

Relaxation theorems in nonlinear elasticity

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Abstract

Relaxation theorems which apply to one, two and three-dimensional nonlinear elasticity are proved. We take into account the fact an infinite amount of energy is required to compress a finite line, surface or volume into zero line, surface or volume. However, we do not prevent orientation reversal.

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1. Main results

1.1. Introduction

Consider an elastic material occupying in a reference configuration $\Omega \subset \mathbb{R}^N$ with $N = 1, 2$ or 3 , where Ω is bounded and open with Lipschitz boundary $\partial\Omega$. The mechanical properties of the material are characterized by a stored-energy function $W : \mathbb{M}^{3 \times N} \rightarrow [0, +\infty]$ (assumed to be Borel measurable) in terms of which the total stored-energy is the integral

$$I(u) := \int_{\Omega} W(\nabla u(x)) \, dx \tag{1}$$

with $\nabla u(x) \in \mathbb{M}^{3 \times N}$ the gradient of u at x , where $\mathbb{M}^{3 \times N}$ denotes the space of all real $3 \times N$ matrices. In order to take into account the fact that an infinite amount of energy is required to compress a finite line ($N = 1$), surface ($N = 2$) or volume ($N = 3$) into zero line, surface or volume, i.e.,

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$$W(\xi) \rightarrow +\infty \quad \text{as} \quad \begin{cases} |\xi| \rightarrow 0 & \text{if } N = 1, \\ |\xi_1 \wedge \xi_2| \rightarrow 0 & \text{if } N = 2, \\ |\det \xi| \rightarrow 0 & \text{if } N = 3, \end{cases} \tag{2}$$

we consider the following conditions:

(C₁) *there exist $\alpha, \beta > 0$ such that for every $\xi \in \mathbb{M}^{3 \times 1}$,*

$$\text{if } |\xi| \geq \alpha \text{ then } W(\xi) \leq \beta(1 + |\xi|^p)$$

for $N = 1$;

(C₂) *there exist $\alpha, \beta > 0$ such that for every $\xi = (\xi_1 \mid \xi_2) \in \mathbb{M}^{3 \times 2}$,*

$$\text{if } |\xi_1 \wedge \xi_2| \geq \alpha \text{ then } W(\xi) \leq \beta(1 + |\xi|^p)$$

for $N = 2$, where $\xi_1 \wedge \xi_2$ denotes the cross product of vectors $\xi_1, \xi_2 \in \mathbb{R}^3$;

(C₃) *for every $\delta > 0$, there exists $c_\delta > 0$ such that for every $\xi \in \mathbb{M}^{3 \times 3}$,*

$$\text{if } |\det \xi| \geq \delta \text{ then } W(\xi) \leq c_\delta(1 + |\xi|^p),$$

where $\det \xi$ denotes the determinant of ξ , and

(C₄) *$W(P\xi Q) = W(\xi)$ for all $\xi \in \mathbb{M}^{3 \times 3}$ and all $P, Q \in \mathbb{SO}(3)$*

for $N = 3$, with $\mathbb{SO}(3) := \{Q \in \mathbb{M}^{3 \times 3} : Q^T Q = Q Q^T = I_3 \text{ and } \det Q = 1\}$, where I_3 denotes the identity matrix in $\mathbb{M}^{3 \times 3}$ and Q^T is the transposed matrix of Q . (In fact, (C₄) is an additional condition which is not related to (2). However, it means that W is frame-indifferent, i.e., $W(P\xi) = W(\xi)$ for all $\xi \in \mathbb{M}^{3 \times 3}$ and all $P \in \mathbb{SO}(3)$, and isotropic, i.e., $W(\xi Q) = W(\xi)$ for all $\xi \in \mathbb{M}^{3 \times 3}$ and all $Q \in \mathbb{SO}(3)$, see for example [12] for more details.)

Fix $p \in]1, +\infty[$, set $W_g^{1,p}(\Omega; \mathbb{R}^3) := \{u \in W^{1,p}(\Omega; \mathbb{R}^3) : u = g \text{ on } \partial\Omega\}$, where g is given continuous piecewise affine function from Ω to \mathbb{R}^3 , define the integral

$$QI(u) := \int_{\Omega} QW(\nabla u(x)) \, dx,$$

where $QW : \mathbb{M}^{3 \times N} \rightarrow [0, +\infty]$ denotes the quasiconvex envelope of W , and consider the following assertions:

(R₁) $\inf\{I(u) : u \in W_g^{1,p}(\Omega; \mathbb{R}^3)\} = \inf\{QI(u) : u \in W_g^{1,p}(\Omega; \mathbb{R}^3)\}$;

(R₂) *if $u_n \rightharpoonup \bar{u}$ with $\{u_n\}_{n \geq 1}$ minimizing sequence for I in $W_g^{1,p}(\Omega; \mathbb{R}^3)$, then \bar{u} is a minimizer for QI in $W_g^{1,p}(\Omega; \mathbb{R}^3)$;*

(R₃) *if \bar{u} is a minimizer for QI in $W_g^{1,p}(\Omega; \mathbb{R}^3)$, then there exists a minimizing sequence $\{u_n\}_{n \geq 1}$ for I in $W_g^{1,p}(\Omega; \mathbb{R}^3)$ such that $u_n \rightharpoonup \bar{u}$,*

where “ \rightharpoonup ” denotes the weak convergence in $W^{1,p}(\Omega; \mathbb{R}^3)$. In this paper we prove (see Section 1.3) the following relaxation theorems:

Theorem 1.1. *($N = 1$) If (C₁) holds and if W is coercive, i.e., $W(\xi) \geq C|\xi|^p$ for all $\xi \in \mathbb{M}^{3 \times N}$ and some $C > 0$, then (R₁), (R₂) and (R₃) hold.*

Theorem 1.2. *($N = 2$) If (C₂) holds and if W is coercive, then (R₁), (R₂) and (R₃) hold.*

Theorem 1.3. *($N = 3$) If (C₃) and (C₄) hold and if W is coercive, then (R₁), (R₂) and (R₃) hold.*

Typically, these theorems can be applied with stored-energy functions W of the form

$$W(\xi) := |\xi|^p + \begin{cases} h(|\xi|) & \text{if } N = 1, \\ h(|\xi_1 \wedge \xi_2|) & \text{if } N = 2, \\ h(|\det \xi|) & \text{if } N = 3, \end{cases}$$

for all $\xi \in \mathbb{M}^{3 \times N}$, where $h : [0, +\infty[\rightarrow [0, +\infty]$ is Borel measurable and such that for every $\delta > 0$, there exists $r_\delta > 0$ such that $h(t) \leq r_\delta$ for all $t \geq \delta$ (for example, $h(0) = +\infty$ and $h(t) = 1/t^s$ if $t > 0$ with $s > 0$).

1.2. Outline of the paper

Let $\mathcal{I} : W^{1,p}(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty]$ be defined by

$$\mathcal{I}(u) := \begin{cases} \int_{\Omega} W(\nabla u(x)) \, dx & \text{if } u \in W_g^{1,p}(\Omega; \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

let $\mathcal{QI} : W^{1,p}(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty]$ be defined by

$$\mathcal{QI}(u) := \begin{cases} \int_{\Omega} \mathcal{Q}W(\nabla u(x)) \, dx & \text{if } u \in W_g^{1,p}(\Omega; \mathbb{R}^3), \\ +\infty & \text{otherwise,} \end{cases}$$

and let $\bar{\mathcal{I}} : W^{1,p}(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty]$ be the lower semicontinuous envelope (or relaxed functional) of \mathcal{I} with respect to the weak topology of $W^{1,p}(\Omega; \mathbb{R}^3)$, i.e.,

$$\bar{\mathcal{I}}(u) := \inf \left\{ \liminf_{n \rightarrow +\infty} \mathcal{I}(u_n) : u_n \rightharpoonup u \right\}.$$

Set $Y :=]0, 1[^N$ and $\text{Aff}_0(Y; \mathbb{R}^3) := \{\phi \in \text{Aff}(Y; \mathbb{R}^3) : \phi = 0 \text{ on } \partial Y\}$, where $\text{Aff}(Y; \mathbb{R}^3)$ denotes the space of all continuous piecewise affine functions from Y to \mathbb{R}^3 , and consider $\mathcal{Z}W : \mathbb{M}^{3 \times N} \rightarrow [0, +\infty]$ given by

$$\mathcal{Z}W(\xi) := \inf \left\{ \int_Y W(\xi + \nabla \phi(y)) \, dy : \phi \in \text{Aff}_0(Y; \mathbb{R}^3) \right\}.$$

Here is the central theorem of the paper:

Theorem 1.4. *If $\mathcal{Z}W$ is of p -polynomial growth, i.e., $\mathcal{Z}W(\xi) \leq c(1 + |\xi|^p)$ for all $\xi \in \mathbb{M}^{3 \times N}$ and some $c > 0$, then $\bar{\mathcal{I}} = \mathcal{QI}$.*

Here $m = 3$ and $N = 1, 2$ or 3 , but the proof of Theorem 1.4 (given in Section 3) does not depend on the integers m and N . This immediately gives the following relaxation result:

Corollary 1.5. *Under the hypotheses of Theorem 1.4, if W is coercive, then (R_1) , (R_2) and (R_3) hold.*

Such results was proved by Dacorogna in [8] when W is continuous and of p -polynomial growth. The distinguishing feature here is that Theorem 1.4 (and so Corollary 1.5) is compatible with (2). More precisely, in Section 4 we prove the following propositions:

Proposition 1.6. *($N = 1$) If (C_1) holds then $\mathcal{Z}W$ is of p -polynomial growth.*

Proposition 1.7. *($N = 2$) If (C_2) holds then $\mathcal{Z}W$ is of p -polynomial growth.*

Proposition 1.8. *($N = 3$) If (C_3) and (C_4) hold then $\mathcal{Z}W$ is of p -polynomial growth.*

Theorem 1.4 follows from Propositions 1.9 and 1.10 below whose proofs are given in Section 3:

Proposition 1.9. *If $\mathcal{Z}W$ is finite then $\mathcal{Q}W = \mathcal{Q}[\mathcal{Z}W] = \mathcal{Z}W$. Furthermore, for $N = 1$ we have $\mathcal{Z}W = W^{**}$, where W^{**} denotes the lower semicontinuous convex envelope of W .*

Proposition 1.10. $\mathcal{J}_0 = \mathcal{J}_1$ with $\mathcal{J}_0, \mathcal{J}_1 : W^{1,p}(\Omega; \mathbb{R}^3) \rightarrow [0, +\infty]$ respectively defined by

$$\mathcal{J}_0(u) := \inf \left\{ \liminf_{n \rightarrow +\infty} I(u_n) : \text{Aff}_g(\Omega; \mathbb{R}^3) \ni u_n \rightharpoonup u \right\}$$

and

$$\mathcal{I}_1(u) := \inf \left\{ \liminf_{n \rightarrow +\infty} \mathcal{Z}I(u_n) : \text{Aff}_g(\Omega; \mathbb{R}^3) \ni u_n \rightharpoonup u \right\},$$

where $\text{Aff}_g(\Omega; \mathbb{R}^3) := \{u \in \text{Aff}(\Omega; \mathbb{R}^3) : u = g \text{ on } \partial\Omega\}$ and

$$\mathcal{Z}I(u) := \int_{\Omega} \mathcal{Z}W(\nabla u(x)) \, dx.$$

Taking Proposition 1.9 into account, from Propositions 1.6, 1.7 and 1.8, we see that stored-energy functions W satisfying (C_1) for $N = 1$, (C_2) for $N = 2$ and (C_3) and (C_4) for $N = 3$, are not quasiconvex, so that the integral $I(u)$ in (1) is not weakly lower semicontinuous on $W^{1,p}(\Omega; \mathbb{R}^3)$ (see [4, Corollary 3.2]). Thus, the Direct Method of the Calculus of Variations cannot be applied to study the existence of minimizers of I in $W_g^{1,p}(\Omega; \mathbb{R}^3)$. For this reason, in the present paper we establish relaxation theorems instead of existence theorems. (In fact, the term “relaxation” means “generalized existence”, see [10,9,6] for a deeper discussion.)

Other related results can be found in [7,5] where we refer the reader. The present work improves our previous one [1] (see also [2,3]). The main new contribution of the present paper is the treatment of the case $N = 3$.

The plan of the paper is as follows. The proofs of Theorems 1.1, 1.2 and 1.3 are given in Section 1.3 (although these can be easily deduced from the previous discussion). Section 2 presents some preliminaries. In Section 3 we prove Propositions 1.9 and 1.10 and Theorem 1.4. Finally, Section 4 contains the proofs of Propositions 1.6, 1.7 and 1.8.

1.3. Proof of Theorems 1.1, 1.2 and 1.3

According to Corollary 1.5, we see that Theorems 1.1, 1.2 and 1.3 are immediate consequences of respectively Propositions 1.6, 1.7 and 1.8.

2. Preliminaries

In this section we recall some (classical) definitions and results. These will be used throughout the paper.

Let $m, N \geq 1$ be two integers. For any bounded open set $D \subset \mathbb{R}^N$, we denote by $\text{Aff}(D; \mathbb{R}^m)$ the space of all continuous piecewise affine functions from D to \mathbb{R}^m , i.e., $u \in \text{Aff}(D; \mathbb{R}^m)$ if and only if u is continuous and there exists a finite family $(D_i)_{i \in I}$ of open disjoint subsets of D such that $|D \setminus \bigcup_{i \in I} D_i| = 0$ and for every $i \in I$, $\nabla u(x) = \xi_i$ in D_i with $\xi_i \in \mathbb{M}^{m \times N}$ (where $|\cdot|$ denotes the Lebesgue measure in \mathbb{R}^N). For any $g \in W^{1,p}(D; \mathbb{R}^m)$ with $p > 1$, we set

$$\text{Aff}_g(D; \mathbb{R}^m) := \{u \in \text{Aff}(D; \mathbb{R}^m) : u = g \text{ on } \partial D\}$$

$(\text{Aff}_0(D; \mathbb{R}^m))$ corresponds to $\text{Aff}_g(D; \mathbb{R}^m)$ with $g = 0$ and

$$W_g^{1,p}(D; \mathbb{R}^m) := \{u \in W^{1,p}(D; \mathbb{R}^m) : u = g \text{ on } \partial D\},$$

(where ∂D denotes the boundary of D). Fix $g \in W^{1,p}(\Omega; \mathbb{R}^m)$ where $\Omega \subset \mathbb{R}^N$ is bounded and open with Lipschitz boundary. The following density theorem will play an essential role in the proof of Theorem 1.4:

Theorem 2.1. (Ekeland and Temam [10]) $\text{Aff}_g(\Omega; \mathbb{R}^m)$ is dense in $W_g^{1,p}(\Omega; \mathbb{R}^m)$ with respect to the strong topology of $W^{1,p}(\Omega; \mathbb{R}^m)$.

Let $f : \mathbb{M}^{m \times N} \rightarrow [0, +\infty]$ be Borel measurable and let $\mathcal{Z}f : \mathbb{M}^{m \times N} \rightarrow [0, +\infty]$ be defined by

$$\mathcal{Z}f(\xi) := \inf \left\{ \int_Y f(\xi + \nabla \phi(x)) \, dx : \phi \in \text{Aff}_0(Y; \mathbb{R}^m) \right\}$$

with $Y :=]0, 1[^N$. To prove Propositions 1.6, 1.7 and 1.8, we will use the following properties of $\mathcal{Z}f$:

Proposition 2.2. (Fonseca [11])

(i) For every bounded open set $D \subset \mathbb{R}^N$ with $|\partial D| = 0$ and every $\xi \in \mathbb{M}^{m \times N}$,

$$\mathcal{Z}f(\xi) = \inf \left\{ \frac{1}{|D|} \int_D f(\xi + \nabla \phi(x)) \, dx : \phi \in \text{Aff}_0(D; \mathbb{R}^m) \right\}.$$

(ii) If $\mathcal{Z}f$ is finite then $\mathcal{Z}f$ is rank-one convex, i.e., for every $\xi, \xi' \in \mathbb{M}^{m \times N}$ with $\text{rank}(\xi - \xi') \leq 1$,

$$\mathcal{Z}f(\lambda \xi + (1 - \lambda)\xi') \leq \lambda \mathcal{Z}f(\xi) + (1 - \lambda)\mathcal{Z}f(\xi').$$

(iii) If $\mathcal{Z}f$ is finite then $\mathcal{Z}f$ is continuous.

(iv) For every bounded open set $D \subset \mathbb{R}^N$ with $|\partial D| = 0$, every $\xi \in \mathbb{M}^{m \times N}$ and every $\phi \in \text{Aff}_0(D; \mathbb{R}^m)$,

$$\mathcal{Z}f(\xi) \leq \frac{1}{|D|} \int_D \mathcal{Z}f(\xi + \nabla \phi(x)) \, dx.$$

Note that Proposition 2.2 is not exactly the one that can found in Fonseca. Nevertheless, it can be proved using the same arguments than the one given by Fonseca (for more details see [2, Remark A.2]).

