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Solutions of an elliptic system with a nearly critical exponent

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Abstract

Consider the problem

$$-\Delta u_\varepsilon = v_\varepsilon^p, \quad v_\varepsilon > 0 \text{ in } \Omega,$$
$$-\Delta v_\varepsilon = u_\varepsilon^{q_\varepsilon}, \quad u_\varepsilon > 0 \text{ in } \Omega,$$
$$u_\varepsilon = v_\varepsilon = 0 \quad \text{on } \partial\Omega,$$

where Ω is a bounded convex domain in $\mathbb{R}^N$, $N > 2$, with smooth boundary $\partial\Omega$. Here $p, q_\varepsilon > 0$, and

$$\varepsilon := \frac{N}{p+1} + \frac{N}{q_\varepsilon + 1} - (N-2).$$

This problem has positive solutions for $\varepsilon > 0$ (with $pq_\varepsilon > 1$) and no non-trivial solution for $\varepsilon \leqslant 0$. We study the asymptotic behavior of *least energy* solutions as $\varepsilon \to 0^+$. These solutions are shown to blow-up at exactly one point, and the location of this point is characterized. In addition, the shape and exact rates for blowing up are given.

Résumé

Considérons le problème

$$-\Delta u_\varepsilon = v_\varepsilon^p, \quad v_\varepsilon > 0 \text{ en } \Omega,$$
$$-\Delta v_\varepsilon = u_\varepsilon^{q_\varepsilon}, \quad u_\varepsilon > 0 \text{ en } \Omega,$$
$$u_\varepsilon = v_\varepsilon = 0 \quad \text{sur } \partial\Omega,$$

où Ω est un domaine convexe et borné de $\mathbb{R}^N$, $N > 2$, avec la frontière régulière $\partial\Omega$. Ici $p, q_\varepsilon > 0$, et

$$\varepsilon := \frac{N}{p+1} + \frac{N}{q_\varepsilon + 1} - (N-2).$$

Ce problème a des solutions positives pour $\varepsilon > 0$ (avec $pq_\varepsilon > 1$) et aucune solution non-triviale pour $\varepsilon \leqslant 0$. Nous étudions le comportement asymptotique de solutions d'*énergie minimale* quand $\varepsilon \to 0^+$. Ces solutions explosent en un seul point, et la position de ce point est caracterisée. De plus, le profil et les vitesses exactes d'explosion sont donnés.

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1. Introduction

We consider the elliptic system

$$-\Delta u_\varepsilon = v_\varepsilon^p, \quad v_\varepsilon > 0 \text{ in } \Omega, \tag{1.1}$$

$$-\Delta v_\varepsilon = u_\varepsilon^{q_\varepsilon}, \quad u_\varepsilon > 0 \text{ in } \Omega, \tag{1.2}$$

$$u_\varepsilon = v_\varepsilon = 0 \quad \text{on } \partial\Omega, \tag{1.3}$$

where Ω is a bounded convex domain in $\mathbb{R}^N$, $N > 2$, with smooth boundary $\partial\Omega$. Here $p, q_\varepsilon > 0$, and

$$\varepsilon := \frac{N}{p+1} + \frac{N}{q_\varepsilon + 1} - (N-2). \tag{1.4}$$

When $\varepsilon \leqslant 0$, there is no solution for (1.1)–(1.3), see [19] and [23]. On the other hand when $\varepsilon > 0$, we can prove existence of solutions obtained by the variational method. In fact, for $\varepsilon > 0$, the embedding $W^{2,\frac{p+1}{p}}(\Omega) \hookrightarrow L^{q_\varepsilon+1}(\Omega)$ is compact for any $q_\varepsilon + 1 > (p+1)/p$, that is $pq_\varepsilon > 1$. Using this, it is not difficult to show that there exists a function $\bar{u}_\varepsilon$ positive solution of the variational problem

$$S_\varepsilon(\Omega) = \inf\bigl\{\|\Delta u\|_{L^{\frac{p+1}{p}}(\Omega)} \mid u \in W^{2,\frac{p+1}{p}}(\Omega),\ \|u\|_{L^{q_\varepsilon+1}(\Omega)} = 1\bigr\}, \tag{1.5}$$

see for example [24]. This solution satisfies $-\Delta\bar{u}_\varepsilon = \bar{v}_\varepsilon^p$, $-\Delta\bar{v}_\varepsilon = S_\varepsilon(\Omega)\bar{u}_\varepsilon^{q_\varepsilon}$, in Ω and $\bar{u}_\varepsilon = \bar{v}_\varepsilon = 0$ on $\partial\Omega$. After changing to suitable multiples of $\bar{u}_\varepsilon$ and $\bar{v}_\varepsilon$, we obtain u_ε and v_ε solving (1.1)–(1.3). The pair $(u_\varepsilon, v_\varepsilon)$ is called a *least energy solution* of (1.1)–(1.3), which by regularity belongs to $C^2(\overline{\Omega}) \times C^2(\overline{\Omega})$. For others existence results, we refer to [4,7,9,16], and [20].

We observe that we can write the system (1.1)–(1.3) only in terms of u_ε, that is

$$-\Delta(-\Delta u_\varepsilon)^{1/p} = u_\varepsilon^{q_\varepsilon}, \quad u_\varepsilon > 0 \text{ in } \Omega, \tag{1.6}$$

$$u_\varepsilon = \Delta u_\varepsilon = 0 \quad \text{on } \partial\Omega. \tag{1.7}$$

Therefore, we refer to u_ε as the *least energy solution* of (1.6)–(1.7).

Concerning least energy solutions of (1.6)–(1.7), in [24] it was proved that $S_\varepsilon(\Omega) \to S$ as $\varepsilon \downarrow 0$, where S is independent of Ω and moreover is the best Sobolev constant for the inequality

$$\|u\|_{L^{q+1}(\mathbb{R}^N)} \leqslant S^{-\frac{p}{p+1}} \|\Delta u\|_{L^{\frac{p+1}{p}}(\mathbb{R}^N)} \tag{1.8}$$

with p, q, N satisfying

$$\frac{N}{p+1} + \frac{N}{q+1} - (N-2) = 0. \tag{1.9}$$

This shows that the sequence $\{u_\varepsilon\}_{\varepsilon>0}$ of least energy solutions of (1.6)–(1.7) satisfy

$$S_\varepsilon(\Omega) = \frac{\int_\Omega |\Delta u_\varepsilon|^{\frac{p+1}{p}}\,\mathrm{d}x}{\|u_\varepsilon\|_{L^{q_\varepsilon+1}(\Omega)}^{\frac{p+1}{p}}} = S + \mathrm{o}(1) \quad \text{as } \varepsilon \to 0. \tag{1.10}$$

Relation (1.9) defines a curve in $\mathbb{R}_+^2$, for the variables p and q. This curve is the so-called *Sobolev Critical Hyperbola* and replaces the notion of critical exponent in the scalar case. This hyperbola first appeared independently in [4] and [20] and later in [7] and [16].

In this article, we shall study in detail the asymptotic behaviors of the variational solution u_ε, of (1.6)–(1.7) as $\varepsilon \downarrow 0$, that is, as q_ε approaches from *below* to q, in the Sobolev Critical Hyperbola (1.9).

By the symmetry of the hyperbola, we assume without restriction that

$$\frac{2}{N-2} < p \leqslant p^* := \frac{N+2}{N-2}. \tag{1.11}$$

For each fixed value of p, the strict inequality gives a lower bound for the dimension, i.e. $N > \max\{2, 2(p+1)/p\}$.

The asymptotic behaviors of Eqs. (1.6)–(1.7) as $\varepsilon \downarrow 0$ has already been studied for the cases $p = p^*$ and $p = 1$. Next we recall some of these results and explain the relation with ours.

