in (1, \frac{N}{N-2})$, one can choose $\alpha$ such that $p^*(a_d, \mathcal{C}_\Omega) = p$. □

**Remark 4.5.** In the above proof Eq. (1.1) has no positive supersolutions at infinity in $\mathcal{C}_\Omega$ in the critical case $p = p^*(a_d, \mathcal{C}_\Omega)$. Next we give an example of Eq. (1.1) with a positive supersolution at infinity in the critical case. Let $\Omega \subset S^{N-1}$ be smooth and $L_{\tilde{d}}$ be as in (4.10) with

$$\tilde{d}(r) = \alpha(\alpha + N - 2) + \frac{2 - N - 2\alpha}{\log(r)} + \frac{2}{\log^2(r)},$$

where $\alpha < 2 - N$. For large enough $R \gg 1$ the operator $L_{\tilde{d}} = -\nabla \cdot a_{\tilde{d}} \cdot \nabla$ is uniformly elliptic on $\mathcal{C}_\Omega^R$. Let $\phi_1 > 0$ be the principal Dirichlet eigenfunction of $-\Delta_\omega$, corresponding to $\lambda_1$. Direct computation shows that the function

$$v_{\phi_1} := \frac{r^\alpha}{\log(r)} \phi_1$$

is a solution to the equation

$$L_{\tilde{d}} v = 0 \quad \text{in } \mathcal{C}_\Omega^R. \tag{4.11}$$

Since $\Omega$ is smooth, the Hopf lemma implies that $v_{\phi_1}$ is a minimal positive solution to (4.11) in $\mathcal{C}_\Omega^R$. Following the lines of the proof of Theorem 1.3, subcritical case, we conclude that $p^*(a_{\tilde{d}}, \mathcal{C}_\Omega) = 1 - 2/\alpha$. On the other hand, one can readily verify that $u = r^\alpha \phi_1$ is a positive supersolution to (1.1) in the critical case $p = 1 - 2/\alpha$.

Note that the value of the critical exponent for $L_{\tilde{d}}$ is the same as $L_d$ due to the fact that $\lim_{r \to \infty}(\tilde{d}(r) - d(r)) = 0$. However the rate of convergence is not sufficient to guarantee the equivalence of the corresponding minimal positive solutions (see, e.g. [2,27] for the related estimates of Green's functions). This explains the nature of the different behavior of the nonlinear equations (1.1) at the critical value of $p$.

## 5. Proof of Theorem 1.6

First we show that for any domain $\Omega \subseteq S^{N-1}$ one has $p^*(a, \mathcal{C}_\Omega) > 1$. Then we prove the second part of Theorem 1.6, saying that if the complement of $\Omega$ has nonempty interior then $p^*(a, \mathcal{C}_\Omega) < \frac{N}{N-2}$. We start with establishing a lower bound on positive solutions of the equation

$$-\nabla \cdot a \cdot \nabla v = 0 \quad \text{in } \mathcal{C}_\Omega. \tag{5.1}$$

**Lemma 5.1.** *Let $\Omega \subseteq S^{N-1}$ be a domain and $\Omega' \Subset \Omega$. Then there exists $\alpha = \alpha(\Omega') \leqslant 2 - N$ such that for any $\rho > 0$ any positive solution $v$ to Eq. (5.1) in $\mathcal{C}_\Omega^\rho$ has a polynomial lower bound*

$$v \geqslant c|x|^\alpha \quad \text{in } \mathcal{C}_{\Omega'}^{2\rho}. \tag{5.2}$$

**Proof.** Set $a = 3/4$, $b = 7/4$. Let $r \geqslant 2\rho$ and $m_r = \inf_{\mathcal{C}_{\Omega'}^{(ra,rb)}} v$. By the strong Harnack inequality $v$ satisfies

$$\inf_{\mathcal{C}_{\Omega'}^{(ra,rb)}} v \geqslant C_S \sup_{\mathcal{C}_{\Omega'}^{(ra,rb)}} v,$$

with the constant $C_S \in (0, 1)$ dependent on $\Omega'$ and not on $r$, as a simple scaling argument shows. Then

$$m_r \leqslant \sup_{\mathcal{C}_{\Omega'}^{(ra,rb)}} v \leqslant \sup_{\mathcal{C}_{\Omega'}^{(ra,2rb)}} v \leqslant C_S^{-1} \inf_{\mathcal{C}_{\Omega'}^{(ra,2rb)}} v \leqslant C_S^{-1} \inf_{\mathcal{C}_{\Omega'}^{(2ra,2rb)}} v = C_S^{-1} m_{2r}. \tag{5.3}$$

