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Elementary abelian 2-subgroups of Sidki-type in finite groups

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Dedicated to Avinoam Mann on the occasion of his retirement, 2006

Abstract. Let G be a finite group. We say that a nontrivial elementary abelian 2-subgroup V of G is of *Sidki-type* in G, if for each involution i in G, $C_V(i) \neq 1$. A conjecture due to S. Sidki (J. Algebra 39, 1976) asserts that if V is of Sidki-type in G, then $V \cap O_2(G) \neq 1$. In this paper we prove a stronger version of Sidki's conjecture. As part of the proof, we also establish weak versions of the saturation results of G. Seitz (Invent. Math. 141, 2000) for involutions in finite groups of Lie type in characteristic 2. Seitz's results apply to elements of order p in groups of Lie type in characteristic p, but only when p is a good prime, and 2 is usually not a good prime.

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Introduction

In the conference honoring the retirement of A. Mann which took place in Jerusalem in May 2006, Said Sidki gave a talk in which he recalled a conjecture that he had made in 1976 ([Si]). In this paper we prove his conjecture:

Theorem 1. *Assume G is a finite group and V is a non-trivial elementary abelian 2-subgroup of G such that for each involution $i \in G$, $C_V(i) \neq 1$. Then $V \cap O_2(G) \neq 1$.*

One way to view Theorem 1 is the following. For a finite group G let $\mathrm{Inv}(G)$ be the set of involutions of G and let $\mathcal{I}$ be the commuting graph on $\mathrm{Inv}(G)$. Thus the vertex set of $\mathcal{I}$ is $\mathrm{Inv}(G)$ with $a, b \in \mathrm{Inv}(G)$ adjacent in $\mathcal{I}$ if a and b commute. Now to

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any graph one can associate a combinatorial simplicial complex: the clique complex, whose simplices are the cliques of the graph. Let $K_2(G)$ be the clique complex of $\mathcal{I}$. Then Theorem 1 says that if there exists a simplex σ in $K_2(G)$ such that each vertex in $\mathcal{I}$ is adjacent to some vertex of σ, then $\langle\sigma\rangle \cap O_2(G) \neq 1$.

Since the clique complex $K_2(G)$ is homotopy equivalent to the Quillen complex $\mathcal{A}_2(G)$ of G at the prime 2 (cf. 5.2 in [A7]), Theorem 1 sheds some light on the Quillen conjecture which asserts that for any prime p, $\mathcal{A}_p(G)$ is contractible if and only if $O_p(G) \neq 1$. Indeed, Theorem 1 shows that the Quillen conjecture at the prime 2 is equivalent to the following statement: $K_2(G)$ is contractible iff G possesses a subgroup of Sidki type.

Since the hypothesis of Theorem 1 does not inherit well to homomorphic images, we prove that $V \cap O_2(G) \neq 1$ under a weaker hypothesis on V:

Theorem 2. *Assume that G is a finite group and V is a nontrivial elementary abelian 2-subgroup of G, such that whenever $v \in V$ inverts an element h of odd order in G then $C_V(h) \neq 1$. Then $V \cap O_2(G) \neq 1$.*

Notice that if h is an element of odd order inverted by $v \in V^\#$ then hv is an involution in G, and $C_V(hv) \neq 1$ iff $C_V(h) \neq 1$. Thus if V satisfies the hypothesis of Theorem 1, then it also satisfies the hypothesis of Theorem 2; consequently Theorem 2 implies Theorem 1.

In [Al], Alperin proved Theorem 1 for $|V| \leq 4$; indeed in this case a short and elementary argument (Lemma 1.4) shows that Theorem 2 holds, but when $|V| > 4$ we know of no such argument to establish Theorem 2. The case $|V| = 8$ is interesting in that Theorem 1 was proved by Sidki in [Si] when $|V| = 8$ using elementary means.

For the general case we show in Lemma 1.1 that if (G, V) satisfies the hypothesis of Theorem 2, and $K \trianglelefteq G$ with $V \cap K = 1$, then $(G/K, VK/K)$ also satisfies this hypothesis. This allows us to reduce the proof of Theorem 2 to the case where G is almost simple (Proposition 1.14). Then we appeal to the classification of the finite simple groups and results on the subgroup structure of such groups to complete the proof.

Call a pair (G, V) that satisfies the hypothesis of Theorem 2 a *Sidki pair*. Let (G, V) be a Sidki pair and suppose $v \in V^\#$ inverts an element $x \in G$ of odd order. Set $D := \langle v, x\rangle$ and $C := C_G(D)$. We observe that

(O1) (H, V) is a Sidki pair for all subgroups $H \leq G$ with $V \leq H$ (obvious);

(O2) $(C, V \cap C)$ is a Sidki pair (Lemma 1.7);

(O3) if (G, V) is a minimal counter example to Theorem 2, and $O_2(C)$ contains a unique involution t, then t is contained in V and $\langle t^{C_G(v)}\rangle$ is elementary abelian (Lemma 1.15 (2)).

These three observations are very useful in the proof of Theorem 2. We now describe this proof in more detail. Suppose (G, V) is a minimal counter example to Theorem 2.

Then G is an almost simple finite group; that is $L = F^*(G)$ is a nonabelian finite simple group. By the classification of the finite simple groups, L is an alternating group, a finite group of Lie type, or one of 26 sporadic groups.

An easy argument (Theorem 2.1) shows that L is not an alternating group. In Sections 3–6 we establish various results about strongly real elements in the automorphism groups of sporadic groups and use observations (O1)–(O3) above to show that L is not sporadic.

This reduces us to the case where L is a group of Lie type and characteristic p for some prime p. We consider two subcases: p is odd and p is even.

In the *odd characteristic case* we employ a different strategy: assume X is an almost simple finite group such that $F^*(X)$ is a group of Lie type and odd characteristic. Let W be a nontrivial elementary abelian 2-subgroup of X. Recall that proper parabolic subgroups P and Q in $F^*(X)$ are called *opposite* if $P \cap Q$ is a Levi factor of both P and Q. Let us denote by $\mathcal{S}(X, W)$ the collection of those W-invariant sets S of proper parabolic subgroups of $F^*(X)$ such that if $P, Q \in S$ are distinct, then P and Q are opposites.

In Lemma 8.1 we show that, in a minimal counter example to Theorem 2, for all $P \in S \in \mathcal{S}(G, V)$, $N_V(P) = 1$. Thus to eliminate the possibility that L is of Lie type in odd characteristic it suffices to establish the following result, which may be of independent interest:

Theorem 3. *Assume X is an almost simple finite group such that $F^*(X)$ is a group of Lie type and odd characteristic. Assume W is a nontrivial elementary abelian 2-subgroup of X. Then:*

(1) *$\mathcal{S}(X, W)$ is nonempty.*

(2) *If $|W| \geq 8$, then there exists $P \in S \in \mathcal{S}(X, W)$ such that $N_W(P) \neq 1$.*

This reduces the proof of Theorem 2 to the case where L is of *Lie type and characteristic* 2. In this case we use observation (O1) and the fact that $|V| > 4$ to "force" certain involutions v of G to belong to V and then we (essentially) argue that for some element $x \in G$ of odd order inverted by v, $D := \langle v, x \rangle$ does not satisfy observation (O2) or (O3). This is done by showing that certain involutions in L satisfy a weak version of the saturation properties established by Seitz in [Se] for elements of order p in groups of Lie type and characteristic p.

Unfortunately the results of Seitz only apply when p is a good prime, which is almost never the case for $p = 2$. Thus we must supply proofs of these weak saturation properties ourselves. These properties are of independent interest and their description is somewhat technical; we first define the set $\mathfrak{S}(Y, i)$.

Let q be a power of 2, and assume that Y is a group such that either

$Y = O^{2'}(Y)$ is a group of Lie type over $\mathbb{F}_q$ with $F^*(Y)$ quasisimple, or

$Y = {}^2F_4(2)'$ is the Tits group.

Let $i \in \mathrm{Inv}(Y)$, and define $\mathfrak{S}(Y, i)$ to be the set of pairs (K, U) such that

$i \in U \leq K \leq Y$;

U is an elementary abelian 2-group of order q and $U \leq Z(C_Y(i))$;

$C_Y(K)$ is a complement to $O_2(C_Y(i))$ in $C_Y(i)$;

Furthermore one of the following holds:

(i) $K \cong L_2(q)$;

(ii) $Y \cong Sz(q)$ or ${}^2F_4(q)$ with $q > 2$, or ${}^2F_4(2)'$, i is a long root involution in Y, and $K \cong Sz(q)$, $Sz(q)$, or D_{10}, respectively;

(iii) $Y \cong \mathrm{Sp}_n(q)$ or $Y \cong F_4(q)$, i is of type c_2, respectively of type tu in Y, and $K \cong L_2(q^2)$; in addition, if $q = 2$, then $C_Y(K) = C_Y(D)$ for each dihedral subgroup $D_{10} \cong D$ of K.

In this definition, and in Theorem 4 below, we use the notation and the description of involutions in groups of Lie type and characteristic 2 from [ASe].

When Y' is of Lie type and characteristic p with p a good prime, Seitz [Se] establishes many properties of elements i' of order p in Y', including the fact that an appropriate analogue of $\mathfrak{S}(Y', i')$ is nonempty.

Theorem 4. *Let q be a power of 2, and assume $Y = O^{2'}(Y)$ is an almost simple group of Lie type over $\mathbb{F}_q$, or $Y = {}^2F_4(2)'$ is the Tits group. Let i be an involution in Y. Then one of the following holds:*

(1) *$\mathfrak{S}(Y, i)$ is nonempty.*

(2) *$Y \cong \mathrm{Sp}_n(q)$ and i is of type c_l for some even integer $l > 2$.*

(3) *$Y \cong \Omega_n^{\varepsilon}(q)$, $n \geq 8$, and i is of type c_l for some even $l > 2$.*

(4) *$Y \cong E_7(q)$ and i is of type u or v.*

(5) *$Y \cong E_8(q)$ and i is of type v.*

The exclusions in (2)–(5) of Theorem 4 may not be necessary. We do not need to consider such involutions in proving Theorem 2, and did not immediately see how to show $\mathfrak{S}(Y, i) \neq \emptyset$ for such involutions (except in case (3)), so we leave the question open. However there do seem to be some serious difficulties involved with case (2).

To conclude the introduction we point out that Theorem 3 is proved in §8, the proof of Theorem 4 is completed in §15, and the proof of Theorem 2 is completed in §16. Our basic references are [A4] for notation and terminology involving finite groups and [GLS3] for notation, terminology, and information about the finite simple groups.

1. Sidki pairs

In this section G is a finite group and V is a nontrivial elementary abelian 2-subgroup of G.

Given a V-invariant subgroup H of G, define $\mathcal{D}(H, V)$ to be the set of $(h, v) \in H \times V$ such that h is of odd order and v inverts h. Let $\mathcal{B}(H, V)$ consist of those $(h, v) \in \mathcal{D}(H, V)$ such that $C_V(h) = 1$. Recall from the Introduction that the pair (G, V) is a *Sidki pair* if $\mathcal{B}(G, V)$ is empty. Write $\mathcal{P}$ for the set of Sidki pairs. A *minimal counter example* to Theorem 2 is a Sidki pair (G, V) such that $O_2(G) \cap V = 1$, $|G|$ is minimal subject to this constraint, and $|V|$ is minimal subject to both constraints. Write $\mathcal{Q}$ for the set of minimal counter examples to Theorem 2.

Lemma 1.1. *Assume $(G, V) \in \mathcal{P}$. Then:*

(1) *If $V \le H \le G$, then $(H, V) \in \mathcal{P}$.*

(2) *Suppose $K \trianglelefteq G$ with $V \cap K = 1$, and set $G^* = G/K$. Then $(G^*, V^*) \in \mathcal{P}$.*

Proof. As $\mathcal{B}(H, V) \subseteq \mathcal{B}(G, V)$, (1) holds. Let $x \in G^*$ be an element of odd order and let $y \in V^*$ be an involution that inverts x. Then $D := \langle x, y\rangle$ is a dihedral group of order $2m$, where m is odd, and there exists $v \in V$ with $v^* = y$. Thus there exists $g \in G$ such that $D = \langle v^*, (v^g)^*\rangle$. Let $E = \langle v, v^g\rangle$. Then there exists $h \in E$ of odd order such that $h^* = x$. Since v inverts h and $(G, V) \in \mathcal{P}$, $C_V(h) \ne 1$. As $V \cap K = 1$, $C_V(h) \cong C_V(h)^* \le C_{V^*}(h^*) = C_{V^*}(x)$, so $C_{V^*}(x) \ne 1$. □

Lemma 1.2. *If $(G, V) \in \mathcal{P}$ and $|V| = 2$, then $V \le O_2(G)$.*

Proof. Assume otherwise. Then by the Baer–Suzuki Theorem (cf. 39.6 in [A4]), a generator v of V inverts a nontrivial element x of odd order, and as $|V| = 2$, $C_V(x) = 1$, contradicting $(G, V) \in \mathcal{P}$. □

The following lemma, and Lemma 1.5 below, should be compared with Theorem 2.1 in [Si].

Lemma 1.3. *Assume $(G, V) \in \mathcal{P}$ and E is of index 2 in V with $(G, E) \notin \mathcal{P}$. Then $V \le O_2(C_G(E))$.*

Proof. Assume otherwise. Then $|V| > 2$ by 1.2. By hypothesis there is $(x, e) \in \mathcal{B}(G, E)$, and as $(G, V) \in \mathcal{P}$, there is $1 \ne v \in C_V(x)$. As $(x, e) \in \mathcal{B}(G, E)$, $C_E(x) = 1$ so $v \in V - E$. Hence by assumption, $v \notin O_2(C_G(E))$, so by the Baer–Suzuki Theorem, there is $g \in C_G(E)$ such that $y := vv^g$ is nontrivial of odd order. Then $(G, V^g) \in \mathcal{P}$ and $(x, e) \in \mathcal{D}(G, V^g)$, so there exists $1 \ne u \in C_{V^g}(x)$. As $C_E(x) = 1$, $u = v^g a$ for some $a \in E$. Then $ya = vv^g a = vu$ centralizes x, and y centralizes a, so also $(ya)^2 = y^2$ centralizes x. Then as $|y|$ is odd, $[x, y] = 1$.

Therefore $|xy|$ is odd and ev inverts xy, so there is $1 \neq w \in C_V(xy)$. If $w \in E$ then $xy = (xy)^w = x^w y$, so $w \in C_E(x) = 1$, a contradiction. Thus $w = bv$ for some $b \in E$. Therefore $xy = (xy)^w = x^b y^{-1}$, so $x^b = xy^2$. Then as $[b, y] = 1$,

$$x = x^{b^2} = (xy^2)^b = x^b y^2 = xy^4,$$

contradicting $|y|$ odd. □

Lemma 1.4. *If $(G, V) \in \mathcal{P}$ and $|V| \leq 4$, then $V \cap O_2(G) \neq 1$.*

Proof. Assume otherwise and choose $v \in V^\#$. By Baer–Suzuki, v inverts a nontrivial element x of odd order in G. As $(G, V) \in \mathcal{P}$, there is $1 \neq e \in C_V(x)$. But by 1.2 and 1.3, $V \leq O_2(C_G(e))$, a contradiction. □

Lemma 1.5. *Assume that $(G, V) \in \mathcal{P}$, but for each maximal subgroup E of V, $(G, E) \notin \mathcal{P}$. Then V centralizes each V-invariant subgroup of G of odd order.*

Proof. Let X be a V-invariant subgroup of G of odd order, and $\mathcal{E}$ the set of maximal subgroups of V. Then (cf. Exercise 8.1 in [A4]) $X = \langle C_X(E) : E \in \mathcal{E}\rangle$. Furthermore, for $E \in \mathcal{E}$ we have $V \leq O_2(C_G(E))$ by hypothesis and 1.3, so $[C_X(E), V] \leq O_2(C_G(E)) \cap X = 1$, and hence V centralizes X. □

Lemma 1.6. *Let $U \leq V$ be a nontrivial subgroup and let H be a U-invariant subgroup of G. Then $(HU, U) \in \mathcal{P}$ iff $\mathcal{B}(H, U) = \emptyset$.*

Proof. As the 2-group U acts on H, $O^2(HU) = O^2(H)$, so $\mathcal{B}(H, U) = \mathcal{B}(HU, U)$. □

Lemma 1.7. *Assume $(G, V) \in \mathcal{P}$, let $v \in V$ and let $x \in G$ be an element of odd order inverted by v. Set $D := \langle v, x\rangle$ and $C := C_G(D)$. Then $(C, C \cap V) \in \mathcal{P}$.*

Proof. Let $W := C \cap V$, and notice that since $(G, V) \in \mathcal{P}$, $W \neq 1$. Pick $1 \neq w \in W$ and let $y \in C$ be an element of odd order inverted by w. Then vw inverts xy, so there exists $u \in V^\#$ such that $u \in C_V(xy)$. Since $w, xy \in C_G(u)$ also $(xy)(xy)^w = xyxy^{-1} = x^2 \in C_G(u)$. But $|x|$ is odd, so $u \in C$ and hence $u \in W$. Note that u centralizes y, so we see that $(C, W) \in \mathcal{P}$. □

Our goal in the remainder of the section is to prove that if (G, V) is a minimal counter example to Theorem 2 then G is almost simple, and to obtain further results on a minimal counter example. Thus in the remainder of the section we assume:

Hypothesis 1.8. (G, V) is a minimal counter example to Theorem 2.

Lemma 1.9. *Let $1 \neq U \leq V$ and assume that H is a nontrivial U-invariant subgroup of G such that $HU \neq G$.*

(1) *If $C_U(H) = 1$ and $O_2(H) = 1$, then $(HU, U) \notin \mathcal{P}$.*

(2) *If $U = V$ and $O_2(H) = 1$, then $C_V(H) \neq 1$.*

Proof. Notice that (2) is an immediate corollary of (1) because by Lemma 1.1 (1), $(HV, V) \in \mathcal{P}$.

Assume the hypothesis of (1) with $(HU, U) \in \mathcal{P}$. Then by minimality of G in Hypothesis 1.8, $U \cap O_2(HU) \neq 1$, and so $[U \cap O_2(HU), H] \leq O_2(HU) \cap H = 1$, contradicting $C_U(H) = 1$. □

Lemma 1.10. (1) *If $V \leq H < G$, then $V \cap O_2(H) \neq 1$.*

(2) *If H is a proper subgroup of G of odd index, then $1 \neq V \cap O_2(K)$ for some $K \in H^G$.*

Proof. Under the hypothesis of (1), $(H, V) \in \mathcal{P}$ by 1.1 (1), so (1) follows from minimality of G. Then (2) follows from (1) and Sylow's Theorem. □

Lemma 1.11. *V centralizes each V-invariant subgroup of G of odd order.*

Proof. This follows from the minimality of V in Hypothesis 1.8, and from 1.5. □

Lemma 1.12. *Let $X_i \leq G$ be V-invariant subgroups of G for $i = 1, 2$, such that $X := \langle X_1, X_2 \rangle = X_1 \times X_2$, $O_2(X) = 1$ and $X_i V \neq G$, then $C_V(X) \neq 1$.*

Proof. Assume that $C_V(X) = 1$ and set

$$V_1 := C_V(X_2).$$

By Lemma 1.9 (2) $V_1 \neq 1$. Since $C_{V_1}(X_1) = 1$, Lemma 1.9 (1) implies that $(X_1V_1, V_1) \notin \mathcal{P}$, so by Lemma 1.6 we may pick $(h_1, v_1) \in \mathcal{B}(X_1, V_1)$. Set

$$V_2 := C_V(h_1).$$

Then $V_2 \cap V_1 = 1$ and hence $C_{V_2}(X_2) = V_2 \cap C_V(X_2) = V_2 \cap V_1 = 1$. Hence, again, we may choose $(h_2, v_2) \in \mathcal{B}(X_2, V_2)$. Note that

$$C_V(h_1h_2) = C_V(h_1) \cap C_V(h_2) = V_2 \cap C_V(h_2) = C_{V_2}(h_2) = 1.$$

However as $X = X_1 \times X_2$ with $h_i \in X_i$, $[h_1, h_2] = 1$ so h_1h_2 is inverted by v_1v_2, contrary to our hypothesis that $(G, V) \in \mathcal{P}$. □

Lemma 1.13. $F(G) = 1$.

Proof. Assume that $O_2(G) \neq 1$. As $(G, V) \in \mathcal{Q}$, $V \cap O_2(G) = 1$. Then we conclude from Lemma 1.1 (2) and minimality of G that $O_2(G/O_2(G)) \neq 1$, which is absurd.

Thus $O_2(G) = 1$. Assume that $K := F(G) \neq 1$, and set $G^* := G/K$. As $O_2(G) = 1$, K is of odd order, so $V \cap K = 1$. Thus $(G^*, V^*) \in \mathcal{P}$ by Lemma 1.1 (2). Then by minimality of G, $1 \neq V^* \cap O_2(G^*)$. Let U be the preimage in V of $V^* \cap O_2(G^*)$. As $E(G)^* \leq E(G^*)$, $[U, E(G)] \leq K$, so U centralizes $E(G)$ (cf. 31.6.3 in [A4]). Further V centralizes K by 1.11, so U centralizes $F^*(G)$, contradicting $O_2(G) = 1$. □

We can now prove

Proposition 1.14. *Assume that (G, V) is a minimal counter example to Theorem 2. Then G is almost simple and $G = VF^*(G)$.*

Proof. We first claim that

(1) V acts transitively via conjugation on the components of G.

Let Ω be the set of components of G and let Ω_1 be an orbit of V on Ω. Assume that $\Omega_2 = \Omega \setminus \Omega_1$ is nonempty. Set $X_i = \langle \Omega_i \rangle$, $i = 1, 2$ and note that we get a contradiction to Lemma 1.12. This establishes (1).

Let $L \in \Omega$. We next claim that

(2) $V_1 := N_V(L) \neq 1$.

Assume (2) fails and let q be an odd prime dividing $|L|$ and $Q_1 \in \mathrm{Syl}_q(L)$. Then $Q := \langle Q_1^V \rangle$ is a V-invariant Sylow q-subgroup of $F^*(G)$, and V is faithful on Q, contrary to 1.11. This establishes (2).

We next show

(3) $(LV_1, V_1) \in \mathcal{P}$.

First, for $1 \neq x \in L$, $C_G(x) \leq N_G(L)$. Thus if x has odd order and is inverted by some $v \in V_1$, then $C_V(x) \neq 1$, and since $C_V(x) \leq N_G(L)$, $C_V(x) \leq V_1$. So, by 1.6, (3) is established.

If $G = LV$ then the proposition holds, so we may assume otherwise. Then by (3) and the minimality of G, $C_{V_1}(L) \neq 1$. However as V is abelian, $C_{V_1}(L) \leq C_{V_1}(F^*(G)) = 1$ by (1), a contradiction. □

Lemma 1.15. *Assume u is an involution in G and x is a nontrivial element of odd order inverted by u. Set $D := \langle u, x \rangle$ and suppose that either*

(1) $O_2(C_G(D)) = 1$, *or*

(2) $O_2(C_G(D))$ *contains a unique involution t, and either $t \notin V$ or $\langle t^{C_G(u)} \rangle$ is not elementary abelian.*

Then $u \notin V$.

Proof. Suppose $u \in V$ and let $H = C_G(u)$, $C = C_G(D)$, and $h \in H$. Set $W_h = C_{V^h}(D)$. By 1.7, $(C, W_h) \in \mathcal{P}$, so by the minimality of G, $W_h \cap O_2(C) \neq 1$ and hence $t \in W_h$. We have

$$t \in \bigcap_{h \in H} W_h \leq \bigcap_{h \in H} V^h =: V_H,$$

so $t \in V$ and as V_H is an elementary abelian normal subgroup of H, $\langle t^H \rangle$ is elementary abelian, a contradiction. □

2. Alternating groups

In this section we prove:

Theorem 2.1. *Assume* (G, V) *is a minimal counter example to Theorem* 2. *Then* G *is not an alternating or symmetric group.*

Let $\Omega = \{1, \dots, n\}$, S the symmetric group on Ω, and A the alternating group on Ω. Assume (G, V) is a minimal counter example to Theorem 2 and $G = S$ or A. For $X \subseteq S$, write $M(X)$ for the set of points of Ω moved by X.

