Monotonicity properties for ground states of the scalar field equation

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Abstract

It is well known that the scalar field equation

$$\Delta u - u + u^p = 0 \quad \text{in } \mathbb{R}^N, \ N \geqslant 3,$$

admits ground state solutions if and only if $1 < p < (N+2)/(N-2)$ and that for each fixed p in this range, there corresponds a unique ground state (up to translation). In this article, we show that the maximum value of such ground states, $\|u\|_\infty$, is an increasing function of p for all $1 < p < (N+2)/(N-2)$. As a consequence of this result we derive a Liouville type theorem ensuring that there exists neither a ground state solution to this equation, nor a positive solution of the Dirichlet problem in any finite ball, with the maximum value less than $e^{N/4}$. Our proof relies on some fine analyses on the first variation of ground states with respect to the initial value and with respect to p. The delicacy of this study can be evidenced by the fact that, on any fixed finite ball, the maximum value of positive solutions to the Dirichlet problem is *never* a monotone function of p, over the whole range $1 < p < (N+2)/(N-2)$.

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1. Introduction

In this article we consider the scalar field equation

$$\Delta u - u + u^p = 0 \quad \text{in } \mathbb{R}^N, \ N \geqslant 3, \tag{1.1}$$

where $u = u(x) \in \mathbb{R}$. This basic semi-linear elliptic partial differential equation arises in various context in physics, as for example in the study of solitary waves for the Klein–Gordon equation and of standing waves for the non-linear Schrödinger equation. It also appears in non-linear optics, laser propagation and cosmology, see references

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in [1]. From the mathematical point of view, this seemingly simple equation poses numerous fundamental questions that have attracted the attention of many authors along the years. After serious advances in the understanding of the structure of solutions to this equation, there are still many interesting and basic questions that remain open.

We are interested in ground states of (1.1), that is, in positive twice differentiable solutions of (1.1) defined in the entire Euclidean space $\mathbb{R}^N$ that satisfy $u(x) \to 0$ as $|x| \to \infty$. It is well known that the scalar field equation admits a ground state if and only if $1 < p < (N + 2)/(N - 2)$. The existence of a ground state was proved by Berestycki and Lions in [1], while the non-existence outside of the range follows from the powerful Pohozaev identity [16].

Further information on ground states u of (1.1) was obtained by Gidas, Ni and Nirenberg in their celebrated paper [7]. They proved that the function $u(x)$ is necessarily radially symmetric and it has a unique critical point, precisely at the center of symmetry, where u achieves its maximum value. If we assume that the center of symmetry is the origin then $u(x) = u(r)$, with $r = |x|$, and u is the unique solution to the initial value problem of the ordinary differential equation

$$u'' + \frac{N-1}{r} u' - u + u^p = 0, \quad u(0) = \alpha > 0, \quad u'(0) = 0, \tag{1.2}$$

where $'$ denotes derivative with respect to r and $\alpha = \|u\|_\infty$ is the maximum value of u.

At this point, the question of uniqueness of ground states for the field equation (1.1) is reduced to the study of all possible solutions $u = u(r, \alpha)$ for (1.2), varying the initial value α. Still, this basic question took many years and the efforts of many authors to be solved, among the main contributions we have the work of Coffman [3], McLeod and Serrin [10], Peletier and Serrin [14,15], Ni and Nussbaum [13] and Ni [12], culminating with the work of Kwong [8]. At the heart of Kwong's proof there is a fine analysis of the derivative of u with respect to α in combination with the Sturm comparison theorem, all aiming to prove the monotonicity of the function $R(\alpha, p)$ with respect to α, where $R = R(\alpha, p)$ denotes the first value of R such that $u(R, \alpha) = 0$, if such an R exists. Actually what matters is a form of such a monotonicity when $R = \infty$. See Theorem 1.3 for the precise statement of uniqueness and corresponding properties.

Once the uniqueness of the ground state of (1.1) has been settled, we see that the maximum value of the solution $\alpha_p = u(0)$ defines a function of p in the range $1 < p < (N + 2)/(N - 2)$. It is natural to ask about the nature of this function, in particular if it defines a one-to-one correspondence and, in this case, if this correspondence is monotone increasing or monotone decreasing in p. It is our purpose in this article to give an answer to this question.

The main goal of this article is to prove the following monotonicity theorem:

Theorem 1.1. *The maximum value α_p of the ground state to the scalar field equation* (1.1) *is a strictly increasing function of p for all $1 < p < (N + 2)/(N - 2)$.*

Our proof relies on some fine analyses of the first variation of ground states with respect to the maximum value α and with respect to p. The delicacy of this study can be evidenced by the fact that, on any fixed finite ball, the maximum value of positive solutions to the Dirichlet problem is *never* a monotone function of p. More precisely, let $R > 0$ and let $B_R = B(0, R)$ be the ball of radius R, and consider the boundary value problem

$$\Delta u - u + u^p = 0 \quad \text{in } B_R, \quad \text{and} \quad u = 0 \quad \text{on } \partial B_R. \tag{1.3}$$

This problem has a positive solution if and only if $1 < p < (N + 2)/(N - 2)$. Moreover, as in the case of $\mathbb{R}^N$, the solution is unique and radially symmetric and its maximum value $\alpha_p(R)$ is achieved at the origin. In contrast with the problem in $\mathbb{R}^N$, we prove in Section 6 that for every $R > 0$, the function $\alpha_p(R)$ is not monotone as a function of p such that $1 < p < (N + 2)/(N - 2)$.

Using the monotonicity of α_p, proved in Theorem 1.1, and analyzing further the asymptotic behavior of α_p, as p approaches the extremes of the interval $(1, \frac{N+2}{N-2})$, we obtain the following Liouville type theorem

Theorem 1.2. *There is no positive solution to Eq.* (1.1)*, nor to Eq.* (1.3) *in a finite ball, satisfying $\|u\|_\infty \leqslant e^{N/4}$.*

Out of our results there come several questions that we do not know how to answer now. First of all, we do not know if the constant $e^{N/4}$ given in Theorem 1.2 is optimal for non-existence of positive solutions to (1.1). Second, the precise picture of the curve $\alpha_p(R)$ versus p is poorly understood and its study becomes a challenging problem.