Quasiconvexity is the correct concept to deal with multiple integral problems in the Calculus of Variations. For the convenience of the reader, we recall its definition:

Definition 2.3. (Morrey [13]) We say that f is quasiconvex if for every $\xi \in \mathbb{M}^{m \times N}$, every bounded open set $D \subset \mathbb{R}^N$ with $|\partial D| = 0$ and every $\phi \in W_0^{1,\infty}(D; \mathbb{R}^m)$,

$$f(\xi) \leq \frac{1}{|D|} \int_D f(\xi + \nabla \phi(x)) \, dx.$$

Remark 2.4. Clearly, if f is quasiconvex then $\mathcal{Z}f = f$.

By the quasiconvex envelope of f , that we denote $\mathcal{Q}f$, we mean the greatest quasiconvex function which less than or equal to f . (Thus, f is quasiconvex if and only if $\mathcal{Q}f = f$.) To prove Proposition 1.9 we will need Theorem 2.5:

Theorem 2.5. (Dacorogna [8,9]) If f is continuous and finite then $\mathcal{Q}f = \mathcal{Z}f$.

Let $\mathcal{F} : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow [0, +\infty]$ be defined by

$$\mathcal{F}(u) := \begin{cases} \int_{\Omega} f(\nabla u(x)) \, dx & \text{if } u \in W_g^{1,p}(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

let $\mathcal{Q}\mathcal{F} : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow [0, +\infty]$ be given by

$$\mathcal{Q}\mathcal{F}(u) := \begin{cases} \int_{\Omega} \mathcal{Q}f(\nabla u(x)) \, dx & \text{if } u \in W_g^{1,p}(\Omega; \mathbb{R}^m), \\ +\infty & \text{otherwise,} \end{cases}$$

and let $\bar{\mathcal{F}} : W^{1,p}(\Omega; \mathbb{R}^m) \rightarrow [0, +\infty]$ be the lower semicontinuous envelope (or relaxed functional) of \mathcal{I} with respect to the weak topology of $W^{1,p}(\Omega; \mathbb{R}^m)$, i.e.,

$$\bar{\mathcal{F}}(u) := \inf \left\{ \liminf_{n \rightarrow +\infty} \mathcal{F}(u_n) : u_n \rightharpoonup u \right\},$$

where “ \rightharpoonup ” denotes the weak convergence in $W^{1,p}(\Omega; \mathbb{R}^m)$. We close this section with the following integral representation theorem that we will use in the proof of Theorem 1.4:

Theorem 2.6. (Dacorogna [8,9]) *If f is continuous and of p -polynomial growth, i.e., $f(\xi) \leq c(1 + |\xi|^p)$ for all $\xi \in \mathbb{M}^{m \times N}$ and some $c > 0$, then $\overline{\mathcal{F}} = \mathcal{Q}\mathcal{F}$.*

3. Proof of Propositions 1.9 and 1.10 and Theorem 1.4

3.1. Proof of Proposition 1.9

Since $\mathcal{Z}W$ is finite, $\mathcal{Z}W$ is continuous by Proposition 2.2(iii). From Theorem 2.5, we deduce that $\mathcal{Q}[\mathcal{Z}W] = \mathcal{Z}[\mathcal{Z}W]$. But $\mathcal{Z}[\mathcal{Z}W] = \mathcal{Z}W$ by Proposition 2.2(iv), hence $\mathcal{Q}[\mathcal{Z}W] = \mathcal{Z}W$. On the other hand, as $\mathcal{Z}W \leq W$ we have $\mathcal{Q}[\mathcal{Z}W] \leq \mathcal{Q}W$. Moreover, as $\mathcal{Q}W$ is quasiconvex, from Remark 2.4 we see that $\mathcal{Z}[\mathcal{Q}W] = \mathcal{Q}W$, and consequently $\mathcal{Q}W \leq \mathcal{Q}[\mathcal{Z}W]$. It follows that $\mathcal{Q}W = \mathcal{Q}[\mathcal{Z}W] = \mathcal{Z}W$.

Assume that $N = 1$. Then, quasiconvexity is equivalent to convexity (see [9, Theorem 1.1, p. 102]). Thus, $\mathcal{Z}W$ is convex (resp. $\mathcal{Z}W^{**} = W^{**}$) since $\mathcal{Z}W = \mathcal{Q}W$ (resp. W^{**} is convex). But $\mathcal{Z}W$ is continuous (resp. $W^{**} \leq W$), hence $\mathcal{Z}W \leq W^{**}$ (resp. $W^{**} \leq \mathcal{Z}W$). It follows that $\mathcal{Z}W = W^{**}$.

3.2. Proof of Proposition 1.10

Clearly $\mathcal{J}_1 \leq \mathcal{J}_0$. We are thus reduced to prove that

$$\mathcal{J}_0 \leq \mathcal{J}_1. \tag{3}$$

We need the following lemma:

Lemma 3.1. *If $u \in \text{Aff}_g(\Omega; \mathbb{R}^3)$ then $\mathcal{J}_0(u) \leq \mathcal{Z}I(u)$.*

Proof of Lemma 3.1. Let $u \in \text{Aff}_g(\Omega; \mathbb{R}^3)$. By definition, there exists a finite family $(\Omega_i)_{i \in I}$ of open disjoint subsets of Ω such that $|\Omega \setminus \cup_{i \in I} \Omega_i| = 0$ and, for every $i \in I$, $\nabla u(x) = \xi_i$ in Ω_i with $\xi_i \in \mathbb{M}^{3 \times N}$. Given any $\delta > 0$ and any $i \in I$, we consider $\phi_i \in \text{Aff}_0(Y; \mathbb{R}^3)$ such that

$$\int_Y W(\xi_i + \nabla \phi_i(y)) \, dy \leq \mathcal{Z}W(\xi_i) + \frac{\delta}{|\Omega|}. \tag{4}$$

Fix any integer $n \geq 1$. By Vitali’s covering theorem, there exists a finite or countable family $(a_{i,j} + \varepsilon_{i,j}Y)_{j \in J_i}$ of disjoint subsets of Ω_i , where $a_{i,j} \in \mathbb{R}^N$ and $0 < \varepsilon_{i,j} < \frac{1}{n}$, such that $|\Omega_i \setminus \cup_{j \in J_i} (a_{i,j} + \varepsilon_{i,j}Y)| = 0$ (and so $\sum_{j \in J_i} \varepsilon_{i,j}^N = |\Omega_i|$). Define $\psi_n : \Omega \rightarrow \mathbb{R}^3$ by

$$\psi_n(x) := \varepsilon_{i,j} \phi_i \left(\frac{x - a_{i,j}}{\varepsilon_{i,j}} \right) \quad \text{if } x \in a_{i,j} + \varepsilon_{i,j}Y.$$

Clearly, for every $n \geq 1$, $\psi_n \in \text{Aff}_0(\Omega; \mathbb{R}^3)$,

$$\|\psi_n\|_{L^\infty(\Omega; \mathbb{R}^3)} \leq \frac{1}{n} \max_{i \in I} \|\phi_i\|_{L^\infty(Y; \mathbb{R}^3)} \quad \text{and} \quad \|\nabla \psi_n\|_{L^\infty(\Omega; \mathbb{M}^{3 \times N})} \leq \max_{i \in I} \|\nabla \phi_i\|_{L^\infty(Y; \mathbb{M}^{3 \times N})},$$

hence (up to a subsequence) $\psi_n \xrightarrow{*} 0$ in $W^{1,\infty}(\Omega; \mathbb{R}^3)$, where “ $\xrightarrow{*}$ ” denotes the weak* convergence in $W^{1,\infty}(\Omega; \mathbb{R}^3)$. Consequently, $\psi_n \rightarrow 0$ in $W^{1,p}(\Omega; \mathbb{R}^3)$. Moreover,

$$\begin{aligned} I(u + \psi_n) &= \sum_{i \in I} \int_{\Omega_i} W(\xi_i + \nabla \psi_n(x)) \, dx \\ &= \sum_{i \in I} \sum_{j \in J_i} \varepsilon_{i,j}^N \int_Y W(\xi_i + \nabla \phi_i(y)) \, dy \\ &= \sum_{i \in I} |\Omega_i| \int_Y W(\xi_i + \nabla \phi_i(y)) \, dy. \end{aligned}$$

As $u + \psi_n \in \text{Aff}_g(\Omega; \mathbb{R}^3)$ for all $n \geq 1$ and $u + \psi_n \rightharpoonup u$ in $W^{1,p}(\Omega; \mathbb{R}^3)$, from (4) we deduce that

$$\mathcal{J}_0(u) \leq \liminf_{n \rightarrow +\infty} I(u + \psi_n) \leq \sum_{i \in I} |\Omega_i| \mathcal{Z}W(\xi_i) + \delta = \mathcal{Z}I(u) + \delta,$$

and the lemma follows. \square

Fix any $u \in W^{1,p}(\Omega; \mathbb{R}^3)$ and any sequence $u_n \rightharpoonup u$ with $u_n \in \text{Aff}_g(\Omega; \mathbb{R}^3)$. Using Lemma 3.1 we have $\mathcal{J}_0(u_n) \leq \mathcal{Z}I(u_n)$ for all $n \geq 1$. Thus,

$$\mathcal{J}_0(u) \leq \liminf_{n \rightarrow +\infty} \mathcal{J}_0(u_n) \leq \liminf_{n \rightarrow +\infty} \mathcal{Z}I(u_n),$$

and (3) follows.