Lemma 2.2. (1) $n \geq 5$.

(2) V *acts on no partition* $\Gamma = \{\Omega_1, \Omega_2\}$ *of* Ω *with* $|\Omega_i| > 4$ *for* $i = 1, 2$.

Proof. Part (1) follows from Proposition 1.14, which says that G is almost simple. Suppose V acts on Γ as in (2), and let $H = N_G(\Gamma)$. Then $V \leq H < G$ with $O_2(H) = 1$, contrary to 1.10. □

Lemma 2.3. (1) $M(v) = \Omega$ *for each* $v \in V^\#$.

(2) *Each orbit of* V *on* Ω *is regular.*

(3) $|V| \geq 8$.

(4) $n \geq 8$.

Proof. Assume (1) fails and pick $v \in V^\#$ so that $M(v) \neq \Omega$, and $m = |M(v)|$ is maximal subject to this constraint. Then v inverts a cycle $h \in G$ of length $m + 1$. Now $(h, v) \in \mathcal{D}(G, V)$, so as $\mathcal{B}(G, V) = \emptyset$, there exists $1 \neq u \in C_V(h)$. Then $M(h) \subseteq \mathrm{Fix}(u)$, so $M(uv) = M(u) \cup M(v)$ is of order $m + |M(u)| > m$. Further uv fixes the fixed point of v on $M(h)$, so $M(uv) \neq \Omega$, contrary to the maximality of m. Thus (1) is established.

Part (1) implies (2), while (3) follows from 1.4. By (2) and (3), $n \equiv 0 \mod 8$, so (4) follows. □

Lemma 2.4. *V is regular on Ω.*

Proof. Let $m = |V|$. By 2.3 (2), V has $r = n/m$ regular orbits of length m on Ω, and we may assume $r > 1$. By 2.3 (3), $m \geq 8$. Let Δ be an orbit of V on Ω, and set $\Sigma = \Omega - \Delta$ and $H = G_\Delta$. Thus H is the alternating or symmetric group on Σ, and as $|\Sigma| \geq m \geq 8$, $O_2(H) = 1$. Hence $C_V(H) \neq 1$ by 1.9 (2). But $C_V(H)$ fixes Σ pointwise, contrary to 2.3 (1). □

Lemma 2.5. $n = 8$.

Proof. Let U be a subgroup of index 2 in V and Γ the set of orbits of U on Ω. As V is regular on Ω, U has two orbits of length $n/2$ and V acts on Γ. Now the lemma follows from 2.2 (2) and 2.3 (4). □

We are now in a position to obtain a contradiction, establishing Theorem 2.1. By 2.4 and 2.5, $G \cong L_4(2)$ and we may choose V to be the group of transvections with a fixed axis P on the natural module M for G. Let $v \in V^\#$ and pick h of order 3 in G inverted by v. Then $\dim([M,h]) = 2$ and $M = [M,h] + P$. However $C_V(h)$ centralizes $[M,h]$ and P, so $C_V(h)$ centralizes $M = [M,h] + P$ and hence $C_V(h) = 1$. Thus $(h, v) \in \mathcal{B}(G, V)$, for our contradiction.

3. Some strongly real elements in sporadic groups

In this section L is a sporadic group. Our notation for conjugacy classes in L come from [GLS3], although we sometimes write 'r' rather than 'rA' if L has a unique class of elements of order r. The information about the normalizers of elements of prime order in L comes from [GLS3] as well; sometimes the appeals to this information are implicit rather than explicit.

For a group H we write $\mathrm{Inv}(H)$ for the set of involutions of H and for $x \in H$, $N_H(x) := N_H(\langle x \rangle)$. Given L-invariant subsets A, B of G, we sometimes write $[A, B] = 1$ to indicate that $[a,b] = 1$ for some $a \in A$ and $b \in B$. We write $t \rightsquigarrow x$ to indicate that the involution t inverts x, and write $A \rightsquigarrow B$ to indicate that $a \rightsquigarrow b$ for some $a \in A$ and $b \in B$.

Lemma 3.1. (1) *Let $M \cong M_{24}$ and let Ω be a set of 24 points permuted transitively by M. Then M has two classes of involutions $2A$ and $2B$ such that:*

(a) *For $a \in 2A$, $C_M(a) \cong L_3(2)/D_8^3$ and $\mathrm{Fix}(a)$ is an octad in Ω.*

(b) *For $b \in 2B$, $C_M(b) \cong S_5/E_{64}$ and b has no fixed point on Ω.*

(c) $2B \rightsquigarrow 11A$.

(2) *Let $L \cong M_{22}$, then $2C \rightsquigarrow 11A$.*

Proof. (1): Parts (a) and (b) appear in [A8, 21.1]. Let $X \leq M$ be of order 11. As the stabilizer $L_3(4)$ in M of 3 points of Ω is an $11'$-group, $|\mathrm{Fix}(X)| = 2$ and X has two orbits Ω_i, $i = 1, 2$, of length 11 on Ω. An involution $t \in M$ that inverts X acts on $\mathrm{Fix}(X)$ and either fixes one point in Ω_i for $i = 1$ and 2, or interchanges Ω_1 and Ω_2. Thus $|\mathrm{Fix}(t)| \leq 4$, so $t \in 2B$.

(2): View $\mathrm{Aut}(L)$ as the global stabilizer in $M = M_{24}$ of two points. Let $c \in \mathrm{Inv}(\mathrm{Aut}(L)) \setminus L$ with $c \rightsquigarrow 11$. By (1), c is in the M-class $2B$. Thus $7 \notin \pi(C_M(c))$, so c is of type $2C$ in $\mathrm{Aut}(L)$. □

Lemma 3.2. *The following table lists groups L and classes $\mathcal{O}$ and $\mathcal{A}$ in* $\mathrm{Aut}(L)$ *such that*

(i) $|\mathrm{Out}(L)| = 2$ *and $\mathcal{A}$ is a class of outer involutions in* $\mathrm{Aut}(L)$*;*

(ii) *for $x \in \mathcal{O}$, $|C_{\mathrm{Aut}(L)}(x)|$ is odd and $\mathcal{A} \rightsquigarrow \mathcal{O}$.*

L	M_{12}	M_{22}	J_3	$O'N$
$\mathcal{O}$	$11A$	$11A$	$19A$	$31A$
$\mathcal{A}$	$2C$	$2C$	$2B$	$2B$

Proof. Observe $p = |x|$ is an odd prime, and from [GLS3], $\langle x\rangle \in \mathrm{Syl}_p(L)$, $|C_L(x)|$ is odd, $|\mathrm{Out}(L)| = 2$, and for each involution $a \in \mathrm{Aut}(L) \setminus L$, $p \notin \pi(C_L(a))$. Thus, by a Frattini argument, some involution $t \in \mathrm{Aut}(L) \setminus L$ acts on $\langle x\rangle$, and then, as $p \notin \pi(C_L(t))$, t inverts x.

Next, if $L \neq M_{22}$, then there is a unique class $\mathcal{A}$ of outer involutions in $\mathrm{Aut}(L)$ so $t \in \mathcal{A}$ and the lemma holds in this case. If $L = M_{22}$ then the lemma follows from Lemma 3.1 (2). □

Lemma 3.3. *The following table lists classes $\mathcal{O}$ and $\mathcal{A}$ in L such that for $x \in \mathcal{O}$, $|C_L(x)|$ is odd and $\mathcal{A} \rightsquigarrow \mathcal{O}$.*

L	J_2	J_3	J_4	HS	McL	Ru	$O'N$
$\mathcal{O}$	$7A$	$17A$	$23A$	$7A$	$5B$	$29A$	$11A$
$\mathcal{A}$	$2B$	$2A$	$2B$	$2B$	$2A$	$2B$	$2A$

Proof. The J_3, McL and $O'N$ entries of the table follow from the fact that these groups have a unique class of involutions and from the structure of the normalizers of subgroups of prime order in these groups.

J_2: Let $L \cong J_2$ and let c be an outer involution in $\mathrm{Aut}(L)$. Then $K = C_L(c) \cong \mathrm{PGL}_2(7)$ has Sylow 2-subgroup $S \cong D_{16}$ and two classes z^K and t^K of involutions, with z 2-central in K. As $Z(S)$ contains a 2-central involution of G, z is 2-central in G, and hence of type $2A$. Then as non-2-central involutions of L of type $2B$ also centralize outer involutions (cf. [GLS3, pg. 268]), $t \in 2B$. Finally $t \rightsquigarrow 7$ in K.

J_4 : $L \cong J_4$ has two classes of elements of order 11 and two classes of involutions. Further, for each $t \in \mathrm{Inv}(L)$, Sylow 11-subgroups of $C_L(t)$ have order 11, so $C_L(t)$ is transitive on its subgroups of order 11. Since each element of order 11 in L is centralized by an involution, we conclude that:

(1) For $X \in \{A, B\}$, there exists a unique class 11_X of elements of order 11 such that $[11_X, 2X] = 1$. Further $11_A \neq 11_B$.

Next let Y be generated by $y \in 11A$. By [GLS3], $N_L(Y) \cong (\mathrm{GL}_2(3)\times\mathbb{Z}_5)/11^{1+2}$ is strongly 11-embedded in L. Let $Q = O_{11}(N_L(Y))$. Then there is an involution $z \in C_L(Y)$ inverting Q/Y, and $3 \in \pi(C_L(y, z))$, so $z \in 2A$, since if $z \in 2B$ then $3 \notin \pi(C_{C_L(z)}(y))$. It follows from (1) that:

(2) $11_X = 11X$ for $X \in \{A, B\}$.

Further as $N_L(Y)$ is strongly 11-embedded in L, each $b \in Q - Y$ is in $11B$, and z inverts some such elements, so:

(3) $2A \rightsquigarrow 11B$.

Let $b \in 11B$ and $v \in C_L(b) \cap 2B$. Then $C_L(v)$ is contained in a maximal subgroup $M \cong M_{24}/E_{2^{11}}$, so it follows from the structure of M_{24} that $b \in N_M(X)$ for some $X \leq M$ of order 23. Finally $N_L(X)$ is Frobenius of order $23 \cdot 22$, so (2) implies that $2B \rightsquigarrow 23$.

HS: Let $L \cong HS$ and let $d \in 2D$. Then $K := C_L(d) \cong S_8$. Consider the covering group $\hat{L}$ of L, and let $\hat{K}$ be the preimage of K in $\hat{L}$. By [GLS3], $Z := Z(\hat{L})$ is of order 2, and the involutions in $K \setminus E(K)$ lift in $\hat{L}$ to elements of order 4. As involutions of type $2A, 2B$ in L lift to involutions, elements of order 4 in $\hat{L}$, respectively, it follows that involutions in $K \setminus E(K)$ are of type $2B$. Since L contains a unique class of elements of order 7, and an element of order 7 in K is inverted by an involution of cycle type 2^3, we are done.

Ru: There are two classes of involutions in *Ru* and only $[2B, 7] = 1$. Next for $x \in 29$, $N_L(x)$ is Frobenius of order $29 \cdot 14$, and hence $\mathrm{Inv}(N_L(x)) \subset 2B$. □

Lemma 3.4. *If $L \cong M_{12}$, then $2A \rightsquigarrow 5A$, $2B \rightsquigarrow 5A$, and for $x \in 5A$, $\mathrm{Inv}(C_L(x)) \subset 2A$.*

Proof. First, L has two classes of involutions, $2A$ and $2B$, and one class $5A$ of elements of order 5. If $t \in 2A$, then t has no fixed points on the set Ω of 12 points permuted by L, while $s \in 2B$ fixes 4 points. Further L contains a transitive subgroup $K \cong L_2(11)$ and for $\omega \in \Omega$, K_ω is a Borel subgroup, so the involutions in K are in the class $2A$ and inside K, $2A \rightsquigarrow 5A$. Next, $2B \rightsquigarrow 5$, inside $L_\omega \cong M_{11}$. Finally, since $5 \notin \pi(C_L(2B))$, $\mathrm{Inv}(C_L(x)) \subset 2A$. □

Lemma 3.5. *If $L \cong J_2$, then $2A \rightsquigarrow 5A$ and for $y \in 5A$, $\mathrm{Inv}(C_L(y)) \subset 2B$.*

Proof. By [GLS3], $[2B, 5A] = 1$, so since $C_L(y) \cong \mathbb{Z}_5 \times A_5$, $\mathrm{Inv}(C_L(y)) \subset 2B$. Let $b \in \mathrm{Inv}(C_L(y))$. By [GLS3], $C_L(b) \cong E_4 \times A_5$, and $\mathrm{Inv}(C_L(b)) \cap 2A = \mathrm{Inv}(E(C_L(b)))$. Thus inside $C_L(b)$, $2A \rightsquigarrow 5A$. □

Lemma 3.6. *Assume $L \cong Co_2$, let $x \in 5B$, and set $N := N_L(x)$. Then:*

(1) *$N \cong F_{20} \times S_5$, where F_{20} is the Frobenius group of order* 20.

(2) *If $s \in \mathrm{Inv}(N)$ and $C_L(s,x) \cong S_5$, then $s \in 2B$.*

(3) *$2C \not\rightsquigarrow 7A$ and for $z \in 7A$, $\mathrm{Inv}(C_L(z)) \subset 2A \cup 2B$.*

Proof. Part (1) appears in [GLS3]. Let t be a transposition in $C_L(x)$. We claim that $t \in 2A$. Set $C := C_L(t)$ and $C^* := C/O_2(C)$. Then x^* is of order 5 in C^* such that $C_{C^*}(x^*) \cong \mathbb{Z}_5 \times S_3$. Now the claim follows from the structure of centralizers of involutions in L. By the claim:

(i) $[5B, 2A] = 1$.

Next from Section 24 in [A8], there exists a subgroup $HS \cong K \leq L$. Note that K contains a Sylow 5-subgroup of L, and then from the structure of the centralizers of elements of order 5 in L and K:

(ii) For $X \in \{5A, 5B\}$, the K-class X is contained in the L-class X.

Similarly for u in the K-class $2B$, $C_K(u)$ contains an $\mathrm{Aut}(A_6)$-section, so:

(iii) The K-class $2B$ is contained in the L-class $2C$.

Next from [GLS3], there exists $S_8 \cong S \leq K$. Let x_S be an element of order 5 in S. Then $3 \in \pi(C_K(x_S))$, so x_S is in the K-class $5B$. Further there exists $t \in \mathrm{Inv}(S)$ centralizing x with $C_S(t) \cong \mathbb{Z}_2 \times S_6$, so t is in the K-class $2B$. Hence from (ii) and (iii):

(iv) $[5B, 2C] = 1$.

Now for each $t \in \mathrm{Inv}(L)$, Sylow 5-subgroups of $C_L(t)$ have order 5, so $C_L(t)$ is transitive on its subgroups of order 5. Hence by (i) and (iv), for $y \in 5A$, $C_L(y) \cap 2A = \emptyset = C_L(y) \cap 2C$, so:

(v) For $y \in 5A$, $\mathrm{Inv}(C_L(y)) \subseteq 2B$.

If i is in the K-class $2B$, then $C_K(i) \cong \mathbb{Z}_2 \times \mathrm{Aut}(A_6)$, so:

(vi) If $t \in \mathrm{Inv}(K)$ is a square in K, then t is in the K-class $2A$.

Next an element y in the K-class $5A$ is centralized in K by an involution i which is a square in $C_K(y)$, so i is in the K-class $2A$ by (vi). By (ii), y is in the L-class $5A$, so by (v), i is in the L-class $2B$. Thus we have shown:

(vii) The K-class $2A$ is contained in the L-class $2B$.

By (ii), we may assume that $x \in K$ and $N_K(x) \cong F_{20} \times A_5$. Let s be an involution in $N_K(x)$ with $C_K(s,x) \cong A_5$. By (vi) and (vii), s is in the L-class $2B$, so (2) follows.

It remains to prove (3). We may assume that $z \in K$. By 3.3, z is inverted by an involution j in the K-class $2B$, so j is in the L-class $2C$ by (iii). Since for $c \in 2C$, $7 \notin \pi(C_L(c))$, (3) follows. □

Lemma 3.7. *Let $L \cong He$. Then:*

(1) *$2B \rightsquigarrow 17$ and $|C_{\mathrm{Aut}(L)}(x)| = 17$, for $x \in 17$.*

(2) *Let $x \in 5A$. Then x is inverted by some element a in $2A$ with $C_L(a,x) \cong A_5$ and $C_{\mathrm{Aut}(L)}(a,x) \cong S_5$.*

Proof. Let $a \in 2A$. Then $C_L(a)$ has a subgroup $E_4 \times L_3(4)$ of index 2, so if $u \in L$ with $|u| = 8$, then $u^4 \in 2B$. Let $y \in 17$. Then $N_L(y)$ is Frobenius of order $17 \cdot 8$, so the first part of (1) follows, and the second is a consequence of [GLS3]. Part (2) follows from 42.14 in [A8]. □

4. Sporadic groups in Theorem 2

In this section we begin to prove the following result.

Theorem 4.1. *If (G, V) is a minimal counter example to Theorem 2, then $F^*(G)$ is not sporadic.*

Assume that (G, V) is a minimal counter example to Theorem 2 with $L = F^*(G)$ sporadic. By Proposition 1.14, $G = VL$. We begin with a series of reductions.

Lemma 4.2. *Assume a is an involution in G and x is an element of odd order inverted by a. Set $D = \langle a, x\rangle$. Then each of the following imply that $a \notin V$:*

(1) $O_2(C_G(D)) = 1$.

(2) $v^G \cap C_G(D) = \emptyset$ *for each* $v \in V^\#$.

Proof. Part (1) is sufficient by 1.15 (1). Further if $a \in V$, then $(x, a) \in \mathcal{D}(G, V)$, so as $(G, V) \in \mathcal{P}$, $C_V(D) \neq 1$; thus (2) is sufficient. □

Lemma 4.3. *Assume a is an involution in G and x is an element of odd order inverted by a. Set $D = \langle a, x\rangle$ and assume $O_2(C_G(D))$ has a unique involution t. Then each of the following imply that $a \notin V$:*

(1) $O_2(C_G(a)) = \langle a\rangle$.

(2) $\langle t^{C_G(a)}\rangle$ *is not elementary abelian.*

Proof. Part (2) is sufficient by 1.15 (2). Then (1) is sufficient by (2). □

Lemma 4.4. $V \cap L \neq 1$.

Proof. This follows as $|\operatorname{Out}(L)| \leq 2$ and $|V| \geq 8$ by 1.4. □

Lemma 4.5. *L is not M_{11}, M_{23}, J_1, Ly or F_3.*

Proof. For each of the groups, $\operatorname{Out}(L) = 1$, and L has one class a^L of involutions. We indicate a class $\mathcal{O} = x^L$ of elements of odd prime order in L satisfying the hypothesis of Lemma 4.2 (1), and appeal to that lemma.

L	M_{11}	M_{23}	J_1	Ly	F_3
$\mathcal{O}$	$5A$	$5A$	$7A$	$31A$	$19A$

□

Lemma 4.6. *If L is M_{12}, J_3, or O'N, then $G = L$.*

Proof. Let $\mathcal{A} = a^L$ and $\mathcal{O} = x^L$ be as in Lemma 3.2. Then, by Lemma 3.2 and Lemma 4.2 (1), $V \cap \mathcal{A} = \emptyset$. But $\mathcal{A}$ is the unique class of outer involutions in $\mathrm{Aut}(L)$, so $V \subset L$. Thus the lemma follows as $G = VL$. □

Lemma 4.7. *L is not J_3 or O'N.*

Proof. Assume otherwise. By Lemma 4.6, $G = L$. Now L has a unique class of involutions $2A$, and Lemma 3.3 shows that the hypotheses of Lemma 4.2 (1) hold for $a \in \mathcal{A} = 2A$ and some $x \in \mathcal{O}$, completing the proof. □

Lemma 4.8. *L is not M_{12}.*

Proof. Assume $L \cong M_{12}$. By Lemma 4.6, $G = L$. Let $x \in 5A$, $a \in 2A$, and $b \in 2B$. Then $C_L(x) \cong \mathbb{Z}_2 \times \mathbb{Z}_5$ and $C_L(a) \cong \mathbb{Z}_2 \times S_5$. By 3.4, we may assume that $a \rightsquigarrow x$ and $b \rightsquigarrow x$. Then by 4.3 (1), $V \cap 2A = \emptyset$. Consequently $V^{\#} \subset 2B$. But by 3.4, $\mathrm{Inv}(C_L(x)) \subseteq 2A$, so Lemma 4.2 (2) supplies a contradiction. □

Lemma 4.9. *L is not M_{22}.*

Proof. Assume $L \cong M_{22}$. By Lemma 3.2 and Lemma 4.2 (1), $V \cap 2C = \emptyset$. Next, $2A$ is the unique class of involutions in L, and $x \in 5A$ is inverted in L, so $2A \rightsquigarrow 5A$. Further $5 \notin \pi(C_L(i))$ for $i \in 2A \cup 2B$, so $\mathrm{Inv}(C_{\mathrm{Aut}(L)}(x)) \subset 2C$. Thus applying Lemma 4.2 (2), $V \cap 2A = \emptyset$, contrary to 4.4. □

Lemma 4.10. *L is not M_{24}.*

Proof. Assume $L \cong M_{24}$. Then $\mathrm{Out}(L) = 1$, so $G = L$. An element of order 11 is self centralizing in L, so by 3.1 (1c) and 4.2 (1), $V \cap 2B = \emptyset$. Let $x \in 3A$. Then $N := N_L(x) \cong S_3 \times L_3(2)$, so if $a \in \mathrm{Inv}(N)$ with $C_N(a, x) \cong L_3(2)$, then a inverts x and $a \in 2A$ because $7 \in \pi(C_L(a))$ but $C_L(b)$ is a $7'$-group for b in the remaining class $2B$ of involutions. It follows from Lemma 4.2 (1) that $V \cap 2A = \emptyset$, contradicting $\mathrm{Inv}(L) = 2A \cup 2B$. □

Lemma 4.11. *L is not J_2.*

Proof. Assume $L \cong J_2$. Let $x \in 7A$. Then $N_L(x)$ is Frobenius of order 42, while $\mathrm{Aut}(L)$ has a unique class $2C$ of outer involutions, and for $c \in 2C$, $C_L(c) \cong \mathrm{PGL}_2(7)$. In particular we may choose $[x, c] = 1$ and for $b \in \mathrm{Inv}(L)$ inverting x, $cb \notin L$, so $cb \in 2C$. Thus $2C \rightsquigarrow 7A$, so $2C \cap V = \emptyset$ by 4.3 (1). Therefore $G = L$.

Next, 3.3 together with Lemma 4.2 (1) imply that $V \cap 2B = \emptyset$. Finally by 3.5 and Lemma 4.2 (2), $V \cap 2A = \emptyset$, which is impossible since $\mathrm{Inv}(L) = 2A \cup 2B$. □

Lemma 4.12. *L is not* J_4.

Proof. Assume $L \cong J_4$. Then $\mathrm{Out}(L) = 1$, so $G = L$. By Lemma 3.3 and Lemma 4.2 (1), $V \cap 2B = \emptyset$.

Next from observation (3) in the treatment of J_4 in 3.3, $2A \rightsquigarrow 11B$, while observations (1) and (2) in that treatment imply that $\mathrm{Inv}(C_L(y)) \subset 2B$, for $y \in 11B$. Thus $V \cap 2A = \emptyset$ by 4.2 (2), contradicting $\mathrm{Inv}(L) = 2A \cup 2B$. □

Lemma 4.13. *L is not* Co_1.

Proof. Assume $L \cong Co_1$. Then $\mathrm{Out}(L) = 1$, so $G = L$.

Let $z \in 2A$ and $H := C_L(z)$. By 25.6 in [A8], $\mathrm{Inv}(O_2(C_L(z))) \subseteq 2A$, so by 1.10 (2), $2A \cap V \neq \emptyset$.