The case $p = p^*$ is equivalent to consider the single equation

$$-\Delta u_\varepsilon = u_\varepsilon^{p^*-\varepsilon} \quad \text{in } \Omega, \quad \text{and} \quad u_\varepsilon = 0 \quad \text{on } \partial\Omega.$$

This problem was studied in [1,10,15,21]. There, exact rates of blow-up were given and the location of blow-up points were characterized. One key ingredient was the *Pohozaev identity* and the observation that the solution u_ε, scaled in the form $\|u_\varepsilon\|^{-1}_{L^\infty(\Omega)} u_\varepsilon$ converges to U solution of

$$-\Delta U = U^{p^*}, \quad U(y) > 0 \quad \text{for } y \in \mathbb{R}^N, \tag{1.12}$$

$$U(0) = 1, \quad U \to 0, \quad \text{as } |y| \to \infty, \tag{1.13}$$

which is unique, explicit, and radially symmetric. For the location of the blow-up and the shape of the solution away of the singularity, it was proved that a scaled u_ε, given by $\|u_\varepsilon\|_{L^\infty(\Omega)} u_\varepsilon$, converges to the Green's function G, solution of $-\Delta G(x, \cdot) = \delta_x$ in Ω, $G(x, \cdot) = 0$ on $\partial\Omega$. The location of blowing-up points are the critical points of $\phi(x) := g(x, x)$ (in fact their minima, see [10]), where $g(x, y)$ is the regular part of $G(x, y)$, i.e.

$$g(x, y) = G(x, y) - \frac{1}{(N-2)\sigma_N |x-y|^{N-2}}.$$

In [6], a similar result was proven in the case $p = 1$ ($N > 4$). There the problem is reduced to study (1.12)–(1.13) with the operator Δ^2 instead of $-\Delta$. Both cases $p = p^*$ and $p = 1$ give the same blow-up rate

$$\varepsilon \|u_\varepsilon\|^2_{L^\infty(\Omega)} \to C \quad \text{as } \varepsilon \to 0^+$$

for some explicit $C := C(p, N, \Omega) > 0$. We can ask ourselves if this behaviors is universal, i.e. holds for all $2/(N-2) < p \leqslant p^*$. We will see later that this is only a coincide; a general result for the blow-up rate is given in Theorems 1.2.

Mimicking the above argument, we will study the asymptotic behaviors of the solution u_ε of (1.6)–(1.7) as $\varepsilon \downarrow 0$. We shall show that $\|u_\varepsilon\|^{-1}_{L^\infty(\Omega)} u_\varepsilon$ converges, as $\varepsilon \downarrow 0$, to the solution U of the problem

$$-\Delta U = V^p, \quad V(y) > 0 \text{ for } y \in \mathbb{R}^N, \tag{1.14}$$

$$-\Delta V = U^q, \quad U(y) > 0 \text{ for } y \in \mathbb{R}^N, \tag{1.15}$$

$$U(0) = 1, \quad U \to 0, \quad V \to 0 \quad \text{as } |y| \to \infty. \tag{1.16}$$

In [5], it was proved that U and V are radially symmetric, if $p \geqslant 1$ and $U \in L^{q+1}(\mathbb{R}^N)$ and $V \in L^{p+1}(\mathbb{R}^N)$. These last integrability conditions hold when considering least energy solutions, see details in Section 2. Thus $U(r) := U(y)$ and $V(r) := V(y)$ with $r = |y|$, moreover U and V are unique, and decreasing in r, see [17,24]. There exist no explicit form of (U, V) for all $p > 2/(N-2)$, however to carry out the analysis it is sufficient to know the asymptotic behaviors of (U, V) as $r \to \infty$, which was studied in [17]. They found

$$\lim_{r\to\infty} r^{N-2} V(r) = a \quad \text{and} \quad \begin{cases} \lim\limits_{r\to\infty} r^{N-2} U(r) = b & \text{if } p > \dfrac{N}{N-2}, \\ \lim\limits_{r\to\infty} \dfrac{r^{N-2}}{\log r} U(r) = b & \text{if } p = \dfrac{N}{N-2}, \\ \lim\limits_{r\to\infty} r^{p(N-2)-2} U(r) = b & \text{if } \dfrac{2}{N-2} < p < \dfrac{N}{N-2}. \end{cases} \tag{1.17}$$

In the following, we restrict further the value of $p \leqslant p^*$ from below:

$$p \geqslant 1 \quad \text{for } N > 4 \quad \text{and} \quad p > \frac{2}{N-2} \quad \text{for } N = 3, 4. \tag{1.18}$$

This restriction is needed since, in different parts of the coming proofs, we use that $p \geqslant 1$, condition automatically satisfies when $N = 3, 4$. We believe however that this restriction is only technical and we conjecture that the results of this paper also hold for $2/(N-2) < p < 1$. In addition, we numerically found (radial) solutions in this range that satisfy Theorem 1.2.

The aim of this paper is to show the following results.

Theorem 1.1. *Let u_ε be a least energy solution of* (1.6)–(1.7) *and* (1.18). *Then*

(a) *there exists $x_0 \in \Omega$ such that, after passing to a subsequence, we have*

$$\text{(i) } u_\varepsilon \to 0 \in C^1\big(\Omega \setminus \{x_0\}\big), \qquad \text{(ii) } v_\varepsilon = |\Delta u_\varepsilon|^{1/p} \to 0 \in C^1\big(\Omega \setminus \{x_0\}\big)$$

as $\varepsilon \to 0^+$ and

$$\text{(iii) } |\Delta u_\varepsilon|^{(p+1)/p} \to \|V\|^{p+1}_{L^{p+1}(\mathbb{R}^N)} \delta_{x_0} \quad \text{as } \varepsilon \to 0^+$$

in the sense of distributions.

(b) *x_0 is a critical point of*

$$\phi(x) := g(x,x) \quad \text{if } p \in \big[N/(N-2), (N+2)/(N-2)\big] \quad \text{and} \tag{1.19}$$

$$\tilde{\phi}(x) := \tilde{g}(x,x) \quad \text{if } p \in \big(2/(N-2), N/(N-2)\big) \tag{1.20}$$

for $x \in \Omega$. The function $\tilde{g}(x,y)$ is defined for $p \in (2/(N-2), N/(N-2))$ by

$$\tilde{g}(x,y) = \widetilde{G}(x,y) - \frac{1}{(p(N-2)-2)(N-p(N-2))(N-2)^p \sigma_N^p |x-y|^{p(N-2)-2}}$$

where $-\Delta \widetilde{G}(x,\cdot) = G^p(x,\cdot)$ in Ω, $\widetilde{G}(x,\cdot) = 0$ on $\partial\Omega$.

This result gives a description of the function whose critical points are the blow-up points. We remark that for $p \in [N/(N-2), (N+2)/(N-2)]$, the critical points remain unchanged and equal to the case of a single equation. Note that if we consider a domain different from a ball, the critical points may change with p in the region $2/(N-2) < p \leqslant N/(N-2)$.

We observe that regularity of $\tilde{\phi}$ is needed to compute its critical points in (b). We show next that $\tilde{\phi}$ is regular. By definition of $\widetilde{G}$, we have

$$\lim_{y \to x} |x-y|^{(p-1)(N-2)} \Delta \tilde{g}(x,y) = -\frac{p g(x,x)}{((N-2)\sigma_N)^{p-1}} \tag{1.21}$$

for $x \in \Omega$. Thus $-\Delta \tilde{g}(x,\cdot) \in L^q(\Omega)$ for any $q \in (N/2, N/(p(N-2)-N+2))$. This implies, by regularity, that $\tilde{g}(x,\cdot) \in L^\infty(\Omega)$ and therefore $\tilde{\phi}(x) = \tilde{g}(x,x)$, $x \in \Omega$ is bounded. In addition, we define

$$\hat{g}(x,y) = \tilde{g}(x,y) + \frac{p g(x,x)|x-y|^{N-p(N-2)}}{(N-p(N-2))(2N-p(N-2)-2)((N-2)\sigma_N)^{p-1}} \tag{1.22}$$

and we have for any $x \in \Omega$ that

$$\lim_{y \to x} |x-y|^{(p-2)(N-2)} \Delta \hat{g}(x,y) = -\frac{p(p-1)g(x,x)}{((N-2)\sigma_N)^{p-2}}. \tag{1.23}$$

Thus $\hat{g}(x,y)$ is regular in y for x fixed. Since $N > p(N-2)$, we take first $y = x$ in (1.22) and then the gradient and we find $\nabla_x \tilde{g}(x,x) = \nabla_x \hat{g}(x,x)$. Hence $\tilde{\phi}(x)$ is regular.