3.3. Proof of Theorem 1.4

Since $\mathcal{Z}W$ is of p -polynomial growth, $\mathcal{Z}W$ is finite, and so $\mathcal{Z}W$ is continuous by Proposition 2.2(iii). From Theorem 2.1 it follows that

$$\mathcal{J}_1(u) = \inf \left\{ \liminf_{n \rightarrow +\infty} \mathcal{Z}I(u_n) : W_g^{1,p}(\Omega; \mathbb{R}^3) \ni u_n \rightharpoonup u \right\}.$$

But $\mathcal{Q}W = \mathcal{Q}[\mathcal{Z}W]$ by Proposition 1.9, hence $\mathcal{J}_1 = \mathcal{Q}\mathcal{I}$ by Theorem 2.6. On the other hand, given any $u \in W^{1,p}(\Omega; \mathbb{R}^3)$ and any $u_n \rightharpoonup u$ with $u_n \in W_g^{1,p}(\Omega; \mathbb{R}^3)$, we have $\mathcal{Z}I(u_n) \leq I(u_n)$ for all $n \geq 1$. Thus,

$$\mathcal{J}_1(u) \leq \liminf_{n \rightarrow +\infty} \mathcal{Z}I(u_n) \leq \liminf_{n \rightarrow +\infty} I(u_n),$$

and so $\mathcal{J}_1 \leq \bar{\mathcal{I}}$. But $\bar{\mathcal{I}} \leq \mathcal{J}_0$ and $\mathcal{J}_0 = \mathcal{J}_1$ by Proposition 1.10, hence $\bar{\mathcal{I}} = \mathcal{J}_1$, and the theorem follows.

4. Proof of Propositions 1.6, 1.7 and 1.8

4.1. Case $N = 1$

In this section we prove Proposition 1.6.

Proof of Proposition 1.6. By (C_1) it is clear that if $|\xi| \geq \alpha$ then $\mathcal{Z}W(\xi) \leq \beta(1 + |\xi|^p)$. Fix any $\xi \in \mathbb{M}^{3 \times 1}$ such that $|\xi| \leq \alpha$ and consider $\phi \in \text{Aff}_0(Y; \mathbb{R}^3)$ given by

$$\phi(x) := \begin{cases} xv & \text{if } x \in]0, \frac{1}{2}[\\ (1-x)v & \text{if } x \in]\frac{1}{2}, 1[\end{cases}$$

with $v \in \mathbb{R}^3$ such that $|v| = 2\alpha$. Then,

$$|\xi + \nabla\phi(x)| = \begin{cases} |\xi + v| & \text{if } x \in]0, \frac{1}{2}[\\ |\xi - v| & \text{if } x \in]\frac{1}{2}, 1[\end{cases}$$

hence $|\xi + \nabla\phi(x)| \geq \min\{|\xi + v|, |\xi - v|\} \geq |v| - |\xi| \geq \alpha$ for all $x \in]0, \frac{1}{2}[\cup]\frac{1}{2}, 1[$, and from (C_1) we deduce that

$$\mathcal{Z}W(\xi) \leq \int_Y W(\xi + \nabla\phi(x)) \, dx \leq \beta \int_Y (1 + |\xi + \nabla\phi(x)|^p) \, dx \leq \beta 2^{2p} \max\{1, \alpha^p\} (1 + |\xi|^p).$$

It follows that $\mathcal{Z}W(\xi) \leq \beta 2^{2p} \max\{1, \alpha^p\} (1 + |\xi|^p)$ for all $\xi \in \mathbb{M}^{3 \times 1}$. \square

4.2. Case $N = 2$

In this section we prove Proposition 1.7. We begin with the following lemma.

Lemma 4.1. *Under (C₂) there exists $\gamma > 0$ such that for all $\xi = (\xi_1 \mid \xi_2) \in \mathbb{M}^{3 \times 2}$,*

$$\text{if } \min\{|\xi_1 + \xi_2|, |\xi_1 - \xi_2|\} \geq \alpha \text{ then } \mathcal{Z}W(\xi) \leq \gamma(1 + |\xi|^p).$$

Proof. Let $\xi = (\xi_1 \mid \xi_2) \in \mathbb{M}^{3 \times 2}$ be such that $\min\{|\xi_1 + \xi_2|, |\xi_1 - \xi_2|\} \geq \alpha$ (with $\alpha > 0$ given by (C₂)). Then, one the three possibilities holds:

- (i) $|\xi_1 \wedge \xi_2| \neq 0$;
- (ii) $|\xi_1 \wedge \xi_2| = 0$ with $\xi_1 \neq 0$;
- (iii) $|\xi_1 \wedge \xi_2| = 0$ with $\xi_2 \neq 0$.

Set $D := \{(x_1, x_2) \in \mathbb{R}^2: x_1 - 1 < x_2 < x_1 + 1 \text{ and } -x_1 - 1 < x_2 < 1 - x_1\}$ and, for each $t \in \mathbb{R}$, define $\varphi_t \in \text{Aff}_0(D; \mathbb{R})$ by

$$\varphi_t(x_1, x_2) := \begin{cases} -tx_1 + t(x_2 + 1) & \text{if } (x_1, x_2) \in \Delta_1, \\ t(1 - x_1) - tx_2 & \text{if } (x_1, x_2) \in \Delta_2, \\ tx_1 + t(1 - x_2) & \text{if } (x_1, x_2) \in \Delta_3, \\ t(x_1 + 1) + tx_2 & \text{if } (x_1, x_2) \in \Delta_4 \end{cases}$$

with

$$\Delta_1 := \{(x_1, x_2) \in D: x_1 \geq 0 \text{ and } x_2 \leq 0\};$$

$$\Delta_2 := \{(x_1, x_2) \in D: x_1 \geq 0 \text{ and } x_2 \geq 0\};$$

$$\Delta_3 := \{(x_1, x_2) \in D: x_1 \leq 0 \text{ and } x_2 \geq 0\};$$

$$\Delta_4 := \{(x_1, x_2) \in D: x_1 \leq 0 \text{ and } x_2 \leq 0\}.$$

Consider $\phi \in \text{Aff}_0(D; \mathbb{R}^3)$ given by

$$\phi := (\varphi_{v_1}, \varphi_{v_2}, \varphi_{v_3}) \quad \text{with} \quad \begin{cases} v = \frac{\xi_1 \wedge \xi_2}{|\xi_1 \wedge \xi_2|} & \text{if (i) is satisfied,} \\ |v| = 1 \text{ and } \langle \xi_1, v \rangle = 0 & \text{if (ii) is satisfied,} \\ |v| = 1 \text{ and } \langle \xi_2, v \rangle = 0 & \text{if (iii) is satisfied} \end{cases}$$

(v_1, v_2, v_3 are the components of the vector v). Then,

$$\xi + \nabla\phi(x) = \begin{cases} (\xi_1 - v \mid \xi_2 + v) & \text{if } x \in \text{int}(\Delta_1), \\ (\xi_1 - v \mid \xi_2 - v) & \text{if } x \in \text{int}(\Delta_2), \\ (\xi_1 + v \mid \xi_2 - v) & \text{if } x \in \text{int}(\Delta_3), \\ (\xi_1 + v \mid \xi_2 + v) & \text{if } x \in \text{int}(\Delta_4) \end{cases}$$

(where $\text{int}(E)$ denotes the interior of the set E). Taking Proposition 2.2(i) into account, it follows that

$$\mathcal{Z}W(\xi) \leq \frac{1}{4}(W(\xi_1 - v \mid \xi_2 + v) + W(\xi_1 - v \mid \xi_2 - v) + W(\xi_1 + v \mid \xi_2 - v) + W(\xi_1 + v \mid \xi_2 + v)). \quad (5)$$

But $|(\xi_1 - v) \wedge (\xi_2 + v)|^2 = |\xi_1 \wedge \xi_2 + (\xi_1 + \xi_2) \wedge v|^2 = |\xi_1 \wedge \xi_2|^2 + |(\xi_1 + \xi_2) \wedge v|^2 \geq |(\xi_1 + \xi_2) \wedge v|^2$, and so

$$|(\xi_1 + v) \wedge (\xi_2 - v)| \geq |(\xi_1 + \xi_2) \wedge v| = |\xi_1 + \xi_2|.$$