Let $x \in 3D$. Then $N_L(x) \cong S_3 \times A_9$, so there is an involution a inverting x with $C_L(a, x) \cong A_9$. Then 3^4 divides $|C_L(a)|$, so from the structure of centralizers of involutions in L it follows that $a \in 2A$. Then since $C_L(a, x) \cong A_9$, Lemma 4.2 (1) implies that $V \cap 2A = \emptyset$, a contradiction. □

Lemma 4.14. *L is not* Co_2.

Proof. Assume $L \cong Co_2$. Then $\mathrm{Out}(L) = 1$, so $G = L$.

Let $x \in 3B$, then $M := N_L(x) \cong S_3 \times \mathrm{Aut}(\mathrm{PSp}_4(3))$. Thus if $a \in \mathrm{Inv}(M)$ with $C_M(a, x) \cong \mathrm{Aut}(\mathrm{PSp}_4(3))$, then a inverts x and as $C_L(a)$ has an $\mathrm{Sp}_4(3)$-section, $a \in 2A$. It follows from Lemma 4.2 (1) that $V \cap 2A = \emptyset$. Similarly by 3.6 (2) there exists $b \in 2B$ inverting $x \in 5B$ such that $C_L(b, x) \cong S_5$, so by Lemma 4.2 (1), $V \cap 2B = \emptyset$. Finally by 3.6 (3), there is $c \in 2C$ inverting $z \in 7A$ with $\mathrm{Inv}(C_L(c, z)) \subseteq 2A \cup 2B$, so $V \cap 2C = \emptyset$ by 4.2 (2), a contradiction. □

Lemma 4.15. *L is not* Co_3.

Proof. Assume $L \cong Co_3$. Since $\mathrm{Out}(L) = 1$, $G = L$. Let $t \in 2A$. Then t is a 2-central involution in L and $C_L(t) \cong 2\mathrm{Sp}_6(2)$, so by 1.10 (2), $V \cap 2A \neq \emptyset$. Let $x \in 5B$, then $N := N_L(x) \cong F_{20} \times A_5$, and hence for $a \in N$, with $C_N(a, x) \cong A_5$, $a \in 2A$, because a is a square in L. But now 4.2 (2) says that $V \cap 2A = \emptyset$, a contradiction. □

Lemma 4.16. *L is not HS.*

Proof. Assume $L \cong HS$. From the proof of 3.6 (2), $x \in 5B$ is inverted by $a \in 2A$ with $C_L(a, x) \cong A_5$, and from [GLS3], $C_{\mathrm{Aut}(L)}(a, x) \cong S_5$. Thus by Lemma 4.2 (1), $V \cap 2A = \emptyset$.

Next by Lemma 3.3 there exists $b \in 2B$ with $b \rightsquigarrow y \in 7A$. Now $C_{\mathrm{Aut}(L)}(y)$ contains a unique involution d, and $S := C_L(d) \cong S_8$. Then $C_L(d, b) \cong E_4 \times S_4$, so $E(C_L(b)) = [E(C_L(b)), d]$. Hence by Lemma 4.3 (2), $V \cap 2B = \emptyset$. Now Lemma 4.4 supplies a contradiction. □

Lemma 4.17. *L is not McL.*

Proof. Let $a \in 2A$. Then $C_L(a) \cong 2A_8 = F^*(C_{\mathrm{Aut}(L)}(a))$. By Lemma 1.10 (2), $V \cap 2A \neq \emptyset$. By 3.3, $a \rightsquigarrow x \in 5B$, and $O_2(C_{\mathrm{Aut}(L)}(x, a)) = 1$. Hence, from Lemma 4.2 (1), it follows that $V \cap 2A = \emptyset$, a contradiction. □

Lemma 4.18. *L is not Suz.*

Proof. Let $z \in 2A$ and $H := C_L(z)$. Then z is a 2-central involution and $Q = O_2(H) = F^*(C_{\mathrm{Aut}(L)}(z)) \cong Q_8^3$, with $H/Q \cong \Omega_6^-(2)$ transitive on the involutions $i \in Q - \langle z \rangle$, so $|C_H(i)|_2 = (|H|_2)/2 = 2^{12}$. Thus as $|C_L(b)|_2 < 2^{12}$ for $b \in 2B$, $\mathrm{Inv}(Q) \subseteq 2A$, so $V \cap 2A \neq \emptyset$ by 1.10 (2).

By [GLS3], H contains $x \in 5A$. As all involutions in $C_L(x)$ are in $E(C_L(x)) \cong A_6$, it follows that $z \rightsquigarrow y \in C_L(x)$ of order 5. Let $D := \langle z, y \rangle$. Then $C_L(D) \cap N_L(x)$ contains an F_{20}-subgroup, so it follows from the structure of centralizers of elements of order 5 in L that $F^*(C_L(D)) = E(C_L(y))$. But now $2A \cap V = \emptyset$ by 4.2 (1), a contradiction. □

Lemma 4.19. *L is not He.*

Proof. Assume $L \cong He$. By 4.4, $V \cap L \neq 1$, while by 3.7 and 4.2 (1), neither of the two classes $2A$ and $2B$ of involutions in L intersect V nontrivially. □

Lemma 4.20. *L is not Ru.*

Proof. Assume $L \cong Ru$. Since $\mathrm{Out}(L) = 1$, $G = L$. By Lemma 3.3 and Lemma 4.2 (1), $V \cap 2B = \emptyset$.

Let $x \in 5B$. From [GLS3], $N := N_L(x) \cong F_{20} \times A_5$. If $a \in N$ is an involution such that $C_L(a, x) \cong A_5$ and a inverts x then $a \in 2A$ since $|C_L(t)|$ is not divisible by 3 for $t \in 2B$. Hence by 4.2 (1), $V \cap 2A = \emptyset$, contradicting $\mathrm{Inv}(L) = 2A \cup 2B$. □

Lemma 4.21. *Assume (G, V) is a minimal counter example to Theorem 2, such that $L = F^*(G)$ is sporadic. Then L is F_5, F_2, F_1, or one of the three Fischer groups F_{22}, F_{23}, or F_{24}.*

Proof. The remaining 20 sporadics were eliminated in earlier lemmas in this section. □

5. Some subgroups of the Monster

In this section M is the Monster F_1. In addition to our usual appeals to [GLS3], we also appeal to information contained in [GMS]. Our notation for the Fischer groups comes from [A9].

First by [GLS3]:

Lemma 5.1. *M has two classes $2A$ and $2B$ of involutions, and for $a \in 2A$, $C_M(a) \cong 2F_2$ is quasisimple, while for $z \in 2B$, $C_M(z) \cong Co_1/D_8^{12}$.*

Lemma 5.2. *Let $a \in 2A$. Then:*

(1) *For $b \in 2A$, $|ab| \leq 6$.*

(2) *Let $3A_a = \{b \in 2A : C_M(a,b) \cong F_{23}\}$. Then $3A_a \neq \emptyset$ and $C_M(a)$ is transitive on $3A_a$.*

(3) *Let $b \in 3A_a$, $x := ab$, and $H := C_M(x)$. Then $x \in 3A$, $H \cong F_{24}/\mathbb{Z}_3$ is quasisimple, and for $t \in \mathrm{Inv}(H)$ with $C_H(t) \cong \mathrm{Aut}(F_{22})/\mathbb{Z}_2$, $t \in 2A$.*

(4) *$2B \leadsto 41A$, and for $y \in 41A$, $\langle y \rangle$ is self centralizing in M.*

(5) *Let $w \in 5A$. Then $C_L(w) = \langle w \rangle \times K$, where $K \cong F_5$. Further for $t \in \mathrm{Inv}(K)$ with $C_K(t) \cong \mathrm{Aut}(HS)/\mathbb{Z}_2$, we have $t \in 2A$.*

Proof. Part (1) is a consequence of 3.4.9 and 3.7 in [GMS]. Part (2) is 3.4.8 in [GMS].

Let $x_0 \in 3A$ and $H_0 := C_M(x_0)$. From [GLS3] it follows that $H_0 \cong F_{24}/\mathbb{Z}_3$ and there is $a_0 \in \mathrm{Inv}(M)$ inverting x_0 and inducing a 3-transposition on H_0. Thus $C_M(x_0, a_0) \cong F_{23}$, so $a_0 \in 2A$ and $a_0 x_0 \in 3A_{a_0}$. Thus the first two remarks in (3) hold. The third remark appears in 3.6.3 of [GMS].

Let $y \in 41A$. From [GLS3], $\langle y \rangle$ is self centralizing in M and inverted by some involution i. By (1), $i \notin 2A$, so $i \in 2B$, establishing (4).

Finally, if $w \in 5A$ then $C_L(w) = \langle w \rangle \times K$ by [GLS3], where $K \cong F_5$. Pick t as in (5) and let $J := C_M(t)$ and $J^* := J/\langle t \rangle$. Then $C_{J^*}(w^*) = \langle w^* \rangle \times C_K(t)^*$, so from the centralizers of elements of order 5 in Co_1, $t \notin 2B$. Thus $t \in 2A$, establishing (5). □

Lemma 5.3. *Let $L \cong F_2$, $a \in 2A$, $b \in 2B$, and $d \in 2D$. Then:*

(1) *a^L is a class of $\{3,4\}$-transpositions of L.*

(2) *$a \leadsto x \in 3A$ with $C_L(a,x) \cong \mathrm{Aut}(F_{22})$.*

(3) *$b \leadsto y \in 5A$ with $C_L(b,y) \cong \mathrm{Aut}(HS)$.*

(4) $\mathrm{Inv}(O_2(C_L(b))) \subseteq 2A \cup 2B \cup 2D$.

(5) *$d \leadsto z \in 19A$ with $C_L(d,z) = \langle u \rangle$, where $u \in 2A$.*

Proof. Part (1) is well known, see [S, (3.14)], and it appears with (2) in the Notes for F_2 in [GLS3]. Let $y \in 5A$. By [GLS3] $N_L(y) \cong F_{20} \times \mathrm{Aut}(HS)$, so there is $i \in \mathrm{Inv}(L)$ inverting y with $C_L(i, y) \cong \mathrm{Aut}(HS)$. By (1), $i \notin 2A$, so as $11 \notin \pi(C_L(i))$ for $i \in 2C \cup 2D$, (3) holds.

Part (4) appears in 3.6.3 in [S]. Let $z \in 19A$. From [GLS3], $N_L(z) = \langle u \rangle \times F$, where u is an involution and F is Frobenius of order $19 \cdot 18$. As $19 \in \pi(C_L(u))$, $u \in 2A$. Let $i \in \mathrm{Inv}(F)$; we claim that $\{i, iu\} \cap 2D \neq \emptyset$. Suppose not and let $Y := C_L(u)$. If $i \in Y'$, then by 3.6.2 in [S], $\{i, iu\} = \{2A, 2B\}$, while if $i \in Y - Y'$, then $\{i, iu\} = \{2A, 2C\}$ by 3.18.1 in [S]. In either case we get a contradiction from (1). Thus (5) holds. □

Lemma 5.4. *Let $L \cong F_5$ and $A = \mathrm{Aut}(L)$. Then:*

(1) *Let c be an involution in $A - L$. Then $c \rightsquigarrow x \in 3A$ and $N_A(x) = \langle c, x \rangle \times S$ with $S \cong S_9$.*

(2) *Let $s \in S$ be a transposition. Then $a = cs \in 2A$ inverts x and $C_L(a, x) \cong S_7$.*

(3) *$2B \rightsquigarrow 11A$, and for $y \in 11A$, $C_L(y) = \langle y \rangle \times \langle u \rangle$, with $u \in 2A$.*

Proof. Let $x \in 3A$. By [GLS3], $N_L(x)$ is of index 2 in $S_3 \times S_9$. Hence by a Frattini argument, either $N_A(x) = T \times S$ with $T \cong S_3$ and $S \cong S_9$, or $C_A(C_L(x))$ is of order 2. The former holds as L is transitive on involutions in $A - L$, and for such an involution c, $C_L(c) \cong S_{10}$. Thus (1) holds. Then by (1), $C_L(a, x) \cong S_7$, so as $C_L(b)$ has no A_7-section for $b \in 2B$, (2) holds.

By 5.2 (5), we may take $L = K \leq M$ as in that lemma, and pick t as in the lemma. By the lemma, t is in the M-class $2A$, so by 5.2 (1), $|tt^l| \leq 6$ for $l \in L$. But by construction, t is in the L-class $2A$, so t inverts no element of L of order 11. Then (3) follows from [GLS3]. □

Lemma 5.5. *Assume (G, V) is a minimal counter example to Theorem 2, such that $L = F^*(G)$ is sporadic. Then L is one of the three Fischer groups F_{22}, F_{23}, or F_{24}.*

Proof. Assume otherwise. Then by 4.21, L is F_1, F_2, or F_5. Suppose that $L \neq G$. If L is F_1 or F_2, then $\mathrm{Out}(L) = 1$, so $L \cong F_5$. But by 5.4 (1) and 4.2 (1), $c \notin V$, for any involution $c \in G - L$, contradicting $G = LV$. Thus $G = L$.

Let I be the set of involutions fused into $O_2(C_L(b))$ for $b \in 2B$ if $L = F_2$, and let $I = \mathrm{Inv}(L)$ otherwise. By 4.4 and 1.10 (2), $V \cap I \neq \emptyset$. Then by 5.2-5.4, for $i \in I$, either there exists x of odd prime order with $i \rightsquigarrow x$ and $O_2(C_L(i, x)) = 1$, or $L \cong F_5$ and $i \in 2B$, or $L \cong F_2$ and $i \in 2D$. We conclude from 4.2 (1) that $L \cong F_5$ or F_2 and $V^\# \subseteq 2B$ or $2D$, respectively. Now by 5.4 (3) and 5.3 (5), $i \rightsquigarrow y \in 11A$ or $19A$ with $C_L(y, i) = \langle u \rangle$, and $u \in 2A$, for $L \cong F_5$ or F_2, respectively. Now 4.2 (2) supplies a contradiction and completes the proof. □

6. The Fischer groups

In this section L is one of the three Fischer groups F_n, $n \in I = \{22, 23, 24\}$. Then $L = F^*(M)$, where $M = M(n)$ is a group generated by a set D of 3-transpositions. Indeed $M = L$ unless $n = 24$, where $|M : L| = 2$ and $M = \mathrm{Aut}(L)$.

In addition to our usual appeals to [GLS3], we also appeal to the description of the 3-transposition group M in [A9]. Further we adopt Fischer's standard notation for 3-transposition groups. For example if $d \in D$, then $d^\perp = C_D(d)$, $D_d = d^\perp - \{d\}$, and $A_d = D - d^\perp$. A subgroup K of M is a D-subgroup if $K = \langle K \cap D \rangle$.

Lemma 6.1. (1) *M has k classes of involutions, j_m, $1 \le m \le k$, where $k = 3$ with $n = 22$ or 23, $k = 4$ if $n = 24$, and j_m consists of the products of m commuting members of D.*

(2) *$j_m = 2A, 2B, 2C$ for $m = 1, 2, 3$ if $n = 22$ or 23, while $j_m = 2C, 2A, 2D, 2B$, for $m = 1, 2, 3, 4$, if $n = 24$.*

(3) *For $d \in D$ and $a \in A_d$, $x = ad \in 3A$, and $F^*(C_{\mathrm{Aut}(L)}(x))$ is a simple orthogonal group over $\mathbb{F}_3$.*

(4) *If $n = 24$, then $j_4 \rightsquigarrow 29A$ and for $y \in 29A$, $\langle y \rangle$ is self centralizing in M.*

(5) *If $n = 22$, then $j_3 \rightsquigarrow 13A$ and for $z \in 13A$, $\langle z \rangle$ is self centralizing in* $\mathrm{Aut}(L)$.

(6) *Let $n = 22$ and $u \in j_2$. Then $u \rightsquigarrow w \in 5A$ with $C_{\mathrm{Aut}(L)}(u, w) = C_L(u, w) \times \langle t \rangle$, where $C_L(u, w) \cong S_5$ and $t \in 2D$ with $\langle t^{C_L(u)} \rangle$ not elementary abelian.*

(7) *If $n = 24$, then $b \in j_2$ inverts $w \in 5A$ with $C_M(b, w) \cong S_9$.*

Proof. Part (1) is 37.4 in [A9]. Then (2) follows from the tables in Chapter 15 of [A9]. Part (3) is part of the standard theory of 3-transposition groups; cf. 15.11 and 15.14 in [A9].

Assume $n = 24$. Then by 5.2 (3), the covering group $\hat{L}$ of L is embedded as a subgroup of the Monster F in such a way that an involution $j \in \hat{L}$, whose image in L is in j_2, is in the F-class $2A$. Thus by 5.2 (1), $|jj^l| \le 6$ for $l \in L$. Thus $y \in 29A$ is not inverted by j. But from [GLS3], there is an involution $i \in L$ with $i \rightsquigarrow y$, so as $\mathrm{Inv}(L) \subseteq j_2 \cup j_4$, $i \in j_4$. Further $\langle y \rangle$ is self centralizing in $\mathrm{Aut}(L)$, so (4) holds.

Suppose $W \le M$ has order 5. Then $N_M(W) = U \times Y$ where U is Frobenius of order 20 and $Y \cong S_9$, with $Y \cap D$ the set of transpositions in Y. In particular the product b of two commuting members of $Y \cap D$ inverts w' of order 5 in Y, and $N_M(W) \cap C_M(w', b) = U \times C_Y(w', b)$, with $C_Y(w', b) \cong S_5$. Thus b centralizes $O^{2'}(C_M(w'))$, so (7) holds.

Assume next that $n = 22$. By 39.1 in [A9], there is a D-subgroup K of L isomorphic to $\Omega_7(3)$. Let W be the natural $\mathbb{F}_3 K$-module and X a Levi factor of the maximal parabolic of K stabilizing a totally singular 3-subspace of W. Then $X \cong L_3(3)$ stabilizes a decomposition $W = W_0 \oplus W_1 \oplus W_2$, where W_0 is a point and W_1 and W_2 are totally singular 3-subspaces. Now X contains $z \in 13A$ and $N_X(z)$ contains g of order 3 such that W_i is a Jordan block of size 3 for g for $i = 1, 2$. Then, in the notation of Section 38 of [A9], $W\theta/W\theta^2$ is of rank 2 with singular points $W_i\theta/W_i\theta^2$, so g is in the class 3^2_+ of K appearing in 38.15 in [A9]. Then by 39.7 in [A9], $C_D(g) = \emptyset$. But for $d \in D$ and $b \in D_d$, $db \in j_2$ and $O^2(C_L(db)) \leq C_L(d)$, so $\mathrm{Inv}(C_L(g)) \subseteq j_3$. Therefore as g centralizes an involution i inverting z, it follows that $j_3 \not\rightsquigarrow 13A$, and (5) follows.

Next for $t \in 2D$, $C_L(t)$ is a D-subgroup isomorphic to $\mathrm{Aut}(\Omega_8^+(2))$. Let $L_t \leq C_L(t)$ with $L_t \cong O_8^+(2)$ and let U be the natural module for L_t. Write $U = U_1 \perp U_2$ where the U_i are nondegenerate 4-subspaces of sign -1. Then the stabilizer in L_t of U_1 is $L_1 \times L_2$ where $L_i = C_{L_t}(U_{3-i}) \cong O(U_i) \cong O_4^-(2)$. In particular there is w_i of order 5 in L_i inverted by $u_i = a_i b_i$, the product of commuting 3-transpositions, so $u_i \in j_2$. Then $N_{L_t}(u_i) = N_L(u_i) \cong F_{20} \times S_5$. Therefore $C_{\mathrm{Aut}(L)}(u_i, w_i) = C_L(u_i, w_i) \times \langle t \rangle$. Finally suppose $T = \langle t^{C_L(u_i)} \rangle$ is elementary abelian. Then $T = \langle t, a_i, b_i \rangle \cong E_8$ as $\langle a_i, b_i \rangle$ is the maximal normal elementary abelian subgroup of $C_L(u_i)$. But then $C_L(u_i)^\infty \leq C_L(t)$, a contradiction. This establishes (6). □

Lemma 6.2. *If (G, V) is a minimal counter example to Theorem 2, then L is not F_{22}, F_{23}, or F_{24}.*

Proof. Assume $L \cong F_n$ for some $n \in I$. By 6.1 (3) and Lemma 4.2 (1), $D \cap V = \emptyset$.

Suppose $n = 23$. Then for $d \in D$, d is 2-central in G and $\langle d \rangle = O_2(C_L(d))$, so $V \cap D \neq \emptyset$ by 1.10 (2), contrary to the previous paragraph. Therefore $n = 22$ or 24. By 4.4, $V \cap L \neq 1$, so as $V \cap D = \emptyset$, it follows from 6.1 that $V \cap X \neq \emptyset$ for $X \in \{j_2, j_3\}$, $X \in \{j_2, j_4\}$ for $n = 22$, 24, respectively. Then it follows from parts (4), (5), and (7) of 6.1 and 4.2 (1), that $n = 22$ and $V \cap L^\# \subseteq j_2$. Then 6.1 (6) and 4.3 (2) supply our final contradiction. □

Notice that Lemma 5.5 and Lemma 6.2 complete the proof of Theorem 4.1.

7. Maximal parabolics

In this section p is a prime and L is a group of Lie type in characteristic p of Lie rank $l \geq 1$. That is $L/O_p(L)$ is the central product of factors which are quasisimple groups of Lie type and characteristic p, $(S)L_2(3)$ if $p = 3$, or $L_2(2)$, $(S)U_3(2)$, ${}^2B_2(2)$, D_{10}, or ${}^2F_4(2)'$ if $p = 2$. Write $R(P)$ for the unipotent radical of a parabolic P of L. Thus $R(P) = O_p(P)$.

Lemma 7.1. *If $L = O^{p'}(L)$ then L is not contained in a group of Lie type of characteristic p and Lie rank less than l.*

Proof. Assume the embedding $L \leq G$ is a counter example with l minimal. Then the Lie rank k of G is less than l. If $l = 1$, then $k = 0$, so $O^{p'}(G) = O_p(G)$, impossible as $L = O^{p'}(L)$ is of Lie rank 1, so $O_p(L) \neq L$. Thus $l > 1$. Passing to $G/R(G)$, we may assume G is reductive.

Let P be a maximal parabolic of L, $R = R(P)$, and $X = O^{p'}(P)$. By the Borel–Tits Theorem (cf. 3.1.3 in [GLS3]), there exists a proper parabolic Q of G such that $P \leq Q$ and $R \leq R(Q)$. Set $Q^* = Q/R(Q)$ and $Y = O^{p'}(Q)$. Then Y^* and X^* are of Lie type and characteristic p. Further $X^* = O^{p'}(X^*)$ is reductive and as P is maximal, the Lie rank of X^* is $l - 1$. As $l > 1$, $l - 1 \geq 1$. Further the Lie rank r of Y^* is at most $k - 1$, so $r < l - 1$. This is contrary to the minimal choice of l, so the lemma is established. □

For $X \leq L$, let $\mathcal{P}(X)$ be the set of proper parabolic subgroups of L containing X.

Lemma 7.2. *Assume L is reductive and $l > 1$. Let $X = O^{p'}(X)$ be a subgroup of L of Lie type and Lie rank $l - 1$. Then:*

(1) *For each $P \in \mathcal{P}(X)$, P is a maximal parabolic and $O_p(X) \leq R(P)$.*

(2) *If $O_p(X) \neq 1$, then $\mathcal{P}(X) = \{P(X)\}$ is a set of size one, $P(X)$ is a maximal parabolic, and $O_p(X) \leq R(P(X))$.*

(3) *If $|\mathcal{P}(X)| > 1$, then $O_p(X) = 1$, and for all distinct $P, Q \in \mathcal{P}(X)$, P and Q are opposite maximal parabolics.*

Proof. Suppose $P \in \mathcal{P}(X)$ and set $Y = O^{p'}(P)$, $R = R(P)$, and $Y^* = Y/R$. Then $X^* = O^{p'}(X^*)$ is of Lie rank $l - 1$ and Y^* is of Lie rank at most $l - 1$, so by 7.1, Y^* is of rank $l - 1$, so P is a maximal parabolic. Suppose $O_p(X^*) \neq 1$. Then by Borel–Tits, there is a proper parabolic P_0 of P such that $X \leq P_0$ and $S := O_p(X) \leq R(P_0)$. But then $O^{p'}(P_0/R(P_0))$ is of Lie rank less then $l - 1$, contrary to 7.1. Thus $O_p(X^*) = 1$, so $O_p(X) \leq R$, establishing (1).