The next two theorems make more precise the behavior of solutions. First we give the rate of blow-up of the maximum of the solutions.

Theorem 1.2. *Let the assumptions of Theorem* 1.1 *be satisfied. Then*

$$\begin{cases} \lim_{\varepsilon\to 0^+} \varepsilon \|u_\varepsilon\|_{L^\infty(\Omega)}^{\frac{N}{p(N-2)-2}+1} = S^{\frac{1-pq}{p(q+1)}} \|U\|^q_{L^q(\mathbb{R}^N)} \|V\|^p_{L^p(\mathbb{R}^N)} |\phi(x_0)| & \text{if } p > \dfrac{N}{N-2}, \\ \lim_{\varepsilon\to 0^+} \varepsilon \dfrac{\|u_\varepsilon\|_{L^\infty(\Omega)}^{\frac{N}{N-2}+1}}{\log \|u_\varepsilon\|_{L^\infty(\Omega)}} = \dfrac{p+1}{N-2} a^{\frac{N}{N-2}} S^{\frac{1-pq}{p(q+1)}} \|U\|^q_{L^q(\mathbb{R}^N)} |\phi(x_0)| & \text{if } p = \dfrac{N}{N-2}, \\ \lim_{\varepsilon\to 0^+} \varepsilon \|u_\varepsilon\|^{p+1}_{L^\infty(\Omega)} = S^{\frac{1-pq}{p(q+1)}} \|U\|^{q(p+1)}_{L^q(\mathbb{R}^N)} |\tilde{\phi}(x_0)| & \text{if } p < \dfrac{N}{N-2}. \end{cases}$$

This theorem gives three regimes of blow-up depending on p. In the case $p > N/(N-2)$, the blow-up rate decreases as $p \downarrow N/(N-2)$ reaching a minimum at $p = N/(N-2)$. There we find a regime with a logarithmic correction. When $p < N/(N-2)$ the blow-up rate increases as $p \downarrow 2/(N-2)$.

Observe that taking $p = q = p^*$, we recover the results in [15,21], that is

$$\varepsilon \|u_\varepsilon\|^2_{L^\infty(\Omega)} \to C \quad \text{as } \varepsilon \to 0^+, \tag{1.24}$$

for some explicitly given $C > 0$. See also [1] for the case when Ω is a ball.

When $N > 4$, we can take $p = 1$, i.e. $q = (N+4)/(N-4)$, recovering the result in [2,6], where they prove that (1.24) holds for some $C > 0$.

The previous theorem is a consequence of the following result, where the behaviors of solutions away from the singularity is given. Here the three regimes also appear and the behaviors of solutions are now given in terms of the Green's function.

Theorem 1.3. *Let the assumptions of Theorem* 1.1 *be satisfied. Then*

$$\lim_{\varepsilon\to 0^+} \|u_\varepsilon\|_{L^\infty(\Omega)} v_\varepsilon(x) = \|U\|^q_{L^q(\mathbb{R}^N)} G(x, x_0), \quad \text{and} \tag{1.25}$$

$$\begin{cases} \lim_{\varepsilon\to 0^+} \|u_\varepsilon\|_{L^\infty(\Omega)}^{\frac{N}{p(N-2)-2}} u_\varepsilon(x) = \|V\|^p_{L^p(\mathbb{R}^N)} G(x, x_0) & \text{if } \quad p > \dfrac{N}{N-2}, \\ \lim_{\varepsilon\to 0^+} \dfrac{\|u_\varepsilon\|_{L^\infty(\Omega)}^{\frac{N}{N-2}}}{\log \|u_\varepsilon\|_{L^\infty(\Omega)}} u_\varepsilon(x) = \dfrac{p+1}{N-2} a^{\frac{N}{N-2}} G(x, x_0) & \text{if } p = \dfrac{N}{N-2}, \\ \lim_{\varepsilon\to 0^+} \|u_\varepsilon\|^p_{L^\infty(\Omega)} u_\varepsilon(x) = \|U\|^{pq}_{L^q(\mathbb{R}^N)} \widetilde{G}(x, x_0) & \text{if } p < \dfrac{N}{N-2}, \end{cases} \tag{1.26}$$

where all the convergences are in $C^{1,\alpha}(\omega)$ with ω any subdomain of Ω not containing x_0. For $p < N/(N-2)$, the convergence in (1.26) *can be improved to $C^{3,\alpha}(\omega)$.*

Let us examine the limit $p \downarrow 2/(N-2)$. In this limit, the exponent of $\|u_\varepsilon\|_{L^\infty(\Omega)}$ in Theorem 1.2 tends to $N/(N-2)$. Next we consider the corresponding behaviors of v_ε. Let $x_\varepsilon \in \Omega$ such that $u_\varepsilon(x_\varepsilon) = \|u_\varepsilon\|_{L^\infty(\Omega)}$. Using (2.7), (2.9) at $y = 0$, and the convergence (2.13), we find

$$\lim_{\varepsilon\to 0} v_\varepsilon(x_\varepsilon) = V(0) \lim_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty(\Omega)}^{\frac{N}{p(N-2)-2}}.$$

This gives

$$\lim_{\varepsilon\to 0} \varepsilon \big[v_\varepsilon(x_\varepsilon)\big]^{\frac{(p+1)[p(N-2)-2]}{N}} = S^{\frac{1-pq}{p(q+1)}} \big[\|U\|^q_{L^q(\mathbb{R}^N)} V(0)^{\frac{(p(N-2)-2)}{N}}\big]^{p+1} |\tilde{\phi}(x_0)|$$

for $p < N/(N-2)$. Note that the exponent of $v_\varepsilon(x_\varepsilon)$ tends to 0^+ as $p \downarrow 2/(N-2)$. Recently in [13] the author studied the limiting case $p = 2/(N-2)$, and found that a positive solution u of $-\Delta(-\Delta u)^{(N-2)/2} = u^q$ in Ω, with $u = \Delta u = 0$ on $\partial\Omega$, remains bounded and develops peak(s) as $q \to \infty$.

We also consider the problem

$$-\Delta(-\Delta u_\varepsilon)^{1/p} = u_\varepsilon^q + \varepsilon u_\varepsilon, \quad u_\varepsilon > 0 \text{ in } \Omega, \tag{1.27}$$

$$u_\varepsilon = \Delta u_\varepsilon = 0 \quad \text{on } \partial\Omega \tag{1.28}$$

for $\varepsilon > 0$. The existence of positive solutions for this problem can be found in [16] and in [20] for the case of a ball. See [14] for related results when $p = 1$. Similarly to the problem (1.6)–(1.7), we can define the least energy solutions for (1.27)–(1.28). Next, we will see a strong link in behaviors between the solutions of the two problems as $\varepsilon \to 0^+$.

The following theorem gives the behaviors of least energy solutions of (1.27)–(1.28) as $\varepsilon \to 0^+$. The blow up rates depend now on the integrability of U^2 in $\mathbb{R}^N$, consequently we divide the result in five cases. The first three cases are the analogous of Theorem 1.2 and there $\|U\|_{L^2(\mathbb{R}^N)} < \infty$ holds. The last two cases are the limiting cases, where we do not have integrability, but we can use the asymptotic behaviors of $U(y)$ as $|y| \to \infty$.

