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Finite Morse index solutions of exponential problems

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Abstract

We prove that the problem $-\Delta u = \mathrm{e}^u$ has no negative finite Morse index solution on $\mathbb{R}^3$ and give some applications to bounded domain problems.

Résumé

Nous montrons que le problème $-\Delta u = \mathrm{e}^u$ ne possède pas de solution négative d'indice de Morse fini sur $\mathbb{R}^3$ et présentons quelques applications à des problèmes sur des domaines bornés.

Keywords: Finite Morse index solutions; Infinitely many bifurcations

0. Introduction

In this short paper, we prove that there are no finite Morse index negative solutions of

$$-\Delta u = \mathrm{e}^u \tag{1}$$

on $\mathbb{R}^N$ if $N = 3$. The key point is to show that, for such a solution, $\|x\|^2 \mathrm{e}^u$ is bounded for large $\|x\|$. This enables us to apply the results of Bidaut-Véron and Véron [3] to obtain a good asymptotic expansion of e^u for large $\|x\|$. Indeed, our results provide a partial answer to establishing which solutions of (1) satisfy that $\|x\|^2 \mathrm{e}^u$ is bounded. The Vérons called this a difficult problem. Central to our proof of this bound is that (1) has no linearized stable solution. This was proved in [9] for $N = 3$. Note that, if $N \geqslant 10$, it follows from [13] that there is a linearized stable negative radial solution of (1). We explain this more in Section 2.

We give two simple applications. In both, Ω is a smooth bounded domain in $\mathbb{R}^N$, where $N = 3$. First assume that $f : \mathbb{R} \to \mathbb{R}$ is C^1, $f(0) \geqslant 0$ and there exists $q \in \mathbb{Z}$ (possibly zero) such that $f'(y) \sim c y^q \exp y$ as $y \to \infty$ where $c > 0$. Then a sequence u_n of positive solutions of

$$-\Delta u = f(u) \quad \text{in } \Omega,$$

$$u = 0 \quad \text{on } \partial\Omega$$

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are bounded in $L^\infty(\Omega)$ if and only if their Morse indices are bounded. This is an analogue of a result in [2] (for positive solutions) where they assumed $f'(y) \sim c|y|^{p-1}$ as $|y| \to \infty$ where $c > 0$ and $1 < p < (N+2)/(N-2)$. They also allowed changing sign solutions. Similar results for some supercritical problems with power growth appear in [8]. The main interest of our result is that the problem is very much supercritical.

Secondly, assume the conditions above hold except we also require that $f(y) > 0$ for $y > 0$, that f is real analytic in a neighbourhood of $[0, \infty)$ and that either $f(0) > 0$, or $f(0) = 0$ and $f'(0) > 0$. Then as in [10] there is a "natural" arc $\widehat{T}$ of positive solutions of

$$-\Delta u = \lambda f(u) \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega$$

bifurcating from $(0,0)$ (or from $(0, f'(0)^{-1}\lambda_1)$ if $f(0) = 0$) such that $-\Delta - \lambda f'(u)I$ (plus the boundary condition) is invertible on $\widehat{T}$ except at isolated points. Here λ_1 is the smallest eigenvalue of $-\Delta$ on Ω for Dirichlet boundary conditions. We can then prove that, if K is a compact subset of $\widehat{T}$, there are infinitely many bifurcation points in $\widehat{T} \setminus K$ where the bifurcation points may be points where the branch merely changes direction. (Indeed, this is what happens for generic Ω, as in [10].) This very considerably generalizes a result of [13] which was for $f(y) = \mathrm{e}^y$ and Ω a ball. We explain their result in more detail below. The result in [13] was proved by making major uses of symmetries of f and ordinary differential equation techniques. There are also results for an annulus in [16]. Our proof is completely different and holds for much more general domains and much more general f. (It is based partly on ideas in [10].) Note that equations of this type appear in many applications (as in [5] or [13]). Note also that the analyticity can be weakened, but we do not know if it can be completely removed.