Similarly, we obtain:

$$|(\xi_1 - v) \wedge (\xi_2 - v)| \geq |\xi_1 - \xi_2|;$$

$$|(\xi_1 + v) \wedge (\xi_2 - v)| \geq |\xi_1 + \xi_2|;$$

$$|(\xi_1 + v) \wedge (\xi_2 + v)| \geq |\xi_1 - \xi_2|.$$

Thus, $|(\xi_1 - \nu) \wedge (\xi_2 + \nu)| \geq \alpha$, $|(\xi_1 - \nu) \wedge (\xi_2 - \nu)| \geq \alpha$, $|(\xi_1 + \nu) \wedge (\xi_2 - \nu)| \geq \alpha$ and $|(\xi_1 + \nu) \wedge (\xi_2 + \nu)| \geq \alpha$, because $\min\{|\xi_1 + \xi_2|, |\xi_1 - \xi_2|\} \geq \alpha$. Using (C₂) it follows that

$$\begin{aligned} W(\xi_1 - \nu \mid \xi_2 + \nu) &\leq \beta(1 + |(\xi_1 - \nu \mid \xi_2 + \nu)|^p) \\ &\leq \beta 2^p(1 + |(\xi_1 \mid \xi_2)|^p + |(-\nu \mid \nu)|^p) \\ &\leq \beta 2^{2p+1}(1 + |\xi|^p). \end{aligned}$$

In the same manner, we have:

$$\begin{aligned} W(\xi_1 - \nu \mid \xi_2 - \nu) &\leq \beta 2^{2p+1}(1 + |\xi|^p); \\ W(\xi_1 + \nu \mid \xi_2 - \nu) &\leq \beta 2^{2p+1}(1 + |\xi|^p); \\ W(\xi_1 + \nu \mid \xi_2 + \nu) &\leq \beta 2^{2p+1}(1 + |\xi|^p), \end{aligned}$$

and, from (5), we conclude that $\mathcal{Z}W(\xi) \leq \beta 2^{2p+1}(1 + |\xi|^p)$. \square

Proof of Proposition 1.7. Let $\xi = (\xi_1 \mid \xi_2) \in \mathbb{M}^{3 \times 2}$. Then, one the four possibilities holds:

- (i) $|\xi_1 \wedge \xi_2| \neq 0$;
- (ii) $|\xi_1 \wedge \xi_2| = 0$ with $\xi_1 = \xi_2 = 0$;
- (iii) $|\xi_1 \wedge \xi_2| = 0$ with $\xi_1 \neq 0$;
- (iv) $|\xi_1 \wedge \xi_2| = 0$ with $\xi_2 \neq 0$.

For each $t \in \mathbb{R}$, define $\varphi_t \in \text{Aff}_0(Y; \mathbb{R})$ by

$$\varphi_t(x_1, x_2) := \begin{cases} tx_2 & \text{if } (x_1, x_2) \in \Delta_1, \\ t(1 - x_1) & \text{if } (x_1, x_2) \in \Delta_2, \\ t(1 - x_2) & \text{if } (x_1, x_2) \in \Delta_3, \\ tx_1 & \text{if } (x_1, x_2) \in \Delta_4 \end{cases}$$

with

$$\begin{aligned} \Delta_1 &:= \{(x_1, x_2) \in Y : x_2 \leq x_1 \leq -x_2 + 1\}; \\ \Delta_2 &:= \{(x_1, x_2) \in Y : -x_1 + 1 \leq x_2 \leq x_1\}; \\ \Delta_3 &:= \{(x_1, x_2) \in Y : -x_2 + 1 \leq x_1 \leq x_2\}; \\ \Delta_4 &:= \{(x_1, x_2) \in Y : x_1 \leq x_2 \leq -x_1 + 1\}. \end{aligned}$$

Consider $\phi \in \text{Aff}_0(Y; \mathbb{R}^3)$ given by

$$\phi := (\varphi_{\nu_1}, \varphi_{\nu_2}, \varphi_{\nu_3}) \quad \text{with} \quad \begin{cases} \nu = \frac{\alpha(\xi_1 \wedge \xi_2)}{|\xi_1 \wedge \xi_2|} & \text{if (i) is satisfied,} \\ |\nu| = \alpha & \text{if (ii) is satisfied,} \\ |\nu| = \alpha \text{ and } \langle \xi_1, \nu \rangle = 0 & \text{if (iii) is satisfied,} \\ |\nu| = \alpha \text{ and } \langle \xi_2, \nu \rangle = 0 & \text{if (iv) is satisfied} \end{cases}$$

(ν_1, ν_2, ν_3 are the components of the vector ν and $\alpha > 0$ is given by (C₂)). Then,

$$\xi + \nabla\phi(x) = \begin{cases} (\xi_1 \mid \xi_2 + \nu) & \text{if } x \in \text{int}(\Delta_1), \\ (\xi_1 - \nu \mid \xi_2) & \text{if } x \in \text{int}(\Delta_2), \\ (\xi_1 \mid \xi_2 - \nu) & \text{if } x \in \text{int}(\Delta_3), \\ (\xi_1 + \nu \mid \xi_2) & \text{if } x \in \text{int}(\Delta_4) \end{cases}$$

(where $\text{int}(E)$ denotes the interior of the set E). Taking Proposition 2.2(iv) into account, it follows that

$$\mathcal{Z}W(\xi) \leq \frac{1}{4}(\mathcal{Z}W(\xi_1 \mid \xi_2 + \nu) + \mathcal{Z}W(\xi_1 - \nu \mid \xi_2) + \mathcal{Z}W(\xi_1 \mid \xi_2 - \nu) + \mathcal{Z}W(\xi_1 + \nu \mid \xi_2)). \tag{6}$$

But

$$|\xi_1 + (\xi_2 + \nu)|^2 = |(\xi_1 + \xi_2) + \nu|^2 = |\xi_1 + \xi_2|^2 + |\nu|^2 = |\xi_1 + \xi_2|^2 + \alpha^2 \geq \alpha^2,$$

hence $|\xi_1 + (\xi_2 + \nu)| \geq \alpha$. Similarly, we obtain $|\xi_1 - (\xi_2 + \nu)| \geq \alpha$, and so

$$\min\{|\xi_1 + (\xi_2 + \nu)|, |\xi_1 - (\xi_2 + \nu)|\} \geq \alpha.$$

In the same manner, we have:

$$\min\{|(\xi_1 - \nu) + \xi_2|, |(\xi_1 - \nu) - \xi_2|\} \geq \alpha;$$

$$\min\{|\xi_1 + (\xi_2 - \nu)|, |\xi_1 - (\xi_2 - \nu)|\} \geq \alpha;$$

$$\min\{|(\xi_1 + \nu) + \xi_2|, |(\xi_1 + \nu) - \xi_2|\} \geq \alpha.$$

Using Lemma 4.1 it follows that

$$\begin{aligned} \mathcal{Z}W(\xi_1 | \xi_2 + \nu) &\leq \gamma(1 + |(\xi_1 | \xi_2 + \nu)|^p) \\ &\leq \gamma 2^p(1 + |(\xi_1 | \xi_2)|^p + |(0 | \nu)|^p) \\ &\leq \max\{1, \alpha^p\} \gamma 2^{p+1}(1 + |\xi|^p). \end{aligned}$$

Similarly, we obtain:

$$\mathcal{Z}W(\xi_1 - \nu | \xi_2) \leq \max\{1, \alpha^p\} \gamma 2^{p+1}(1 + |\xi|^p);$$

$$\mathcal{Z}W(\xi_1 | \xi_2 - \nu) \leq \max\{1, \alpha^p\} \gamma 2^{p+1}(1 + |\xi|^p);$$

$$\mathcal{Z}W(\xi_1 + \nu | \xi_2) \leq \max\{1, \alpha^p\} \gamma 2^{p+1}(1 + |\xi|^p),$$

and, from (6), we conclude that $\mathcal{Z}W(\xi) \leq \max\{1, \alpha^p\} \gamma 2^{p+1}(1 + |\xi|^p)$. \square

4.3. Case $N = 3$

In this section we prove Proposition 1.8. We begin with three lemmas.