Next suppose (2) fails, and choose a counter example X such that S is maximal (recall $S = O_p(X)$). Let $K = N_L(S)$. By Borel–Tits, there exists $P \in \mathcal{P}(K)$ with $S \leq R = R(P)$. To complete the proof of (2), it suffices to show $\mathcal{P}(X) = \{P\}$, so assume otherwise and let $Q \in \mathcal{P}(X) - \{P\}$ and $T = R(Q) \cap P$. Then X acts on T, so $XT = O^{p'}(XT)$ is of Lie type with P, Q distinct members of $\mathcal{P}(XT)$. Therefore $T = S$ by maximality of S.

Suppose $S = R(Q)$. By (1), P and Q are maximal parabolics, so as $S = R(Q)$ we have $Q = K$. Then reversing the roles of P and Q, by symmetry $S = R \cap Q$, and as $P \neq Q$, $S \neq R$. But then $S < N_R(S) \leq R \cap Q = S$, a contradiction.

Therefore $S \neq R(Q)$, so $S < N_{R(Q)}(S) \leq R(Q) \cap P = S$. This contradiction establishes (2).

Finally suppose P, Q are distinct members of $\mathcal{P}(X)$. Then $O_p(X) = 1$ by (2). Suppose P and Q are not opposites. Then $1 \neq S := R(P) \cap Q$ is X-invariant, so

$Y = SX$ is of Lie type with $Y = O^{p'}(Y)$ and $S = O_p(Y) \neq 1$. As $P, Q \in \mathcal{P}(Y)$, this contradicts (2). Thus (3) holds. □

Let $\mathcal{T}(L)$ be the set of sets T of maximal parabolic subgroups of L such that for all distinct P, Q in T, P and Q are opposites.

Lemma 7.3. *Assume L is reductive, $l > 1$, and $X = O^{p'}(X)$ is a reductive subgroup of L of Lie type and Lie rank l. Let $\mathcal{Y}$ be the set of subgroups $Y = O^{p'}(Q)$ such that Q is a maximal parabolic of X. Then:*

(1) *For each $Y \in \mathcal{Y}$, $\mathcal{P}(Y) = \{P(Y)\}$ is a set of size one, $P(Y)$ is a maximal parabolic, and $O_p(Y) \leq R(P(Y))$.*

(2) *Suppose $T \in \mathcal{T}(X)$. Then $S = \{P(Y) : Y \in T\} \in \mathcal{T}(L)$.*

Proof. Let $Y \in \mathcal{Y}$. Then $Y = O^{p'}(Y) \leq L$ is of Lie type and Lie rank $l - 1$ with $O_p(Y) \neq 1$, so (1) follows from 7.2 (2).

Assume the hypothesis and notation of (2). Let $Q_1, Q_2 \in T$ be distinct, and set $Y_i := O^{p'}(Q_i)$. Then $Y_{1,2} := Y_1 \cap Y_2 = O^{p'}(Q_1 \cap Q_2)$, where $Q_1 \cap Q_2$ is a common Levi factor of Q_1 and Q_2. Now $P(Y_i) \in \mathcal{P}(Y_{1,2})$ for $i = 1, 2$, so by 7.2 (3), either $P(Y_1) = P(Y_2)$ or $P(Y_1)$ and $P(Y_2)$ are opposites. Thus (2) holds. □

Lemma 7.4. *Assume $X = O^{p'}(X)$ is a subgroup of L of Lie type and characteristic p of the same Lie rank as $O^{p'}(L)$. Then $R(X) \leq R(L)$.*

Proof. First $X = O^{p'}(X) \leq O^{p'}(L)$, and if $O^{p'}(L) = R(L)$ the lemma is trivial, so we may assume the Lie rank of $O^{p'}(L)$ is positive. Thus replacing L by $O^{p'}(L)$, we may assume $L = O^{p'}(L)$, so by hypothesis, X is of Lie rank l. Similarly replacing L by $L/R(L)$, we may assume L is reductive and $R = R(X) \neq 1$, and it remains to derive a contradiction.

By the Borel–Tits Theorem, there is $P \in \mathcal{P}(X)$. Now applying 7.1 to X in the role of L, we have a contradiction. □

8. Chev (p), p odd

In this section we assume G is an almost simple finite group such that $L = F^*(G) \in \mathrm{Chev}(p)$ is of Lie type and odd characteristic p, and $G = LV$ for some nontrivial elementary abelian 2-subgroup V of G.

Let $\mathcal{S} = \mathcal{S}(L, V)$ be the set of nonempty V-invariant sets S of parabolic subgroups of L, such that for all distinct $P, Q \in S$, P and Q are opposites. In particular $O_p(P) \cap Q = 1$. Let $\widehat{\mathcal{S}} = \widehat{\mathcal{S}}(L, V)$ consist of those $S \in \mathcal{S}$ such that for some $P \in S$, $N_V(P) \neq 1$.

Lemma 8.1. *Suppose $S \in \mathcal{S}$ and there exists $P \in S$ with $N_V(P) \neq 1$. Then there exists $(h, v) \in \mathcal{B}(G, V)$ with h a p-element, so $(G, V) \notin \mathcal{P}$.*

Proof. Assume otherwise and let R be the radical of P and $U = N_V(P)$. Then U is faithful on R, so (e.g. by 1.14 and 1.6) there is $(h, u) \in \mathcal{B}(R, U)$, and h is a p-element. As $(h, u) \notin \mathcal{B}(G, V)$ there is $1 \neq v \in C_V(h)$. As $(h, u) \in \mathcal{B}(R, U)$, $v \notin U$, so $P \neq P^v$. Then $h = h^v \in R \cap R^v$, impossible as $P \neq P^v$, so $R \cap R^v = 1$ as P and P^v are opposites. □

Lemma 8.2. *Assume L is of Lie rank* 1. *Then:*

(1) *The set S of Borel subgroups of L is in $\mathcal{S}$.*

(2) *If $m_2(V) > 2$, then $S \in \widehat{\mathcal{S}}$.*

Proof. Part (1) is trivial. Suppose $S \notin \widehat{\mathcal{S}}$, and let $\mathcal{I}$ be the set of involutions in $\mathrm{Aut}(L)$ acting on no member of S. Then $V^\# \subseteq \mathcal{I}$. But if L is ${}^2G_2(q)$ or $U_3(q)$, then $\mathcal{I}$ is empty. Hence $L \cong L_2(q)$. Therefore $\mathcal{I}$ consists of the involutions in L if $q \equiv -1 \mod 4$, while $\mathcal{I}$ consists of the involutions inducing outer-diagonal automorphisms on L if $q \equiv 1 \mod 4$. In particular, $m_2(V) \leq m_2(L) = 2$, completing the proof of (2). □

Theorem 8.3. (1) $\mathcal{S}(L, V) \neq \emptyset$.

(2) *If $m_2(V) \geq 3$, then $\widehat{\mathcal{S}}(L, V) \neq \emptyset$.*

(3) $(G, V) \notin \mathcal{P}$.

In the remainder of the section assume that (G, V) is a counter example to Theorem 8.3 with G of minimal order. Observe that (2), 8.1, and 1.4 imply (3), so (1) or (2) fails.

As (1) or (2) fails it follows from 8.2 that:

Lemma 8.4. *The Lie rank of L is greater than* 1.

Let $V \leq T \in \mathrm{Syl}_2(G)$. See [A2] for the definition of the set of *fundamental subgroups* of L, and the set $\mathrm{Fun}(T)$ of fundamental subgroups of L associated to T. By 8.4 and [A2], $\mathrm{Fun}(T)$ is nonempty and T-invariant. Also, from [A2] we have:

Lemma 8.5. *Let $K \in \mathrm{Fun}(T)$. Then:*

(1) *$K \cong \mathrm{SL}_2(q)$, where L is defined over $\mathbb{F}_q$.*

(2) *$\mathrm{Syl}_p(K)$ consists of centers of long root subgroups of L. Thus for $X \in \mathrm{Syl}_p(K)$, $N_L(X)$ is a proper parabolic of L.*

(3) *If L is not $L_n(q)$ with $n > 2$, then for distinct $X, Y \in \mathrm{Syl}_p(K)$, $N_L(X)$ and $N_L(Y)$ are opposite maximal parabolics.*

(4) *If L is $L_n(q)$ with $n > 2$, then for distinct $X, Y \in \mathrm{Syl}_p(K)$, $M(X)$ and $M(Y)$ are opposite maximal parabolics, where $M(Z)$ is the stabilizer in L of the center of Z in the action of L on its projective geometry.*

Lemma 8.6. *V acts on no member of* $\mathrm{Fun}(T)$.

Proof. Assume V acts on $K \in \mathrm{Fun}(T)$ and set $S = \{M(X) : X \in \mathrm{Syl}_p(K)\}$, where $M(X)$ is defined in 8.5 (4) if L is $L_n(q)$, and $M(X) = N_L(X)$ otherwise. By 8.5, $S \in \mathcal{S}$, and if $m_2(V) > 2$, then $1 \neq N_V(M(X))$ for some $X \in \mathrm{Syl}_p(K)$ by 8.2, so $S \in \hat{\mathcal{S}}$. But this contradicts the choice of V. □

Lemma 8.7. *L is not $L_3^{\varepsilon}(q)$, $G_2(q)$, ${}^3D_4(q)$, or $E_7(q)$.*

Proof. Assume otherwise. Then by Theorem 2 in [A2], either T acts on some member of $\mathrm{Fun}(T)$, or $L \cong G_2(q)$, with q an odd power of 3, and $V \not\leq L$. By 8.6, the latter holds. Let $v \in V - L$ and $Y = C_L(v)$. Then $Y \cong {}^2G_2(q)$. For $Q \in \mathrm{Syl}_3(Y)$ let $B(Q) = N_L(O^{3'}(C_L(Z(Q))))$. Then $B(Q)$ is a Borel subgroup of L. Further for $P \in \mathrm{Syl}_3(Y) - \{Q\}$, $B(P)$ and $B(Q)$ are opposites. Namely $A = R(B(Q)) \cap R(B(P))$ centralizes $\langle Z(Q), Z(P)\rangle = Y$, so the remark follows as $C_G(Y) = \langle v\rangle$. Thus $S = \{B(Q) : Q \in \mathrm{Syl}_3(Y)\} \in \hat{\mathcal{S}}$, contrary to the choice of V. □

Lemma 8.8. *L is not $L_n^{\varepsilon}(q)$ or $\mathrm{PSp}_n(q)$ for $n > 4$.*

Proof. Assume otherwise and let M be the natural module for $\hat{L} = \mathrm{SL}_n^{\varepsilon}(q)$ or $\mathrm{Sp}_n(q)$. For $K \in \mathrm{Fun}(T)$, let $\hat{K}$ be the preimage of K in $\hat{L}$ and $\tilde{K} = O^{p'}(\hat{K})$. Then $\tilde{K}$ is a fundamental subgroup of $\hat{L}$, and we abuse notation and write K for $\tilde{K}$. Let $\Delta = \{K_1, \ldots, K_m\}$ be an orbit of V on $\mathrm{Fun}(T)$. By 8.6, $m > 1$. Let U be a complement to the kernel of the action of V on Δ, $D = \langle\Delta\rangle$, and $J = O^{p'}(C_D(U))$. From [A2], $[M, D] = M_1 \perp \cdots \perp M_m$ and $M_i = [M, K_i]$ is the natural module for K_i. Then $J \cong \mathrm{SL}_2(q)$ is a full diagonal subgroup of D with $[M, J] = [M, D]$, and for $X \in \mathrm{Syl}_p(J)$, $[M, X] = [M_1, X] \perp \cdots \perp [M_m, X]$ is an m-dimensional totally singular subspace of M. Set $P(X) = N_L([M, X])$. Then $P(X)$ is a parabolic with $N_G(X) \leq N_G(P(X))$. Further for distinct $X, Y \in \mathrm{Syl}_p(J)$, $[M, X] \cap C_M(Y) = 0$, so $P(X)$ and $P(Y)$ are opposites.

As V acts on D and V is abelian, V acts on J. Thus $S = \{P(X) : X \in \mathrm{Syl}_p(J)\} \in \hat{\mathcal{S}}$, contrary to the choice of V. □

Lemma 8.9. *L is not $P\Omega_n^{\varepsilon}(q)$ for $n \geq 5$.*

Proof. Assume otherwise, and let L be the image in $\mathrm{PGL}(M)$ of the orthogonal group $\hat{L}$ acting on its natural module M, regarded as an orthogonal space over $\mathbb{F}_q$. Let $\Gamma = \Gamma O(M)$ be the group of semilinear maps preserving the orthogonal space M,

$G^*(M)$ the group of similarities of M, and $GO(M)$ the group of isometries of M. These groups are discussed in Section 15 of [A1]. Further (cf. 15.1 in [A1]) as $G = LV$ for some 2-group V, G is the image of some $\widehat{G} \leq \Gamma$ in $P\Gamma(M)$. As in the proof of the previous lemma, we abuse notation and identify each fundamental subgroup K of L with the fundamental subgroup of $\hat{L}$ mapping onto K under the natural map. We recall some facts from Section 15 of [A1].

Let $K \in \mathrm{Fun}(T)$ and $z = z(K)$ the involution in K. Then there exists a unique $K' \in \mathrm{Fun}(T) - \{K\}$ with $z(K') = z(K)$. Indeed $[M, z]$ is a 4-dimensional space of sign $+$ and $KK' = O^{p'}(O([M, z]))$.

Suppose $v \in G$ is an involution with $K^v = K'$, and let v be the image of $\hat{v} \in \Gamma$. Then $O^{p'}(C_{KK'}(v)) = J \cong L_2(q)$. Let $X \in \mathrm{Syl}_p(J)$.

Assume first that $\hat{v} \in G^*(M)$. Then from 15.8 in [A1], the action of v on KK' agrees with that of some involution in $GO(M)$, and J acts as $\Omega_3(q)$ on $[M, z]$. Thus $[M, z] = [M, J] \perp M_0$ is the orthogonal direct sum of a nonsingular point M_0 with a 3-dimensional orthogonal space $[M, J]$, X is a short root subgroup of L, and $M(X) := C_{[M,J]}(X) = [M, X, X]$ is a singular point of M, so $P(X) = N_L(M(X))$ is a parabolic subgroup of L with $N_G(X) \leq N_G(P(X))$. Further for distinct $X, Y \in \mathrm{Syl}_p(J)$, $M(X)$ is not orthogonal to $M(Y)$, so $P(X)$ and $P(Y)$ are opposites.

So assume instead that $\hat{v} \notin G^*(M)$. Then $\hat{v} = \sigma t$ where σ induces an involutory field automorphism on K and K', and t is a reflection with $K^t = K'$. In this case J acts as $\Omega_4^-(q^{1/2})$ on the $\mathbb{F}_{q^{1/2}}$-subspace $C_{[M,z]}(\hat{v})$ of $[M, z]$, so $M(X) := C_M(X)$ is again a singular point of M, and $P(X)$ has the same properties as in the previous case.

Let $\Delta = \{K_1, \ldots, K_m\}$ be an orbit of V on $\mathrm{Fun}(T)$. By 8.6, $m > 1$. Let U be a complement to the kernel of the action of V on Δ, $D = \langle\Delta\rangle$, and $J = O^{p'}(C_D(U))$. There are two cases:

(i) For $1 \leq i \leq m$, $K_i' \notin \Delta$.

(ii) For $1 \leq i \leq m$, $K_i' \in \Delta$.

From the discussion above, $[M, D] = M_1 \perp \cdots \perp M_r$, where $r = m$ in case (i), and in case (ii), $r = m/2$ and we can choose notation so that $K_i' = K_{i+r}$, where the indices are read modulo r. Further $M_i = [M, K_i]$ is a 4-dimensional orthogonal space of sign $+1$, and the sum of two natural modules for K_i. In case (ii), $M_i = [M, K_i']$.

In case (i), $J \cong \mathrm{SL}_2(q)$ is a full diagonal subgroup of D with $[M, J] = [M, D]$, and for $X \in \mathrm{Syl}_p(J)$, $[M, X] = [M_1, X] \perp \cdots \perp [M_m, X]$ is a $2m$-dimensional totally singular subspace of M. Set $P(X) = N_L([M, X])$. Then $P(X)$ is a parabolic with $N_G(X) \leq N_G(P(X))$. Further for distinct $X, Y \in \mathrm{Syl}_p(J)$, $[M, X] \cap [M, Y]^\perp = 0$, so $P(X)$ and $P(Y)$ are opposites.

In case (ii), there is $w \in U$ such that for each $1 \leq i \leq r$, $K_i^w = K_i'$. Let $J_i = O^{p'}(C_{K_i K_i'}(w))$. From our earlier discussion, $J_i \cong L_2(q)$. Let $E = \langle J_i : 1 \leq i \leq r\rangle$. Then J is a full diagonal subgroup of E. Let $X \in \mathrm{Syl}_p(J)$, X_i the projection of X on J_i, and write $M(X_i)$ for the singular point of M_i defined above. Then $M(X) = M(X_1) \perp \cdots \perp M(X_r)$ is an r-dimensional totally singular subspace of M. Set $P(X) = N_L(M(X))$. Again $N_G(X) \leq N_G(P(X))$, and for distinct $X, Y \in \mathrm{Syl}_p(J)$, $P(X)$ and $P(Y)$ are opposites.

As V acts on D and V is abelian, V acts on J. Thus $S = \{P(X) : X \in \mathrm{Syl}_p(J)\} \in \mathcal{S}$. Then as $1 \neq U \leq N_G(P(X))$, $S \in \hat{\mathcal{S}}$, contrary to the choice of V. This completes the proof of the lemma. □

Lemma 8.10. *L is $F_4(q)$, $E_6^{\varepsilon}(q)$, or $E_8(q)$.*

Proof. By 8.3, the Lie rank of L is greater than 1. Hence if L is $L_n^{\varepsilon}(q)$ or $\mathrm{PSp}_n(q)$, then by 8.7, $n \geq 4$. Then by 8.8, $n = 4$. Thus L is $P\Omega_m^{\varepsilon}(q)$ for $m = 5$ or 6, contrary to 8.9. Thus if L is classical, then L is $P\Omega_n^{\varepsilon}(q)$ for some $n > 6$, contrary to 8.9. Therefore L is exceptional. Now the lemma follows from 8.7. □

We recall from Definition 4.1.8 in [GLS3] that an involution $z \in L$ is of *parabolic type* if $O^{p'}(C_L(z)) = O^{p'}(P \cap P')$ for some pair P, P' of opposite maximal parabolics. Also z is of *equal rank type* if $O^{p'}(C_L(z))$ has the same Lie rank as L.

Lemma 8.11. (1) *L has a unique class z^L of 2-central involutions.*

(2) *Let $L_z = C_L(z)$. Then either*

(i) *L is $F_4(q)$ or $E_8(q)$, z is of equal-rank type, L_z is quasisimple, $\langle z \rangle = Z(L_z)$, and L_z is $\mathrm{Spin}_9(q)$ or $\Omega_{16}^{+}(q)$, respectively.*

(ii) *L is $E_6^{\varepsilon}(q)$, z is of parabolic type, and $O^{p'}(L_z) \cong \mathrm{Spin}_{10}^{\varepsilon}(q)$.*

(3) *We may assume $V \leq C_G(z)$.*

Proof. Parts (1) and (2) are a consequence of Theorem 4.5.1 in [GLS3]. By (1), z is 2-central in G, so (3) follows. □

Lemma 8.12. *L is not $E_6^{\varepsilon}(q)$.*

Proof. Assume otherwise. Choose z as in 8.11 and let $L_z = C_L(z)$. Thus $V \leq G_z = C_G(z)$. By 8.11 (2), z is of parabolic type. Let $S = \mathcal{P}(O^{p'}(L_z))$, in the notation of Section 7. Then S is V-invariant, and by 7.2 (3), $S \in \hat{\mathcal{S}}$, contrary to the choice of V. □

Lemma 8.13. *L is not $F_4(q)$ or $E_8(q)$.*

Proof. Assume otherwise. Choose z as in 8.11 and let $L_z = C_L(z)$. Thus $V \leq G_z = C_G(z)$. By 8.11 (2), z is of equal-rank type. By minimality of $|G|$, there is $S_z \in \hat{\mathcal{S}}(VL_z, V)$. Indeed from 8.11 (2i), L_z is $\mathrm{Spin}_9(q)$ or $\Omega_{16}^{+}(q)$, while from the treatment of these groups in 8.9, the members of S_z are maximal parabolics of L_z. Thus for $Y \in S_z$, $\mathcal{P}(Y) = \{P(Y)\}$ is a set of size one by 7.3 (1). Set $S = \{P(Y) : Y \in S_z\}$. By 7.3 (2), $S \in \hat{\mathcal{S}}$, contrary to the choice of V. □

Observe that 8.10, 8.12, and 8.13 complete the proof of Theorem 8.3. Further, Theorem 8.3 implies Theorem 3, and deals with those minimal counter examples to Theorem 2 whose generalized Fitting subgroup is of Lie type and odd characteristic.

9. Classical groups in characteristic 2

In this section $q \geq 2$ is a power of 2, and M is an n-dimensional vector space over $F := \mathbb{F}_q$.

Given an involution $u \in \mathrm{GL}(M)$ and $a \in F$, define

$$U(a) := 1 + a(u + 1) \in \mathrm{End}(M),$$

and set

$$Rt(u) = Rt_{\mathrm{GL}(M)}(u) = \{U(a) : a \in F\}.$$

Observe that:

Lemma 9.1. (1) *For each involution $u \in \mathrm{GL}(M)$, the map $a \mapsto U(a)$ is an isomorphism of the additive group of F with the subgroup $Rt(u)$ of* $\mathrm{GL}(M)$.

(2) $Rt(u) \leq Z(C_{\mathrm{GL}(M)}(u))$.

As in Section 4 of [ASe], for $1 \leq l \leq n/2$, write j_l for the set of all involutions $u \in \mathrm{GL}(M)$ such that $\dim([M, u]) = l$. From 4.1 in [ASe]:

Lemma 9.2. *j_l, $1 \leq l \leq n/2$, is the set of conjugacy classes of involutions in* $\mathrm{SL}(M)$.

Pick $1 \leq l \leq n/2$ and let V, W be F-spaces of dimension 2, l, respectively. Set $K := \mathrm{SL}(V)$, $L_1 := \mathrm{GL}(W)$, $M_1 := V \otimes W$, and let $\pi : K \times L_1 \to \mathrm{GL}(M_1)$ be the tensor product representation. Let u be an involution in K and U the radical of the Borel subgroup of K containing u. Observe that $Rt_{\mathrm{GL}(V)}(u) = U$, and that an easy calculation from linear algebra shows $U\pi = Rt_{\mathrm{GL}(M_1)}(u\pi)$.