In more detail, Joseph and Lundgren [13] prove that, if Ω is a ball and $f(y) = \mathrm{e}^y$, the branch of positive solutions (that is, solutions with $u > 0$ on Ω and $\lambda \geqslant 0$) has the following structure. If $N \geqslant 10$, it consists of a simple continuous curve u_λ^1 for $0 \leqslant \lambda < \alpha$ where $u_0^1 = 0$ and there are no non-negative bounded solutions if $\lambda \geqslant \alpha$. Moreover $\|u_\lambda^1\|_\infty \to \infty$ as $\lambda \to \alpha$. If $N = 1, 2$, the solutions are u_λ^1, u_λ^2 for $0 < \lambda \leqslant \alpha$, where u_λ^i depend continuously on λ, $u_\lambda^1 < u_\lambda^2$ on Ω if $\lambda < \alpha$, $u_\alpha^1 = u_\alpha^2$, $u_\lambda^1 \to 0$ as $\lambda \to 0$ and $\|u_\lambda^2\|_\infty \to \infty$ as $\lambda \to 0$. If $3 \leqslant N \leqslant 9$ the solutions are a single unbounded smooth arc in $C(\Omega) \times [0, \infty)$ starting from $(0,0)$ with infinitely many changes of direction (that is, turning points). Note that the solutions here all satisfy $\lambda \leqslant \beta$ for some β (and thus it is $\|u\|_\infty$ which becomes unbounded). Moving plane techniques ensure that the solutions must be radial and that in the first two cases u_λ^1 is linearized stable when it exists (in the sense to be defined below).

While our proofs resemble in structure the proof of the corresponding result in the supercritical power case in [8], the details are quite different.

This paper and [8] seem to suggest that one can very often understand the finite Morse index solutions rather well if there are very few linearized stable solutions.

We prove the finite Morse index result in Section 1 and briefly consider applications in Section 2.

1. Finite Morse index solutions for $f(y) = \mathrm{e}^y$

The purpose of this section is to prove that (1) has no negative solution u on $\mathbb{R}^3$ of finite Morse index. Here a solution is said to be of finite Morse index if the negative spectrum of the closed linear operator $-\Delta - \mathrm{e}^u I$ on $L^2(\mathbb{R}^3)$ consists of a finite number of points of finite multiplicity. (Note that the linear operator is selfadjoint.) In fact, by scaling, it would suffice to assume u is bounded above. This result is the key result of this paper. Note that the result is false if $N \geqslant 10$. We explain this briefly in Section 2. Note also that the case $N = 3$ is the case of key interest in physical applications.

Theorem 1.1. $-\Delta u = \mathrm{e}^u$ *has no negative locally bounded solution on* $\mathbb{R}^3$ *of finite Morse index.*

We prove this by a series of lemmas. First note that a solution u of (1) is said to be linearized stable if

$$\int_{\mathbb{R}^N} \left(|\nabla\phi|^2 - \mathrm{e}^u\phi^2\right) \geqslant 0$$

for all $\phi \in C_0^\infty(\mathbb{R}^N)$. This was called stability in [9]. On a domain Ω, we require this inequality for $\phi \in C_0^\infty(\Omega)$.

It is sometimes more convenient to instead discuss positive solutions of

$$\Delta v = \mathrm{e}^{-v} \tag{2}$$

on $\mathbb{R}^3$.

Lemma 1.2. (2) *has no positive linearized stable solution on* $\mathbb{R}^3$.

Proof. This is Theorem 1 in [9] and essentially uses some ideas from Ambrosio and Cabré [1]. □

Lemma 1.3. *There exists $\mu > 1$ such that, if v is a positive solution of* (2) *on B_k, then*

$$v(x) \leqslant \mu(v(0) + K) \quad \text{on } B_{k/2},$$

where K depends only on k.