Lemma 4.2. *If (C_3) holds then $\mathcal{Z}W$ is finite.*

Proof. Clearly, if $\xi \in \mathbb{M}_*^{3 \times 3}$ then $\mathcal{Z}W(\xi) < +\infty$ with $\mathbb{M}_*^{3 \times 3} := \{\xi \in \mathbb{M}^{3 \times 3} : \det \xi \neq 0\}$. We are thus reduced to prove that $\mathcal{Z}W(\xi) < +\infty$ for all $\xi \in \mathbb{M}^{3 \times 3} \setminus \mathbb{M}_*^{3 \times 3}$. Fix $\xi = (\xi_1 | \xi_2 | \xi_3) \in \mathbb{M}^{3 \times 3} \setminus \mathbb{M}_*^{3 \times 3}$ where $\xi_1, \xi_2, \xi_3 \in \mathbb{R}^3$ are the columns of ξ . Then $\text{rank}(\xi) \in \{0, 1, 2\}$ (where $\text{rank}(\xi)$ denotes the rank of the matrix ξ).

Step 1. We prove that if $\text{rank}(\xi) = 2$ then $\mathcal{Z}W(\xi) < +\infty$. Without loss of generality we can assume that there exist $\lambda, \mu \in \mathbb{R}$ such that $\xi_3 = \lambda \xi_1 + \mu \xi_2$. Given any $s \in \mathbb{R}^$, consider $D \subset \mathbb{R}^3$ given by*

$$D := \text{int} \left(\bigcup_{i=1}^8 \Delta_i^s \right) \tag{7}$$

(where $\text{int}(E)$ denotes the interior of the set E) with:

$$\Delta_1^s := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \text{ and } x_1 + x_2 + s x_3 \leq 1\};$$

$$\Delta_2^s := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \leq 0, x_2 \geq 0, x_3 \geq 0 \text{ and } -x_1 + x_2 + s x_3 \leq 1\};$$

$$\Delta_3^s := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \leq 0, x_2 \leq 0, x_3 \geq 0 \text{ and } -x_1 - x_2 + s x_3 \leq 1\};$$

$$\Delta_4^s := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq 0, x_2 \leq 0, x_3 \geq 0 \text{ and } x_1 - x_2 + s x_3 \leq 1\};$$

$$\Delta_5^s := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq 0, x_2 \geq 0, x_3 \leq 0 \text{ and } x_1 + x_2 - s x_3 \leq 1\};$$

$$\Delta_6^s := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \leq 0, x_2 \geq 0, x_3 \leq 0 \text{ and } -x_1 + x_2 - s x_3 \leq 1\};$$

$$\Delta_7^s := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \leq 0, x_2 \leq 0, x_3 \leq 0 \text{ and } -x_1 - x_2 - s x_3 \leq 1\};$$

$$\Delta_8^s := \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 \geq 0, x_2 \leq 0, x_3 \leq 0 \text{ and } x_1 - x_2 - s x_3 \leq 1\}.$$

Clearly, D is bounded, open and $|\partial D| = 0$. For each $t \in \mathbb{R}$, define $\varphi_{s,t} \in \text{Aff}_0(D; \mathbb{R})$ by

$$\varphi_{s,t}(x_1, x_2, x_3) := \begin{cases} -t(x_1 + 1) - tx_2 - tsx_3 & \text{if } (x_1, x_2, x_3) \in \Delta_1^s, \\ tx_1 - t(x_2 + 1) - tsx_3 & \text{if } (x_1, x_2, x_3) \in \Delta_2^s, \\ t(x_1 - 1) + tx_2 - tsx_3 & \text{if } (x_1, x_2, x_3) \in \Delta_3^s, \\ -tx_1 + t(x_2 - 1) - tsx_3 & \text{if } (x_1, x_2, x_3) \in \Delta_4^s, \\ -t(x_1 + 1) - tx_2 + tsx_3 & \text{if } (x_1, x_2, x_3) \in \Delta_5^s, \\ tx_1 - t(x_2 + 1) + tsx_3 & \text{if } (x_1, x_2, x_3) \in \Delta_6^s, \\ t(x_1 - 1) + tx_2 + tsx_3 & \text{if } (x_1, x_2, x_3) \in \Delta_7^s, \\ -tx_1 + t(x_2 - 1) + tsx_3 & \text{if } (x_1, x_2, x_3) \in \Delta_8^s. \end{cases} \tag{8}$$

(Note that $\varphi_{s,t}$ is simply the only function, affine on every Δ_i^s , null on ∂D , such that $\varphi_{s,t}(0) = -t$.) Fix

$$s \in \mathbb{R}^* \setminus \{\lambda - \mu, -(\lambda - \mu), \lambda + \mu, -(\lambda + \mu)\}$$

and consider $\phi \in \text{Aff}_0(D; \mathbb{R}^3)$ given by

$$\phi := (\varphi_{s,v_1}, \varphi_{s,v_2}, \varphi_{s,v_3}) \quad \text{with } v := \frac{\xi_1 \wedge \xi_2}{|\xi_1 \wedge \xi_2|^2},$$

(v_1, v_2, v_3 are the components of the vector v). Then,

$$\xi + \nabla\phi(x) = \begin{cases} (\xi_1 - v \mid \xi_2 - v \mid \xi_3 - sv) & \text{if } x \in \text{int}(\Delta_1^s), \\ (\xi_1 + v \mid \xi_2 - v \mid \xi_3 - sv) & \text{if } x \in \text{int}(\Delta_2^s), \\ (\xi_1 + v \mid \xi_2 + v \mid \xi_3 - sv) & \text{if } x \in \text{int}(\Delta_3^s), \\ (\xi_1 - v \mid \xi_2 + v \mid \xi_3 - sv) & \text{if } x \in \text{int}(\Delta_4^s), \\ (\xi_1 - v \mid \xi_2 - v \mid \xi_3 + sv) & \text{if } x \in \text{int}(\Delta_5^s), \\ (\xi_1 + v \mid \xi_2 - v \mid \xi_3 + sv) & \text{if } x \in \text{int}(\Delta_6^s), \\ (\xi_1 + v \mid \xi_2 + v \mid \xi_3 + sv) & \text{if } x \in \text{int}(\Delta_7^s), \\ (\xi_1 - v \mid \xi_2 + v \mid \xi_3 + sv) & \text{if } x \in \text{int}(\Delta_8^s). \end{cases}$$

As $\det \xi = 0$, $\xi_1 \wedge \xi_3 = \mu(\xi_1 \wedge \xi_2)$ and $\xi_2 \wedge \xi_3 = \lambda(\xi_2 \wedge \xi_1)$ we have

$$|\det(\xi + \nabla\phi(x))| = \begin{cases} |s + (\lambda - \mu)| & \text{if } x \in \text{int}(\Delta_1^s) \cup \text{int}(\Delta_7^s), \\ |s - (\lambda + \mu)| & \text{if } x \in \text{int}(\Delta_2^s) \cup \text{int}(\Delta_8^s), \\ |s - (\lambda - \mu)| & \text{if } x \in \text{int}(\Delta_3^s) \cup \text{int}(\Delta_5^s), \\ |s + (\lambda + \mu)| & \text{if } x \in \text{int}(\Delta_4^s) \cup \text{int}(\Delta_6^s). \end{cases}$$

It follows that for a.e. $x \in D$, $|\det(\xi + \nabla\phi(x))| \geq \min\{|s + (\lambda - \mu)|, |s - (\lambda + \mu)|, |s - (\lambda - \mu)|, |s + (\lambda + \mu)|\} =: \delta$ ($\delta > 0$). Taking Proposition 2.2(i) into account and using (C₃), we see that there exists $c_\delta > 0$ such that

$$\mathcal{Z}W(\xi) \leq \frac{1}{|D|} \int_D W(\xi + \nabla\phi(x)) \leq c_\delta + \frac{c_\delta}{|D|} \|\xi + \nabla\phi\|_{L^p(D; \mathbb{R}^3)}^p,$$

which implies that $\mathcal{Z}W(\xi) < +\infty$.