Regard M_1 as a subspace of M and choose a complement M_2 to M_1 in M. Extend π to a representation $\rho : K \times L_1 \to \mathrm{GL}(M)$ by decreeing that $g\rho_{|M_2} = 1$ for $g \in K \times L_1$. Then as $U\pi = Rt_{\mathrm{GL}(M_1)}(u\pi)$, also $U\rho = Rt_{\mathrm{GL}(M)}(u\rho)$. Identify K and L_1 with their images in $\mathrm{GL}(M)$ under the injection ρ. Then K centralizes L_1, and each of these subgroups centralizes

$$L_2 := C_{\mathrm{GL}(M)}(M_1) \cap N_{\mathrm{GL}(M)}(M_2).$$

By construction, $u \in j_l$. Further, from 4.3 in [ASe], $L_u := L_1 L_2 \cap \mathrm{SL}(M)$ is a Levi factor of $C_{\mathrm{SL}(M)}(u)$. Next $C_{\mathrm{GL}(M)}(K)$ acts on $M_1 = [M, K]$ and $M_2 = C_M(K)$, so $C_{\mathrm{SL}(M)}(K) = L_2 C_G(K)$, where $G := C_{\mathrm{GL}(M)}(M_2) \cap N_{\mathrm{GL}(M)}(M_1)$. By 27.14 in [A4], $L_1 = C_{\mathrm{GL}(M_1)}(K)$. Together with 9.1 (2) these observations imply:

Lemma 9.3. *$u \in j_l$ and $(K, U) \in \mathfrak{S}(\mathrm{SL}(M), u)$.*

Recall that $\mathfrak{S}(Y, i)$ was defined in the Introduction, for Y a group of Lie type in characteristic 2, and i an involution in Y. Indeed $C_{\mathrm{GL}(M)}(L_2) = Z(L_2)G$, and by 27.14 in [A4], $C_{\mathrm{GL}(M_1)}(L_1) = KZ(\mathrm{GL}(M_1))$, so:

Lemma 9.4. *$KZ(\mathrm{SL}(M)) = C_{\mathrm{SL}(M)}(L_u)$.*

Therefore

$$K = K(u) := O^{2'}(C_{\mathrm{SL}(M)}(L_u))$$

is canonically defined by u, up to conjugation in $C_{\mathrm{SL}(M)}(u)$.

Let $A := \mathrm{Aut}(F)$ and pick a basis B_X for each $X \in \{V, W, M_2\}$. Then $B = B_V \otimes B_W \cup B_{M_2}$ is a basis for M, and if $\Sigma := \{\sigma_a : a \in A\}$ is the group of field automorphisms of $\mathrm{GL}(M)$ determined by B then Σ acts on K, L_1, and L_2, and we may choose notation so that Σ centralizes u. Further Σ induces a group of field automorphisms on K, L_1, and L_2.

Similarly let τ be the transpose-inverse map on $\mathrm{GL}(M)$ determined by B. Then τ is a graph automorphism of $\mathrm{SL}(M)$ acting on K, L_1 and L_2, and τ induces an inner automorphism on K, and graph automorphisms on L_1 and L_2. We conclude:

Lemma 9.5. *$C_{\mathrm{Aut}(\mathrm{SL}(M))}(K) = \langle t \rangle L_u$, where $t \in \tau L_u$ induces a graph automorphism on L_u.*

Given a form f on a vector space N, write $O(N, f)$ for the isometry group of the form.

Next let f_X be a bilinear or sesquilinear form on X for $X \in \{V, W, M_2\}$. Then $f_1 = f_V \otimes f_W$ is the unique form on M_1 such that

$$f_1(v_1 \otimes w_1, v_2 \otimes w_2) = f_V(v_1, v_2) f_W(w_1, w_2),$$

for $v_i \in V$ and $w_i \in W$ (cf. Section 9 in [A3]). Indeed by parts (2) and (3) of 9.1 in [A3], if f_X is symplectic, unitary, for $X \in \{V, W\}$, then so is f_1. Further if f_X is symmetric or unitary and nondegenerate then by 9.1.1 in [A3], so is f_1. If f_V is symplectic, then

$$f_1(v \otimes w, v \otimes w) = f_V(v, v) f_W(w, w) = 0 \cdot f_W(w, w) = 0,$$

so f_1 is symplectic. Let $f_M = f_1 + f_{M_2}$, and observe by construction M_1 is nondegenerate and $M_2 = M_1^{\perp}$. We have shown:

Lemma 9.6. (1) *If f_V, f_W, and f_{M_2} are unitary, then so is f_M.*

(2) *If f_V and f_{M_2} are symplectic, and f_W is symmetric and nondegenerate, then f_M is symplectic.*

(3) *In either case M_1 is nondegenerate and $M_2 = M_1^{\perp}$.*

If f_V is symplectic let $K_f = K$, and observe that $K = \mathrm{Sp}(V, f_V)$ preserves f_V. If $q = q_0^2$ is a square and f_V is unitary, let $K_f = K \cap O(V, f_V)$. Then $K_f \cong L_2(q_0)$ and we choose notation so that $u \in K_f$. Let $L_{1,f} = L_1 \cap O(W, f_W)$ and $L_{2,f} = L_2 \cap O(M_2, f_{M_2})$. Then by 9.2 in [A3]:

Lemma 9.7. *K_f, $L_{1,f}$, and $L_{2,f}$ are subgroups of $O(M, f_M)$.*

First assume f_M is unitary. Then $O(M, f_M) = \mathrm{GU}(M)$ is the general unitary group on M, and $\mathrm{SU}(M) = \mathrm{SL}(M) \cap \mathrm{GU}(M)$ is the special unitary group. We abuse notation and write j_m for $j_m \cap \mathrm{GU}(M)$. By 6.1 in [ASe]:

Lemma 9.8. *j_l, $1 \le l \le n/2$, is the set of conjugacy classes of involutions in $\mathrm{SU}(M)$, and $u \in j_l$.*

Let $U_f := U \cap K_f$. Thus U_f is a Sylow 2-subgroup of $K_f \cong L_2(q_0)$ and from 9.1 (2), $U_f = Rt_{\mathrm{GU}(M)}(u)$, so $U_f \le Z(C_{\mathrm{GL}(U)}(u))$. By construction, K_f centralizes $L_{i,f}$ for $i = 1, 2$, $L_{1,f} = \mathrm{GU}(W, f_W) \cong \mathrm{GU}_l(q_0)$, and $L_{2,f} = \mathrm{GU}(M_2, f_{M_2}) \cong \mathrm{GU}_{n-2l}(q_0)$. Then arguing as in the proof of 9.3, using 6.2 in [ASe] in place of 4.3 in [ASe], we conclude:

Lemma 9.9. *If $u \in j_l$, then $(K_f, U_f) \in \mathfrak{S}(\mathrm{SU}(M), u)$.*

Further $\mathrm{GU}(M)$ is the centralizer in $\mathrm{GL}(M)$ of the graph-field automorphism $\tau\sigma$, where σ is the involution in Σ. Hence $\Sigma \cong \mathrm{Out}(\mathrm{SL}(M))$, $\langle\sigma\rangle = C_\Sigma(K_f)$, $\Sigma/\langle\sigma\rangle$ induces the group of field automorphisms on K_f, and (in the language of 2.5.13 in [GLS3]) σ induces a graph automorphism on $\mathrm{SU}(M)$.

Next assume f_V, f_W, and f_{M_2} satisfy the hypothesis of 9.6 (2), so that f_M is symplectic by that lemma. Then $O(M, f_m) = \mathrm{Sp}(M)$ is the symplectic group on M, and in particular contained in $\mathrm{SL}(M)$. From 7.7 in [ASe]:

Lemma 9.10. *The set of conjugacy classes of involutions in $\mathrm{Sp}(M)$ is a_l, c_l, b_k, $1 \le l, k \le n/2$, l even, k odd.*

The notation is explained in Section 7 of [ASe]. In particular $\dim([i, M]) = l$ for $i \in x_l$ and $x \in \{a, b, c\}$. Thus $u \in x_l$ for some $x \in \{a, b, c\}$. Moreover $x \in \{b, c\}$ iff there exists $x \in M$ such that $f_M(x, xu) \ne 0$.

Suppose $x_2 \in V - C_V(u)$ so that $x_2 u = x_2 + x_1$ for some $x_1 \in [V, u]$ and $f_V(x_1, x_2) \ne 0$. Let $w \in W$. Then $(x_2 \otimes w)u = x_2 \otimes w + x_1 \otimes w$, and $f_M(x_2 \otimes w, (x_2 \otimes w)u) \ne 0$ iff

$$0 \ne f_M(x_2 \otimes w, x_1 \otimes w) = f_V(x_2, x_1) f_W(w, w),$$

so as $f_V(x_1, x_2) \ne 0$, $u \in a_l$ iff $f_W(w, w) = 0$ for all $w \in W$ iff f_W is symplectic.

We now make a choice of the form f_W. We just saw that $u \in a_l$ iff f_W is symplectic. Further $L_{2,f} = O(M_2, f_{M_2}) = \mathrm{Sp}(M_2) \cong \mathrm{Sp}_{n-2l}(q)$, and $L_{1,f} = O(W, f_W) = \mathrm{Sp}(W) \cong \mathrm{Sp}_l(q)$ as f_W is symplectic. Then arguing as in the proof of 9.3, using 7.9 in [ASe] in place of 4.3 in [ASe], we conclude:

Lemma 9.11. *If $u \in a_l$, then $(K, U) \in \mathfrak{S}(\mathrm{Sp}(M), u)$.*

To obtain elements of b_l and c_l we choose f_W to have an orthonormal basis B_W. Then f_W is not symplectic, so by an earlier remark, $u \in b_l$, c_l for l odd, even, respectively. Set $z := \sum_{w \in B_W} w$. If l is odd then z is nonsingular and $W = z \oplus z^\perp$ with $z^\perp$ symplectic, so $L_{1,f} \cong O(W, f_W) \cong \mathrm{Sp}(z^\perp) \cong \mathrm{Sp}_{l-1}(q)$. Then arguing as in the proof of 9.3, using 7.10 in [ASe] in place of 4.3 in [ASe], we conclude:

Lemma 9.12. *If $u \in b_l$, then $(K, U) \in \mathfrak{S}(\mathrm{Sp}(M), u)$.*

Finally if l is even, then z is singular with $z^\perp/\langle z \rangle$ symplectic, so $O(W, f_W) \cong \mathrm{Sp}_{l-2}(q)/E_{q^{l-1}}$, and hence $O_2(C_{\mathrm{Sp}(M)}(K)) \neq 1$. Thus in this case $(K, U) \notin \mathfrak{S}(\mathrm{Sp}(M), u)$, so we must look elsewhere for members of $\mathfrak{S}(\mathrm{Sp}(M), u)$ when $u \in c_l$. However as we are only interested in proving Theorem 4, we need only consider involutions in c_2.

Take $l = 2$ and let E be a quadratic extension of F. As in Section 7 of [A3], there is an E-structure M_E on M_1 and a sympletic form f_E on M_E, such that $f_V = Tr_F^E \circ f_E$. Let $K_E = O(M_E, f_E) \cong \mathrm{Sp}_2(E) \cong L_2(q^2)$. Then from 7.2.6 in [A3], $K_E \leq \mathrm{Sp}(M_1)$. For u_E an involution in K_E, $\dim([M, u_E]) = 2$ and u_E inverts an element of order $q^2 + 1$, so $u_E \notin a_2$ since the root elements in a_2 only invert elements of odd order dividing $q^2 - 1$. Therefore $u_E \in c_2$, so we may take $u \in K_E$. Indeed if $Rt_{\mathrm{SL}(M_E)}(u) = \{U_E(e) : e \in E\}$ then $U = Rt_K(u) = \{U_E(a) : a \in F\} \leq K_E$. Further K_E is irreducible on $M_1 = [M, K_E]$, so $C_{\mathrm{Sp}(M)}(K_E) = L_{2,f} \cong \mathrm{Sp}_{n-4}(q)$. Finally if $q = 2$ and $u \in D \leq K_E$ with $D \cong D_{10}$, then $\mathrm{Sp}(M_1) \cong S_6$ so $C_{\mathrm{Sp}(M_1)}(D) = 1$, and hence $C_{\mathrm{Sp}(M)}(D) = L_{2,f}$. Then, as usual, arguing as in the proof of 9.3, using 7.11 in [ASe] in place of 4.3 in [ASe], we conclude:

Lemma 9.13. *If $u \in c_2$, then $(K_E, U) \in \mathfrak{S}(\mathrm{Sp}(M), u)$.*

If $n \neq 4$, then from [GLS3], $\Sigma \cong \mathrm{Out}(\mathrm{Sp}(M))$. Further Σ induces a group of field automorphisms on K, K_E, $L_{1,f}$, and $L_{2,f}$. When $n = 4$, $\mathrm{Out}(\mathrm{Sp}(M))$ is cyclic with Σ of index 2, and for $t \in \mathrm{Aut}(\mathrm{Sp}(M))$ whose image is not in Σ, t is nontrivial on the Dynkin diagram of $\mathrm{Sp}(M)$.

Finally we consider the orthogonal groups. Let Q be a quadratic form on M with associated symplectic form f_M. Then $O(M, Q) = O^\varepsilon(M)$, where ε is the sign of Q. Note that in particular, $O^\varepsilon(Q) \leq \mathrm{Sp}(M)$. Now n is even, and we may assume $n \geq 6$.

Let $G := \Omega^\varepsilon(M)$ be the commutator group of $O^\varepsilon(M)$. Then $G \cong \Omega_n^\varepsilon(q)$ is simple of index 2 in $O^\varepsilon(M) \cong O_n^\varepsilon(q)$. We abuse notation and write x_l for the intersection $x_l \cap O^\varepsilon(M)$ of the class x_l in $\mathrm{Sp}(M)$ with $O^\varepsilon(M)$, for $x \in \{a, b, c\}$. Then from 8.5, 8.11, and 8.12 in [ASe]:

Lemma 9.14. (1) *The set of conjugacy classes of involutions in $O^\varepsilon(M)$ is a_l, c_l, b_k, $1 \le l, k \le n/2$, l even, k odd, with $\varepsilon = +1$ for $a_{n/2}$.*

(2) *a_l and c_l are contained in $\Omega^\varepsilon(M)$, but b_k is not.*

(3) *a_l and c_l are classes in $\Omega^\varepsilon(M)$, except that $a_{n/2}$ splits into two $\Omega^+(M)$-classes, with t and s conjugate in $\Omega^+(M)$ iff $[M, s]$ and $[M, t]$ are conjugate.*

In Theorem 4 we are only concerned with involutions in the simple group $\Omega^\varepsilon(M)$, which by 9.14 are of type a_l and c_l. Further in Theorem 4 we do not need to consider c_l for $l > 2$.

First we may take u of type a_l; hence l is even and f_W is symplectic. By 9.1.4 in [A3], there is a unique quadratic form Q_1 on M_1 with associated bilinear form f_1 such that $Q_1(v \otimes w) = 0$ for all $v \in V$ and $w \in W$. By 9.5 in [A3], (M_1, Q_1) is of sign $+1$. Pick a quadratic form Q_2 on M_2 of sign ε, and let $Q = Q_1 + Q_2$. Then Q is a quadratic form on M of sign ε with symplectic form f_M. Note if $n = 2l$, then $Q = Q_1$ is of sign $+1$, which is forced by 9.14 (1). By 9.2.4 in [A3], $K \times L_{1,f} \cong L_2(q) \times \mathrm{Sp}_l(q)$ preserves Q and hence is contained in $\Omega^\varepsilon(M)$. Let $L_{2,Q} = \Omega(M_2, Q_2) \cong \Omega_{n-2l}^\varepsilon(q)$, so that $L_{2,Q} \le \Omega^\varepsilon(M)$ centralizes K and $L_{1,f}$. As usual, arguing as in the proof of 9.3, using 8.6 in [ASe] in place of 4.3 in [ASe], we conclude:

Lemma 9.15. *If $u \in a_l$, then $(K, U) \in \mathfrak{S}(O^\varepsilon(M), u)$.*

This time $\mathrm{Aut}(\Omega^\varepsilon(M))$ is $\Gamma O^\varepsilon(M)$, which is $O^\varepsilon(M)$ extended by Σ, unless $\dim(M) = 8$ and $\varepsilon = +1$, where $\mathrm{Aut}(\Omega^\varepsilon(M)) = \Gamma O^\varepsilon(M)\langle\xi\rangle$, for a graph automorphism ξ of order 3. Further Σ induces field automorphisms on K, $L_{1,f}$ and $L_{2,Q}$, and when $l = 2$, a suitable transvection in $O^\varepsilon(M)$ centralizes $KL_{1,f}$ and induces a transvection on $L_{2,Q}$. Thus for $u = a_2$, $O_2(C_{\Gamma O^\varepsilon(M)}(K)) = 1$.

Lemma 9.16. *Let $u \in c_2$, M_* a nonsingular point of $[M, u]$, and $Y := N_G(M_*)$. Assume $q > 2$ or $n > 6$.*

(1) *There is $(K_*, U_*) \in \mathfrak{S}(Y, u)$.*

(2) *$(K_*, U_*) \in \mathfrak{S}(G, u)$.*

(3) *$[M, K_*]$ is 3-dimensional with f_M-radical M_*, and for $x \in M = M_*^\perp$, $M_x = \langle M_*, x\rangle$ is a nondegenerate 4-dimensional orthogonal space, such that for each involution $t_x \in O^\varepsilon(M)$ inducing a transvection on M_x with center M_*, t_x centralizes K_*.*

(4) $C_{\Gamma O^{\varepsilon}(M)}(K_* C_G(K_*)) = \langle t_* \rangle$, *where* $t_* \in O^{\varepsilon}(M)$ *is the transvection with center* M_*.

Proof. First, Y acts faithfully as $\mathrm{Sp}(\widetilde{M}) \cong \mathrm{Sp}_{n-2}(q)$ on $\widetilde{M} := M_*^{\perp}/M_*$. In particular u is of type b_1 in Y, so (1) follows from 9.12, and from the proof of that lemma, $Y_* = C_Y(K_*) \cong \mathrm{Sp}_{n-4}(q)$.

Next $C_G(u) \leq G_0$, where G_0 is the stabilizer in G of the unique singular point M_0 in $[M, u]$. Further $R := R(G_0)$ is abelian and $C_G(u) = RL_0$ where $L_0 \cong \mathrm{Sp}_{n-4}(q)$. Thus $Y_* \cong L_0$, so $U_* \leq C_R(Y_*) = C_R(L_0) = Z(C_G(u))$. Let $Y_1 := C_G(K_*)$. Then $R_1 := O_2(Y_1)$ is Y_*-invariant, so as $C_G(u) = Y_* R$, $R_1 \leq R$. Then as $q > 2$ or $n > 6$, each nontrivial Y_*-submodule of R intersects U_* nontrivially. Thus either $R_1 = 1$ or $U_1 = U_* \cap R_1 \neq 1$. In the latter case $K_* = \langle U_1^{K_*} \rangle \leq R_1$, a contradiction. Thus (2) holds.

Further by construction, the first remark in (3) holds. Thus for $x \in M = M_*^{\perp}$, $M_x = \langle M_*, x \rangle$ is a nondegenerate 4-dimensional orthogonal space, and for each involution $t_x \in O^{\varepsilon}(M)$ inducing a transvection on M_x with center M_*, t_x centralizes the stabilizer K_* of M_* in $\Omega(M_x)$, completing the proof of (3). Then (4) follows from an easy calculation. □

Observe that, collecting the results in this section, we have shown:

Theorem 9.17. *Theorem* 4 *holds when* $F^*(Y)$ *is a classical group.*

We close this section with two results useful in the proof of Theorem 2.

Lemma 9.18. *Assume* $G = O^{\varepsilon}(M)$ *with* $n \geq 6$, *and let* $u \in c_2$. *Then* $u \in D \leq G$ *with* $D \cong D_{2(q^2+1)}$ *such that* $C_{\Gamma O^{\varepsilon}(M)}(D) \cong O_{n-4}^{-\varepsilon}(q)$.

Proof. Let $M = M_1 \perp M_2$ where M_1 is a nondegenerate 4-dimensional subspace of sign -1. Then $M_2 = M_1^{\perp}$ is of dimension $n - 4$ and sign $-\varepsilon$, and $K_1 := C_G(M_2) \cong O(M_1, Q) \cong O_4^-(q)$ is $L_2(q^2)$ extended by a field automorphism. Further $\mathrm{Inv}(E(K_1)) \subseteq c_2$, so we may take $u \in D \leq K_1$ with $D \cong D_{2(q^2+1)}$. As D is irreducible on M_1, $C_G(D) = C_G(M_1) \cong O(M_2, Q) \cong O_{n-4}^{-\varepsilon}(q)$. Hence the lemma holds. □

Lemma 9.19. *Assume* $L = \mathrm{SL}(M) \cong L_n(2)$ *with* $n \in \{5, 6\}$. *Let* $G = \mathrm{Aut}(L)$ *and* $\tau \in G$ *a graph automorphism of* L *such that* $C_L(\tau) \cong \mathrm{Sp}_4(q)$, $\mathrm{Sp}_6(q)$, *for* $n = 5, 6$, *respectively. Then* $\tau \in D \leq G$ *such that* $D \cong D_{2m}$, *and:*

(1) *If* $n = 5$, *then* $m = 31$ *and* $C_G(D) = 1$.

(2) *If* $n = 6$, *then* $m = 7$ *and* $C_G(D) \cong L_2(8)$.

Proof. Let E be a field extension of $F = \mathbb{F}_2$ of degree r, where $r = 5, 3$, for $n = 5, 6$, respectively. Then M admits the structure M_E of an n/r-dimensional E-space, and the stabilizer Y in L of that structure is isomorphic to $\Gamma L(M_E)$.

Suppose first that $n = 5$. Then Y is $X \cong E^\# \cong \mathbb{Z}_{31}$ extended by $\mathrm{Aut}(E) \cong \mathbb{Z}_5$. Further $X \in \mathrm{Syl}_{31}(L)$ and $Y = N_L(X)$, so by a Frattini argument, $G = LN_G(X)$. By 19.8 and 19.9 in [ASe], all involutions in $G - L$ are conjugate to τ. Then as $31 \notin \pi(C_L(\tau))$, some conjugate of τ inverts X, so that (1) holds.

So assume $n = 6$. Here $Y_0 := O^2(Y) = X \times W$, with $X \cong E^\# \cong \mathbb{Z}_7$ and $W = E(Y) \cong L_2(8)$. Further Y is Y_0 extended by an element f of order 3 inducing a field automorphism on X and W. Now L is transitive on such subgroups, and $\mathrm{Aut}(W) = W\langle f\rangle$, so by a Frattini argument, there exists an involution t inverting X and centralizing $W\langle f\rangle$. Finally from 19.8 and 19.9 in [ASe], all involutions in $G - L$ centralizing an $L_2(8)$-subgroup of L are conjugate to τ, so (2) holds. □

10. Chev (2)

In this section we assume the following hypothesis:

Hypothesis 10.1. G is an almost simple finite group with $F^*(G) = L$ a group of Lie type $G(q)$ over a field of even order q, or L is the Tits group and $q = 2$. Let Φ be a root system for L and l the Lie rank of L. Given $\alpha \in \Phi$, let $\widehat{U}_\alpha$ be the root group of α, $U_\alpha = \Omega_1(\widehat{U}_\alpha)$, $K(\alpha) = \langle U_\alpha, U_{-\alpha}\rangle$, and $X(\alpha) = C_L(K(\alpha))$.

Lemma 10.2. *Assume $l > 1$ and let $\alpha \in \Phi$. Assume either α is a long root, or $L \cong \mathrm{Sp}_{2l}(q)$ and α is a short root. Set $K = K(\alpha)$, $X = X(\alpha)$, and pick $u \in U_\alpha^\#$. Then:*

(1) *Either $K \cong L_2(q)$ or $L \cong {}^2F_4(q)$ and $K \cong Sz(q)$, or L is the Tits group and $K \cong D_{10}$. In any event $H_K = N_K(U_\alpha) \cap N_K(U_{-\alpha}) \cong \mathbb{Z}_{q-1}$ is a Cartan subgroup of the Borel subgroup $N_K(U_\alpha)$ of K.*

(2) *$P = P(\alpha) = N_L(U_\alpha)$ is a parabolic subgroup of L, and if L is not $L_n(q)$ then P is a maximal parabolic.*

(3) *$X \times H_K = P \cap P(-\alpha)$ is a Levi factor of P.*

(4) *$C_L(u) = R(P)X$ and $(K, U_\alpha) \in \mathfrak{S}(L, u)$.*

(5) *Assume $G = LV$ for some elementary abelian 2-group V. Then either*

  (i) *$O_2(C_G(K)) = 1$, or*

  (ii) *$L \cong L_3(q)$ with $q \le 4$, or $L_4(2)$, and $O_2(C_G(K)) = \langle\tau\rangle$, where τ induces a graph automorphism on L.*

Proof. Parts (1) and (2) are well known; for instance see Example 3.2.6 in [GLS3].