Proof. Let $\bar{v}$ be the average value of v on the sphere S_r. Then, as in [9, p. 964], we easily see that $r^{-2}(r^2\bar{v}'(r))'$ is the average of e^{-v} on S_r and hence $r^{-2}(r^2\bar{v}'(r))' \leqslant 1$ (since $v \geqslant 0$). Now $\bar{v}'(0) = 0$ by the Taylor expansion of v at 0 and $\bar{v}(0) = v(0)$. Hence it follows easily by integrating this differential inequality that $\bar{v}(r) \leqslant v(0) + K$ on B_k. The result then follows from Theorem 3.19 in Hayman and Kennedy [12] if we choose $\rho = \frac{1}{2}r$. □

Lemma 1.4. *Given $s > 0$, there exists k positive such that there is no linearized stable solution v of* (2) *on B_k such that $v \geqslant 0$ on B_k and $v(0) \leqslant s$.*

Proof. Suppose by way of contradiction that there existed linearized stable solutions v_n of (2) on B_n for all n such that $v_n \geqslant 0$ on B_n and $v_n(0) \leqslant s$. If $r > 0$, Lemma 1.3 implies that v_n are uniformly bounded on B_r for large n and thus, by the equation for v_n, Δv_n are uniformly bounded on B_r. Thus, by standard local estimates, $\{v_n\}$ is uniformly bounded in $C^{2,\alpha}$ on $B_{r/2}$. Hence we can choose a subsequence converging uniformly on $B_{r/2}$. Thus, by a standard diagonalization argument, we can choose a subsequence of v_n which converges uniformly on compact sets to a non-negative solution $\tilde{v}$ of $\Delta\tilde{v} = \mathrm{e}^{-\tilde{v}}$ on $\mathbb{R}^3$. Moreover, because the integral in the definition of linearized stability is an integral over compact sets, it follows easily that $\tilde{v}$ is linearized stable on $\mathbb{R}^3$ (since a subsequence of v_n converges uniformly to $\tilde{v}$ on compact sets). This contradicts Lemma 1.2. □

The following lemma is the key lemma. It combines Lemma 1.4 and rescaling with a careful choice of origin.

Lemma 1.5. *If v is a finite Morse index positive solution of* (2) *on $\mathbb{R}^3$, then $\|x\|^2\mathrm{e}^{-v}$ is bounded on $\mathbb{R}^3$.*

Proof. It clearly suffices to find $C > 0$ such that

$$v(x) \geqslant 2\ln\|x\| - C \tag{3}$$

for $\|x\|$ large.

First note that since v has finite Morse index, it is easy to see that there is an $\widetilde{m} > 0$ such that v is linearized stable on $\mathbb{R}^3 \setminus \overline{B}_{\widetilde{m}}$. (A very similar argument appears at the beginning of the proof of Theorem 4 in [7].)

Next note that $v(x) \to \infty$ as $\|x\| \to \infty$. To see this, suppose by way of contradiction that $\|x_i\| \to \infty$ as $i \to \infty$ and $v(x_i) \leqslant \tilde{s}$ for all i. Then $w(x) = v(x - x_i)$ are linearized stable positive solutions of (2) on large balls with $w(0) \leqslant \tilde{s}$. This contradicts Lemma 1.4.

Let $m(t) := \inf\{v(x)\colon\ \|x\| = t\}$. Then m is continuous and $m(t) \to \infty$ as $t \to \infty$. We will have proved (3) if we show that $m(t) - 2\ln t$ is bounded below on $(0, \infty)$. Suppose that this is false. If C is large negative (where C will be chosen later), there exists $\hat{\tau} > 0$ such that

$$m(t) - 2\ln t \geqslant C \quad \text{if } 0 < t \leqslant \hat{\tau} \tag{4}$$

and equality holds if $t = \hat{\tau}$. (Note that $m(t) - 2\ln t \to \infty$ as $t \to 0^+$.) Note also that $\hat{\tau}$ will be large if C is large negative.