Finally, we should mention that the whole structure of the set of solutions to (1.1) and (1.3) is not well understood, since a complete analysis of changing sign solutions is still missing, including uniqueness of solutions with a pre-scribed number of zeros and the relation between the different parameters involved, that is, p, α, R and N. With this work we expect to contribute to a better understanding of this problem providing answers to an open question and continuing with the development of a methodology that serves as a tool to treat this and other related problems.

We complete this introduction with the statement of some known facts for future reference. The existence and uniqueness of ground states or positive solutions of (1.3) can be obtained by the characterization of the solution set of (1.2) which we summarize in

Theorem 1.3. *For each given $1 < p < (N+2)/(N-2)$, there exists a finite number $\alpha_p > 1$ such that*

(i) *if $0 < \alpha < \alpha_p$, then $u(r, \alpha)$ is a positive, oscillatory function in $(0, \infty)$: It assumes an increasing sequence of minimum values, and a decreasing sequence of maximum values, with both sequences approaching one at infinity.*

(ii) *If $\alpha = \alpha_p$, then $u(r, \alpha)$ is a ground state: It is positive in $(0, \infty)$ and for any $\epsilon \in (0, 1)$,*

$$\limsup_{r \to \infty} u(r) e^{\sqrt{1-\epsilon}\, r} = 0 \quad and \quad \lim_{r \to \infty} \frac{u'(r)}{u(r)} = -1. \tag{1.4}$$

(iii) *If $\alpha > \alpha_p$, then $u(r, \alpha)$ is a crossing function: It is positive in $(0, R)$ and vanishes at R with $u'(R) < 0$ for some finite $R = R(\alpha) > 0$. Moreover, $R(\alpha)$ is a decreasing function of α with*

$$\lim_{\alpha \downarrow \alpha_p} R(\alpha) = \infty \quad and \quad \lim_{\alpha \to \infty} R(\alpha) = 0.$$

Remark 1.1. While the oscillatory solution of course changes monotonicity and concavity infinitely many times, the ground state or crossing solution remains decreasing for all $0 < r < R$ and changes concavity exactly once, at some $r_c > 0$ where $u > 1$ and u changes from concave down to concave up as r increases across r_c. This fact was proved recently in the more general setting of Pucci's operators in [6].

We conclude this introduction with some comments on the case when p is not included in Theorem 1.3. As it was already mentioned, when $p \geqslant (N+2)/(N-2)$, then there is no ground state, and no positive solutions of (1.3), as a consequence of Pohozaev identity [16].

When $0 < p < 1$, then the constant solution $u \equiv 1$ behaves like a *repeller* that pushes all solutions of (1.2) away: If $\alpha > 1$, then $u(r, \alpha)$ is an unbounded increasing function defined for all $r \geqslant 0$; if $\alpha < 1$, then $u(r, \alpha)$ is a decreasing function and vanishes at some finite $r = R$. Therefore, there can be no ground states of (1.1). However, one can show that the Dirichlet problem (1.3) has exactly one solution for each finite ball. The analysis of these questions is much simpler for this sub-linear non-linearity. The linear case $p = 1$ is completely a different story and will not be discussed here.

This paper is organized as follows: In Section 2, we give a proof of a key lemma that not only leads to the uniqueness of positive solutions, but also gives detailed characterization of the variation of u with respect to its maximum value. In Sections 3 and 4, we shall discuss in details the properties of the first variation of u with respect to p. Those properties will be used in Section 5 to complete the proof of Theorem 1.1. Finally, in Section 6 we prove Theorem 1.2 and we present the arguments revealing that the maximum value of the positive solutions of (1.3) is *never* a monotonic function of p over the whole range $1 < p < (N+2)/(N-2)$, on whatever finite ball B in $\mathbb{R}^N$.

2. A key lemma for the uniqueness

Historically, most technical complexities in establishing the uniqueness of ground states and positive solutions in finite balls have been involved with the study of the variation

$$v(r) = \frac{\partial u(r; \alpha, p)}{\partial \alpha}$$

which solves the second order linear equation

$$v'' + \frac{N-1}{r}v' + f'(u)v = 0, \quad v(0) = 1, \quad v'(0) = 0, \tag{2.1}$$

where $f'(u) = -1 + pu^{p-1}$. In the original work of Kwong [8] and the subsequent studies of McLeod [11], Kwong and Zhang [9] and Chen, C.S. Lin [2], tremendous efforts have been devoted into showing that u and v do not vanish simultaneously if u is a crossing solution, and v does not approach zero if u is a ground state. Several ingenious ideas have been developed in their rather involved techniques, which are nevertheless not strong enough to determine how many zeros the function v has when u remains positive.

   In a fundamental paper Tang [19] proved that v vanishes exactly once when $u \geqslant 0$. We shall recall the result of Tang, together with a self-contained proof. Moreover, we shall prove a new result which asserts that at the unique zero of v, the solution u is larger than 1. In addition, we will establish the positivity of the important function

$$\zeta(r) = r^N \big[u'v' + f(u)v \big] + (N-2)r^{N-1}u'v. \tag{2.2}$$

These results will play key roles in our study.

Lemma 2.1. *Let* $1 < p < (N+2)/(N-2)$. *Suppose* u *is either a ground state of* (1.1)*, or a positive solution of* (1.3) *in a ball of radius* $R > 0$. *Then there exists a number* $\tau_v > 0$ *such that*

$$v(r) > 0 \quad for \; 0 \leqslant r < \tau_v, \quad v(\tau_v) = 0, \quad and \quad v(r) < 0 \quad for \; \tau_v < r < R, \tag{2.3}$$

where we allow $R = \infty$ *when* u *is a ground state. Furthermore, we have*

  (i) $u(\tau_v) > 1$.
 (ii) *The ratio* $v(r)/u(r)$ *is strict decreasing over* $(0, R)$.
(iii) *The function* $\zeta(r)$ *is positive in* $(0, R)$ *and*

$$\lim_{r \to R} \zeta(r) > 0.$$

Consequently, $v(R) < 0$ *when* $R < \infty$*, and* $v \to -\infty$ *as* $r \to \infty$ *when* $R = \infty$.