Let $P' = P(-\alpha)$. We claim that P and P' are opposite parabolics. Namely conjugating in the Weyl group of L, we may assume α is the root of highest height, or $L \cong \mathrm{Sp}_{2l}(q)$ and α is the short root of highest height. Then, in the notation of Section 47 of [A4], and in particular in the notation before 47.4 in [A4], $P = P_J$ for some subset J of the set π of simple roots for some ordering of Φ, $R(P) = V_J = \langle U_\gamma : \gamma \in \psi_J\rangle$, and $L_J = \langle U_\beta, U_{-\beta} : \beta \in \psi_J\rangle$. Then $P' = V_{-J}L_{-J}$ and $L_{-J} = L_J$, while $V_J \cap V_{-J} = 1$, so using the Bruhat decomposition (cf. 47.2 in [A4])

$$\begin{aligned} P \cap P' &= L_J H_K V_J \cap L_J H_K V_{-J} = L_J H_K (L_J H_K V_J \cap V_{-J}) \\ &= L_J H_K (V_J \cap V_{-J}) = L_J H_K = Y, \end{aligned}$$

where $Y := L_J H_K$ is a Levi factor of P and P'. This establishes the claim.

As $U = U_\alpha$ and $U' = U_{-\alpha}$ are normal in P and P', respectively, Y acts on $K = \langle U, U'\rangle$. Further $H_K \leq Y$ and $H_K \cong \mathrm{Aut}_L(U)$, so $Y = H_K C_Y(U)$. Then as $C_{\mathrm{Aut}(K)}(U) \cap N_{\mathrm{Aut}(K)}(U') = 1, C_Y(U) = C_Y(K) = Y \cap X$. Thus $Y = H_K(X \cap Y)$. On the other hand $X \leq N_L(U) \cap N_L(U') = P \cap P' = Y$, so $X = X \cap Y$, completing the proof of (3).

By choice of α and the Chevalley commutator relations, U is in the center of the Sylow 2-subgroup $S = \langle U_\beta : \beta \in \Phi^+\rangle$ of L. Thus (cf. 47.7 in [A4]) setting $L_u := O^{2'}(C_L(u))$, $L_u H$ is a parabolic subgroup of L. By construction, $P = R(P)XH_K \leq L_u H$, so if P is maximal then $P = L_u H$ and $L_u = C_P(u) = R(P)X$. If P is not maximal, then $L \cong L_n(q)$ by (2), where it is well known that $U = Rt_L(u)$ so that $P = L_u H$ and $L_u R(P)X$. Therefore $U \leq Z(C_L(u))$, so (4) follows from (3).

We next prove (5), so assume $G = LV$ for some elementary abelian 2-group V. We adopt the terminology in 2.5.13 of [GLS3] when discussing involutory automorphisms of L. As $O_2(X) = 1$, $O_2(C_L(K)) = 1$ by (4), so we may assume $V \not\leq L$. If G contains a field or graph-field automorphism (excluding L of type B_2 or F_4), then some involutory field or graph-field automorphism σ acts as a field automorphism on K, and hence $C_{L\langle\sigma\rangle}(K) = X$. Therefore (5) (i) holds unless G contains a graph automorphism τ (or graph-field in the case of B_2 and F_4). Therefore we may assume L is $L_n^\varepsilon(q)$, $\mathrm{Sp}_4(q)$, $D_m^\varepsilon(q), m \geq 4$, $F_4(q)$, or $E_6^\varepsilon(q)$. If L is $\mathrm{Sp}_4(q)$ or $F_4(q)$ then $N_{L\langle\tau\rangle}(K) \leq L$, and hence (5) (i) holds. Thus we may assume we are in one of the remaining cases, where some involutory graph automorphism τ centralizes K.

Suppose $L \cong L_3^\varepsilon(q)$. As $l > 1$, $\varepsilon = +1$. Then $K = C_L(\tau)$, so τ inverts X and (5) (i) holds unless $X = 1$. In that event $q \leq 4$ and (5) (ii) holds.

In the remaining cases, $O^{2'}(X) = M$ is of Lie type over $\mathbb{F}_q$, and described in 12.1 in [ASe]. Further τ induces a graph automorphism on M, so τ induces an outer automorphism on M unless $L \cong L_4^\varepsilon(q)$, where $M \in K^L$ is centralized by τ. In the first case (5) (i) holds, so we may assume the latter. Then from 19.9 in [ASe], $C_L(\tau) \cong \mathrm{Sp}_4(q)$, so $M = C_X(\tau)$. Therefore τ inverts $O(X)$, so (5) (i) holds unless

$O(X) = 1$. As $O(X)$ is of order $q-\varepsilon$, we conclude that (5) (ii) holds when $O(X) = 1$, completing the proof of (5). □

Lemma 10.3. *Suppose L is $Sz(q)$, let u be an involution in L, and $U = Z(R(B))$ where B is a Borel subgroup of L containing u. Then $(L, U) \in \mathfrak{S}(L, u)$.*

Proof. This is essentially immediate from the definitions. □

Lemma 10.4. *Assume L is exceptional and either $l = 2$ or L is the Tits group. Then:*

(1) *$L \cong G_2(q)$, ${}^3D_4(q)$, ${}^2F_4(q)$, or the Tits group.*

(2) *L has two classes of involutions; the long and short root involutions.*

(3) *Let $u \in U_\alpha$ be a short root involution. If L is not ${}^3D_4(q)$ set $K = K(\alpha)$ and $U = U_\alpha$. If L is ${}^3D_4(q)$ let $U = \{U_\alpha(a) : a \in \mathbb{F}_q\}$ and $K = \langle U, s_\alpha\rangle$, where s_α is the reflection through α. Then $(K, U) \in \mathfrak{S}(L, u)$.*

(4) *Let $G = LV$ for some elementary abelian 2-group V. Then $O_2(C_G(K)) = 1$.*

Proof. Part (1) follows by inspection of the list of groups of Lie type, and (2) appears in Section 18 of [ASe].

The proof of (3) is similar to that of parts (3) and (4) of 10.2. First choose α to be the short root of highest height. From Section 18 of [ASe], $C_L(u) \le P = P_J$ is a maximal parabolic of L such that $P = R(P)H_0L_J$, where $L_J \cong L_2(q)$ centralizes U, and H_0 is a Cartan subgroup of $K(\alpha)$. Then $L_J = \langle U_\beta, U_{-\beta}\rangle$ centralizes $K(\alpha) = \langle U_\alpha, U_{-\alpha}\rangle$. On the other hand as in the proof of 10.2 (3), P and $P' = P_{-J}$ are opposite parabolics, so $C_L(K) \le P \cap P' = L_J H_0$, and hence $C_L(K) = L_J C_{H_0}(U) = L_J$. This establishes (3).

Assume the hypothesis of (4). By (3) we may assume $|\mathrm{Out}(L)|$ is even, so L is not ${}^2F_4(q)$. Similarly (4) holds for the Tits group, as its automorphism group is ${}^2F_4(2)$. Finally if L is $G_2(q)$ or ${}^3D_4(q)$ and $|\mathrm{Out}(L)|$ is even then from 19.1 in [ASe], $|G : L| = 2$ and each involution in $G - L$ is a field automorphism. In particular some field automorphism of order 2 induces a field automorphism of order 2 on K, so (4) holds. □

During much of the remainder of the paper we will assume the following hypothesis:

Hypothesis 10.5. Hypothesis 10.1 holds with L an exceptional group and $l \ge 4$; that is L is $F_4(q)$, $E_6^\varepsilon(q)$, $E_7(q)$, or $E_8(q)$. Adopt the notation on page 5 of [ASe] for labeling the simple roots α_i, $i \in I = \{1, \dots, l\}$ in Φ, and adopt the notation on page 4 of [ASe] for labeling parabolics P_S, $S \subseteq I$. In particular $Q_S = R(P_S)$ is the radical of P_S, and $L_S = O^{2'}(\bar{L}_S)$, where $\bar{L}_S$ is a Levi factor of P_S. Let ξ be the root of highest height and let $U = U_\xi$, $K = K(\xi)$, and $X = X(\xi)$. Let H_K be the Cartan subgroup of K defined in 10.2 (1).

Lemma 10.6. *Assume $\alpha \in \Phi$ is a long root and let $U = U_\alpha$, $K = K(\alpha)$, and $X = X(\alpha)$. Let $u \in U^\#$ and $t \in X$ be involutions, and $t \in K_t \leq X$ with $K \cong K_t$. Set $v = ut$, let K_v be the full diagonal subgroup of $K \times K_t$, and $Y = C_X(K_t)$. Let P be a parabolic subgroup of L such that $C_L(v) \leq P$, set $R = R(P)$, and assume that*

(1) *$O^{2'}(C_P(v)/O_2(C_R(v)))$ and $O^{2'}(Y)$ are reductive groups of Lie type and even characteristic of the same Lie rank.*

(2) *$U \leq Z(C_{R(P)}(v))$.*

Then $O_2(C_L(K_v)) = 1$.

Proof. By definition, $X = C_L(K)$, so as K_v is a full diagonal subgroup of $K \times K_t$ and $K_t \leq X$, we have $C_X(K_v) = C_X(K_t) = Y$. In particular $Y \leq H = C_L(K_v)$. Suppose $Q = O_2(H) \neq 1$. Then $UYQ \leq C_L(v) = C_P(v)$. Set $M = O^{2'}(C_P(v))$, $M^* = M/C_R(v)$ and $Y_0 = O^{2'}(Y)$. By (1), Y_0 is reductive, so $O_2(Y_0) = 1$ and hence $Y_0^* \cong Y_0$. By (1), M^* is reductive so $O_2(M^*) = 1$ and hence $O_2(C_L(v)) = C_R(v)$. Further M^* and Y_0^* have the same Lie rank by (1), so as Q^* is a Y-invariant 2-subgroup of M^*, it follows from 7.4 that $Q^* = 1$. Thus $Q \leq C_R(v)$, so by (2), u centralizes Q. Then as $C_L(u) = C_L(U)$, U centralizes Q, so $K = \langle U^{K_v} \rangle$ centralizes Q. Therefore $Q \leq C_L(K) = X$, so $Q \leq O_2(C_X(K_v)) = O_2(Y)$, contradicting Y reductive. □

11. $E_6^\varepsilon(q)$

In this section we assume Hypothesis 10.5 with $L = E_6^\varepsilon(q)$ and adopt the following notation.

Notation 11.1. Let $F = \mathbb{F}_q, \mathbb{F}_{q^2}$, for $\varepsilon = +1, -1$, respectively. Let $\hat{L}$ be the universal group of type $E_6(F)$ and M a faithful 27-dimensional $F\hat{L}$-module. The module M is described in detail in [A5]. Let $\widehat{G}$ be $\hat{L}$ extended by field automorphisms, so that $\widehat{G} \leq \Gamma L(M)$. Regard L as the image of the subgroup $\hat{L}_\sigma$ of fixed points on $\hat{L}$ of $\sigma \in \mathrm{Aut}(\hat{L})$ under the projective map, where $\sigma = 1$ if $\varepsilon = +1$ and σ is a graph-field automorphism if $\varepsilon = -1$. Let $\widehat{K}$ be a fundamental subgroup of $\hat{L}$ and $\widehat{X} = C_{\hat{L}}(\widehat{K})$. Then K, X are the image of $\widehat{K}_\sigma$, $\widehat{X}_\sigma$ in L, respectively.

Lemma 11.2. (1) $P(\xi) = P_r$, *where* $r = 2, 1$ *for* $\varepsilon = +1, -1$, *respectively. Thus* $L_1 = X \cong \mathrm{SL}_6^\varepsilon(q)$.

(2) *$Q_r \cong q^{1+20}$ is special with center U.*

(3) *If $\varepsilon = +1$, then $L_6 \cong \Omega_{10}^+(q)$ and $Q_6 \cong E_{q^{16}}$ is a spin module for L_6. Further $Q_2 \cap Q_6$ is of rank* 11, *and $L_{2,6} \cong \mathrm{SL}_5(q)$ acts naturally on $Q_6/(Q_2 \cap Q_6)$.*

(4) *If* $\varepsilon = -1$, *then* $L_4 \cong \Omega_8^-(q)$, $Q_4 \cong q^{8+16}$ *is special,* $Z(Q_4)$ *is the natural module for* L_4, *and* $Q_4/Z(Q_4)$ *is a spin module.*

Proof. From the discussion in Sections 14 and 15 in [ASe], $P_r = P(\xi)$, $Q_r \cong q^{1+20}$, and $L_r \cong L_6^{\varepsilon}(q)$. Then $L_1 = X$ by 10.2 (3) establishing (1) and (2). Part (4) follows from 4.6 in [CKS], and the first statement in (3) appears in 3.14 in [A5]. The second follows as the radical of $P_{2,6} \cap X$ is the natural module for $L_{2,6} \cong \mathrm{SL}_5(q)$, and $Q_{2,6} = Q_2 Q_6$. □

Lemma 11.3. (1) *L has three classes of involutions with representatives* u_i, $1 \le i \le 3$, *denoted by x, y, z in* 15.1 *of* [ASe] *when* $\varepsilon = +1$, *and t, u, v in* 14.1 *of* [ASe] *when* $\varepsilon = -1$, *respectively. Further* $u_1 \in U$.

(2) *Let* $m_i = \dim([M, u_i])$. *Then* $m_1 = 6$, $m_2 = 10$, *and* $m_3 = 12$.

(3) *The involutions in X are of type* j_i, $1 \le i \le 3$.

(4) u_i *is L-conjugate to a member of* j_i *for* $1 \le i \le 3$.

(5) *Let* J_i, $1 \le i \le 3$, *be commuting fundamental subgroups of X*, $J_4 = K$, *and* $\Delta = \{J_i : 1 \le i \le 4\}$. *Then* $N_L(\Delta)$ *induces* $\mathrm{Sym}(\Delta)$ *on* Δ.

(6) *Let* N_6 *be the natural module for X and for i an involution in X and* $g \in L$, *set* $Rt_L(i^g) = Rt_{\mathrm{GL}(N_6)}(i)^g$. *Then* $Rt_L(i^g)$ *is well defined and* $Rt_L(i^g) \le Z(C_L(i^g))$.

Proof. Part (1) is 14.1 and 15.1 in [ASe], while (3) follows from 11.1 (1), 9.2, and 9.8.

Define Δ as in (5), and let $X_i = C_L(J_i)$. Then $\Delta = \{J_i\} \cup (\Delta \cap X_i)$, and $N_{X_i}(\Delta)$ induces the symmetric group S_3 on $\Delta - \{J_i\}$, so (5) follows.

Assume for the moment that $\varepsilon = +1$. We abuse notation and write K, X for $\hat{K}, \hat{X}$. As observed in Section 4 of [A6], the members i of j_1 are the root involutions in X, and $Rt_L(i)$ is the root group of i in L. Then (6) holds for the involutions $i \in j_m$ for $m = 2, 3$ by 4.4 and 4.5 in [A6]. From page 167 in [A5], $[K, M]$ is the sum of six copies of the natural module for K, so $m([M, i]) = 6$ for i an involution in K, and hence $m_1 = 6$ by (5). Also $m_2 = 10$ and $m_3 = 12$ by 4.1 and 4.2 in [A6]. Thus (2) holds in this case. From 4.3 in [A6], each involution in L is conjugate to an involution in j_m for a unique m. Following Section 4 in [A6], call such involutions *involutions of type m*. The isomorphism type of the centralizers of involutions of type 2 appears in 4.6 of [A6], and comparing this to 15.5 in [ASe], the involution y in 15.1 of [ASe] is of type 2. Then as involutions of type 1 and the involution x in 15.1 of [ASe] are root involutions, it follows that the involution z in 15.1 of [ASe] is of type 3. Thus (4) holds in this case.

We have established the lemma when $\varepsilon = +1$. When $\varepsilon = -1$, we may choose σ to act on $\hat{K}$, $\hat{X}$, and $\hat{K}_i$, and then to centralize $u_m \in j_m$. Then $u_m \in \hat{L}_\sigma$ is in the class j_m in $X = \hat{X}_\sigma \cong \mathrm{SU}_6(q)$, so (2) and (4) hold when $\varepsilon = -1$, modulo verifying the correspondence between the u_m and t, u, v in (1). As $\dim([u_m, M]) \ne \dim([u_k, M])$ for $m \ne k$, u_m, $1 \le m \le 3$ are representatives for the three classes of involutions in L. The graph-field automorphism σ induces a field automorphism on $C_{\hat{L}}(u_2)/O_\infty(C_{\hat{L}}(u_2)) \cong \mathrm{Sp}_6(q^2)$, so $C_L(u_2)/O_\infty(C_L(u_2)) \cong \mathrm{Sp}_6(q)$, and hence u_2 is conjugate to the involution u in 14.3 of [ASe]. This verifies the correspondence between our two labelings of involutions, and completes the proof of (1). Part (6) follows as $Rt_L(i) = Rt_{\hat{L}}(i)_\sigma$. □

Lemma 11.4. *For* $1 \le m \le 3$, *let* $u_m \in j_m$ *and* $U_m = Rt_L(u_m)$. *Then:*

(1) *There exist* $K_m \le X$ *with* $(K_m, U_m) \in \mathfrak{S}(X, u_m)$.

(2) $C_L(u_m) = C_L(K_m)O_2(C_L(u_m))$, *so if* $O_2(C_K(u_m)) = 1$, *then* $(K_m, U_m) \in \mathfrak{S}(L, u_m)$.

(3) $O_2(C_G(K_m)) \le L$.

Proof. Part (1) was established in 9.16. In particular $K_m \cong L_2(q)$ and $u_m \in U_m \cong E_q$. By 11.3 (6), $U_m \le Z(C_L(u_m))$. Thus to establish (2), it remains to show that $C_L(K_m)$ is a supplement to $O_2(C_L(u_m))$ in $C_L(u_m)$. By 10.2 (4), we may assume $m = 2$ or 3.

First take $m = 3$. Then the discussion in Section 9 (or 4.3 and 6.2 in [ASe]) says that $C_X(K_3) \cong L_3^\varepsilon(q)$. Set $B_3 := KC_X(K_3)$. Then $B_3 \cong L_2(q) \times L_3^\varepsilon(q)$, and by 14.3 and 15.5 in [ASe], B_3 is a complement to $O_2(C_L(u_3))$ in $C_L(u_3)$, so (2) holds when $m = 3$.

Next suppose that $m = 2$. Then from the discussion in Section 9 (or 4.3 and 6.2 in [ASe]), $C_X(K_2) \cong \mathrm{GL}_2(q)\times L_2(q)$, so $B_2 := KC_X(K_2) \cong \mathrm{GL}_2(q)\times L_2(q)^2$. Let $Y = O^{2'}(C_L(u_2))$ and $Y^* = Y/O_2(Y)$. By 14.3 and 15.5 in [ASe], $Y^* \cong \mathrm{Sp}_6(q)$. Let $\Delta = \{J_i : 1 \le i \le 4\}$ be as in 11.3 (5). From the discussion in Section 9, we may take K_2 to be a full diagonal subgroup of J_1J_2. Then by 11.3 (5), there is $g \in N_L(\Delta)$ with K_2^g a full diagonal subgroup of $J_1J_4 = KJ_1$. Let $Y_0^g = C_X(K_2^g)$. Then $Y_0^g = C_X(J_1) \cong \mathrm{GL}_4^\varepsilon(q)$ from Section 9. Hence $Y_0^* \cong \Omega_6^\varepsilon(q)$, and $N_{Y^*}(Y_0^*) \cong O_6^\varepsilon(q)$ is the unique maximal overgroup of Y_0^* in Y^*. Also $B_0 = O^2(O^{2'}(B_2)) \not\le Y_0O_2(C_L(u_2))$ as B_0^g does not centralize KJ_1. Therefore $C_L(K_2)^* = Y^*$, and hence $C_L(K_2)$ is a supplement to $O_2(C_L(u_2))$ in $C_L(u_2)$, completing the proof of (2).

Suppose $\varepsilon = +1$. Then $\mathrm{Aut}(L)$ is L extended by $\Lambda \times \langle\tau\rangle$, where Λ is a group of field automorphisms and τ is a graph automorphism. Further we may choose Λ to induce a group of field automorphisms on K_m, and from 19.7 in [ASe], $C_L(\tau) \cong F_4(q)$ and we may choose τ to centralize K and induce a graph automorphism on X with $J_i \le C_X(\tau) \cong \mathrm{Sp}_6(q)$ for each $J_i \in \Delta$. Then from 9.5, τ centralizes K_m and induces a graph automorphism on $C_X(K_m)$, and on Y_0 when $m = 2$. Thus (3) holds in this case. On the other hand when $\varepsilon = -1$, L is the image of $\hat{L}_\sigma$ and $\sigma = \tau\lambda$, where λ induces an involutory field automorphism on $\hat{L}$. Then $\mathrm{Out}(L)$ is cyclic with τ inducing the involutory graph automorphism on L, so from the discussion above, τ is faithful on $C_L(K_m)$, and hence (3) holds again. □

Lemma 11.5. *$(K_2, U_2) \in \mathfrak{S}(L, u_2)$ and $O_2(C_G(K_2)) = 1$.*

Proof. By 11.4 it suffices to show that $O_2(C_L(u_2)) = 1$.

As we saw during the proof of 11.4, $K_2^g \leq KJ_1$ for some $g \in N_L(\Delta)$, so $v_2 := u_2^g = w_1 w_4$ for some $w_i \in J_i$. Suppose for the moment that $\varepsilon = +1$. Choose notation so that $w_1 \in V = R(P_{2,6}) \cap X$. From 11.2 (3), V is a complement to $Q_2 \cap Q_6$ in Q_6. Thus $v_2 \in Q_6$, and from the proof of 11.4, $C_X(K_2^g) = C_X(J_1) \cong L_4(q)$. From 15.4 and 15.5 in [ASe], $C_L(v_2) \leq P \in P_6^L$, and $O^{2'}(C_L(v_2))/O_2(C_L(v_2)) \cong \mathrm{Sp}_6(q)$. Then as $C_X(J_1)$ and $\mathrm{Sp}_6(q)$ both are of Lie rank 3, and $C_X(J_1)$ acts on $Q_6 \leq C_L(v_2)$, it follows from 7.4 that $Q_6 \leq O_2(C_L(v_2))$. Then as Q_6 is weakly closed in P_6 with respect to L, $C_L(v_2) \leq P_6$. Now 10.6 completes the proof in this case.

Finally suppose $\varepsilon = -1$. Then σ induces a field automorphism on $\widehat{K}_2$ and $O^{2'}(C_{\hat{L}}(\widehat{K}_2))$, so

$$O^{2'}(C_L(K_2)) = O^{2'}(C_{\hat{L}}(\widehat{K}_2)_\sigma) \cong \mathrm{Sp}_6(q)$$

as $\widehat{K}_2 = \langle K_2, \widehat{U}_2 \rangle$ and $\widehat{U}_2 \leq Z(C_{\hat{L}}(u_2))$. □

Lemma 11.6. *Let $s = 4, 2$ for $\varepsilon = +1, -1$, respectively, $g \in P_s - P_r$, $I = UU^g$, $E = Q_r \cap Q_r^g$, and $Y := \langle Q_r, Q_r^g \rangle$. Then:*

1. *$Q_s = (Q_r \cap Q_s)(Q_r^g \cap Q_s)$ and $Y/Q_s \cong L_2(q)$.*
2. *$L_s = (Y \cap L_s) \times Y_{r,s}$ with $Y_s := Y \cap L_s \cong L_2(q)$ and $L_{r,s} \cong \mathrm{SL}_3(q) \times \mathrm{SL}_3(q)$ or $\mathrm{SL}_3(q^2)$ for $\varepsilon = +1, -1$, respectively.*
3. *$I = Z(Q_s)$ is the natural module for Y_s.*
4. *$[Y, E] = I$ and E/I is a tensor product module of order q^9 for $L_{r,s}$.*
5. *Q_s/E is the tensor product of the natural module for Y_s and the dual of E/I for $L_{1,6}$.*
6. *There exists $v_3 \in u_3^L$ with $v_3 \in E - I$, such that $C_{Q_s}(v_3) = O_2(C_L(v_3))$, $C_{L_s}(v_3) = Y_s \times C_{L_{r,s}}(v_3)$, and $C_{L_{r,s}}(u_3) \cong \mathrm{SL}_3^\varepsilon(q)$.*
7. *$(K_3, U_3) \in \mathfrak{S}(L, u_3)$ and $O_2(C_G(K_3)) = 1$.*

Proof. Parts (1)–(5) follow from the standard theory of large special groups; see for example 8.15 in [A8]. From 15.4 and 14.2 in [ASe], there is $v_3 \in u_3^L$ such that $C_L(v_3) \leq P_s$ with $O_2(C_L(v_3)) = C_{Q_s}(v_3)$, and $O^{2'}(C_L(v_3))/O_2(C_L(v_3))$ as described in (6). From the action of L_s on Q_s described in (1)–(5), it follows that $v_3 \in E$.