Now $\inf\{v(x)\colon \|x\| \geqslant \hat{\tau}\}$ is achieved since $v(x) \to \infty$ as $\|x\| \to \infty$. Suppose it is achieved at y, where $\|y\| = \tau \geqslant \hat{\tau}$. If $\hat{\tau} \leqslant \|x\| \leqslant \tau$, $m(\|x\|) \geqslant m(\tau)$ by the choice of τ. Hence, since log is increasing,

$$m\big(\|x\|\big) - 2\ln\big(\|x\|\big) \geqslant m(\tau) - 2\ln\tau.$$

Hence (4) will continue to hold if we replace $\hat{\tau}$ by τ provided we decrease C so that equality holds for $t = \tau$.

Hence we can assume that

$$m(t) - 2\ln t \geqslant C \quad \text{for } 0 < t \leqslant \tau,$$

equality holds for $t = \tau$ and

$$m(t) \geqslant m(\tau) \quad \text{for } t \geqslant \tau. \tag{5}$$

Choose $s > 2$ and let $s = 2\ln q$. Thus $q > 1$. If $\tau q^{-1} \leqslant t \leqslant \tau$,

$$m(t) - 2\ln t \geqslant m(\tau) - 2\ln\tau,$$

that is,

$$\begin{aligned} m(t) &\geqslant m(\tau) + 2\ln\big(t\tau^{-1}\big) \\ &\geqslant m(\tau) + 2\ln q^{-1} \\ &= m(\tau) - 2\ln q = m(\tau) - s. \end{aligned} \tag{6}$$

Let $\alpha = m(\tau) - s$. Then it is easy to check that $w(x) \equiv -\alpha + v(e^{\frac{1}{2}\alpha}(x - y) + y)$ is a solution of (2) on $\mathbb{R}^3$ and

$$w(y) = -\alpha + v(y) = -\big(m(\tau) - s\big) + m(\tau) = s.$$

Now w is non-negative when

$$v\big(e^{\frac{1}{2}\alpha}(x - y) + y\big) \geqslant m(\tau) - s. \tag{7}$$

We show that this is true if

$$\big\|e^{\frac{1}{2}\alpha}(x - y)\big\| \leqslant \big(1 - q^{-1}\big)\|y\|. \tag{8}$$

To prove this, first note that, by the triangle inequality, (8) implies that

$$\big\|y + e^{\frac{1}{2}\alpha}(x - y)\big\| \geqslant q^{-1}\|y\|.$$

There are two cases to consider. If

$$q^{-1}\|y\| \leqslant \big\|y + e^{\frac{1}{2}\alpha}(x - y)\big\| \leqslant \|y\| = \tau,$$

then (6) implies that

$$v\big(e^{\frac{1}{2}\alpha}(x - y) + y\big) \geqslant m(\tau) - s,$$

as required. If

$$\big\|y + e^{\frac{1}{2}\alpha}(x - y)\big\| \geqslant \|y\| = \tau,$$

then (5) implies that

$$v\big(y + e^{\frac{1}{2}\alpha}(x - y)\big) \geqslant m(\tau) = v(y) \geqslant m(\tau) - s.$$

Hence we have proved that w is non-negative on $W := \{x\colon \|e^{\frac{1}{2}\alpha}(x - y)\| \leqslant (1 - q^{-1})\|y\|\}$. On W,

$$\|x - y\| \leqslant e^{\frac{1}{2}(s - m(\tau))}\big(1 - q^{-1}\big)\|y\|.$$

However $m(\tau) - 2\ln\|y\| = C$ by (4) and the choice of α, and $s = 2\ln q$. Thus, on W,

$$\|x - y\| \leqslant e^{-\frac{1}{2}C} e^{\frac{1}{2}s}\big(1 - e^{-\frac{1}{2}s}\big).$$