Proof. We first show that v must vanish before u reaches 1. Define r_1 to be the unique number for which $u(r_1) = 1$. Suppose for contradiction that $v > 0$ for $0 < r < r_1$, then

$$u''(r_1)v'(r_1) - u'(r_1)v''(r_1) = -f(1)v'(r_1) + f'(1)u'(r_1)v(r_1)$$
$$= (p-1)u'(r_1)v(r_1) < 0.$$

On the other hand, using the identity

$$\big[r^{n-1}(u''v' - u'v'') \big]' = r^{n-1}f''(u)u'^2 v,$$

introduced by Tang in [20], we find that

$$r_1^{n-1} \big[u''(r_1)v'(r_1) - u'(r_1)v''(r_1) \big] = p(p-1)\int_0^{r_1} r^{n-1}u^{p-2}(r)u'^2(r)v(r)\,\mathrm{d}r > 0.$$

This gives a contradiction. Therefore, at the first zero of v, denoted by τ_v, we have $u(\tau_v) > 1$, and (i) is proved.

   We then show that the ratio $v(r)/u(r)$ is strict decreasing over $(0, R)$. For this purpose, we consider the Wronskian of u and v

$$\xi(r) = r^{n-1}(u'v - uv'). \tag{2.4}$$

In our case $f(u) = -u + u^p$, we find that

$$\xi'(r) = r^{n-1}\big[uf'(u) - f(u) \big]v = (p-1)r^{n-1}u^p v. \tag{2.5}$$

This clearly implies that $\xi(r) > 0$ for all $0 < r < \tau_v$, over which $v(r)/u(r)$ is thus strictly decreasing. If $\xi(r)$ is not always positive in the interval $(0, R)$, then there would be a number $\tau_\xi > \tau_v$ such that

$$\xi(r) > 0 \quad \text{for} \quad 0 < r < \tau_\xi, \quad \text{and} \quad \xi(\tau_\xi) = 0. \tag{2.6}$$

We should also have

$$v(\tau_\xi) < 0 \tag{2.7}$$

since at any possibly subsequent zero of v adjacent to τ_v, it would be true that $\xi < 0$ there. To continue, we recall the important function defined by Tang in [19]

$$T(r) = g(u)\xi(r) - \zeta(r), \quad g(u) = \frac{2f(u)}{uf'(u) - f(u)} = \frac{2}{p-1}(1 - u^{1-p}), \tag{2.8}$$

where ζ was defined in (2.2). Because

$$\zeta'(r) = 2r^{n-1}f(u)v, \tag{2.9}$$

we obtain

$$T'(r) = g'(u)u'(r)\xi(r) = 2u^{-p}u'(r)\xi(r). \tag{2.10}$$

Integration of T over $(0, \tau_\xi)$ gives

$$T(\tau_\xi) = 2\int_0^{\tau_\xi} u^{-p}u'(r)\xi(r) < 0$$

by (2.6). Hence we find that

$$\zeta(\tau_\xi) = -T(\tau_\xi) > 0. \tag{2.11}$$

However, by the definition of $\zeta(r)$ we calculate

$$\begin{aligned}
\zeta(\tau_\xi) &= \tau_\xi^n\left[u'v' + f(u)v\right] + (n-2)\tau_\xi^{n-1}u'v \\
&= \left[\tau_\xi^n\left(u'v'u/v + f(u)u\right) + (n-2)\tau_\xi^{n-1}u'u\right]v/u \\
&= \left[\tau_\xi^n\left(u'^2 + f(u)u\right) + (n-2)\tau_\xi^{n-1}u'u\right]v/u \\
&= Q(\tau_\xi)v(\tau_\xi)/u(\tau_\xi),
\end{aligned}$$

where

$$Q(r) = r^n\left[u'^2 + uf(u)\right] + (n-2)r^{n-1}uu'.$$

So by (2.7) and (2.11) we arrive at

$$Q(\tau_\xi) < 0. \tag{2.12}$$

We will show that this is impossible. Define

$$P(r) = r^n\left[u'^2 + 2F(u)\right] + (n-2)r^{n-1}uu', \quad F(u) = \int_0^u f(s)\,ds. \tag{2.13}$$

The well-known Pohozaev identity gives

$$P'(r) = r^{n-1}\left[2nF(u) - (n-2)uf(u)\right] = 2r^{n-1}\left(\sigma u^{p+1} - u^2\right) \tag{2.14}$$

where

$$\sigma = \frac{2n - (n-2)(p+1)}{2(p+1)} > 0.$$

Since $P(R) > 0$ when $R < \infty$, and $P(r) \to 0$ as $r \to \infty$ when u is a ground state which vanishes exponentially at infinity, it is evident that $P(r)$ increases from 0 to a finite number less than R, and then decreases thereafter. Therefore, for all $0 < r < R$, one has $P(r) > 0$ and

$$Q(r) = P(r) + r^n \big[u f(u) - 2F(u) \big] > \frac{p-1}{p+1} r^n u^{p+1} > 0,$$

yielding a contradiction of (2.12). Hence we claim that $\xi(r)$ is positive in $(0, R)$ and part (ii) is proved.

Clearly, part (ii) also indicates that v can only vanish once within $(0, R)$. Hence (2.3) is proved.

Finally, by (2.10) and positivity of ξ we conclude that $T(r) < 0$ for all $r \in (0, R)$. Since $g(u(r)) > 0$ when $r \leqslant r_1$, it holds that $\zeta(r) > 0$ for $r \leqslant r_1$. Moreover, by (2.9) one has

$$\zeta(r) > \zeta(r_1) > 0, \quad r > r_1,$$

implying that

$$\zeta(r) > 0 \quad \text{in } (0, R), \quad \text{and} \quad \lim_{r \to R} \zeta(r) > 0. \tag{2.15}$$

When $R < \infty$, one sees immediately that $v(R) < 0$. For a ground state, it can be easily shown that, as $r \to \infty$, either $v \to 0$ which contradicts (2.15), or $v \to -\infty$, which must be true. This establishes part (iii) and completes the proof of the lemma. $\square$

3. The first variation with respect to p

To understand how the maximum value of ground states changes as p varies, we consider another variation given by

$$\phi(r) = \frac{\partial u(r; \alpha, p)}{\partial p}.$$

This function can be useful in characterizing the solutions of the initial value problem (1.2) when α is kept a constant but p varies. The variation with respect to p was first used by Felmer and Quaas in [5], in the study of critical exponents for the Pucci operator.