Suppose for the moment that $\varepsilon = +1$ and consider the parabolic $X_4 = X \cap P_4$ of X. Then $W = R(X_4) \cong E_{q^9}$ is the tensor product module for $L_{2,4} \cong \mathrm{SL}_3(q) \times \mathrm{SL}_3(q)$. From 11.3 (4), we may pick $u_3 \in W \cap j_3$, and from Section 4 in [ASe],

$B_4 := C_{L_{2,4}}(u_3) \cong \mathrm{SL}_3(q)$ and W is the adjoint module for B_4. Then KB_4 is contained in a Levi factor $L'_{2,4}$ in $P'_4 \in P_4^L$ with $C_L(u_3) \leq P'_4$, completing the proof of (6) in this case.

Further $K_2 = O^{2'}(C_X(B_4))$, $O^{2'}(C_X(K_3)) = B_4$, and K_3 centralizes KB_4. Suppose $O_2(C_L(K_3)) = R \neq 1$. Then as R is normalized by KB_4, $R \leq O_2(C_L(u_3))$. If $1 \neq I' \cap R$, then as K is irreducible on I', $I' \leq R$. But then $K_3 \leq C_L(I') \leq P'$, contradicting $u_3 \in Q'_4$. Hence $I' \cap R = 1$. Similarly $C_W(K_3) = 1$ and W and I' contain all proper KB_4-submodules of E', so $E' \cap R = 1$. Then as KB_4 is irreducible on $Q'' = C_{Q'_4}(u_3)/E'$, it follows that $Q'' = C_{Q'_4}(K_3)E$, so $Q'' \leq C_L(K_3)$ as $E' = \Phi(Q'')$. This contradiction shows that $O_2(C_L(K_3)) = 1$, and then (7) follows from 11.4.

Finally suppose $\varepsilon = -1$. Then taking fixed points of σ on $\hat{L}$, we conclude (6) and (7) also hold in this case. □

12. $F_4(q)$

In this section we assume Hypothesis 10.5 with $L = F_4(q)$, and adopt the following notation.

Notation 12.1. As in 11.1, let $F = \mathbb{F}_q$, $\hat{L}$ be the universal group of type $E_6(F)$, and M a faithful 27-dimensional $F\hat{L}$-module. We regard L as the image of the fixed points $\hat{L}_\tau \cong Z(\hat{L}) \times F_4(q)$ of a graph automorphism τ of $\hat{L}$. Let $\widehat{K}$ be a fundamental subgroup of $\hat{L}$, $\widehat{X} = C_{\hat{L}}(\widehat{K})$, and define $\hat{\Delta} = \{\hat{J} : 1 \leq i \leq 4\}$ as in 11.3 (5). Choose notation so that τ centralizes $\hat{\Delta}$. Then τ induces a graph automorphism on $\widehat{X}$ and K, J_i, X, Δ are the images of $\widehat{K}$, $\hat{J}_i$, $\widehat{X}_\tau$, $\hat{\Delta}$ in L, respectively. Recall from 11.3 (6) that N_6 is the natural module for $\widehat{X} = \mathrm{SL}(N_6)$, and observe $X = \mathrm{Sp}(N_6)$.

Lemma 12.2. (1) *$P(\xi) = P_1$, so $L_1 = X \cong \mathrm{Sp}_6(q)$.*

(2) *$Q_1 = EJ$, where $E \cong E_{q^7}$ is the orthogonal module for $X \cong SO_7(q)$, $J \cong q^{1+8}$ is special with $[J, E] = 1$ and $Z(J) = U = C_E(X)$, and Q_1/E is the spin module for X.*

Proof. From the discussion after 13.1 in [ASe], $P_1 = P(\xi)$ and $L_1 \cong \mathrm{Sp}_6(q)$. Then $L_1 = X$ by 10.2 (3), establishing (1). Part (2) follows from 4.5 in [CKS]. □

Lemma 12.3. (1) *L has four classes of involutions with representatives u_l, u_s, u_c, v, denoted by t, u, tu, v in* 13.1 *of* [ASe], *respectively. Further $u_l \in U$ and u_s are long and short root involutions, respectively.*

(2) *The involutions in X are of type b_1, a_2, c_2, and b_3.*

(3) *The L-conjugates of u_l, u_s, u_c, v in X,are in b_1, a_2, c_2, b_3, respectively.*

(4) $N_L(\Delta)$ *induces* $\mathrm{Sym}(\Delta)$ *on* Δ.

(5) *Let* $r = r_1 \dots r_4$ *with* $r_i \in J_i$ *and* $r_i^2 = 1$. *Set* $\omega(r) = |\{i : r_i \neq 1\}|$. *Then* r *is conjugate in* L *to* u_l, u_c, v *for* $\omega(r) = 1, 2, 3$, *respectively.*

Proof. The first part of (1) is 13.1 in [ASe], while (2) is 9.10. Moreover b_1, a_2 are the long, short root involutions of X, and from Table 1 on page 5 of [ASe], these root involutions are also long, short root involutions in L, in the respective case. This completes the proof of (1).

The proof of (4) is the same as that of 11.3 (5).

Let $X_i := C_X(J_i)$, U_i a root group in J_i, $V = R(C_X(U_1))$ and $V' = R(C_{X_1}(U))$, and $Y = C_X(J_1)$. Then $N_{J_1}(U_1)VY$ and $H_K V'Y$ are parabolics in X and X_1, respectively. Set $P_1^* = P_1/Q_1$. From 12.2 (2), $V'' = C_{Q_1}(J_1) = C_E(J_1) \cong E_{q^5}$. On the other hand the Lie rank of $J_1 Y$ and L_1 is 3, so $(V')^* = 1$ by 7.4. Thus $V' \leq V''$ and hence $V' = V''$ as $|V'| = q^5 = |V''|$. Then from the representation of Y on V', the involutions in V' of type c_2 in X_1 are in the third class of 2-central involutions in P_1, i.e. those which are not root involutions. As u_c belongs to this class, it follows that $c_2 \subseteq u_c^L$.

From 12.1, $X = \widehat{X}_\tau$ with $b_1 \subseteq j_1$, $a_2, c_2 \subseteq j_2$, and $b_3 \subseteq j_3$. Define m_i as in 11.3 (2). Then as $m_3 \neq m_1$ or m_2, b_3 is in the fourth L-class v^L of involutions of L, completing the proof of (3).

Finally (5) follows from (3) and (4). □

Lemma 12.4. *For* $x \in \{u_c, v\}$ *let* $U_x = Rt_L(x)$. *Then for* $x \in \{u_c, v\}$:

(1) *There exist* $K_x \leq X$ *with* $(K_x, U_x) \in \mathfrak{S}(X, x)$.

(2) $(K_x, U_x) \in \mathfrak{S}(L, x)$.

(3) $O_2(C_G(K_x)) = 1$.

Proof. Part (1) was established in 9.16. In particular $K_v \cong L_2(q)$, $K_{u_c} \cong L_2(q^2)$, and $x \in U_x \cong E_q$. From the construction of K_x in Section 9, $U_x = Rt_{\mathrm{Sp}(N_6)}(x)$, and from 12.1, $Rt_{\mathrm{Sp}(N_6)}(x) = Rt_{\mathrm{SL}(N_6)}(x)$. Thus from 11.3 (6), $U_x = Rt_{\hat{L}}(x) \leq Z(C_L(x))$. Thus to establish (2), it remains to show that $C_L(K_m)$ is a complement to $O_2(C_L(x))$ in $C_L(x)$. Set $R := O_2(C_L(K_x))$.

First take $x = v$. Then the discussion in Section 9 (or 7.10 in [ASe]) says that $C_X(K_v) \cong L_2(q)$. Set $B_v := KC_X(K_v)$. Then $B_v \cong L_2(q) \times L_2(q)$, and by 13.3 in [ASe], B_v is a complement to $O_2(C_L(v))$ in $C_L(v)$. Thus to complete the proof of (2) in this case, we may assume $R \neq 1$ and it remains to derive a contradiction. By Borel–Tits, $N_L(R) \leq P$ a parabolic of L. As $L_2(q)^3 \cong K_v B_v \leq N_L(R)$, we conclude P is a conjugate of P_1 or P_4. Thus $K_v B_3$ centralizes some root group U_0, so

$$U_0 \leq C_L(K_v B_3) \leq C_L(K) \cap C_L(K_v C_X(K_v)) = C_X(K_v C_X(K_v)) = 1$$

as $K_v \in \mathfrak{S}(X, v)$ and $C_X(K_v) \cong L_2(q)$. This completes the proof of (2) when $x = v$.

Next take $x = u_c$. Now (cf. Table 4.1 in [GL]) there exists a subgroup Y of L isomorphic to $\mathrm{Sp}_8(q)$ and generated by root groups. Conjugating in $\mathrm{Aut}(L)$ we may assume long root groups of Y are long root groups of L, and then that K is such a root group. Then $C_Y(K) \cong \mathrm{Sp}_6(q) \cong X$, so $C_Y(K) = X$. Hence $K_x \leq X \leq Y$ and x is of type c_2 in Y by 12.3. Hence (cf. 9.13), $B_x := C_Y(K_x) \cong \mathrm{Sp}_4(q)$. Then by 13.3 in [ASe] B_x is a complement to $O_2(C_L(x))$ in $C_L(x)$. If $R \neq 1$, then by Borel–Tits, $K_x B_x \leq N_L(R) \leq P$ for some proper parabolic P of L. This is impossible as no proper parabolic has an $(L_2(q^2) \times \mathrm{Sp}_4(q))$-section. Hence (2) is established.

Finally we prove (3). By (2) we may assume $G \neq L$. From 19.3 in [ASe], $\mathrm{Out}(L)$ is cyclic, and then as $G = LV$, 19.5 in [ASe] says that $G = L\langle t\rangle$ where t is an involution inducing a field or graph-field automorphism on L, with q a square in the first case. If t is a field automorphism, then $\mathrm{Aut}_G(K_x)$ contains a field automorphism not in $\mathrm{Aut}_L(K_x)$, so $C_G(K_x) \leq L$. If t is a graph-field automorphism, then from 19.5 in [ASe], $C_L(t) \cong {}^2F_4(2)$ is of Lie rank 2, so it does not contain $K_x C_L(K_x)$ of Lie rank 3 by 7.1, so again $C_L(K_x C_L(K_x)) = 1$. □

Lemma 12.5. *For each involution $x \in L$, there exist $(K_x, U_x) \in \mathfrak{S}(L, x)$, and $O_2(C_G(K_x)) = 1$.*

Proof. The lemma follows from 10.2 if x is a long root involution, and hence also when x is a short root involution, as long and short root involutions are fused in $\mathrm{Aut}(L)$. Thus by 12.3, we may assume $x \in \{u_c, v\}$, where the lemma follows from 12.4. □

13. $E_7(q)$

In this section we assume Hypothesis 10.5 with $L = E_7(q)$.

Lemma 13.1. (1) *$P(\xi) = P_1$. Thus $L_1 = X \cong \Omega^+_{12}(q)$.*

(2) *$Q_1 \cong q^{1+32}$ is special with center U.*

(3) *$L_7 \cong E_6(q)$ and $Q_7 \cong E_{q^{27}}$ is the 27-dimensional $\mathbb{F}_q$-module for L_7, discussed in* 11.1. *Further $Q_1 \cap Q_7 \cong E_{q^{17}}$, and $L_{1,7} \cong \Omega^+_{10}(q)$ acts naturally on the complement $W = Q_7 \cap X = R(P_{1,7} \cap X)$ to $Q_1 \cap Q_7$ in Q_7.*

Proof. From the discussion in Section 16 in [ASe], $P_1 = P(\xi)$, $Q_1 \cong q^{1+32}$, and $L_1 \cong \Omega^+_{12}(q)$. Then $L_1 = X$ by 10.2 (3), establishing (1) and (2). From the Dynkin diagram for L, $L_7 \cong E_6(q)$, and then as $|E_6(q)|_2 = q^{36}$ and $|E_7(q)| = q^{63}$, $|Q_7| = q^{27}$. Then as 27 is the minimal dimension of a faithful L_7 $\mathbb{F}_q$-module, and all such modules are quasiequivalent, the first statement in (3) holds. The second follows as the radical of $P_{1,7} \cap X$ is the natural module for $L_{1,7} \cong \Omega^+_{10}(q)$. □

Lemma 13.2. (1) *L has five classes of involutions.*

(2) *P_7 controls fusion in Q_7.*

(3) *P_7 has three orbits on involutions in Q_7 with representatives u_i, $1 \leq i \leq 3$, denoted by x, y, u in* 16.1 *of* [ASe].

(4) *$u_1 \in U$ and U is a singular point in Q_7.*

(5) *u_2 and u_3 are in brilliant and dark points of Q_7, respectively.*

(6) *Let $W_1 = U$ and W_i, $i = 2, 3$ be root groups in the orthogonal space W for $L_{1,7}$ in* 13.1 (3)*, which are not orthogonal in that space. Let w_i be an involution in W_i. Then we may pick $u_1 = w_1$, $u_2 = w_1 w_2$, and $u_3 = w_1 w_2 w_3$.*

(7) *$C_L(u_3) = C_{P_7}(u_3)$ contains Q_7 and $C_L(u_3)/Q_7 \cong F_4(q)$.*

Proof. Part (1) is 16.1 in [ASe]. As the radical Q_7 is weakly closed and abelian, (2) follows from 7.7 in [A8].

By 13.1 (3), Q_7 is the 27-dimensional module for $L_7 \cong E_6(q)$. That module is described in detail in [A5]; we adopt the terminology from [A5] in discussing the module. In particular the first statement in (3) follows from 3.16.1 in [A5], which says P_7 has three orbits on the $\mathbb{F}_q$-points of the $\mathbb{F}_q$-module Q_7, namely the singular, brilliant, and dark points. The torus H_K is transitive on vectors in each point. The $\Omega_{10}^+(q)$-parabolic $P_{1,7}$ stabilizes a singular point, which is therefore U. Thus from 16.20 in [ASe], $u_1 \in U$ is the involution denoted by x in that lemma. From 13.1 (3), $Q_1 \cap Q_7$ is of dimension 17; this is the subspace $U\Delta$ of [A5]. Let $U = W_1$ and pick root groups W_i, $i = 2, 3$ in Q_7 such that $\{W_1, W_2, W_3\}$ is *special* as defined on page 164 of [A5]. In particular $W_i \not\leq W_j\Delta$ for $i \neq j$. We may choose W_i, $i = 1, 2$, to be root groups in the complement W to $U\Delta$ defined in 13.1 (3). The condition $W_3 \not\leq W_2\Delta$ is equivalent to W_2 and W_3 not orthogonal in the orthogonal space W for $L_{1,7}$. Set $u_2 = u_1 w_2$ for some involution $w_2 \in W_2$; this is the involution denoted by y in 16.20 of [ASe], since u_2 is diagonal in the product of two commuting root groups U and W_2 such that $W_2 \not\leq O_2(N_L(U))$, and (cf. the proof of 12.1 in [ASe]) L is transitive on such involutions. From [A5], u_2 is contained in a brilliant point of Q_7.

Finally let $u_3 = u_1 w_2 w_3$ for some involution $w_3 \in W_3$. From [A5], u_3 is contained in a dark point of Q_7, and hence (cf. 8.14 in [A5]) $C_{P_7}(u_3)/Q_7 \cong F_4(q)$. From 16.20 in [ASe], only the centralizer of the involution denoted by u in that lemma contains an $F_4(q)$-section, so u_3 is in that class. Then by 16.19 in [ASe], $C_L(u_3) = C_{P_7}(u_3)$.

We have verified (1)–(7), so the proof of the lemma is complete. □

Lemma 13.3. *Let M be the* 12*-dimensional orthogonal space over $\mathbb{F}_q$ for X, and $M = M_1 \perp M_2 \perp M_3$ an orthogonal decomposition with each M_i a* 4*-dimensional nondegenerate subspace of sign* -1*. For $1 \leq i \leq 3$, let J_i, J_i' be the fundamental* *subgroups of X with $M_i = [M, J_i] = [M, J_i']$, and let $\Delta = \{J_i, J_i', K : 1 \leq i \leq 3\}$, and $D = \langle \Delta \rangle$. Then:*

(1) *$D = \prod_{J \in \Delta} J$ is the direct product of members of Δ and $N_L(D)/D$ acts faithfully as $L_3(2)$ on Δ preserving a projective plane Γ on Δ in which $\{K, J_i, J_i'\}$, $1 \leq i \leq 3$ are the lines through K.*

(2) *Let $r = r_1 \dots r_7$ with $r_i \in D_i$, $\Delta = \{D_1, \dots, D_7\}$, and $r_i^2 = 1$. Set $\delta(r) = \{D_i : r_i \neq 1\}$. Then r is L-conjugate to u_1, u_2 if $|\delta(r)| = 1, 2$ respectively, and is conjugate to u_3 if $\delta(r)$ is a line in Γ.*

Proof. Working in $K \times X$, D is the direct product of the seven copies of $L_2(q)$ in Δ. Further $X = C_L(K)$ and from the structure of $X = \Omega_{12}^+(q)$, $C_X(D \cap X) = 1$, so $C_L(D) = 1$. Next $N_X(D)/(D \cap X)$ acts faithfully as S_4 on $\Delta - \{K\}$, and preserves the partition $\{\{J_i, J_i'\}: 1 \leq i \leq 3\}$, and similarly $N_L(J_1) \cap N_L(D)$ is transitive on $\Delta - \{J_1\}$, so (1) follows. Then (2) follows from (1) and the description of u_i in 13.2 (6). □

Lemma 13.4. *Pick u_3 as in* 13.2 (6) *and notation as in* 13.3. *Pick $W_2 \leq J_1$ and $W_3 \leq J_1'$. Then:*

(1) *There exists $S_3 \cong S \leq N_L(D) \cap C_L(u_3)$ faithful on $\delta := \{K, J_1, J_1'\}$ such that the involution $s \in S$ fixing K is in X.*

(2) *Set $K_3 := C_{KJ_1J_1'}(S)$ and let $u_3 \in U_3 \in$ $\mathrm{Syl}_2(K_3)$. Then $(K_3, U_3) \in \mathfrak{S}(L, u_3)$.*

(3) $O_2(C_L(KC_L(K))) = 1$.

Proof. Observe that $N_X(D)$ is the wreath product of J_1 with S_4. Then pick S to be a conjugate of an S_3-subgroup of a wreath complement to $D \cap X$ in $N_X(D)$ such that S is transitive on δ; this is possible by 13.3. This establishes (1).

Let $Y := C_L(u_3)$. By 13.2 (7), $Q_7 = O_2(Y)$ and $Y^* := Y/Q_7 \cong F_4(q)$. Next $C_X(K_3) = C_X(K_s)$, where K_s is the projection of K_3 on $J_1 J_1'$. Thus $K_s = C_{J_1 J_1'}(s)$. From 9.17, $B_K := C_X(K_s) \cong \mathrm{Sp}_8(q)$. As S is transitive on δ there is also an S-conjugate B_1 of B_X in $C_L(J_1)$ centralizing K_3, and as B_K centralizes W_1 but not W_1, $B_X^* \neq B_1^*$. Then as $B_X^* \cong \mathrm{Sp}_8(q)$ is maximal in $Y^* \cong F_4(q)$, $Y^* = \langle B_X, B_1 \rangle^* \leq Y_3^*$, where $Y_3 := C_L(K_3)$. Set $R := O_2(Y_3)$. Then $R \leq Q_7$ and $U_3 \leq C_{Q_7}(Y_3)$, so as $C_{Q_7}(Y_3)$ is of order q and Y_3 is irreducible on $Q_7/C_{Q_7}(Y_3)$, it follows that $U_3 = Z(Y)$ and either $R = 1$ or R is a complement to U_3 in Q_7. But in the latter case K_3 acts on $Q_7 = C_L(R)$, contradicting $U_3 \leq Q_7$. This completes the proof of (2).

Finally from 19.2 in [ASe], all involutions in $\mathrm{Aut}(L) - L$ are field automorphisms, and such an involution induces a field automorphism on Y^*, so (3) holds. □

Lemma 13.5. (1) *$L_6 \cong L_2(q) \times \Omega^+_{10}(q)$ and $Q_6 \cong q^{10+32}$ is special.*

(2) *$Z_6 = Z(Q_6) \cong E_{q^{10}}$ admits the structure of a 10-dimensional orthogonal space over $\mathbb{F}_q$, preserved by L_6, and L_6 induces $\Omega^+_{10}(q)$ on this space.*

(3) *We may take $u_2 = u_1w_2$ where W_2 is the root group of X such that $X_6 = X \cap P_6 = N_X(W_2)$, and $w_2 \in W_2$.*

(4) *u_2 is a nonsingular point in the orthogonal space Z_6 and $C_L(u_2) = Q_6C_{L_6}(u_2)$ with $C_{L_6}(u_2) \cong L_2(q) \times \mathrm{Sp}_8(q)$.*

(5) *Let $W_2 \leq J_1$, s an involution in $N_L(\Delta)$ with cycle (K, J_1), $K_2 = C_{KJ_1}(s)$, and $u \in U_2 \in \mathrm{Syl}_2(K_2)$. Then $(K_2, U_2) \in \mathfrak{S}(L, u_2)$.*

(6) *$O_2(C_G(K_2)) = 1$.*

Proof. Parts (1) and (2) are established during the discussion on page 60 of [ASe]. Part (3) follows from 13.2 (6). Arguing as usual, $R(X_6) \cong q^{1+12}$ is a complement to $Q_1 \cap Q_6$ in Q_6, so $W_2 = \Phi(R(X_6)) \leq \Phi(Q_6) = Z_6$. Indeed W_2 is a complement to $Q_1 \cap Z_6$ in Z_6, and $Q_1 \cap Z_6$ is the subspace of the orthogonal space Z_6 orthogonal to U. Thus $u_2 = u_1w_2$ is a nonsingular point in Z_6, so $C_{L_6}(u_2) \cong L_2(q) \times \mathrm{Sp}_8(q)$. From 13.2 (3), u_2 is the involution y of 16.1 in [ASe], so from the description of $C_L(y)$ in 16.20 of [ASe], $Y := C_L(u_2) = C_{P_6}(u_2)$. Thus (4) holds.

Adopt the notation of (5). Then $C_X(K_2) = C_X(J_1) \cong L_2(q) \times \Omega^+_8(q)$ by 10.2 (3). Then $C_X(K_2)$ and $Y^* := Y/Q_6 \cong L_2(q) \times \mathrm{Sp}_8(q)$ have the same Lie rank, so $O_2(C_L(K_2)) = 1$ by 10.6.

Next by 13.3, there is $g \in N_L(\Delta)$ with $K_2^g \leq J_1J_1'$. Then u_2^g is of type c_2 in X and $s^g \in X$ centralizes K_2^g, so by 9.14, $C_X(K_2^g) \cong \mathrm{Sp}_8(q)$. It follows from (4) that $KC_X(K_2)$ is a complement to Q_6 in $C_L(u_2)$. Further from the action of $B := C_{L_6}(u_2)$ on Q_6, $C_{Q_6}(B) \leq Z_6$, so $U_3 \leq Z(C_L(u_2))$, completing the proof of (5).