Step 2. We prove that if $\text{rank}(\xi) = 1$ then $\mathcal{Z}W(\xi) < +\infty$. Without loss of generality we can assume that there exist $\lambda, \mu \in \mathbb{R}$ such that $\xi_2 = \lambda\xi_1$ and $\xi_3 = \mu\xi_1$. Consider $D \subset \mathbb{R}^3$ given by (7) with $s \in \mathbb{R}^* \setminus \{-\mu, \mu\}$, and define $\phi \in \text{Aff}_0(D; \mathbb{R}^3)$ by $\phi := (\varphi_{s,v_1}, \varphi_{s,v_2}, \varphi_{s,v_3})$ with $v = (v_1, v_2, v_3) \in \mathbb{R}^3 \setminus \{0\}$ such that $\langle v, \xi_1 \rangle = 0$, where, for every $i \in \{1, 2, 3\}$, φ_{s,v_i} is defined by (8) with $t = v_i$. By Proposition 2.2(iv) we have

$$\begin{aligned} \mathcal{Z}W(\xi) \leq & \frac{1}{8} (\mathcal{Z}W(\xi_1 - v \mid \xi_2 - v \mid \xi_3 - sv) + \mathcal{Z}W(\xi_1 + v \mid \xi_2 - v \mid \xi_3 - sv) \\ & + \mathcal{Z}W(\xi_1 + v \mid \xi_2 + v \mid \xi_3 - sv) + \mathcal{Z}W(\xi_1 - v \mid \xi_2 + v \mid \xi_3 - sv) \\ & + \mathcal{Z}W(\xi_1 - v \mid \xi_2 - v \mid \xi_3 + sv) + \mathcal{Z}W(\xi_1 + v \mid \xi_2 - v \mid \xi_3 + sv) \\ & + \mathcal{Z}W(\xi_1 + v \mid \xi_2 + v \mid \xi_3 + sv) + \mathcal{Z}W(\xi_1 - v \mid \xi_2 + v \mid \xi_3 + sv)). \end{aligned}$$

Noticing that $s \in \mathbb{R}^* \setminus \{-\mu, \mu\}$ it is easy to see that:

$$\begin{aligned} \text{rank}(\xi_1 - \nu \mid \xi_2 - \nu \mid \xi_3 - s\nu) &= 2; \\ \text{rank}(\xi_1 + \nu \mid \xi_2 - \nu \mid \xi_3 - s\nu) &= 2; \\ \text{rank}(\xi_1 + \nu \mid \xi_2 + \nu \mid \xi_3 - s\nu) &= 2; \\ \text{rank}(\xi_1 - \nu \mid \xi_2 + \nu \mid \xi_3 - s\nu) &= 2; \\ \text{rank}(\xi_1 - \nu \mid \xi_2 - \nu \mid \xi_3 + s\nu) &= 2; \\ \text{rank}(\xi_1 + \nu \mid \xi_2 - \nu \mid \xi_3 + s\nu) &= 2; \\ \text{rank}(\xi_1 + \nu \mid \xi_2 + \nu \mid \xi_3 + s\nu) &= 2; \\ \text{rank}(\xi_1 - \nu \mid \xi_2 + \nu \mid \xi_3 + s\nu) &= 2, \end{aligned}$$

and using Step 1 we deduce that $\mathcal{Z}W(\xi) < +\infty$.

Step 3. We prove that $\mathcal{Z}W(0) < +\infty$. This follows from Step 2 by using Proposition 2.2(iv) with $D \subset \mathbb{R}^3$ given by (7) with $s \in \mathbb{R}^*$, and $\phi \in \text{Aff}_0(D; \mathbb{R}^3)$ defined by $\phi := (\varphi_{s,\nu_1}, \varphi_{s,\nu_2}, \varphi_{s,\nu_3})$ with $(\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3 \setminus \{0\}$, where, for every $i \in \{1, 2, 3\}$, φ_{s,ν_i} is defined by (8) with $t = \nu_i$. \square

Lemma 4.3. *Under (C₃) there exists $c > 0$ such that for every $\xi \in \mathbb{M}^{3 \times 3}$, if ξ is diagonal then $\mathcal{Z}W(\xi) \leq c(1 + |\xi|^p)$.*

Proof. Combining Lemma 4.2 with Proposition 2.2(ii), we deduce that $\mathcal{Z}W$ is continuous, and so there exists $c_0 > 0$ such that $\mathcal{Z}W(\xi) \leq c_0$ for all $\xi \in \mathbb{M}^{3 \times 3}$ with $|\xi|^2 \leq 3$. Moreover, it is obvious that $\mathcal{Z}W(\xi) \leq c_1(1 + |\xi|^p)$ for all $\xi \in \mathbb{M}^{3 \times 3}$ such that $|\det \xi| \geq 1$, where $c_1 > 0$ is given by (C₃) with $\delta = 1$. We are thus led to consider $\xi \in \mathbb{M}^{3 \times 3}$ such that ξ is diagonal, $|\det \xi| \leq 1$ and $|\xi|^2 \geq 3$, i.e., $\xi = (\xi_{ij})$ with $\xi_{ij} = 0$ if $i \neq j$,

$$|\xi_{11}\xi_{22}\xi_{33}| \leq 1 \quad \text{and} \quad |\xi_{11}|^2 + |\xi_{22}|^2 + |\xi_{33}|^2 \geq 3.$$

Then, one the six possibilities holds:

- (i) $|\xi_{11}| \leq 1, |\xi_{22}| \geq 1$ and $|\xi_{33}| \geq 1$;
- (ii) $|\xi_{22}| \leq 1, |\xi_{33}| \geq 1$ and $|\xi_{11}| \geq 1$;
- (iii) $|\xi_{33}| \leq 1, |\xi_{11}| \geq 1$ and $|\xi_{22}| \geq 1$;
- (iv) $|\xi_{11}| \geq 1, |\xi_{22}| \leq 1$ and $|\xi_{33}| \leq 1$;
- (v) $|\xi_{22}| \geq 1, |\xi_{33}| \leq 1$ and $|\xi_{11}| \leq 1$;
- (vi) $|\xi_{33}| \geq 1, |\xi_{11}| \leq 1$ and $|\xi_{22}| \leq 1$.

Claim 1. *There exists $c_2 > 0$ such that if ξ is diagonal with $|\det \xi| \leq 1$ and satisfies either (i), (ii) or (iii), then $\mathcal{Z}W(\xi) \leq c_2(1 + |\xi|^p)$.*

Consider $D \subset \mathbb{R}^3$ given by (7) with $s = 1$, and define $\phi \in \text{Aff}_0(D; \mathbb{R}^3)$ by $\phi := (\varphi_{1,\nu_1}, \varphi_{1,\nu_2}, \varphi_{1,\nu_3})$, where

$$(\nu_1, \nu_2, \nu_3) = \begin{cases} (2, 0, 0) & \text{if (i) is satisfied,} \\ (0, 2, 0) & \text{if (ii) is satisfied,} \\ (0, 0, 2) & \text{if (iii) is satisfied,} \end{cases}$$

and, for every $i \in \{1, 2, 3\}$, φ_{1,ν_i} is defined by (8) with $s = 1$ and $t = \nu_i$. It is then easy to see that for a.e. $x \in D$,

$$|\det(\xi + \nabla\phi(x))| \geq \begin{cases} |2|\xi_{22}||\xi_{33}| - |\det \xi| & \text{if (i) is satisfied,} \\ |2|\xi_{11}||\xi_{33}| - |\det \xi| & \text{if (ii) is satisfied,} \\ |2|\xi_{11}||\xi_{22}| - |\det \xi| & \text{if (iii) is satisfied,} \end{cases}$$

so that $|\det(\xi + \nabla\phi(x))| \geq 1$. Taking Proposition 2.2(i) into account, using (C₃) and noticing that $|\nabla\phi(x)| = 2\sqrt{3}$ for a.e. $x \in D$, we deduce that $\mathcal{Z}W(\xi) \leq c_2(1 + |\xi|^p)$ with $c_2 := c_1 2^p (1 + (2\sqrt{3})^p)$.

Claim 2. *There exists $c_3 > 0$ such that if ξ is diagonal and satisfies either (iv), (v) or (vi), then $\mathcal{Z}W(\xi) \leq c_3(1 + |\xi|^p)$.*

Let $\zeta \in \mathbb{M}^{3 \times 3}$ be a rank-one diagonal matrix defined by

$$\begin{aligned} \zeta_{11} &:= \begin{cases} \xi_{22} + \text{sign}(\xi_{22}) & \text{if (iv) is satisfied,} \\ 0 & \text{if either (v) or (vi) is satisfied;} \end{cases} \\ \zeta_{22} &:= \begin{cases} \xi_{33} + \text{sign}(\xi_{33}) & \text{if (v) is satisfied,} \\ 0 & \text{if either (iv) or (vi) is satisfied;} \end{cases} \\ \zeta_{33} &:= \begin{cases} \xi_{11} + \text{sign}(\xi_{11}) & \text{if (vi) is satisfied,} \\ 0 & \text{if either (iv) or (v) is satisfied,} \end{cases} \end{aligned}$$

where $\text{sign}(r) = 1$ if $r \geq 0$ and $\text{sign}(r) = -1$ if $r < 0$. Then, $\xi^+ := \xi + \zeta$ and $\xi^- := \xi - \zeta$ are diagonal matrices such that:

- (a) $|\xi_{11}^+| \geq 1, |\xi_{22}^+| \geq 1, |\xi_{33}^+| \leq 1$ and $|\xi_{11}^-| \geq 1, |\xi_{22}^-| \geq 1, |\xi_{33}^-| \leq 1$ if (iv) is satisfied;
- (b) $|\xi_{11}^+| \leq 1, |\xi_{22}^+| \geq 1, |\xi_{33}^+| \geq 1$ and $|\xi_{11}^-| \leq 1, |\xi_{22}^-| \geq 1, |\xi_{33}^-| \geq 1$ if (v) is satisfied;
- (c) $|\xi_{11}^+| \geq 1, |\xi_{22}^+| \leq 1, |\xi_{33}^+| \geq 1$ and $|\xi_{11}^-| \geq 1, |\xi_{22}^-| \leq 1, |\xi_{33}^-| \geq 1$ if (vi) is satisfied.