Let $r_n = 2^n \rho$ and $n \in \mathbb{N}$. Iterating (5.3) we obtain $m_{r_n} \geqslant C_S^{n-1} m_{2\rho}$. Choosing $n$ such that $ar_n \leqslant |x| < 2ar_n$ one can see that

$$v \geqslant c|x|^\alpha \quad \text{in } \mathcal{C}_{\Omega'}^{2a\rho},$$

where $\alpha = \log_2 C_S$ and $c = c(\rho) = (a\rho)^{-\alpha} C_S^{-1} m_{2\rho}$. Taking into account (2.9) we conclude that $\alpha \leqslant 2 - N$.   $\square$

**Remark 5.2.** A similar argument was used before by Pinchover [28, Lemma 6.5]. Observe that in the same way one can get a rough polynomial upper bound on positive solutions of (5.1).

The lower bound (5.2) allows us to prove nonexistence of positive solutions to (1.1) exactly by the same argument as was used in the proof of Theorem 1.3 in the subcritical case.

**Proposition 5.3.** *Let* $\Omega \subseteq S^{N-1}$ *be a domain. Then* $p^*(a, \mathcal{C}_\Omega) \geqslant 1 - 2/\alpha$ *where* $\alpha \leqslant 2 - N$ *is the exponent in the lower bound* (5.2).

**Proof.** Assume that $u \geqslant 0$ is a supersolution to (1.1) in $\mathcal{C}_\Omega^\rho$ with exponent $p < 1 - 1/\alpha$. By Lemma 2.3 and (5.2) we conclude that for any subdomain $\Omega' \subset \Omega$ there exists $c = c(\Omega) > 0$ such that

$$u \geqslant c|x|^\alpha \quad \text{in } \mathcal{C}_{\Omega'}^{2\rho+2}.$$

Therefore $u$ is a supersolution to

$$-\nabla \cdot a \cdot \nabla u = Vu \quad \text{in } \mathcal{C}_{\Omega'}^{2\rho+2},$$

where $V(x) := u^{p-1}(x)$ satisfies the inequality

$$V(x) \geqslant c'|x|^{\alpha(p-1)} \quad \text{in } \mathcal{C}_{\Omega'}^{2\rho+2}$$

with $\alpha(p-1) > -2$. Then Lemma 2.7 implies that $u \equiv 0$ in $\mathcal{C}_\Omega^\rho$. Since $\alpha > 0$ does not depend on $\rho$, we conclude that $p^*(a, \mathcal{C}_\Omega) \geqslant 1 - 1/\alpha$. $\quad\square$

Our next step is to obtain a polynomial upper bound on the minimal positive solutions to the equation

$$-\nabla \cdot a \cdot \nabla v - Vv = 0 \quad \text{in } \mathcal{C}_\Omega,$$

with a special potential $V$ which will be specified later. In order to do this we need the notion of a Green bounded potential. Let $\Gamma_a(x, y)$ be the positive minimal Green function to

$$-\nabla \cdot a \cdot \nabla v = 0 \quad \text{in } \mathbb{R}^N.$$

We say that a potential $0 \leqslant V \in L^1_{\text{loc}}(\mathbb{R}^N)$ is Green bounded and write $V \in GB$ if

$$\|V\|_{GB,a} := \sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} \Gamma_a(x, y) V(y) \, dy < \infty,$$

which is equivalent up to a constant factor to the condition $\sup_{x \in \mathbb{R}^N} \int_{\mathbb{R}^N} |x - y|^{2-N} |V(y)| \, dy < \infty$, but we will use below the numerical value of $\|V\|_{GB,a}$. One can see, e.g. by the Stein interpolation theorem, that if $V \in GB$ then $V$ is form bounded in the sense of (2.2). We will use the following important property of Green bounded potentials, which was proved in [16], see also [23,24,27,29].

**Lemma 5.4.** *Let* $V \in GB$ *and* $\|V\|_{GB,a} < 1$. *Then there exists a solution* $w > 0$ *to the equation*

$$-\nabla \cdot a \cdot \nabla w - Vw = 0 \quad \text{in } \mathbb{R}^N, \tag{5.4}$$

*such that* $0 < c < w < c^{-1}$ *in* $\mathbb{R}^N$.