Since these last two inequalities are equivalent we see that if C is large negative, w is non-negative on the ball $B_k(y)$ of radius k and centre y. Moreover, if C is large negative, w is linearized stable on $B_k(y)$. This follows since if $x \in B_k(y)$,

$$\begin{aligned}\left\| y+\mathrm{e}^{\frac{1}{2}\alpha}(x-y)\right\| &\geqslant \|y\|-\mathrm{e}^{\frac{1}{2}\alpha}k\\ &=\tau-\mathrm{e}^{\frac{1}{2}(m(\tau)-s)}k\\ &=\tau-\mathrm{e}^{\frac{1}{2}(2\ln\tau+C)}\mathrm{e}^{-\frac{1}{2}s}k \quad \text{by (4)}\\ &=\tau\big(1-\mathrm{e}^{\frac{1}{2}C}\mathrm{e}^{-\frac{1}{2}s}k\big)\\ &\geqslant \widetilde{m}\end{aligned}$$

since, if C is large negative, τ is large.

Hence we have a linearized stable positive solution w on $B_k(y)$ with $w(y)=s$. This contradicts Lemma 1.4 (after a translation) and hence we have proved Lemma 1.5. □

It is now easier to revert to (1) so as to make it more convenient to quote [3].

Lemma 1.6. *If u is a finite Morse index negative solution of* (1)*, then $\|x\|^2\mathrm{e}^{u(x)} \to 2\mathrm{e}^{2w(\theta)}$ as $\|x\|\to\infty$, uniformly in $\theta\in S^2$, and $-\Delta' w=\mathrm{e}^{2w}-1$ on S^2, where Δ' is the Laplace–Beltrami operator on S^2.*

Proof. This is simply Theorem 2.2 in [3]. Lemma 1.5 proves the necessary condition to apply their theorem. □

Proof of Theorem 1.1. If w is a solution of the w equation, it follows by integrating the equation for w over S^2 that the average of e^{2w} over S^2 is 1. Now the functional for the linearized operator at a solution u is

$$J(\phi)=\int_{\mathbb{R}^3}\big(|\nabla\phi|^2-\mathrm{e}^u\phi^2\big).$$

If ϕ is radial, this becomes

$$4\pi\int_0^\infty\Big(\big(\phi'(r)\big)^2-\overline{\mathrm{e}^u}\big(\phi(r)\big)^2\Big)r^2\,\mathrm{d}r,$$

where '$-$' denotes the average on S^2. By the asymptotics in Lemma 1.6, $r^2\overline{\mathrm{e}^u}\to 2(\overline{\mathrm{e}^{2w}})=2$ as $r\to\infty$. Hence we see that u will have infinite Morse index if, given $m\in\mathbb{Z}^+$, we can find $\beta<2$ but close to 2 and m smooth radial functions $\{\phi_i\}_{i=1}^m$ of disjoint compact support so that the ϕ_i have support in r large (where $r^2\overline{\mathrm{e}^u}\geqslant\beta$) and

$$J_1(\phi_i)=\int_0^\infty\Big(\big(\phi_i'(r)\big)^2-\beta r^{-2}\big(\phi_i(r)\big)^2\Big)r^2\,\mathrm{d}r<0$$

for each i. To do this, let us consider the equation $-(r^2z'(r))'=\beta_1 z(r)$, where $\beta_1<\beta$ but close to β. This is an Euler equation which has solutions $r^\alpha\sin(\tilde\tau(r-r_0))$ if $\tilde s=\alpha+\mathrm{i}\tilde\tau$ is a solution of $s(s+1)+\beta_1=0$. The roots of this equation are complex since $\beta_1>1/4$. (Recall β_1 is close to 2.) Thus by choosing a sequence of r_0's tending to ∞, we can find a sequence of solutions ϕ_i of $-(r^2z'(r))'=\beta_1 z(r)$ for $r_i\leqslant r\leqslant\tilde r_i$ vanishing at $r=r_i$ and at $\tilde r_i$ where $r_i\to\infty$ as $i\to\infty$. We extend ϕ_i to be zero outside this range. By choosing a subsequence of ϕ_i's we can arrange that the ϕ_i have disjoint support, and their support is in r large for each i. By a simple integration by parts, $J_1(\phi_i)<0$ for each i. Hence, by our earlier comments, we have proved that u has infinite Morse index. This contradicts our assumptions. Hence no such u exists, as claimed. □

Remark 1.7.