The function ϕ is the unique solution to the initial value problem of the ordinary differential equation

$$\phi'' + \frac{N-1}{r} \phi' - \phi + p u^{p-1} \phi + u^p \log u = 0, \qquad \phi(0) = \phi'(0) = 0. \tag{3.1}$$

In what follows in this section, we prove a series of lemmas to understand the behavior of the function ϕ which is relevant for our analysis.

Lemma 3.1. *If $u(0) = \alpha > 1$, then ϕ is negative for small $r > 0$.*

Proof. Using the L'Hospital's rule we have

$$\lim_{r \to 0} \frac{\phi'(r)}{r} = \phi''(0).$$

Thus, taking limit of Eq. (3.1) gives

$$N\phi''(0) + \alpha^p \log \alpha = 0.$$

For $\alpha > 1$, we then have

$$\phi''(0) = -\frac{1}{N} \alpha^p \log \alpha < 0. \tag{3.2}$$

Because $\phi(0) = \phi'(0) = 0$, this clearly implies that ϕ is negative for small $r > 0$. $\square$

Denote the unique zero of the v function by τ_v, then we have

Lemma 3.2. *If u is a ground state, then $\phi(r)$ stays negative in $(0, \tau_v]$, and vanishes at most once over (τ_v, ∞). In other words, either $\phi(r) < 0$ for all $0 < r < \infty$, or there exists a number $\tau_\phi > \tau_v$ where ϕ vanishes and*

$$\phi(r) < 0 \quad \text{for } 0 < r < \tau_\phi, \qquad \phi(r) > 0 \quad \text{for } r > \tau_\phi. \tag{3.3}$$

Proof. Let $u = u(r; \alpha, p)$ be a ground state, and v, ϕ the associated variations defined as above. Introduce the Wronskian type function

$$\xi_v(r) = r^{N-1}\big[v'(r)\phi(r) - v(r)\phi'(r)\big]. \tag{3.4}$$

Apparently, $\xi_v(0) = 0$. Using Eqs. (2.1) and (3.1) we can also verify

$$\xi_v'(r) = r^{N-1}u^p v \log u. \tag{3.5}$$

More importantly, it can be shown that

$$\xi_v(r) > 0 \quad \text{for all } r > 0. \tag{3.6}$$

To prove (3.6), we need an important function

$$T_p(r) = \xi_v(r) - h\big(u(r)\big)\zeta(r), \tag{3.7}$$

where

$$h(u) = \frac{u^p \log u}{2(-u + u^p)}, \quad \text{for } u > 0, \quad u \neq 1, \tag{3.8}$$

$$h(1) = \lim_{u \to 1} h(u) = \frac{1}{2(p-1)},$$

and $\zeta(r)$ is defined in (2.2). We note that $h(u)$ is continuously differentiable for $u > 0$, because, when $u \neq 1$,

$$h'(u) = \frac{1}{2(-u + u^p)^2}\big[\big(pu^{p-1}\log u + u^{p-1}\big)\big(-u + u^p\big) - u^p \log u\big(-1 + pu^{p-1}\big)\big]$$

$$= \frac{u^p}{2(-u + u^p)^2}\big[u^{p-1} - 1 - (p-1)\log u\big],$$

and

$$\lim_{u \to 1} h'(u) = \frac{1}{4}.$$

Furthermore, since

$$u^{p-1} - 1 - (p-1)\log u > 0 \quad \text{for all } u > 0 \text{ and } u \neq 1,$$

we see that $h(u)$ is a strictly increasing function of $u > 0$. Taken together, we have

$$h(u) > 0 \quad \text{and} \quad h'(u) > 0 \quad \text{for all } u > 0. \tag{3.9}$$

Combining (3.5), (2.9) and (2.15) we find that

$$T_p'(r) = -h'(u)u'(r)\zeta(r) > 0.$$

Therefore, we have, for all $r > 0$, $T_p(r) > T_p(0) = 0$. Hence

$$\xi_v(r) > h\big(u(r)\big)\zeta(r) > 0.$$

This implies (3.6). Now, suppose for contradiction that ϕ vanishes somewhere in $(0, \tau_v]$. Then there is a number $\tau_\phi \in (0, \tau_v]$ such that ϕ is negative in $(0, \tau_\phi)$, and equals zero at τ_ϕ. Thus it also holds that $\phi'(\tau_\phi) \geqslant 0$. Since v is either positive at τ_ϕ when $\tau_\phi < \tau_v$, or zero if $\tau_\phi = \tau_v$, we must have

$$\xi_v(\tau_\phi) = \tau_\phi^{N-1}\big[v'(\tau_\phi)\phi(\tau_\phi) - v(\tau_\phi)\phi'(\tau_\phi)\big] = -\tau_\phi^{N-1}v(\tau_\phi)\phi'(\tau_\phi) \leqslant 0.$$

This leads to a contradiction of (3.6). Therefore, the function $\phi(r)$ could only vanish after v becomes negative. Because v stays negative in (τ_v, ∞), a similar argument as above can show that ϕ cannot take a second zero, should it vanish somewhere in (τ_v, ∞). The proof is completed. $\quad\square$

Lemma 3.3. *If $\phi(r)$ has a zero in $(0, \infty)$, then $\lim_{r \to \infty} \phi(r) = \infty$.*

Proof. Let $\tau_\phi > \tau_v$ be the unique zero of ϕ as asserted in Lemma 3.2. Take an arbitrary number $r_c > \tau_\phi$ and put

$$-c = \phi(r_c)/v(r_c) < 0.$$

Because $\phi(r)/v(r)$ is a decreasing function by (3.6), and $v < 0$ for all $r > \tau_v$, we conclude that $\phi(r) > -cv(r) > 0$ for large r. Furthermore, since $v(r) \to -\infty$ as $r \to \infty$, there holds that $\lim_{r \to \infty} \phi(r) = \infty$.   □