Theorem 1.4. *Let u_ε be a least energy solution of* (1.27)–(1.28) *and* (1.18). *Then the conclusions of Theorem* 1.1 *and Theorem* 1.3 *hold, and for $N > 4$ we have*

$$\lim_{\varepsilon\to 0^+} \varepsilon \|u_\varepsilon\|_{L^\infty(\Omega)}^{2\frac{(N-3)p-3}{(N-2)p-2}} = \|U\|_{L^2(\mathbb{R}^N)}^{-2} \|U\|_{L^q(\mathbb{R}^N)}^q \|V\|_{L^p(\mathbb{R}^N)}^p \left|\phi(x_0)\right| \quad \text{if } p > \frac{N}{N-2}, \tag{1.29}$$

$$\lim_{\varepsilon\to 0^+} \varepsilon \frac{\|u_\varepsilon\|_{L^\infty(\Omega)}^{2\frac{(N-3)p-3}{(N-2)p-2}}}{\log \|u_\varepsilon\|_{L^\infty(\Omega)}} = \frac{p+1}{N-2} a^{\frac{N}{N-2}} \|U\|_{L^2(\mathbb{R}^N)}^{-2} \|U\|_{L^q(\mathbb{R}^N)}^q \left|\phi(x_0)\right| \quad \text{if } p = \frac{N}{N-2}, \tag{1.30}$$

$$\lim_{\varepsilon\to 0^+} \varepsilon \|u_\varepsilon\|_{L^\infty(\Omega)}^{\frac{2[(N-3)p-3]-N}{(N-2)p-2}+p} = \|U\|_{L^2(\mathbb{R}^N)}^{-2} \|U\|_{L^q(\mathbb{R}^N)}^{q(p+1)} \left|\tilde\phi(x_0)\right| \quad \text{if } \frac{N+4}{2(N-2)} < p < \frac{N}{N-2}$$
$$\text{and } p > \frac{\sqrt{(N-4)^2 + (N-2)(N+6)} - (N-4)}{N-2} \tag{1.31}$$

for $N \geqslant 8$, we have

$$\lim_{\varepsilon\to 0^+} \varepsilon \|u_\varepsilon\|_{L^\infty(\Omega)}^{\frac{N-8}{2(N-2)}} \log \|u_\varepsilon\|_{L^\infty(\Omega)} = b^{-2} \|U\|_{L^q(\mathbb{R}^N)} \left|\tilde\phi(x_0)\right| \quad \text{if } p = \frac{N+4}{2(N-2)} \tag{1.32}$$

and for $N = 4$, we have

$$\lim_{\varepsilon\to 0^+} \varepsilon \log \|u_\varepsilon\|_{L^\infty(\Omega)} = b^{-2} \|U\|_{L^q(\mathbb{R}^N)}^q \|V\|_{L^p(\mathbb{R}^N)}^p \left|\phi(x_0)\right| \quad \text{if } p = q = p^* = 3. \tag{1.33}$$

We can check in the three first cases that the corresponding exponents of $\|u_\varepsilon\|_{L^\infty(\Omega)}$ are positive. In particular the case (1.31) with $p = 1$, yields

$$\lim_{\varepsilon\to 0^+} \varepsilon \|u_\varepsilon\|_{L^\infty(\Omega)}^{\frac{2(N-8)}{N-4}} = \|U\|_{L^2(\mathbb{R}^N)}^{-2} \|U\|_{L^{\frac{N+4}{N-4}}(\mathbb{R}^N)}^{\frac{2N+8}{N-4}} \left|\tilde\phi(x_0)\right| \quad \text{for } N > 8.$$

In (1.32) the exponent of $\|u_\varepsilon\|_{L^\infty(\Omega)}$ is non-negative, in fact in the limiting case $N = 8$ and $p = 1$, we have

$$\lim_{\varepsilon\to 0^+} \varepsilon \log \|u_\varepsilon\|_{L^\infty(\Omega)} = b^{-2} \|U\|_{L^3(\mathbb{R}^N)} \left|\tilde\phi(x_0)\right|.$$

Note that for $p = 1$, the function U is known so the constants in the last two cases can be calculated explicitly.

The cases (1.29) (for $p = p^*$) and (1.33) have been found in [15]. Note that in these cases U is a known function and equal to V, so the constants can be computed.

2. Preliminaries

Before proving the main theorem, we need some properties of u_ε. Using that u_ε is a solution of (1.1)–(1.3), we have

$$\int_\Omega (\Delta u_\varepsilon)^{\frac{p+1}{p}} \,\mathrm{d}x = \int_\Omega v_\varepsilon \Delta u_\varepsilon \,\mathrm{d}x = \int_\Omega u_\varepsilon \Delta v_\varepsilon \,\mathrm{d}x = \int_\Omega u_\varepsilon^{q_\varepsilon+1} \,\mathrm{d}x.$$

Then $[S+\mathrm{o}(1)]\|u_\varepsilon\|_{L^{q_\varepsilon+1}(\Omega)}^{\frac{p+1}{p}}=\|u_\varepsilon\|_{L^{q_\varepsilon+1}(\Omega)}^{q_\varepsilon+1}$ implies

$$\lim_{\varepsilon\to 0}\int_\Omega u_\varepsilon^{q_\varepsilon+1}\,\mathrm{d}x=S^{\frac{pq-1}{p(q+1)}}. \tag{2.1}$$

Lemma 2.1. *The minimizing sequence u_ε of* (1.10) *is such that*

$$\|u_\varepsilon\|_{L^\infty(\Omega)}\to\infty$$

moreover $\|(-\Delta u_\varepsilon)^{1/p}\|_{L^\infty(\Omega)}=\|v_\varepsilon\|_{L^\infty(\Omega)}\to\infty$ *as* $\varepsilon\to 0$.

Proof. If $\|u_\varepsilon\|_{L^\infty(\Omega)}\to\infty$ then by regularity, we find $\|v_\varepsilon\|_{L^\infty(\Omega)}\to\infty$, see [12, Theorem 3.7]. Now, assume that $\|u_\varepsilon\|_{L^\infty(\Omega)}\leqslant M$ and $\|v_\varepsilon\|_{L^\infty(\Omega)}\leqslant M$, by elliptic regularity, we have that

$$\|v_\varepsilon\|_{C^{2+\alpha}(\overline{\Omega})}\leqslant M\quad\text{and}\quad\|u_\varepsilon\|_{C^{2+\alpha}(\overline{\Omega})}\leqslant M$$

with $\alpha\in(0,1)$ and some constant M. This implies that there exists $u^*,v^*\in C^2(\overline{\Omega})$, such that

$$u_\varepsilon\to u^*\quad\text{in } C^2(\overline{\Omega}),\quad v_\varepsilon\to v^*\quad\text{in } C^2(\overline{\Omega})\quad\text{as }\varepsilon\to 0.$$

Hence u^* satisfies

$$0\neq\int_\Omega(\Delta u^*)^{\frac{p+1}{p}}\,\mathrm{d}x=S\left[\int_\Omega(u^*)^{q+1}\,\mathrm{d}x\right]^{\frac{(p+1)}{p(q+1)}}$$

which contradicts that S cannot be achieved by a minimizer in a bounded domain, see [24]. In other words there exists no non-trivial solution for

$$-\Delta u^*=(v^*)^p,\quad v>0\text{ in }\Omega, \tag{2.2}$$
$$-\Delta v^*=(u^*)^q,\quad u>0\text{ in }\Omega, \tag{2.3}$$
$$u^*=v^*=0\quad\text{on }\partial\Omega \tag{2.4}$$

in a convex bounded domain, with p,q satisfying (1.9), see [19,23]. □

To simplify notation, we denote

$$\alpha=\frac{N}{q+1}\quad\text{and}\quad\beta=\frac{N}{p+1},$$

so the Sobolev Critical Hyperbola (1.9) takes the form $\alpha+\beta=N-2$.

For any $\varepsilon>0$, let $(u_\varepsilon,v_\varepsilon)$ be a solution of (1.1)–(1.3). By the Pohozaev identity, see [19] or [23], we have for any $\tilde\alpha,\tilde\beta\in\mathbb{R}$ that

$$\left(\frac{N}{q_\varepsilon+1}-\tilde\alpha\right)\int_\Omega u_\varepsilon^{q_\varepsilon+1}\,\mathrm{d}x+\left(\frac{N}{p+1}-\tilde\beta\right)\int_\Omega v_\varepsilon^{p+1}\,\mathrm{d}x+(N-2-\tilde\alpha-\tilde\beta)\int_\Omega(\nabla u_\varepsilon,\nabla v_\varepsilon)\,\mathrm{d}x$$
$$=-\int_{\partial\Omega}(\nabla u_\varepsilon,n)(\nabla v_\varepsilon,x-y)\,\mathrm{d}s. \tag{2.5}$$

We can choose $\tilde\alpha+\tilde\beta=N-2$, $\tilde\alpha=\alpha$ and so $\tilde\beta=\beta$. This implies that

$$\varepsilon\int_\Omega u_\varepsilon^{q_\varepsilon+1}\,\mathrm{d}x=-\int_{\partial\Omega}\frac{\partial u_\varepsilon}{\partial n}\frac{\partial v_\varepsilon}{\partial n}(n,x-y)\,\mathrm{d}s. \tag{2.6}$$

Since u_ε becomes unbounded as $\varepsilon\to 0$ we choose $\mu=\mu(\varepsilon)$ and $x_\varepsilon\in\Omega$ such that

$$u_\varepsilon(x_\varepsilon)=\mu^{-\alpha_\varepsilon}=\|u_\varepsilon\|_{L^\infty(\Omega)} \tag{2.7}$$

where $\alpha_\varepsilon = N/(q_\varepsilon + 1)$. Note that $\mu \to 0$ as $\varepsilon \to 0$.