(1) If one examines the argument, we see that it clearly suffices that u is a solution for $\|x\|>\widetilde{R}$ and has finite Morse index there.

(2) The last part could also be proved by using that the best constant in Hardy's inequality is not achieved.
(3) It would be of interest to generalize Theorem 1.1 to higher dimensions. The main difficulty seems to be in generalizing Lemma 1.2. Our arguments certainly show that, if Lemma 1.2 holds, then Lemma 1.5 holds. In particular, it seems likely that our ideas and those of Moschini [14] can be used to prove Theorem 1.1 for $N = 4$.

2. Applications

In this section we indicate two simple applications of Theorem 1.1. Essentially, we combine Theorem 1.1 with known techniques from [2,8,10,11].

Theorem 2.1. *Assume that v_n are positive solutions of $-\Delta u = \exp u$ in Ω, $u = 0$ on $\partial\Omega$ where Ω is a smooth bounded domain in $\mathbb{R}^3$. Then $\{\|v_n\|_\infty\}$ is bounded if and only if $\{m(v_n)\}$ is bounded, where $m(v_n)$ is the Morse index of the solution v_n.*

Proof. If $\{\|v_n\|_\infty\}$ is bounded, it is easy to see that $\{m(v_n)\}$ is bounded. The converse is by a simple blow up argument combined with Theorem 1.1. We assume by way of contradiction that $\|v_n\|_\infty \to \infty$ as $n \to \infty$ but $\{m(v_n)\}$ is bounded. The blow up argument is almost identical to the proof of Theorem 2 in [9] in the case of exponential growth. We let $v_i(x) = v_{i,m} + w_i(x)$ where $v_{i,m}$ is the maximum of v_i on Ω which occurs at x_i and $w_i \leqslant 0$ on Ω. If we rescale x as there, we find that a subsequence of the w_i (rescaled) converges uniformly on compact sets to a non-positive solution w of (1) on $\mathbb{R}^3$. As there, we find that the blow up never yields the half-space problem (essentially because of the effect of the scaling on the boundary conditions). Moreover since the Morse index of v_n (and thus w_n) are uniformly bounded, a similar argument to that in the proof of Theorem 8 in [7] shows that w has finite Morse index. This contradicts Theorem 1.1. □

This generalizes results of [2] and [8] to the exponential case. ([2] was the subcritical case.) As in [8] or [6], we could handle nonlinearities which are asymptotically $y^q e^y$ as $y \to \infty$ and could prove some results for sign changing solutions.

We can use the ideas in the proof of Theorem 2.1 to prove the existence of a linearized stable negative solution of (1) on $\mathbb{R}^N$ if $N \geqslant 10$. We sketch this. As λ increases to α, a rescaling of $u_\lambda^1 - \|u_\lambda^1\|_\infty$ converges uniformly on compact sets to a linearized stable radial solution of (1) on $\mathbb{R}^N$ (since linearized stability is preserved under uniform convergence on compact sets).