4. Further properties of the first variation

In this section, we will derive some further properties for the ϕ function, which can be very helpful in dealing with the delicate case when $\phi(r)$ stays negative for all $0 < r < \infty$. For this purpose, we introduce, analogous to the ζ function,

$$\eta(r) = r^N \left[u'\phi' + f(u)\phi \right] + (N - 2)r^{N-1}u'\phi. \tag{4.1}$$

Differentiation gives

$$\eta'(r) = 2r^{N-1}f(u)\phi - r^N u^p u' \log u. \tag{4.2}$$

If we write

$$\eta_1(r) = \frac{\phi}{ru'} - h(u), \tag{4.3}$$

then

$$\eta'(r) = 2r^N f(u)u'\eta_1(r). \tag{4.4}$$

To easy our calculation later, we evaluate the derivative of ϕ/ru' as follows:

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\phi}{ru'} \right) = \frac{\eta(r)}{r^{N+1}u'^2(r)}. \tag{4.5}$$

This can be derived by differentiation and the definition of η, see (4.1):

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{\phi}{ru'} \right) &= \frac{1}{r^2 u'^2}\left[ru'\phi' - \phi(ru')' \right] \\
&= \frac{1}{r^2 u'^2}\left[ru'\phi' + rf(u)\phi + (N - 2)u'\phi \right] \\
&= \frac{\eta(r)}{r^{N+1}u'^2(r)}.
\end{aligned}$$

Lemma 4.1. $\eta(0) = 0$ *and* $\eta(r) < 0$ *for small* $r > 0$.

Proof. That $\eta(0) = 0$ is obvious. However, the evaluation of η for small $r > 0$ is non-trivial. We calculate

$$\begin{aligned}
l &= \lim_{r \to 0} \frac{\eta(r)}{r^{N+4}} \\
&= \lim_{r \to 0} \frac{2r^N f(u(r))u'(r)\eta_1(r)}{(N + 4)r^{N+3}} \quad \text{by (4.4)} \\
&= \frac{2f(\alpha)}{N + 4} \lim_{r \to 0} \frac{u'(r)}{r} \frac{\eta_1(r)}{r^2} \\
&= \frac{f(\alpha)u''(0)}{N + 4} \lim_{r \to 0} \frac{2\eta_1(r)}{r^2}.
\end{aligned}$$

To continue, we first note that

$$\lim_{r \to 0} \frac{\phi(r)}{ru'(r)} = \lim_{r \to 0} \frac{\phi(r)}{r^2} \frac{r}{u'(r)} = \frac{\phi''(0)}{2u''(0)} = h(\alpha),$$

by (3.2) and the formula $u''(0) = -f(\alpha)/N$. Thus $\lim_{r \to 0} \eta_1(r) = 0$, and so by the L'Hospital's rule we obtain

$$\lim_{r \to 0} \frac{2\eta_1(r)}{r^2} = \lim_{r \to 0} \frac{1}{r} \left(\frac{\eta(r)}{r^{N+1}u'^2} - h'(u)u'(r) \right) \quad \text{by (4.3), (4.5)}$$

$$= \lim_{r \to 0} \frac{\eta(r)}{r^{N+4}} \frac{r^2}{u'^2(r)} - \lim_{r \to 0} \frac{h'(u)u'(r)}{r}$$

$$= \frac{l}{u''^2(0)} - h'(\alpha)u''(0).$$

Therefore, we have

$$l = -\frac{N}{N+4}l - \frac{f^3(\alpha)h'(\alpha)}{N^2(N+4)},$$

and so, by substituting the expression of $h'(\alpha)$,

$$l = -\frac{f^3(\alpha)h'(\alpha)}{2N^2(N+2)}$$

$$= -\frac{\alpha^p f(\alpha)}{4N^2(N+2)} \left[\alpha^{p-1} - 1 - (p-1)\log\alpha \right] < 0.$$

Thus $\eta(r)$ is negative for small $r > 0$. $\quad \square$

Next, we show that η will remain negative at least until the ground state solution reaches one. More precisely, let r_1 be the unique number satisfying

$$u(r_1) = 1.$$

Then we have

Lemma 4.2. $\eta(r) < 0$ *for all* $0 < r \leqslant r_1$.

Proof. We first prove the identity

$$\phi(r)\zeta(r) - ru'(r)\xi_v(r) = v(r)\eta(r) \tag{4.6}$$

that relates the various functions used so far. This can be verified by the definitions of these functions in (2.2), (3.4), and (4.1) that give rise to

$$\phi(r)\zeta(r) - ru'(r)\xi_v(r) = r^N \phi \left[u'v' + f(u)v \right] + (N-2)r^{N-1}u'v\phi - r^N(u'v'\phi - u'v\phi')$$

$$= r^N f(u)\phi v + (N-2)r^{N-1}u'\phi v + r^N u'\phi'v$$

$$= v\eta.$$

Next, we recall that both $T_p(r)$ and $\zeta(r)$ are positive for all $r > 0$, see the proof of Lemma 3.2. Therefore,

$$h(u) = \frac{u^p \log u}{2f(u)} < \frac{\xi_v(r)}{\zeta(r)} \quad \text{for all } r > 0.$$

If $0 < r < r_1$, then $u > 1$ and $f(u) > 0$, implying further that

$$u^p \log u < \frac{2f(u)\xi_v(r)}{\zeta(r)}.$$

Now, by (4.2) and (4.6) we obtain, provided that $0 < r < r_1$,

$$\eta'(r) = 2r^{N-1}f(u)\phi - r^N u^p u' \log u$$

$$< 2r^{N-1}f(u)\phi - r^N u' \frac{2f(u)\xi_v(r)}{\zeta(r)}$$

$$= 2r^{N-1}f(u)(\phi\zeta - ru'\xi_v)/\zeta(r)$$

$$= 2r^{N-1}f(u)v\eta(r)/\zeta(r).$$

This implies that, at any possible zero of η within the interval $(0, r_1]$, it holds that $\eta' < 0$. However, since $\eta < 0$ for small $r < 0$, η must stay negative in the whole range $(0, r_1]$. The proof is completed. $\quad\square$

By this lemma, we can easily prove

Lemma 4.3. *If* $\phi(r) < 0$ *in* $(0, r_1]$, *then* $\phi'(r_1) > 0$.