Combining Lemma 4.2 with Proposition 2.2(ii) we deduce that $\mathcal{Z}W$ is rank-one convex, so that

$$\mathcal{Z}W(\xi) \leq \frac{1}{2}(\mathcal{Z}W(\xi^+) + \mathcal{Z}W(\xi^-)). \tag{9}$$

According to (a), (b) and (c), from Claim 1 we see that $\mathcal{Z}W(\xi^+) \leq c_2(1 + |\xi^+|^p)$ (resp. $\mathcal{Z}W(\xi^-) \leq c_2(1 + |\xi^-|^p)$) if $|\det \xi^+| \leq 1$ (resp. $|\det \xi^-| \leq 1$). On the other hand, by (C₃) we have $\mathcal{Z}W(\xi^+) \leq c_1(1 + |\xi^+|^p)$ (resp. $\mathcal{Z}W(\xi^-) \leq c_1(1 + |\xi^-|^p)$) if $|\det \xi^+| \geq 1$ (resp. $|\det \xi^-| \geq 1$). Noticing that $|\xi^+|^p \leq 2^{2p}(1 + |\xi|^p)$ (resp. $|\xi^-|^p \leq 2^{2p}(1 + |\xi|^p)$) and using (9), we deduce that $\mathcal{Z}W(\xi) \leq c_3(1 + |\xi|^p)$ with $c_3 := 2^{2p} \max\{c_1, c_2\}$.

From Claims 1 and 2, it follows that for every $\xi \in \mathbb{M}^{3 \times 3}$, if ξ is diagonal with $|\xi|^2 \geq 3$ and $|\det \xi| \leq 1$ then $\mathcal{Z}W(\xi) \leq c_4(1 + |\xi|^p)$ with $c_4 := \max\{c_2, c_3\}$. Setting $c := \max\{c_0, c_4\}$ we conclude that $\mathcal{Z}W(\xi) \leq c(1 + |\xi|^p)$ for all $\xi \in \mathbb{M}^{3 \times 3}$ such that ξ is diagonal. \square

Lemma 4.4. *If (C₄) holds then $\mathcal{Z}W(P\xi Q) = \mathcal{Z}W(\xi)$ for all $\xi \in \mathbb{M}^{3 \times 3}$ and all $P, Q \in \mathbb{SO}(3)$.*

Proof. It suffices to show that

(i) $\mathcal{Z}W(P\xi Q) \leq \mathcal{Z}W(\xi)$ for all $\xi \in \mathbb{M}^{3 \times 3}$ and all $P, Q \in \mathbb{SO}(3)$.

Indeed, given $\xi \in \mathbb{M}^{3 \times 3}$ and $P, Q \in \mathbb{SO}(3)$, we have $\xi = P^T(P\xi Q)Q^T$, and using (i) we obtain $\mathcal{Z}W(\xi) \leq \mathcal{Z}W(P\xi Q)$. Moreover, (i) is equivalent to

(ii) $\mathcal{Z}W(P\xi) \leq \mathcal{Z}W(\xi)$ for all $\xi \in \mathbb{M}^{3 \times 3}$ and all $P \in \mathbb{SO}(3)$ and

(iii) $\mathcal{Z}W(\xi Q) \leq \mathcal{Z}W(\xi)$ for all $\xi \in \mathbb{M}^{3 \times 3}$ and all $Q \in \mathbb{SO}(3)$.

Indeed, (ii) (resp. (iii)) follows from (i) with $Q = I_3$ (resp. $P = I_3$), where I_3 is the identity matrix in $\mathbb{M}^{3 \times 3}$. On the other hand, given $\xi \in \mathbb{M}^{3 \times 3}$ and $P, Q \in \mathbb{SO}(3)$, by (ii) (resp. (iii)) we have $\mathcal{Z}W(P(\xi Q)) \leq \mathcal{Z}W(\xi Q)$ (resp. $\mathcal{Z}W(\xi Q) \leq \mathcal{Z}W(\xi)$), and so $\mathcal{Z}W(P\xi Q) \leq \mathcal{Z}W(\xi)$. We are thus reduced to prove (ii) and (iii).

Proof of (ii). Fix any $\phi \in \text{Aff}_0(Y; \mathbb{R}^3)$ and set $\varphi := P\phi$. Then, $\varphi \in \text{Aff}_0(Y; \mathbb{R}^3)$ and $\nabla\varphi = P\nabla\phi$, hence

$$\mathcal{Z}W(P\xi) \leq \int_Y W(P\xi + \nabla\varphi(x)) \, dx = \int_Y W(P(\xi + P^T\nabla\varphi(x))) \, dx = \int_Y W(P(\xi + \nabla\phi(x))) \, dx.$$

From (C₄) we deduce that

$$\mathcal{Z}W(P\xi) \leq \int_Y W(\xi + \nabla\phi(x)) \, dx$$

for all $\phi \in \text{Aff}_0(Y; \mathbb{R}^3)$, which implies that $\mathcal{Z}W(P\xi) \leq \mathcal{Z}W(\xi)$. \square

Proof of (iii). By Vitali's covering theorem, there exists a finite or countable family $(a_i + \varepsilon_i Q^T Y)_{i \in I}$ of disjoint subsets of Y , where $a_i \in \mathbb{R}^3$ and $0 < \varepsilon_i < 1$, such that $|Y \setminus \bigcup_{i \in I} (a_i + \varepsilon_i Q^T Y)| = 0$ (and so $\sum_{i \in I} \varepsilon_i^3 = 1$). Fix any $\phi \in \text{Aff}_0(Y; \mathbb{R}^3)$ and define $\varphi \in \text{Aff}_0(Y; \mathbb{R}^3)$ by

$$\varphi(x) = \varepsilon_i \phi \left(Q \frac{x - a_i}{\varepsilon_i} \right) \quad \text{if } x \in a_i + \varepsilon_i Q^T Y.$$

Then,

$$\mathcal{Z}W(\xi Q) \leq \int_Y W(\xi Q + \nabla \varphi(x)) \, dx = \sum_{i \in I} \varepsilon_i^3 \int_Y W(\xi Q + \nabla \varphi(x) Q) \, dx = \int_Y W((\xi + \nabla \phi(x)) Q) \, dx.$$

From (C₄) we deduce that

$$\mathcal{Z}W(\xi Q) \leq \int_Y W(\xi + \nabla \phi(x)) \, dx$$

for all $\phi \in \text{Aff}_0(Y; \mathbb{R}^3)$, which implies that $\mathcal{Z}W(\xi Q) \leq \mathcal{Z}W(\xi)$. \square

Proof of Proposition 1.8. Fix any $\xi \in \mathbb{M}_*^{3 \times 3}$ (with $\mathbb{M}_*^{3 \times 3} := \{\xi \in \mathbb{M}^{3 \times 3} : \det \xi \neq 0\}$) and consider $P \in \mathbb{S}\mathbb{O}(3)$ given by $P := \xi M^{-1}$ with

$$M := \begin{cases} \sqrt{\xi^T \xi} & \text{if } \det \xi > 0, \\ -\sqrt{\xi^T \xi} & \text{if } \det \xi < 0. \end{cases}$$

As M is symmetric, there exist $Q \in \mathbb{S}\mathbb{O}(3)$ and $\zeta \in \mathbb{M}^{3 \times 3}$ such that ζ is diagonal and $M = Q^T \zeta Q$, hence $\xi = P Q^T \zeta Q$. Consequently, $\mathcal{Z}W(\xi) = \mathcal{Z}W(\zeta)$ by Lemma 4.4. Noticing that $|\zeta| = |\xi|$, from Lemma 4.3 we deduce that there exists $c > 0$ such that $\mathcal{Z}W(\xi) \leq c(1 + |\xi|^p)$ for all $\xi \in \mathbb{M}_*^{3 \times 3}$. Combining Lemma 4.2 with Proposition 2.2(iii), we see that $\mathcal{Z}W$ is continuous, and using the fact that $\mathbb{M}_*^{3 \times 3}$ is dense in $\mathbb{M}^{3 \times 3}$, we conclude that $\mathcal{Z}W(\xi) \leq c(1 + |\xi|^p)$ for all $\xi \in \mathbb{M}^{3 \times 3}$. \square

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