Our second result concerns the problem

$$\begin{aligned} -\Delta u &= \lambda f(u) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{9}$$

where Ω is a smooth bounded domain in $\mathbb{R}^3$, $f:\mathbb{R} \to \mathbb{R}$ is C^1, $f(y) > 0$ for $y > 0$ and either $f(0) > 0$, or $f(0) = 0$ and $f'(0) > 0$. Moreover, we assume that $f'(y) \sim y^q e^y$ as $y \to \infty$ (where q may be zero) and that f is real analytic in a neighbourhood of $[0,\infty)$ (or more generally f generates a real analytic map of $\{u \in C^1(\overline{\Omega})\colon u > 0 \text{ in } \Omega, \frac{\partial u}{\partial n} \neq 0$ on $\partial\Omega\}$, with the C^1 norm into $C(\Omega)$). As proved in [10], it is a consequence of analytic bifurcation theory [4] that there is an unbounded connected arc of positive solutions $\widehat{T}$ of (9) such that $(0, \lambda_1(f'(0))^{-1}) \in \widehat{T}$ (or $(0,0) \in \widehat{T}$ if $f(0) > 0$) and $\|u(s)\|' + |\lambda(s)| \to \infty$ as $s \to \infty$ (where $\widehat{T} = \{(u(s), \lambda(s))\colon s \geqslant 0\}$) and $-\Delta - \lambda(s) f'(u(s)) I$ (plus the boundary condition) is invertible except at isolated points. Here $\|\cdot\|'$ is the usual C^1 norm. Thus the implicit function theorem applies on $\widehat{T}$ except at isolated points. Moreover the construction of $\widehat{T}$ is "almost" natural. (This is explained more in [10].)

Theorem 2.2. *Assume that the above conditions hold and S is a bounded subset of $C[0,1] \times \mathbb{R}$. Then $\widehat{T} \setminus S$ contains infinitely many bifurcation points.*

Remark 2.3. These bifurcation points may merely be turning points where the branch bends back. This is always the case where Ω is a ball.

Proof. This is an easy consequence of Theorem 2.1 here, the comments after its proof and the proof of Theorem 4 in [8]. One also needs to note that if (u_n, λ_n) are solutions then standard estimates imply that $\{\|u_n\|\}_\infty$ is bounded if $\{\lambda_n(\|u_n\|_\infty)^q \exp(\|u_n\|_\infty)\}$ is bounded and that the blow up argument in [9] still applies if $\lambda_n(\|u_n\|_\infty)^q \exp(\|u_n\|_\infty) \to \infty$ as $n \to \infty$ and $\|u_n\|_\infty \geqslant 1$ (as can be seen by a simple perusal of the proof there). □

Remark 2.4.

(1) The result still holds for other branches which have the properties we mentioned above on the branch $\widehat{T}$. As in [10], it follows from [15] that if f is C^2, then for "generic" Ω, the branch of positive solutions is a smooth curve (though possibly not connected). Thus $\widehat{T}$ is a smooth curve with no secondary bifurcations off $\widehat{T}$. In particular, for "generic" Ω the analyticity is unnecessary.
(2) Our result contrasts with the supercritical power nonlinearity where it is noted in [8] that the theorem fails on some annuli. The difference seems to come from the very different scaling laws in the two cases.
(3) The results in [13] show that Theorem 2.2 fails on a ball if $N \geqslant 10$.
(4) We do not know if there is a λ where there are infinitely many solutions. This is true for the example in [13] (which was for rather particular f's on a ball).
(5) Note that by the results in [11], the real analyticity assumption is satisfied if

$$f(y) = f_0(y) + \sum_{i=1}^{k} y^{p_i} f_i(y)$$

where f_i are real analytic in a neighbourhood of $[0, \infty)$ and $p_i > 1$ for each i.
(6) It can be shown that an analogue of Theorem 2.2 holds if we replace the conditions at zero by $f(0) = 0$ and $f'(y) \sim q y^{q-1}$ as $y \to 0^+$ where $1 < q \leqslant N/(N-2)$ provided that all the positive solutions of $-\Delta u = u^q$ in Ω, $u = 0$ on $\partial\Omega$ are non-degenerate. It can be shown that this condition is always true for "generic" Ω or for q close to 1.

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