First we claim that x_ε stays away from the boundary. This is a consequence of the moving plane method and interior estimates [8,11]: let ϕ_1 the positive eigenfunction of $(-\Delta, H_0^1(\Omega))$, normalized to $\max_{x\in\Omega}\phi_1(x) = 1$. Since $p \geqslant 1$, multiplying by ϕ_1 we obtain

$$\lambda_1 \int_\Omega u_\varepsilon\phi_1 = \int_\Omega v_\varepsilon^p\phi_1 \geqslant 2\lambda_1 \int_\Omega v_\varepsilon\phi_1 - C\int_\Omega \phi_1,$$
$$\lambda_1 \int_\Omega v_\varepsilon\phi_1 = \int_\Omega u_\varepsilon^{q_\varepsilon}\phi_1 \geqslant 2\lambda_1 \int_\Omega u_\varepsilon\phi_1 - C\int_\Omega \phi_1$$

for some $C = C(p, q, \lambda_1) > 0$. Hence $\int_\Omega u_\varepsilon\phi_1 \leqslant (C/\lambda_1)\int_\Omega \phi_1$ which implies $\int_{\Omega'} u_\varepsilon \leqslant C(\Omega')$ with $\Omega' \subset \Omega$ and $\int_{\Omega'} v_\varepsilon \leqslant C(\Omega')$. Using the moving planes method [11], we find that there exist $t_0\alpha > 0$ such that

$$u_\varepsilon(x - t\nu) \quad \text{and} \quad v_\varepsilon(x - t\nu) \quad \text{are non-decreasing for } t \in [0, t_0],$$

$\nu \in \mathbb{R}^N$ with $|\nu| = 1$, and $(\nu, n(x)) \geqslant \alpha$ and $x \in \partial\Omega$. Therefore we can find γ, δ such that for any $x \in \{z \in \overline{\Omega}\colon d(z, \partial\Omega) < \delta\} = \Omega_\delta$ there exists a measurable set Γ_x with (i) $\operatorname{meas}(\Gamma_x) \geqslant \gamma$, (ii) $\Gamma_x \subset \Omega \setminus \overline{\Omega}_{\delta/2}$, and (iii) $u_\varepsilon(y) \geqslant u_\varepsilon(x)$ and $v_\varepsilon(y) \geqslant v_\varepsilon(x)$ for any $y \in \Gamma_x$. Then for any $x \in \Omega_\delta$, we have

$$u_\varepsilon(x) \leqslant \frac{1}{\operatorname{meas}(\Gamma_x)}\int_{\Gamma_x} u_\varepsilon(y)\,\mathrm{d}y \leqslant \frac{1}{\gamma}\int_{\Omega_\delta} u_\varepsilon \leqslant C(\Omega_\delta), \quad \text{and}$$
$$v_\varepsilon(x) \leqslant \frac{1}{\operatorname{meas}(\Gamma_x)}\int_{\Gamma_x} v_\varepsilon(y)\,\mathrm{d}y \leqslant \frac{1}{\gamma}\int_{\Omega_\delta} v_\varepsilon \leqslant C(\Omega_\delta).$$

Hence if $u_\varepsilon(x_\varepsilon) \to \infty$, this implies that x_ε will stay out of Ω_δ a neighborhood of the boundary. This proves the claim.

Let $x_\varepsilon \to x_0 \in \Omega$. We define a family of rescaled functions

$$u_{\varepsilon,\mu}(y) = \mu^{\alpha_\varepsilon} u_\varepsilon\big(\mu^{1-\varepsilon/2}y + x_\varepsilon\big), \tag{2.8}$$
$$v_{\varepsilon,\mu}(y) = \mu^{\beta} v_\varepsilon\big(\mu^{1-\varepsilon/2}y + x_\varepsilon\big) \tag{2.9}$$

and find using the definitions of ε, α_ε and β, that

$$-\Delta u_{\varepsilon,\mu} = v_{\varepsilon,\mu}^p \mu^{\alpha_\varepsilon+2-\varepsilon-p\beta} = v_{\varepsilon,\mu}^p \quad \text{in } \Omega_\varepsilon, \tag{2.10}$$
$$-\Delta v_{\varepsilon,\mu} = u_{\varepsilon,\mu}^{q_\varepsilon} \mu^{\beta+2-\varepsilon-q_\varepsilon\alpha_\varepsilon} = u_{\varepsilon,\mu}^{q_\varepsilon} \quad \text{in } \Omega_\varepsilon, \tag{2.11}$$
$$u_{\varepsilon,\mu} = v_{\varepsilon,\mu} = 0 \quad \text{on } \partial\Omega_\varepsilon. \tag{2.12}$$

By equicontinuity and using Arzela–Ascoli, we have that

$$u_{\varepsilon,\mu} \to U \quad \text{and} \quad v_{\varepsilon,\mu} \to V \quad \text{as } \varepsilon \to 0. \tag{2.13}$$

in $C^2(K)$ for any K compact in $\mathbb{R}^N$, where (U, V) satisfies (1.14)–(1.16). Now extending $u_{\varepsilon,\mu}$ and $v_{\varepsilon,\mu}$ by zero outside Ω_ε and using (2.1), by the argument in [22] or [24], we have that $u_{\varepsilon,\mu} \to \overline{U}$ strongly (up to a subsequence) in $W^{2,\frac{p+1}{p}}(\mathbb{R}^N)$. In the limit $\overline{U} \in L^{q+1}(\mathbb{R}^N)$ and $\overline{V} := (-\Delta\overline{U})^{\frac{1}{p}} \in L^{p+1}(\mathbb{R}^N)$, and they satisfy (1.14)–(1.16). Since $p \geqslant 1$, the solution $(\overline{U}, \overline{V})$ is unique and radially symmetric, see [5]. In addition the radial solutions are unique [17,24], so $\overline{U} \equiv U$ and $\overline{V} \equiv V$, consequently

$$\int_{\mathbb{R}^N} [u_{\varepsilon,\mu} - U]^{q+1}(y)\,\mathrm{d}y \to 0, \qquad \int_{\mathbb{R}^N} [v_{\varepsilon,\mu} - V]^{p+1}(y)\,\mathrm{d}y \to 0. \tag{2.14}$$

Lemma 2.2. *There exists $\delta > 0$ such that*

$$\delta \leqslant \mu^\varepsilon \leqslant 1.$$

Proof. Since $\mu \to 0$, we have $\mu^\varepsilon \leqslant 1$. By (2.14), we get $\int_{B_1} u_{\varepsilon,\mu}^{q_\varepsilon+1}\,\mathrm{d}x \geqslant M$, but

$$M \leqslant \int_{B_1} u_{\varepsilon,\mu}^{q_\varepsilon+1}\,\mathrm{d}x = \mu^{\varepsilon N/2} \int_{|y-x_\varepsilon|\leqslant \mu^{1-\varepsilon/2}} u_\varepsilon^{q_\varepsilon+1}(y)\,\mathrm{d}y \leqslant \mu^{\varepsilon N/2} \int_\Omega u_\varepsilon^{q_\varepsilon+1}(y)\,\mathrm{d}y. \tag{2.15}$$

Using the convergence (2.1), we obtain the result. □

Lemma 2.3. *There exists $K > 0$ such that the solution $(u_{\varepsilon,\mu}, v_{\varepsilon,\mu})$ satisfies*

$$u_{\varepsilon,\mu}(y) \leqslant KU(y), \quad v_{\varepsilon,\mu}(y) \leqslant KV(y) \quad \forall y \in \mathbb{R}^N. \tag{2.16}$$

We prove this lemma in Section 4.