Recall from 19.3 in [ASe] that each involution in $\mathrm{Aut}(L) - L$ induces a field automorphism on L, and hence also on Y^*, so (6) follows. □

14. $E_8(q)$

In this section we assume Hypothesis 10.5 with $L = E_8(q)$.

Lemma 14.1. (1) *$P(\xi) = P_8$. Thus $L_8 = X \cong E_7(q)$.*

(2) *$Q_8 \cong q^{1+56}$ is special with center U and Q_8/U is the faithful 56-dimensional $\mathbb{F}_qX$-module.*

Proof. From the discussion in Section 17 in [ASe], $P_8 = P(\xi)$, $Q_8 \cong q^{1+56}$, and $L_8 \cong E_7(q)$. Then $L_8 = X$ by 10.2 (3), establishing (1). Up to quasiequivalence, $E_7(q)$ has a unique faithful 56-dimensional $\mathbb{F}_q$-module, so (2) follows. □

Lemma 14.2. (1) *L has four classes of involutions.*

(2) *Each involution in Q_8 is conjugate in L to u_i for some $1 \le i \le 3$, denoted by x, y, z in* 17.1 *of* [ASe], *respectively.*

(3) $u_1 \in U$.

Proof. Part (1) is 17.1 in [ASe]. Let $\widetilde{Q}_8 = Q_8/U$. By 14.1 (2), $\widetilde{Q}_8$ is the 56-dimensional module for $X \cong E_7(q)$. The orbits of X on $\widetilde{Q}_8$, and centralizers in X of representatives, are listed in 4.3 of [LS]. In particular if $i \in Q_7 - U$ is an involution, then each nonabelian composition factor of $C_X(\tilde{i})$ is a section of $C_L(i)$. We compare the factors from [LS] of $C_X(\tilde{i})$ to the factors of $C_L(i)$ in 17.15 of [ASe], and conclude only conjugates of x, y, and z can be contained in Q_8, since the only nonabelian composition factor of $C_L(u)$ is $\mathrm{Sp}_8(q)$, which does not contain the nonabelian composition factor of $C_X(j)$ for any $j \in \widetilde{Q}_8$. Thus (2) holds.

From the definition of x in Section 17 of [ASe], $x = u_1$ is a root element, so (3) holds. □

Lemma 14.3. *Let Δ' be the set of seven fundamental subgroups of X defined in* 13.3 *and set $\Delta := \Delta' \cup \{K\}$. For $1 \le i \le 3$ let I_i be the set of involutions in $\langle \Delta \rangle$ projecting nontrivially on exactly i members of Δ. Then $N_L(\Delta)$ is 3-transitive on Δ and transitive on I_i for each $1 \le i \le 3$.*

Proof. From 13.3, $N_X(\Delta')$ is 2-transitive on Δ', so $N_L(\Delta)$ is 3-transitive on Δ by our usual argument. Hence the lemma follows. □

Lemma 14.4. (1) *$L_1 \cong \Omega^+_{14}(q)$ and $Q_1 \cong q^{14+64}$ is special.*

(2) *$Z_1 = Z(Q_1) \cong E_{q^{14}}$ admits the structure of a 14-dimensional orthogonal space over $\mathbb{F}_q$, preserved by L_1, and L_1 induces $\Omega^+_{14}(q)$ on this space.*

(3) *We may take $u_2 = u_1 w_2$ where W_2 is the root group of X such that $X_1 = X \cap P_1 = N_X(W_2)$, and $w_2 \in W_2$. In particular, in the language of* 14.3, *we may choose $W_2 \le J \in \Delta'$, so $u_2 \in I_2$.*

(4) *u_2 is a nonsingular point in the orthogonal space Z_1 and $C_L(u_2) = Q_1 C_{L_1}(u_2)$ with $C_{L_1}(u_2) \cong \mathrm{Sp}_{12}(q)$.*

(5) *Let $K_2 = C_{KJ}(s)$ for s an involution in $N_L(\Delta)$ with cycle (K, J), and $u_2 \in U_2 \in \mathrm{Syl}_2(K_2)$. Then $(K_2, U_2) \in \mathfrak{S}(L, u_2)$.*

(6) $O_2(C_G(K_2)) = 1$.

Proof. Parts (1) and (2) are established during the discussion on page 69 of [ASe].

From 14.2 (2), u_2 is conjugate to the involution y of 17.1 in [ASe], and from the definition of y in Section 17 of [ASe], u_2 is diagonal in the product of two commuting root groups U and W_2 such that $W_2 \not\le O_2(N_L(U))$. Thus as L is transitive on such involutions (cf. the proof of 12.1 in [ASe]), we may choose u_2 as in (3). Arguing as usual, $R(X_1) \cong q^{1+32}$ is a complement to $Q_1 \cap Q_8$ in Q_1, so $W_2 = \Phi(R(X_1)) \leq \Phi(Q_1) = Z_1$. Indeed W_1 is a complement to $Q_8 \cap Z_6$ in Z_1, and $Q_8 \cap Z_6$ is the subspace of the orthogonal space Z_1 orthogonal to U. Thus $u_2 = u_1 w_2$ is a nonsingular vector in Z_1, so $C_{L_1}(u_2) \cong \mathrm{Sp}_{12}(q)$. From the description of $C_L(y)$ in 17.15 of [ASe], $Y := C_L(u_2) = C_{P_1}(u_2)$. Thus (4) holds.

Define (K_2, U_2) as in (5) and let $B_2 := C_L(K_2)$. Then $C_X(K_2) = C_X(J) \cong \Omega^+_{12}(q)$ is of Lie rank 6, as is $Y^* := Y/O_2(Y) \cong \mathrm{Sp}_{12}(q)$, so $O_2(B_2) = 1$ by 10.6. By 14.3, there is $g \in N_L(\Delta)$ with $K_2^g \leq JJ_2$ for some $J_2 \in \Delta' - \{J\}$. As $K^g = C_{JJ^g}(s^g)$, $K^g \in \mathfrak{S}(X, u_2^g)$ so $X_2 := C_X(K_2^g) \cong L_2(q) \times \mathrm{Sp}_8(q)$ by 13.5. Then $B := (KX_2)^{g^{-1}} \leq B_2$ and $N_{Y^*}(C_X(K_2)^*) \cong O^+_{12}(q)$ is the unique maximal overgroup in Y^* of $C_X(K_2)^* \cong \Omega^+_{12}(q)$. Hence as $O^+_{12}(q)$ contains no copy of $B \cong L_2(q)^2 \times \mathrm{Sp}_8(q)$, it follows that $B_2 = C_L(K_2)$. As $C_{Q_1}(C_{L_1}(u_2))$ is of order q it follows that $U_2 = C_{Q_1}(B_2) = Z(C_L(u_2))$, so (5) holds. Finally by 19.2 in [ASe], each involution in $\mathrm{Aut}(L) - L$ induces a field automorphism on L, so (6) follows as usual. □

Lemma 14.5. *Let $g \in P_7 - P_8$, $I = UU^g$, $E = Q_8 \cap Q_8^g$, and $J = \langle Q_8, Q_8^g \rangle$. Then:*

(1) *$Q_7 = (Q_8 \cap Q_7)(Q_8^g \cap Q_7)$ and $J/Q_7 \cong L_2(q)$.*

(2) *$L_7 = (J \cap L_7) \times L_{7,8}$ with $J_7 = J \cap L_7 \cong L_2(q)$ and $L_{7,8} \cong E_6(q)$.*

(3) *$I = Z(Q_7)$ is the natural module for J_7.*

(4) *$[J, E] = I$ and E/I is the 27-dimensional module for $L_{7,8}$.*

(5) *Q_7/E is the tensor product of the natural module for J_7 and the dual of E/I for $L_{7,8}$.*

(6) *We may choose $u_3 \in E - I$, such that $C_{P_7}(u_3) = C_L(u_3)$, $C_{L_7}(u_3) = J_7 \times C_{L_{7,8}}(u_3)$, and $C_{L_{7,8}}(u_3) \cong F_4(q)$.*

(7) *There exists $K_3 \in \mathfrak{S}(L, u_3)$.*

(8) *$O_2(C_G(K_3)) = 1$.*

Proof. The proof is similar to that of 11.6. Parts (1)–(5) follow from the standard theory of large special groups; see for example 8.15 in [A8]. By 14.2 (2), u_3 is fused to the involution z of 17.1 in [ASe]. Thus from 17.14 in [ASe], we may take $C_L(u_3) \leq P_7$ with $O_2(C_L(u_3)) = C_{Q_7}(u_3)$, and $O^{2'}(C_L(u_3))/O_2(C_L(u_3))$ as described in (6). From the action of L_s on Q_s described in (1)–(5), it follows that $u_3 \in E$, completing the proof of (6).

Let $X_7 = X \cap P_7$ and $W = R(X_7)$. As $L_{7,8} \cong E_6(q)$, from 13.2 (7) there is an involution $v_3 \in W$ such that $C_X(v_3) = C_{X_7}(v_3)$ with $C_X(v_3)/W \cong F_4(q)$. Further by 13.4, there is $K_v \in \mathfrak{S}(X, v_3)$. Thus $C_L(K_v)$ contains $KC_X(K_v) \cong L_2(q) \times F_4(q)$.

Now from the structure of $C_X(v_3)$ and the list of centralizers in 17.15 in [ASe], $v_3 \in u_3^L$. Thus there exists $g \in L$ with $v_3^g = u_3$. Let $K_3 := K_v^g$. We have shown that $C_L(K_3)$ contains a subgroup $Y = K_4 \times Y_0$ where $K_4 = K^g$ is a fundamental subgroup of L, and $Y_0 = C_X(K_v)^g \cong F_4(q)$. Hence K_4Y_0 is a complement to $O_2(C_L(u_3))$ in $C_L(u_3)$ by (6).

Suppose $O_2(C_L(K_3)) = R \neq 1$. As the complement Y to $O_2(C_L(u_3))$ acts on R, $R \leq Q_7$. From the description of the action of L_7 on Q_7 in (1)–(5) and the description of the action of E_6 on its 27-dimensional module in [A5]:

(i) K_4 is irreducible on I.

(ii) $\widetilde{E} = E/I = \widetilde{W}^g$, and $W^g = U_3 \times W_3$ and Y_0 is irreducible on W_3.

(iii) $Q_7^* = Q_7/E = C_{Q_7}(u_3)^* \times Q^*$ where $|Q^*| = q^2$ and Y is irreducible on both factors.

We argue as in the proof of 11.6 to derive a contradiction: First $I \cap R = 1$ by (i) and as $K_3 \not\leq P_7 = N_L(I)$. Second, as all Y-submodules of E not containing I either are contained in U_3 or contain W_3, $R \cap E = 1$ as $R \cap W^g = 1$. Therefore third, by (iii), $R^* = C_{Q_7}(u_3)^*$, so as $\Phi(C_{Q_7}(u_3)) \leq E$, $C_{Q_7}(u_3) = R$, a contradiction.

Hence $O_2(C_L(K_3)) = 1$. Moreover we have also shown that $U_3 = Z(C_L(u_3))$, so (7) holds. Finally (8) follows from (7) as usual. □

15. The proof of Theorem 4

In this section we complete the proof of Theorem 4. Thus we assume the hypothesis of that Theorem: $G = O^{2'}(G)$ is an almost simple group of Lie type over $\mathbb{F}_q$, or the Tits group. Set $L := F^*(G)$ and let i be an involution in L.

If L is a classical group, then Theorem 4 holds by Theorem 9.17. Thus we may assume L is exceptional. In particular the pair (G, L) satisfies Hypothesis 10.1, with l the Lie rank of L.

If $l = 1$ then $L \cong Sz(q)$, a case handled in 10.3. Then 10.4 handles the case $l = 2$. Thus we may assume $l > 3$, so L satisfies Hypothesis 10.5.

If i is a long root involution, then the pair (L, i) satisfies Theorem 4 by 10.2 (4). Suppose L is $E_6^{\varepsilon}(q)$. Then by 11.2, L has three classes of involutions with representatives u_i, $1 \leq i \leq 3$. The involution u_1 is a root involution, a case already treated, while Theorem 4 holds for the involutions u_2 and u_3 by 11.5 and 11.6.

Theorem 4 holds when L is $F_4(q)$ by 12.5. Suppose L is $E_7(q)$. By 13.2 (1), L has five classes of involutions, described in 16.1 of [ASe]. The classes with representatives z and v appear in case (4) of Theorem 4. By 13.2 (3), the remaining classes have representatives u_i, $1 \leq i \leq 3$, corresponding to the classes x, y, u of 16.1 in [ASe], respectively. By 13.2 (4), u_1 is a root involution, while Theorem 4 holds for u_2 and u_3 by 13.5 and 13.4.

This leaves the case $L \cong E_8(q)$. By 14.2 (1), L has four classes of involutions, described in 17.1 of [ASe]. The class with representative v appears in case (5) of Theorem 4. By 14.2 (2), the remaining classes have representatives u_i, $1 \leq i \leq 3$, corresponding to the classes x, y, z of [ASe]. By 14.2 (3), u_1 is a root involution, while Theorem 4 holds for u_2 and u_3 by 14.4 and 14.5.

This completes the proof of Theorem 4.

16. The proof of Theorem 2

In this section we complete the proof of Theorem 2. Thus we assume the pair (G, V) is a minimal counter example to Theorem 2, as defined in Section 1. Then by Proposition 1.14:

Lemma 16.1. *G is almost simple and $G = LV$, where $L = F^*(G)$.*

Lemma 16.2. *L is group of Lie type over $\mathbb{F}_q$, for some power q of 2. Hence the pair (G, L) satisfies Hypothesis* 10.1.

Proof. By Theorem 2.1, G is not an alternating or symmetric group. By Theorem 4.1, L is not sporadic. Finally by Theorem 8.3 (3), L is not a group of Lie type and odd characteristic. Thus the lemma follows from the classification of the finite simple groups. □

Lemma 16.3. *L is not $L_3(2)$, $L_4^\varepsilon(2)$, $\mathrm{Sp}_4(2)'$, or $G_2(2)'$.*

Proof. As $A_8 \cong L_4(2)$, L is not $L_4(2)$ by 2.1. Similarly as $L_3(2) \cong L_2(7)$, $\mathrm{Sp}_4(2)' \cong L_2(9)$, $G_2(2)' \cong U_3(3)$, and $U_4(2) \cong \mathrm{PSp}_4(3)$, L is none of these groups by 8.3. □

Lemma 16.4. (1) $V \cap L \neq 1$.

(2) *Suppose i is an involution in L, $(K, U) \in \mathfrak{S}(L, i)$, and $O_2(C_G(K)) \leq L$. Then $i \notin V$.*

(3) *There exists an involution $u \in V \cap L$ such that for each $K \in \mathfrak{S}(L, u)$, $O_2(C_G(K)) \not\leq L$.*

Proof. From 2.5.12 in [GLS3], $m_2(\mathrm{Out}(L)) \leq 2$. On the other hand, by 1.4 we have $m_2(V) > 2$, so (1) holds.

Assume the hypothesis of (2). Then from the definition of $\mathfrak{S}(L, i)$ in the Introduction, one of the following holds:

(a) $K \cong L_2(q)$ or $Sz(q)$, or

(b) $K \cong L_2(q^2)$ with $q > 2$, or

(c) $q = 2$ and $K \cong D_{10}$, or

(d) i is of type c_2 in $\mathrm{Sp}_n(2)$ or $F_4(2)$ and $K \cong L_2(4)$.

Pick D to be a dihedral subgroup of K of order $2m$ containing u, where $m > 1$ is odd in case (a), $m = q^2 + 1$ in case (b), and $m = 5$ in case (c) and (d). We first claim that $C_G(D) = C_G(K)$. In case (d) this holds by case (iii) of the definition of $\mathfrak{S}(L, i)$. In case (c) $D = K$. In the remaining cases, $K = \langle U, D \rangle$ and $C_L(i) = C_L(U)$, so the claim holds in those cases too.

Next, by definition of $\mathfrak{S}(L, i)$, $H = C_L(K)$ is a complement to $O_2(C_L(i))$ in $C_L(i)$, so $O_2(H) = 1$. Thus as $O_2(C_G(K)) \leq L$, also $O_2(C_G(K)) = 1$. Then $O_2(C_G(D)) = 1$ by the claim, so (2) follows from 1.15 (1).

Finally (1) and (2) imply (3). □

Lemma 16.5. (1) *V acts on a parabolic P of L, where*

(i) *P is the stabilizer of a singular point in the natural module for L if L is an orthogonal group $\Omega_n^\varepsilon(q)$, $n \geq 8$;*

(ii) *P is a maximal V-invariant parabolic of L if L is $\mathrm{Sp}_4(q)$ or $F_4(q)$ and G is nontrivial on the Dynkin diagram of L;*

(iii) *$O^{2'}(P/R(P)) \cong E_6(q)$ if L is $E_7(q)$;*

(iv) *$P = N_L(U_\alpha)$ for $\alpha \in \Phi$ a long root in the remaining cases.*

(2) *$V \cap R(P) \neq 1$.*

(3) *If $L \cong \mathrm{Sp}_4(q)$ and G is nontrivial on the Dynkin diagram of L then all involutions in $V - L$ are graph-field involutions, and all those in $V \cap L$ are of type c_2.*

(4) *If $L \cong L_n^\varepsilon(q)$, then V contains an involution of type j_2.*

(5) *If L is symplectic and G is trivial on the Dynkin diagram of L, or if $L \cong \Omega_n^\varepsilon(q)$ with $n \geq 8$, then V contains an involution of type c_2.*

Proof. The normalizer in G of the parabolic listed in (1) contains a Sylow 2-subgroup of G, so (1) follows.

Because $l > 1$ and the groups in 16.3 are excluded by that lemma, by inspection, either $L \cong L_3(4)$ or $O_2(N_G(P)) = O_2(P) = R(P)$. In the former case, (2) follows from 16.4 (1), and in the latter case, (2) follows from 1.10 (1).

In the remaining parts, we adopt the notation for involutions in L from earlier sections, and for outer involutions from [GLS3]. Assume the hypothesis of (3). Then since $G = LV$, $V \not\leq L$. Now (cf. 19.5 in [ASe]) all involutions in $G - L$ are graph-field automorphisms, so V contains a graph-field automorphism τ. Then as $C_L(\tau) \cong Sz(q)$ has all involutions in c_2, (3) follows.

We claim that V contains no long root involutions, and if L is symplectic, V contains no short root involutions. For if i is such an involution, then by 10.2 (4) there is $(K, U) \in \mathfrak{S}(L, i)$, so $O_2(C_G(K)) \not\leq L$ by 16.4 (2). Then by 10.2 (5) and 16.3, $L \cong L_3(4)$ and $O_2(C_G(K)) = \langle \tau \rangle$, where τ induces a graph automorphism on L. But $\langle \tau^{C_L(i)} \rangle$ is not elementary abelian, contrary to 1.15. Thus the claim is established.

In cases (4) and (5), we will show that all other involutions in $R = R(P)$ are of type j_1 and c_2, in the respective case, to complete the proof of the lemma. Namely let M be the natural module for $\hat{L}$, the covering group of L, and identify involutions v in L with those in $\hat{L}$, so that we can consider $\dim([M, v])$. If $L \cong L_n(q)$ then R is generated by transvections, so $\dim([M, r]) \leq 2$ for all $r \in R$. Then as j_m is the class of involutions j with $\dim([M, j]) = m$, and the root involutions are the transvections, the lemma follows in this case. In the remaining cases, P is the parabolic stabilizing a singular point M_0 of M, and R centralizes the chain $0 < M_0 < M_0^\perp < M$, so $\dim([M, r]) \leq 2$ for $r \in R$. Moreover from Section 9, the involutions j in L with $\dim([M, j]) \leq 2$ are of type j_1 and j_2 if L is unitary, and of type b_1, a_2, and c_2 if L is symplectic or orthogonal, with no involutions in L of type b_1 when L is orthogonal. Further when L is unitary, the long root involutions are the transvections, when L is symplectic the root involutions are those of type b_1 and a_2, while if L is orthogonal the root involutions are of type a_2. Hence (4) and (5) follow. □

Lemma 16.6. *L is exceptional.*

Proof. Assume otherwise; then L is classical. If $L \cong \mathrm{Sp}_4(q)$ and V is nontrivial on the Dynkin diagram, then by 16.5 (3), V contains an involution u of type c_2 and each involution in $G - L$ is a graph-field automorphism. Then by Theorem 4, there is $(K, U) \in \mathfrak{S}(L, u)$. From the proof of 16.4 (2), $C_G(K) = C_G(D)$ for a suitable dihedral subgroup D of order $2m$ containing u with m odd, so by 1.15, there is $t \in V - L$ centralizing K. This is impossible as t induces a graph-field automorphism, so $C_L(t) \cong Sz(q)$, and $Sz(q)$ contains no copy of $L_2(q^2)$.

Thus the hypothesis of part (4) or (5) of 16.5 is satisfied, so from that lemma, V contains u of type j_2 or c_2. By Theorem 4, there is $(K, U) \in \mathfrak{S}(L, u)$, and from the discussion in Section 9, $O_2(C_G(K)) \leq L$ unless $L \cong L_4^\varepsilon(q)$, $L_n(2)$, $n \in \{5, 6\}$, or $\Omega_n^\varepsilon(q)$, and $C_G(KJ) = \langle \tau \rangle$, where $J = C_L(K)$ and τ is a graph automorphism of L in the first two cases, and a transvection in $O_n^\varepsilon(q)$ in the last case. By 1.15, $\tau \in V$.

As $L_4^\varepsilon(q) \cong \Omega_6^\varepsilon(q)$, we can subsume this case in the last case. So consider the last case. From 9.18, $\tau \in D \cong D_{2(q^2+1)}$ with $C_G(D) \cong O_{n-2}^{-\varepsilon}(q)$, so 1.15 contradicts $\tau \in V$.

Similarly if $L \cong L_n(2)$ for $n = 5$ or 6, then by 9.19, $\tau \in D \cong D_{2m}$ with $m = 31$ or 7, and $C_G(D) \cong 1$ or $\mathrm{Aut}(L_2(8))$, for $n = 5, 6$, respectively, contrary to 1.15. This completes the proof. □

Lemma 16.7. *V does not contain a long root involution.*

Proof. Let u be a long root involution. By 10.2 (3) there is $(K, U) \in \mathfrak{S}(L, u)$, and by 16.5 and 10.2 (5), $O_2(C_G(K)) = 1$. Now 16.4 (2) completes the proof. □

Lemma 16.8. *L is not exceptional.*

Proof. Assume L is exceptional. Suppose first L is not $E_7(q)$ or $E_8(q)$. In this case we show:

(a) For each involution u of L, which is not a long root involution, there is $K \in \mathfrak{S}(L,u)$ with $O_2(C_G(K)) = 1$.

Observe that this suffices by 16.4 (3) and 16.7.

If $l = 1$, then $L \cong Sz(q)$ and all involutions in L are long root involutions, so (a) holds trivially. If $l = 2$, then (a) follows from 10.4. If L is $F_4(q)$ then (a) follows from 12.5. Finally if L is $E_6^\varepsilon(q)$ then by 11.3 (1) all involutions of L which are not root involutions are conjugate to u_2 or u_3, so (a) follows from 11.5 and 11.6 (7).

Therefore L is $E_7(q)$ or $E_8(q)$. Adopt the notation in 10.5. Let $P = P_l$ and $Q = Q_l$. Then $N_G(P) = N_G(Q)$ contains a Sylow 2-subgroup of G, so we may take $V \leq N_G(Q)$, and then as $Q = O_2(N_G(P))$, $V \cap Q \neq 1$ by 1.10 (1). Then by 16.4 (2) and 16.7, it suffices to show:

(b) For each involution $u \in Q$ which is not a root involution, there is $K \in \mathfrak{S}(L,u)$ with $O_2(C_G(K)) \leq L$.

From 13.2 (3) and 14.2 (2), the involutions in Q are conjugate to u_i for some $1 \leq i \leq 3$. Further u_1 is a root involution. Then 13.4, 13.5, 14.4 and 14.5 show that (b) holds. This completes the proof of the lemma. □

Observe that 16.6 and 16.8 supply a contradiction, which completes the proof of Theorem 2.

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