Proof. By Lemma 4.2, we have $\eta(r_1) < 0$. Since $f(u(r_1)) = f(1) = 0$, there results

$$r_1^N u'(r_1)\phi'(r_1) + (N-2)r_1^{N-1}u'(r_1)\phi(r_1) < 0.$$

Because $u'(r_1) < 0$, it holds that

$$\phi'(r_1) > -\frac{N-2}{r_1}\phi(r_1) > 0.$$

This proves the lemma. $\quad\square$

5. Proof of Theorem 1.1

For a pair of numbers p and $\bar{p}$ satisfying

$$1 < p < \bar{p} < (N+2)/(N-2), \tag{5.1}$$

there correspond a pair of ground states, denoted by u and $\bar{u}$, respectively, of the scalar field equation. Let α and $\bar{\alpha}$ be the maximum value of these two ground states. The radial function $u(r)$ is then the unique solution of the initial value problem (1.2). Similarly, $\bar{u}(r)$ is the unique solution of the initial value problem

$$u'' + \frac{N-1}{r}u' - u + u^{\bar{p}} = 0, \quad u(0) = \bar{\alpha}, \quad u'(0) = 0. \tag{5.2}$$

We need to prove

$$\alpha < \bar{\alpha}. \tag{5.3}$$

To this end, we introduce the third function $\hat{u}(r)$, that solves

$$u'' + \frac{N-1}{r}u' - u + u^{\bar{p}} = 0, \quad u(0) = \alpha, \quad u'(0) = 0. \tag{5.4}$$

$u(r)$ and $\hat{u}(r)$ have the *same* initial data, but solve *different* equations; whereas $\hat{u}(r)$ and $\bar{u}$ have *different* initial data, but solve the *same* equation. To prove (5.3), it suffices to show that $\hat{u}$ must be an oscillatory function over $(0, \infty)$, or equivalently, $\hat{u}$ is neither a ground state itself nor a crossing solution, when p and $\bar{p}$ are sufficiently close.

Because $\phi(r) < 0$ for small $r > 0$, see Lemma 3.1, we can further assume that $\hat{u}(r) < u(r)$ when $r > 0$ is small. There are two possibilities: either $\hat{u}$ and u meet at some large value of r, or $\hat{u} < u$ for all $r > 0$. These will be discussed separately below. For the first case when u and $\hat{u}$ intersect, our discussion is relatively simpler and will be given first. Next we will rely on some delicate analyses to rule out the second possibility that $\hat{u} < u$ for all $r > 0$.

Case I: *The curves* $u(r)$ *and* $\hat{u}(r)$ *intersect at some* $r > 0$.

If $\hat{u}$ is an oscillatory function, then our proof is done. We thus assume that $\hat{u}(r)$, like $u(r)$, is a decreasing function. As above, we denote by r_1 the unique number that makes the ground state $u = 1$. We let $r^* > r_1$ be the unique number at which

$$pu^{p-1}(r^*) = 1. \tag{5.5}$$

If furthermore $\phi(r)$ has a zero $\tau_\phi > 0$, then by Lemma 3.3 we can thus pick up a large number

$$\hat{r} > r^*$$

such that

$$\phi(\hat{r}) > 0 \quad \text{and} \quad \phi'(\hat{r}) > 0.$$

Write $w(r) = u(r) - \hat{u}(r)$. Provided that $\bar{p} > p$ is sufficiently close to p, one has

$$p\hat{u}^{p-1}(\hat{r}) < 1, \quad w(\hat{r}) < 0 \quad \text{and} \quad w'(\hat{r}) < 0. \tag{5.6}$$

For any $r > \hat{r}$ for which $\hat{u} > u$, we have

$$\begin{aligned}
w''(r) + \frac{N-1}{r} w'(r) &= -\hat{u}(r) + \hat{u}^{\bar{p}}(r) + u(r) - u^p(r) \\
&< -\hat{u}(r) + \hat{u}^p(r) + u(r) - u^p(r) \quad \text{since } \hat{u}(r) < 1 \text{ and } \bar{p} > p \\
&= -(\hat{u} - u)\left(1 - \frac{\hat{u}^p - u^p}{\hat{u} - u}\right) \\
&= -(\hat{u} - u)\left(1 - p u_*^{p-1}\right),
\end{aligned}$$

where $u_* \in (u, \hat{u})$. Because $u_* < \hat{u}(r) < \hat{u}(\hat{r})$, from (5.6) it follows that

$$w''(r) + \frac{N-1}{r} w'(r) < 0.$$

Let r_∞ be the supremum of r for which $\hat{u}(r) > u(r)$ for all $r \in (\hat{r}, r_\infty)$. Then the function $r^{N-1} w'(r)$ is strictly decreasing in $(\hat{r}, r_\infty)$, implying that

$$\lim_{r \to r_\infty} r^{N-1} w'(r) < \hat{r}^{N-1} w'(\hat{r}) < 0. \tag{5.7}$$

Now, if r_∞ is a finite number, then necessarily $w(r_\infty) = 0$; since $w(r) < 0$ for $\hat{r} < r < r_\infty$, we have $w'(r_\infty) \geqslant 0$. This contradicts (5.7). Thus $r_\infty = \infty$, and so $\hat{u}(r) > u(r)$ for all $r > \hat{r}$. Evidently, $\hat{u}$ cannot be a crossing solution. We can also demonstrate that $\hat{u}$ is not a ground state either, since otherwise, $\hat{u}$ has to decay exponentially as $r \to \infty$, leading to $\lim_{r \to \infty} r^{N-1} w'(r) = 0$, and a contradiction of (5.7) again.