Using this result we first prove the required upper bound in the case of the "half–space" cone $\mathcal{C}_+ = \{x_N > 0\}$ with the cross-section $S_+ = \{|x| = 1, x_N > 0\}$. For a given uniformly elliptic matrix $a$ and a potential $V$ defined on $\mathcal{C}_+$ we denote by $\bar{a}$ and $\bar{V}$ the extensions of $a$ and $V$ from $\mathcal{C}_+$ to $\mathbb{R}^N$ by reflection, so that $\bar{a}(\cdot, -x_N) = \bar{a}(\cdot, x_N)$ and $\bar{V}(\cdot, -x_N) = \bar{V}(\cdot, x_N)$. Thus the matrix $\bar{a}$ is uniformly elliptic on $\mathbb{R}^N$ with the same ellipticity constant as $a$.

**Lemma 5.5.** *Let $0 \leqslant V \in L^1_{\text{loc}}(\mathcal{C}_+)$ be a potential such that $\|\bar{V}\|_{GB,\bar{a}} < 1$. Let $v_\psi > 0$ be a minimal positive solution in $\mathcal{C}^1_+$ to the equation*

$$-\nabla \cdot a \cdot \nabla v - V v = 0 \quad in \ \mathcal{C}_+,$$

*as constructed in (2.6), (2.7). Then there exists $\gamma \in (0, 1)$ such that*

$$0 < v_\psi \leqslant c|x|^{2-N-\gamma} \quad in \ \mathcal{C}^1_+. \tag{5.5}$$

**Proof.** Let $\bar{v}$ denote the extension of $v_\psi$ from $\mathcal{C}^1_+$ to $\bar{B}^c_1$ by reflection, that is $\bar{v}(\cdot, x_N) = -v_\psi(\cdot, -x_N)$. Thus $\bar{v}(x)$ is a solution to the equation

$$-\nabla \cdot \bar{a} \cdot \nabla \bar{v} - \bar{V} \bar{v} = 0 \quad in \ \bar{B}^c_1.$$

Let $w$ be a solution to (5.4) given by Lemma 5.4. One can check by direct computation (see [17, Lemma 3.4]), that $v_1 := \bar{v}/w$ is a solution to the equation

$$-\nabla \cdot (w^2 \bar{a}) \cdot \nabla v_1 = 0 \quad in \ \bar{B}^c_1, \tag{5.6}$$

where the matrix $w^2 \bar{a}$ is clearly uniformly elliptic. Let $\Gamma(x) := \Gamma_{w^2\bar{a}}(x, 0)$ be the positive minimal Green function to the equation $-\nabla \cdot (w^2\bar{a}) \cdot \nabla u = 0$ in $\mathbb{R}^N$. By the classical estimate [21] one has

$$c_1|x|^{2-N} \leqslant \Gamma(x) \leqslant c_2|x|^{2-N} \quad in \ \bar{B}^c_1. \tag{5.7}$$

Applying Lemma 2.3 to $v_1$ and $\Gamma$ on $\mathcal{C}^1_+$ and by the construction of $v_1$ we conclude that

$$|v_1(x)| \leqslant c_3 \Gamma(x) \quad on \ \bar{B}^c_1. \tag{5.8}$$

Applying the Kelvin transformation $y = y(x) = x/|x|^2$ and $x = x(y) = y/|y|^2$ to (5.6) we see that the function $\tilde{v}_1(y) = v_1(x(y))/\Gamma(x(y))$, $\tilde{v} \in L^\infty(B_1)$, solves the equation

$$-\nabla \cdot \tilde{a} \cdot \nabla \tilde{v}_1 = 0 \quad in \ B_1,$$

where the matrix $\tilde{a}(y)$ is uniformly elliptic on $B_1$. It follows that $\tilde{v}_1 \in H^1_{\text{loc}}(B_1)$ (see, e.g., [31]). Then, by the De Giorgi–Nash regularity result [15], $\tilde{v}_1 \in C^{0,\gamma}(B_1)$ for some $\gamma \in (0, 1)$. Notice that