Lemma 2.4. *There exists a constant $C > 0$ such that*

$$\varepsilon \leqslant C\mu^{N-2}h(\mu) \quad \textit{with } h(\mu) = \begin{cases} 1 & \textit{for } p > N/(N-2), \\ |\log(\mu)| & \textit{for } p = N/(N-2), \\ \mu^{(p(N-2)-N)} & \textit{for } p < N/(N-2). \end{cases} \tag{2.17}$$

Proof. We will establish the following

$$\int_{\partial\Omega} \frac{\partial u_\varepsilon}{\partial n}\frac{\partial v_\varepsilon}{\partial n}(n,x)\,\mathrm{d}x \leqslant C\mu^{N-2}h(\mu)$$

and from here the result follows applying (2.6). We claim that

$$\left|\frac{\partial u_\varepsilon}{\partial n}\right| \leqslant C\mu^{\alpha_\varepsilon}, \qquad \left|\frac{\partial v_\varepsilon}{\partial n}\right| \leqslant C\mu^{\beta}h(\mu).$$

In the following M is a positive constant that can vary from line to line and we shall use systematically Lemma 2.2.

For $p > N/(N-2)$, using that $-p\beta + N = \beta$, we have

$$\int_\Omega v_\varepsilon^p(x)\,\mathrm{d}x \leqslant M\mu^{-p\beta+N(1-\varepsilon/2)} \int_{\mathbb{R}^N} V^p(y)\,\mathrm{d}y \leqslant M\mu^\beta$$

and by (2.16) there exists $M > 0$ such that

$$v_\varepsilon^p(x) \leqslant M\frac{\mu^{\beta+p(N-2)-N-p(N-2)\varepsilon/2}}{|x-x_0|^{p(N-2)}} \tag{2.18}$$

for $x \neq x_0$. Using that $\beta < \beta + p(N-2) - N$, by Lemma 5.1 we find $|\partial v_\varepsilon/\partial n| \leqslant C\mu^\beta$. For u_ε, using that $-q_\varepsilon\alpha_\varepsilon + N = \alpha_\varepsilon$,

$$\int_\Omega u_\varepsilon^{q_\varepsilon}\,\mathrm{d}x \leqslant M\mu^{-q_\varepsilon\alpha_\varepsilon+N(1-\varepsilon/2)} \int_{\mathbb{R}^N} U^q(y)\,\mathrm{d}y \leqslant M\mu^{\alpha_\varepsilon}$$

and by (2.16) there exist $M > 0$ such

$$u_\varepsilon^{q_\varepsilon}(x) \leqslant M\frac{\mu^{-q_\varepsilon\alpha_\varepsilon+q_\varepsilon(N-2)-q_\varepsilon(N-2)\varepsilon/2}}{|x-x_0|^{q_\varepsilon(N-2)}} \tag{2.19}$$

for $x \neq x_0$. Using that $\alpha_\varepsilon < \alpha_\varepsilon - N + q_\varepsilon(N-2)$, by Lemma 5.1, we obtain $|\partial u_\varepsilon/\partial n| \leqslant C\mu^{\alpha_\varepsilon}$.

For $p < N/(N-2)$, we have

$$\int_\Omega v_\varepsilon^p\,\mathrm{d}x \leqslant M\mu^{-p\beta+p(N-2)(1-\varepsilon/2)} \lim_{\mu\to 0}\frac{1}{\mu^{(p(N-2)-N)(1-\varepsilon/2)}} \int_{B_{1/\mu^{1-\varepsilon/2}}(x_\varepsilon)} V^p(y)\,\mathrm{d}y \tag{2.20}$$

$$\leqslant M\mu^{\beta+(p(N-2)-N)} \tag{2.21}$$

and pointwise for v_ε, we have (2.18) for $x \neq x_0$. Now for u_ε, we have

$$\int_\Omega u_\varepsilon^{q_\varepsilon} \leqslant M\mu^{-q_\varepsilon\alpha_\varepsilon+N(1-\varepsilon/2)} \int_{\mathbb{R}^N} U^q(y)\,\mathrm{d}y \leqslant M\mu^{\alpha_\varepsilon}$$

and by (2.16) there exist $M > 0$ such that

$$u_\varepsilon^{q_\varepsilon}(x) \leqslant M \frac{\mu^{-q_\varepsilon\alpha_\varepsilon+q_\varepsilon(p(N-2)-2)-q_\varepsilon(p(N-2)-2)\varepsilon/2}}{|x-x_0|^{q_\varepsilon(p(N-2)-2)}} \tag{2.22}$$

for $x \neq x_0$. From these estimates we prove the claim applying Lemma 5.1 and noting that $\alpha_\varepsilon < \alpha_\varepsilon - N + q_\varepsilon(p(N-2)-2) + (p+1)\varepsilon/\alpha_\varepsilon$. For the case $p = N/(N-2)$, we proceed as before noting that

$$\int_\Omega v_\varepsilon^p\,\mathrm{d}x \leqslant M\mu^{-p\beta+N(1-\varepsilon/2)}\big|\log(\mu)\big| \lim_{\mu\to 0}\frac{1}{|\log(\mu)|}\int_{B_{1/\mu^{1-\varepsilon/2}}(x_\varepsilon)} V^p(y)\,\mathrm{d}y \leqslant M\big|\log(\mu)\big|\mu^\beta$$

and for $x \neq x_0$ we have (2.18). Similarly to (2.22), we obtain that for $x \neq x_0$, there exist $M > 0$ such that

$$u_\varepsilon^{q_\varepsilon}(x) \leqslant M\frac{\mu^{-q_\varepsilon\alpha_\varepsilon+q_\varepsilon(N-2)-q_\varepsilon(N-2)\varepsilon/2}}{|x-x_0|^{q_\varepsilon(N-2)}}\log\big(|x-x_0|\mu^{-1+\varepsilon/2}\big)^{q_\varepsilon}. \tag{2.23}$$

Using this and proceeding as before we prove the claim and the lemma follows. □

Lemma 2.5.

$$|\mu^\varepsilon - 1| = \mathrm{O}\big(\mu^{N-2}h(\mu)\log\mu\big).$$

Proof. By the theorem of the mean $|\mu^\varepsilon - 1| = |\mu^{s\varepsilon}\varepsilon\log\mu|$ for some $s \in (0,1)$ and therefore (2.17) gives the result. □

3. Proof of the theorems

We shall give only the proof of Theorems 1.1, 1.2 and 1.3. The proof of Theorem 1.4 is almost identical to the first three theorems. In fact the main difference is the Pohozaev identity (2.6), which now reads

$$\varepsilon\int_\Omega u_\varepsilon^2\,\mathrm{d}x = -\int_{\partial\Omega}\frac{\partial u_\varepsilon}{\partial n}\frac{\partial v_\varepsilon}{\partial n}(n, x-y)\,\mathrm{d}s. \tag{3.1}$$

Proof of Theorem 1.3. We start by proving the case $p > N/(N-2)$. We have

$$-\Delta\big(\|u_\varepsilon\|_{L^\infty(\Omega)}^{\beta/\alpha}u_\varepsilon\big) = \|u_\varepsilon\|_{L^\infty(\Omega)}^{\beta/\alpha}v_\varepsilon^p \quad \text{in } \Omega, \tag{3.2}$$

$$-\Delta\big(\|u_\varepsilon\|_{L^\infty(\Omega)}v_\varepsilon\big) = \|u_\varepsilon\|_{L^\infty(\Omega)}u_\varepsilon^{q_\varepsilon} \quad \text{in } \Omega, \tag{3.3}$$

$$u_\varepsilon = v_\varepsilon = 0 \quad \text{on } \partial\Omega. \tag{3.4}$$

We integrate the right-hand side of (3.2)

$$\int_\Omega \|u_\varepsilon\|_{L^\infty(\Omega)}^{\beta/\alpha}v_\varepsilon^p\,\mathrm{d}x = \mu^{-(p+1)\beta+N+N\varepsilon/2}\int_{\Omega_\varepsilon} v_{\varepsilon,\mu}^p(y)\,\mathrm{d}y.$$