Next we consider the case when $\phi(r) < 0$ *for all* $r > 0$. Denote by $r_p > 0$ the r value where the curves $u(r)$ and $\hat{u}(r)$ first meet beyond the origin; in other words,

$$w(r) = u(r) - \hat{u}(r) > 0 \quad \text{for } 0 < r < r_p, \quad \text{but } w(r_p) = 0. \tag{5.8}$$

Then r_p tends to infinite as $\bar{p} \to p$. Therefore, we can assume that

$$r_p > r^*.$$

By (5.8) $w'(r_p) \leqslant 0$. If $w'(r_p) = 0$, then

$$w''(r_p) = -\hat{u}(r_p) + \hat{u}^{\bar{p}}(r_p) + u(r_p) - u^p(r_p) = u^{\bar{p}}(r_p) - u^p(r_p) < 0,$$

since $\hat{u}(r_p) = u(r_p) < 1$ and $\bar{p} > p$, which forces r_p to be a maximum point of w, this is a contradiction. Therefore, similar to (5.6) we have

$$p\hat{u}^{p-1}(r_p) < 1, \quad w(r_p) = 0 \quad \text{and} \quad w'(r_p) < 0. \tag{5.9}$$

Moreover, the solution curves $u(r)$ and $\hat{u}(r)$ intersect transversally at r_p, and $\hat{u} > u$ for r slightly larger than r_p. For any $r > r_p$ for which $\hat{u} > u$, a similar argument as above can show that

$$w''(r) + \frac{N-1}{r} w'(r) < 0.$$

Thus by the same reasoning as above we can show that $\hat{u}$ can only be an oscillatory function again. This completes the discussion of Case I.

Case II: $u(r) > \hat{u}(r)$ *whenever* $r > 0$ *and* $\hat{u}(r) > 0$.

Obviously, should this case occur, $\hat{u}$ would not be an oscillatory function. We need to prove that $\hat{u}$ is neither a crossing solution, nor a ground state too. We note further that this case could only occur when the ϕ function stays negative for all $r > 0$. Here we have to rely on a somewhat indirect approach based on ideas due to Peletier and Serrin [14], subsequently developed by Cortázar, Elgueta and Felmer [4], Serrin and Tang [17] and Tang [18,19]. For the decreasing functions $u(r)$ and $\hat{u}(r)$, we let $r(u)$ and $s(u)$ denote their respective inverse functions. Thus $r(u)$ and $s(u)$ are both defined and positive for $u \in (0, \alpha)$. Since

$$u_r = \frac{1}{r_u}, \qquad u_{rr} = -\frac{r_{uu}}{r_u^3},$$

the function $r(u)$ satisfies the equation

$$r_{uu} = \frac{N-1}{r}r_u^2 + f(u)r_u^3. \tag{5.10}$$

A similar equation holds for $s(u)$. If $0 < u_c < 1$ is a critical point of $r - s$, then using Eq. (5.10) and the corresponding equation of $s(r)$ we obtain

$$(r-s)''(u_c) = (N-1)r'^2(u_c)\left(\frac{1}{r(u_c)} - \frac{1}{s(u_c)}\right) + r'^3(u_c)(u_c^p - u_c^{\bar{p}}) < 0, \quad ' = \frac{\partial}{\partial u}$$

because $r'(u_c) < 0$, $r(u_c) > s(u_c) > 0$, $u_c < 1$ and $\bar{p} > p$. Hence the difference function $r - s$ cannot assume a positive minimum value in $0 < u < 1$.

By Lemma 4.3, we have

$$\phi'(r_1) = \frac{\partial^2 u}{\partial r \partial p}(r_1) > 0.$$

This gives

$$\frac{\partial^2 r(1)}{\partial p \partial u} = \frac{\partial}{\partial p}\left(\frac{1}{u_r(r_1)}\right) = -\frac{u_{rp}(r_1)}{u_r^2(r_1)} = -\frac{\phi'(r_1)}{u_r^2(r_1)} < 0.$$

Thus, provided that $\bar{p} > p$ is sufficiently close to p, we can assume

$$s'(1) < r'(1) < 0. \tag{5.11}$$

Assume for contradiction that $\hat{u}(r)$ is a crossing solution that vanishes at some $r > 0$, where $\hat{u}'$ is negative and finite. Because u is a ground state, there results

$$\lim_{u \downarrow 0}[r'(u) - s'(u)] = -\infty.$$

Thus the difference $r(u) - s(u)$ is decreasing for small $u > 0$, and is increasing near $u = 1$ by (5.11). Consequently, $r - s$ has to assume a positive minimum value within $u \in (0, 1)$, which contradicts the maximum principle just established above.

Next we assume for contradiction that $\hat{u}$ is itself a ground state. Then $\hat{u}(r) < u(r)$ for all $r > 0$, and so $r - s > 0$ for all $u \in (0, \alpha)$. By (5.11) and the maximum principle it follows that $r - s$ is strictly increasing in $u \in (0, 1)$. Thus

$$r(u) > s(u) > 0, \quad \text{and} \quad s'(u) < r'(u) < 0 \quad \text{for all } 0 < u < 1. \tag{5.12}$$

To derive a contradiction, we consider the function

$$B(u) = \frac{1}{r'^2(u)} - \frac{1}{s'^2(u)}, \qquad B(u) > 0, \tag{5.13}$$

and ideas of Serrin and Tang [17]. We first calculate

$$0 \leqslant \lim_{u \downarrow 0} s^{N-1}(u)B(u) \leqslant \lim_{u \downarrow 0} r^{N-1}(u)B(u)$$

$$\leqslant \lim_{u \downarrow 0} \frac{r^{N-1}(u)}{r'^2(u)} = \lim_{r \to \infty} r^{N-1}u'^2(r) = 0,$$

since the ground state u decays exponentially at infinite. Therefore

$$\lim_{u \downarrow 0} s^{N-1}(u) B(u) = 0. \tag{5.14}$$

Next, we compute, for *small* $u > 0$,

$$\begin{aligned}
\frac{1}{2} B'(u) &= -\frac{r''(u)}{r'^3(u)} + \frac{s''(u)}{s'^3(u)} \\
&= -\frac{N-1}{rr'} + u - u^p + \frac{N-1}{ss'} - u + u^{\bar{p}} \\
&< -\frac{N-1}{rr'} + \frac{N-1}{ss'}.
\end{aligned}$$