$$\tilde{v}_1(y) = 0 \quad in \ \{y \in B_1, \ y_N = 0\}$$

by the construction. Therefore $\tilde{v}_1(0) = 0$, hence

$$\left|\tilde{v}_1(y)\right| \leqslant c|y|^\gamma \quad in \ B_1.$$

We conclude that

$$|\bar{v}| \leqslant c_3 \left|\bar{v}_1(x)\right| \leqslant c_4|x|^{2-N-\gamma} \quad in \ \bar{B}^c_1,$$

as required. $\quad\square$

**Lemma 5.6.** *Let $\Omega \subset S^{N-1}$ be a domain such that $S^{N-1} \setminus \Omega$ has nonempty interior. Let*

$$W_\epsilon(x) := \frac{\epsilon}{(|x|^2 \log^2 |x|) \vee 1}.$$

*Then there exists $\epsilon > 0$ and $\beta = \beta(\epsilon) < 2 - N$ such that any minimal positive solution $v_\psi$ in $\mathcal{C}^1_\Omega$ to the equation*

$$-\nabla \cdot a \cdot \nabla v - W_\epsilon v = 0 \quad in \ \mathcal{C}_\Omega$$

*has the polynomial upper bound*

$$v_\psi \leqslant c|x|^\beta \quad in \ \mathcal{C}^1_\Omega. \tag{5.9}$$

**Proof.** If $\mathcal{C}_\Omega \subseteq \mathcal{C}_+$ then (5.9) follows from (5.5) by Lemma 2.3. We shall consider the case $\mathcal{C}_\Omega \nsubseteq \mathcal{C}_+$.

Without loss of generality we can assume that $(0, \ldots, 0, -1) \notin \Omega$. Set $\hat{x} = (x_1, \ldots, x_{N-1})$ and $\sigma = \inf\{|\hat{x}|: x \in \Omega, x_N < 0\}$. Let $D_\sigma = \{x \in S^{N-1}: |\hat{x}| \leqslant \sigma, x_N < 0\}$ and $\hat{D}_\sigma := S^{N-1} \setminus D_\sigma$. Then $\mathcal{C}_\Omega \subseteq \mathcal{C}_{\hat{D}_\sigma}$. Extend the matrix $a$ by *id* from $\mathcal{C}_\Omega$ to $\mathcal{C}_{\hat{D}_\sigma}$. Let $w_\psi$ be a minimal positive solution in $\mathcal{C}^1_{\hat{D}_\sigma}$ to the equation

$$-\nabla \cdot a \cdot \nabla w - W_\epsilon w = 0 \quad \text{in } \mathcal{C}_{\hat{D}_\sigma}.$$

To complete the proof we need only to show that $w_\psi$ satisfies (5.9) in $\mathcal{C}^1_{\hat{D}_\sigma}$. Then the same bound on minimal positive solutions in $\mathcal{C}^1_\Omega$ follows from Lemma 2.3.

Consider the transformation

$$y = y(x) = \left(x_1, \ldots, x_{N-1}, x_N + k|\hat{x}|\right),$$

where $k = \sqrt{\sigma^{-2} - 1}$. Then $y : \mathcal{C}_{\hat{D}_\sigma} \to \mathcal{C}_+$ is a bijection, the Jacobian of $y(x)$ is nondegenerate and has the determinant equal to 1 everywhere. Moreover, $|x| \leqslant |y(x)| \leqslant \kappa |x|$ for all $x \in \mathcal{C}_{\hat{D}_\sigma}$, where $\kappa = \sqrt{2 + k^2}$. Therefore $\hat{w}(y) := w_\psi(x(y))$ solves the equation

$$-\nabla \cdot \hat{a} \cdot \nabla \hat{w} - \hat{W}_\epsilon \hat{w} = 0 \quad \text{in } \mathcal{C}^\kappa_+,$$

with the uniformly elliptic matrix $\hat{a}(y) := a(x(y))$ and $\hat{W}_\epsilon(y) := W_\epsilon(x(y))$. One can easily check by direct computation that $\hat{W}_\epsilon \in GB$. Fix $\epsilon > 0$ such that $\|\bar{\hat{W}}_\epsilon\|_{GB,\tilde{a}} < 1$. Then by Lemma 5.5 we conclude that $\hat{w}(y)$ satisfies (5.5). Therefore $w_\psi(x)$ obeys (5.9) with $\beta := 2 - N - \gamma$ as required. $\quad\square$

**Proposition 5.7.** *Let $\Omega \subset S^{N-1}$ be a domain such that $S^{N-1} \setminus \Omega$ has nonempty interior. Then $p^*(a, \mathcal{C}_\Omega) \leqslant 1 - 2/\beta$, where $\beta < 2 - N$ is from the upper bound* (5.9).