But $N-(p+1)\beta = 0$, so using (2.16) by dominated convergence and Lemma 2.5, we get

$$\lim_{\varepsilon\to 0}\int_\Omega \|u_\varepsilon\|_{L^\infty(\Omega)}^{\beta/\alpha}v_\varepsilon^p\,\mathrm{d}x = \int_{\mathbb{R}^N} V^p(y)\,\mathrm{d}y = \|V\|_{L^p(\mathbb{R}^N)} < \infty.$$

Similarly, now using

$$\int_\Omega \|u_\varepsilon\|_{L^\infty(\Omega)} u_\varepsilon^{q_\varepsilon}\,\mathrm{d}x = \mu^{-(q_\varepsilon+1)\alpha_\varepsilon+N+N\varepsilon/2}\int_{\Omega_\varepsilon} u_{\varepsilon,\mu}^{q_\varepsilon}\,\mathrm{d}x \to \|U\|_{L^q(\mathbb{R}^N)} < \infty \tag{3.5}$$

as $\varepsilon \to 0$. Also using the bound (2.16), we find

$$\|u_\varepsilon\|_{L^\infty(\Omega)}^{\beta/\alpha} v_\varepsilon^p(x) \leqslant \frac{M\mu^{-(p+1)\beta+p(N-2)-p(N-2)\varepsilon/2}}{|x-x_0|^{p(N-2)}}$$

for $x \neq x_0$ and some $M > 0$. But $-(p+1)\beta + p(N-2) > 0$ and Lemma 2.2 then $\|u_\varepsilon\|_{L^\infty(\Omega)}^{\beta/\alpha} v_\varepsilon^p(x) \to 0$ for $x \neq x_0$. Also we have

$$\|u_\varepsilon\|_{L^\infty(\Omega)} u_\varepsilon^{q_\varepsilon}(x) \leqslant \frac{M\mu^{-(q_\varepsilon+1)\alpha_\varepsilon+q_\varepsilon(N-2)-q_\varepsilon(N-2)\varepsilon/2}}{|x-x_0|^{q_\varepsilon(N-2)}}$$

for $x \neq x_0$ and some $M > 0$. But $-(q_\varepsilon+1)\alpha_\varepsilon + q_\varepsilon(N-2) > 0$ and Lemma 2.2 then $\|u_\varepsilon\|_{L^\infty(\Omega)} u_\varepsilon^{q_\varepsilon}(x) \to 0$ for $x \neq x_0$.

From here we have

$$-\Delta\big(\|u_\varepsilon\|_{L^\infty(\Omega)}^{\beta/\alpha} u_\varepsilon\big) \to \|V\|_{L^p(\mathbb{R}^N)}^p \delta_{x=x_0} \quad \text{and} \quad -\Delta\big(\|u_\varepsilon\|_{L^\infty(\Omega)} v_\varepsilon\big) \to \|U\|_{L^q(\mathbb{R}^N)}^q \delta_{x=x_0}$$

in the sense of distributions in Ω, as $\varepsilon \to 0$. Let ω be any neighborhood of $\partial\Omega$ not containing x_0. By regularity theory, see Lemma 5.1, we find

$$\big\| \|u_\varepsilon\|_{L^\infty(\Omega)}^{\beta/\alpha} u_\varepsilon \big\|_{C^{1,\alpha}(w)} \leqslant C\big[\big\| \|u_\varepsilon\|_{L^\infty(\Omega)}^{\beta/\alpha} v_\varepsilon^p \big\|_{L^1(\Omega)} + \big\| \|u_\varepsilon\|_{L^\infty(\Omega)}^{\beta/\alpha} v_\varepsilon^p \big\|_{L^\infty(w)}\big]$$

and a similar bound for $\| \|u_\varepsilon\|_{L^\infty(\Omega)} v_\varepsilon\|_{C^{1,\alpha}(w)}$. Consequently

$$\|u_\varepsilon\|_{L^\infty(\Omega)}^{\beta/\alpha} u_\varepsilon \to \|V\|_{L^p(\mathbb{R}^N)}^p G \quad \text{in } C^{1,\alpha}(w) \text{ as } \varepsilon \to 0 \tag{3.6}$$

and

$$\|u_\varepsilon\|_{L^\infty(\Omega)} v_\varepsilon \to \|U\|_{L^q(\mathbb{R}^N)}^q G \quad \text{in } C^{1,\alpha}(w) \text{ as } \varepsilon \to 0. \tag{3.7}$$

For the case $p < N/(N-2)$, we proceed as before and we have (3.5) and the bound

$$\|u_\varepsilon\|_{L^\infty(\Omega)} u_\varepsilon^{q_\varepsilon}(x) \leqslant \frac{M\mu^{-(q_\varepsilon+1)\alpha_\varepsilon+q_\varepsilon(p(N-2)-2)-q_\varepsilon(p(N-2)-2)\varepsilon/2}}{|x-x_0|^{q_\varepsilon(p(N-2)-2)}}$$

for $x \neq x_0$ and some $M > 0$. Using that $-(q_\varepsilon+1)\alpha_\varepsilon + q(p(N-2)-2) = 2(p+1) > 0$ and Lemma 2.2, we get $\|u_\varepsilon\|_{L^\infty(\Omega)}^{\beta/\alpha} u_\varepsilon^{q_\varepsilon}(x) \to 0$ for $x \neq x_0$ and hence

$$\|u_\varepsilon\|_{L^\infty(\Omega)} v_\varepsilon \to \|U\|_{L^q(\mathbb{R}^N)}^q G \quad \text{in } C^{1,\alpha}(w) \text{ as } \varepsilon \to 0. \tag{3.8}$$

Now we claim that

$$\|u_\varepsilon\|_{L^\infty(\Omega)}^{\frac{1}{\alpha}(\beta+p(N-2)-N)} u_\varepsilon \to \|U\|_{L^q(\mathbb{R}^N)}^{pq} \widetilde{G} \quad \text{in } C^{1,\alpha}(w) \text{ as } \varepsilon \to 0. \tag{3.9}$$

We have

$$-\Delta\big(\|u_\varepsilon\|_{L^\infty(\Omega)}^{\frac{1}{\alpha}(\beta+p(N-2)-N)} u_\varepsilon\big) = \|u_\varepsilon\|_{L^\infty(\Omega)}^{\frac{1}{\alpha}(\beta+p(N-2)-N)} v_\varepsilon^p = \|u_\varepsilon\|_{L^\infty(\Omega)}^p v_\varepsilon^p.$$

Since the last term converges to $(\|U\|_{L^q(\mathbb{R}^N)}^q G)^p$ in $C^{1,\alpha}(\omega)$ as $\varepsilon \to 0$ and $p \geqslant 1$, we have

$$\|u_\varepsilon\|_{L^\infty(\Omega)}^{\frac{1}{\alpha}(\beta+p(N-2)-N)} u_\varepsilon \to \|U\|_{L^q(\mathbb{R}^N)}^{pq} \widetilde{G} \quad \text{in } C^{3,\alpha}(w) \text{ as } \varepsilon \to 0.$$

For the remaining case $p = N/(N-2)$, we have as $\varepsilon \to 0$, the convergence

$$\int_\Omega \frac{\|u_\varepsilon\|_{L^\infty(\Omega)}^{\beta/\alpha}}{|\log(\|u_\varepsilon\|_{L^\infty(\Omega)})|} v_\varepsilon^p\,\mathrm{d}x = \frac{\mu^{-(p+1)\beta+N+N\varepsilon/2}}{\alpha_\varepsilon|\log(\mu)|}\int_{\Omega_\varepsilon} v_{\varepsilon,\mu}^p\,\mathrm{d}y \to \frac{1}{\alpha}\lim_{r\to\infty} V(r)^{\frac{N}{N-2}} r^N = \frac{a^{\frac{N}{N-2}}}{\alpha}$$

and the pointwise bound for $x \neq x_0$

$$\frac{\|u_\varepsilon\|_{L^\infty(\Omega)}^{\beta/\alpha}}{|\log(\|u_\varepsilon\|^{\beta/\alpha})|} v_\varepsilon^p(x) \leqslant \frac{M\mu^{-p(N-2)\varepsilon/2}}{\log(\mu)|x-x_0|^{p(N-2)}}.$$