Now from Young's inequality follows

$$\begin{aligned}
-\frac{s(u) B'(u)}{s'(u)} &< 2(N-1)\left(\frac{s}{rr's'} - \frac{1}{s'^2}\right) \\
&= 2(N-1)\left(\frac{s}{r} \cdot \frac{1}{|r'||s'|} - \frac{1}{s'^2}\right) < 2(N-1)\left(\frac{1}{|r'||s'|} - \frac{1}{s'^2}\right) \\
&\leqslant (N-1)\left(\frac{1}{r'^2} + \frac{1}{s'^2} - \frac{2}{s'^2}\right) = (N-1)\left(\frac{1}{r'^2} - \frac{1}{s'^2}\right) \\
&= (N-1) B(u).
\end{aligned}$$

Therefore, as $s'(u) < 0$, we obtain

$$s(u) B'(u) + (N-1) s'(u) B(u) < 0.$$

Hence the function $s^{N-1}(u) B(u)$ is decreasing for $0 < u < 1$. By (5.14) we have that $B(u) < 0$ for small $u > 0$, which is impossible in view of (5.13), completing Case II. Thus the monotonicity of $\|u\|_\infty$ is established, finishing the proof of Theorem 1.1.

6. Proof of Theorem 1.2 and properties in finite balls

In this section we prove the Liouville type theorem announced in the introduction. For this purpose we obtain the asymptotic behavior of $\alpha_p = \|u\|_\infty$ when p approaches 1 and $(N+2)/(N-2)$. We also prove the non-monotonicity of the maximum value for solutions of (1.3).

We start with the Pohozaev identity

$$\begin{aligned}
P(r) &= r^N u'^2(r) + 2r^N F\big(u(r)\big) + (N-2) r^{N-1} u(r) u'(r) \\
&= 2 \int_0^r t^{N-1}\big[\sigma u^{p+1}(t) - u^2(t)\big] \, dt, \quad \sigma = \frac{2N - (N-2)(p+1)}{2(p+1)}.
\end{aligned}$$

Because u decays exponentially at infinity, $P(r) \to 0$ as $r \to \infty$. It implies that

$$\alpha_p^{p-1} > \frac{1}{\sigma} = \frac{2(p+1)}{2N - (N-2)(p+1)},$$

yielding at once that

$$\lim_{p \uparrow (N+2)/(N-2)} \alpha_p = \infty.$$

On the other hand, using the monotonicity of α_p proved in Theorem 1.1, we have for any $1 < p < (N+2)/(N-2)$,

$$
\begin{aligned}
\alpha_p &> \lim_{p \downarrow 1} \alpha_p \geqslant \lim_{p \downarrow 1} \sigma^{-1/(p-1)} \\
&= \lim_{p \downarrow 1} \left(\frac{2(p-1)+4}{4-(N-2)(p-1)} \right)^{1/(p-1)} \\
&= \lim_{q \to \infty} \left(1 + \frac{N}{4q+2-N} \right)^q \qquad q = \frac{1}{p-1} \\
&= \lim_{q \to \infty} \left(1 + \frac{1}{\hat{q}} \right)^{n q \hat{q}/(4q+2-N)} \qquad \hat{q} = \frac{4q+2-N}{N} \\
&= e^{N/4}.
\end{aligned}
$$

This yields the lower bound for α_p and the Liouville type theorem for ground states of the scalar field equation. Finally, since the maximum value of a positive solution to the Dirichlet boundary value problem in any finite ball is necessarily larger than α_p, the Liouville type theorem in finite balls also follows. This completes the proof of Theorem 1.2.

To end this section we establish the non-monotonicity in a ball of radius $R > 0$. We know that for each given $1 < p < (N+2)/(N-2)$, there corresponds exactly one positive solution of (1.3). Let $\alpha_p(R)$ be the maximum value of this solution then by the next lemma the non-monotonicity follows

Lemma 6.1.

$$
\lim_{p \to 1^+} \alpha_p(R) = \lim_{p \to ((N+2)/(N-2))^-} \alpha_p(R) = \infty, \quad \forall R > 0. \tag{6.1}
$$

Proof. First, it holds that

$$
\alpha_p(R) > \alpha_p,
$$

where α_p is the maximum value of the ground state. By the fact that α_p approaches infinity as $p \uparrow (N+2)/(N-2)$, we find that

$$
\lim_{p \uparrow (N+2)/(N-2)} \alpha_p(R) = \infty. \tag{6.2}
$$

Second, we can prove

$$
\lim_{p \downarrow 1} \alpha_p(R) = \infty. \tag{6.3}
$$

Indeed, for any finite $\beta > 1$, we can find $p_\beta > 1$ sufficiently close to one such that

$$
\max_{0 < u \leqslant \beta} |f(u)| = \max_{0 < u \leqslant \beta} |-u + u^p| \leqslant \frac{N}{R^2}.
$$

Hence for $p = p_\beta$, and $1 < \alpha \leqslant \beta$, the solution $u(r, \alpha)$ of the initial value problem (1.2) satisfies

$$
\left| r^{N-1} u'(r) \right| = \left| \int_0^r s^{N-1} f(u(s)) \, ds \right| \leqslant \frac{r^N}{R^2}.
$$

This gives, for $0 < r < R$,

$$
|u'(r)| \leqslant \frac{r}{R^2} < \frac{1}{R}
$$

and so $|u(R) - u(0)| < 1$. Consequently, $u(R) > 0$, and $\alpha_p(R) > \beta$ for $p = p_\beta$, from which (6.3) follows. $\quad\square$

Remark 6.1. The limits of (6.2) and (6.3) indicate that $\alpha_p(R)$ is decreasing near $p = 1$ and increasing near $p = (N+2)/(N-2)$. Therefore there exists a critical exponent $p_R \in (1, (N+2)/(N-2))$ at which the least maximum value is achieved. We do not know if this is the only critical point of $\alpha_p(R)$ as a function of p.

Acknowledgements

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