**Proof.** Fix $p > p_0 = 1 - 2/\beta$ and set $\delta = p - p_0$. Let $w_\psi > 0$ be a minimal positive solution in $\mathcal{C}^1_\Omega$ to

$$-\nabla \cdot a \cdot \nabla w - W_\epsilon w = 0 \quad \text{in } \mathcal{C}_\Omega$$

where $\epsilon > 0$ is from Lemma 5.6. Then by (5.9) for some $\bar{\tau} = \bar{\tau}(\delta) > 0$ small enough the function $\bar{\tau} w_\psi$ satisfies

$$(\bar{\tau} w_\psi)^{p-1} \leqslant \bar{\tau}^{p-1}\left(c|x|^\beta\right)^{p-1} \leqslant \frac{\bar{\tau}^{p-1} c_1}{|x|^{2+\delta|\beta|}} \leqslant \frac{\epsilon}{|x|^2 \log^2(|x|+2)} = W_\epsilon(x) \quad \text{in } \mathcal{C}^1_\Omega.$$

Therefore

$$-\nabla \cdot a \cdot \nabla(\bar{\tau} w_\psi) = W_\epsilon(\bar{\tau} w_\psi) \geqslant (\bar{\tau} w_\psi)^{p-1}(\bar{\tau} w_\psi) = (\bar{\tau} w_\psi)^p \quad \text{in } \mathcal{C}^1_\Omega,$$

that is, $\bar{\tau} w_\psi > 0$ is a supersolution to (1.1) in $\mathcal{C}^1_\Omega$. $\quad\square$

*Concluding remarks.* The proofs of Propositions 5.3 and 5.7 rely only on the polynomial lower and upper bounds (5.2) and (5.9). Namely, given $\alpha \leqslant \beta < 2 - N$ in (5.2) and (5.9) we conclude that

$$1 - \frac{2}{\alpha} \leqslant p^*(a, \mathcal{C}_\Omega) \leqslant 1 - \frac{2}{\beta}.$$

By the next example we show that the (optimal) constants $\alpha$ and $\beta$ might be actually different.

Let $\Omega \subset S^{N-1}$ be smooth and $L_d$ be as in (4.10) with

$$d(r) = A(r)\left(A(r) + N - 2\right) + R(r),$$

where

$$A(r) = \gamma + \delta\big[\sin\big(k\log\log(r)\big) + k\cos\big(k\log\log(r)\big)\big],$$
$$R(r) = k\delta\big[\cos\big(k\log\log(r)\big) - k\sin\big(k\log\log(r)\big)\big]\log^{-1}(r),$$

$\gamma < 2 - N$, $\delta > 0$ and $k > 0$ such that $\gamma + \delta\sqrt{k^2+1} < 2 - N$. Thus for large enough $R \gg 1$ the operator $L_d = -\nabla \cdot a_d \cdot \nabla$ is uniformly elliptic on $\mathcal{C}_\Omega^R$. Let $\phi_1 > 0$ be the principal Dirichlet eigenfunction of $-\Delta_\omega$, corresponding to $\lambda_1$. Direct computation and the Hopf Lemma show that the function

$$v_{\phi_1} := r^{\gamma + \delta\sin(k\log\log(r))}\phi_1$$

is a minimal positive solution to the equation $L_d v = 0$ in $\mathcal{C}_\Omega^R$. Clearly any $\alpha$ and $\beta$ ($\alpha < \beta < 2 - N$) could be represented as $\alpha = \gamma - \delta$ and $\beta = \gamma + \delta$ for an appropriate choice of parameters $\gamma$, $\delta$ and $k$. Therefore one cannot expect a sharp polynomial asymptotics of minimal solutions to the equation $-\nabla \cdot a \cdot \nabla v = 0$ in cone-like domains without additional restrictions on the matrix $a(x)$.

It is an interesting open problem to determine the value of the critical exponent $p^*(a, \mathcal{C}_\Omega)$ in the case of minimal solutions oscillating at infinity between two different polynomials.

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