By Lemma 2.2,

$$\frac{\|u_\varepsilon\|_{L^\infty(\Omega)}^{\beta/\alpha}}{|\log(\|u_\varepsilon\|^{\beta/\alpha})|} v_\varepsilon^p(x) \to 0$$

for $x \neq x_0$. Writing

$$-\Delta\left(\frac{\|u_\varepsilon\|_{L^\infty}^{\beta/\alpha}}{|\log(\|u_\varepsilon\|^{\beta/\alpha})|} u_\varepsilon\right) = \frac{\|u_\varepsilon\|_{L^\infty}^{\beta/\alpha}}{|\log(\|u_\varepsilon\|^{\beta/\alpha})|} v_\varepsilon^p,$$

we observe that the last term converges to $\delta_{x=x_0}$. By Lemma 5.1, we have

$$\frac{\|u_\varepsilon\|_{L^\infty}^{\beta/\alpha}}{|\log(\|u_\varepsilon\|^{\beta/\alpha})|} u_\varepsilon \to \frac{a^{N/(N-2)}}{\alpha} G \quad \text{in } C^{1,\alpha}(w) \text{ as } \varepsilon \to 0,$$

and clearly we have (3.7) using (2.23). This completes the proof of the theorem. □

Proof of Theorem 1.2. For $p > N/(N-2)$ we have

$$\varepsilon\|u_\varepsilon\|^{(N-2)/\alpha} \int_\Omega u_\varepsilon^{q_\varepsilon+1}\,\mathrm{d}x = \int_{\partial\Omega} \big(\|u_\varepsilon\|_{L^\infty(\Omega)}^{\beta/\alpha}\nabla u_\varepsilon, n\big)\big(\|u_\varepsilon\|_{L^\infty(\Omega)}\nabla v_\varepsilon, n\big)(n, x-y)\,\mathrm{d}s.$$

By (3.6) and (3.7),

$$\lim_{\varepsilon\to 0} \varepsilon\|u_\varepsilon\|^{(N-2)/\alpha} \int_\Omega u_\varepsilon^{q_\varepsilon+1}\,\mathrm{d}x = \|V\|_{L^p(\mathbb{R}^N)}^p \|U\|_{L^q(\mathbb{R}^N)}^q \int_{\partial\Omega} \frac{\partial G(x,x_0)}{\partial n}\frac{\partial G(x,x_0)}{\partial n}(n, x-x_0)\,\mathrm{d}s.$$

Also for the case $p < N/(N-2)$, using

$$\varepsilon\|u_\varepsilon\|_{L^\infty(\Omega)}^{\frac{1}{\alpha}(p(N-2)-2)} \int_\Omega u_\varepsilon^{q_\varepsilon+1}\,\mathrm{d}x = \int_{\partial\Omega} \big(\|u_\varepsilon\|_{L^\infty(\Omega)}^{\frac{1}{\alpha}(\beta+p(N-2)-N)}\nabla u_\varepsilon, n\big)\big(\|u_\varepsilon\|_{L^\infty(\Omega)}\nabla v_\varepsilon, n\big)(n, x-y)\,\mathrm{d}s$$

and (3.9) and (3.8), we get

$$\lim_{\varepsilon\to 0} \varepsilon\|u_\varepsilon\|_{L^\infty(\Omega)}^{\frac{1}{\alpha}(p(N-2)-2)} \int_\Omega u_\varepsilon^{q_\varepsilon+1}\,\mathrm{d}x = \|U\|_{L^q(\mathbb{R}^N)}^{q(p+1)} \int_{\partial\Omega} \frac{\partial \widetilde{G}(x,x_0)}{\partial n}\frac{\partial G(x,x_0)}{\partial n}(n, x-x_0)\,\mathrm{d}s.$$

The case $p = N/(N-2)$ is analogous.

The proof of the theorems follows from the next lemma. □

Lemma 3.1. *We have the following identities*

$$\text{(i)} \int_{\partial\Omega} \frac{\partial G(x,x_0)}{\partial n}\frac{\partial G(x,x_0)}{\partial n}(n, x-x_0)\,\mathrm{d}s = -(N-2)g(x_0,x_0)$$

and

$$\text{(ii)} \int_{\partial\Omega} \frac{\partial \widetilde{G}(x,x_0)}{\partial n}\frac{\partial G(x,x_0)}{\partial n}(n, x-x_0)\,\mathrm{d}s = -\frac{N}{q+1}\tilde{g}(x_0,x_0).$$

Proof. (i) was proven in [3], see also [15]. To prove (ii) we follow a similar procedure. From [19,23], for any $y \in \mathbb{R}^N$, we have the following identity

$$\int_{\Omega'} \Delta u(x-y, \nabla v) + \Delta v(x-y, \nabla u) - (N-2)(\nabla u, \nabla v)\,\mathrm{d}x$$
$$= \int_{\partial\Omega'} \frac{\partial u}{\partial n}(x-y, \nabla v) + \frac{\partial v}{\partial n}(x-y, \nabla u) - (\nabla u, \nabla v)(x-y, n)\,\mathrm{d}s,$$

where $\Omega' = \Omega \setminus B_r$ with $r > 0$. For a system $-\Delta v = 0$ and $-\Delta u = v^p$, in Ω', the identity takes the form

$$\int_{\Omega'} \frac{N}{p+1} v^{p+1} - \bar{a} v^{p+1}\,\mathrm{d}x = \int_{\partial\Omega'} \frac{1}{p+1} v^{p+1}(x-y, n)\,\mathrm{d}s + \int_{\partial\Omega'} \frac{\partial u}{\partial n}\big[(x-y, \nabla v) + \bar{a} v\big]$$
$$+ \frac{\partial v}{\partial n}\big[(x-y, \nabla u) + \bar{b} u\big] - (\nabla u, \nabla v)(x-y, n)\,\mathrm{d}s \tag{3.10}$$

with $\bar{a} + \bar{b} = N - 2$. Let $y = 0$, choose $\bar{a} = N/(p+1)$ and take $v = G(x,0)$ and $u = \widetilde{G}(x,0)$. Using that $u = v = 0$ on $\partial\Omega$, and so $\nabla u = (\nabla u, n)n$ and $\nabla v = (\nabla v, n)n$ on $\partial\Omega$, we obtain

$$\int_{\partial\Omega} \frac{\partial \widetilde{G}}{\partial n} \frac{\partial \widetilde{G}}{\partial n}(x,n)\,\mathrm{d}s = \int_{\partial B_r} \frac{1}{p+1} G^{p+1}(x,n) + \frac{\partial \widetilde{G}}{\partial n}\left[(x, \nabla G) + \frac{N}{p+1} G\right]\mathrm{d}s$$
$$+ \int_{\partial B_r} \frac{\partial G}{\partial n}\left[(x, \nabla \widetilde{G}) + \frac{N}{q+1}\widetilde{G}\right] - (\nabla\widetilde{G}, \nabla G)(x,n)\,\mathrm{d}s.$$

Let $k = p(N-2)$ and $\Gamma = \sigma_N(N-2)$. For $|x| = r$, we have

$$\nabla\widetilde{G} = -\frac{1}{\Gamma^p(N-k)}|x|^{-k}x + \nabla\tilde{g}, \qquad \nabla G = -\frac{1}{\sigma_N}|x|^{-N}x + \nabla g,$$
$$\frac{\partial\widetilde{G}}{\partial n} = -\frac{1}{\Gamma^p(N-k)}|x|^{1-k} + (\nabla\tilde{g}, n), \qquad \frac{\partial G}{\partial n} = -\frac{1}{\sigma_N}|x|^{1-N} + (\nabla g, n),$$
$$\big(x, \nabla\widetilde{G}\big) + \frac{N}{q+1}\widetilde{G} = \left(\frac{N}{(q+1)(k-2)} - 1\right)\frac{1}{\Gamma^p(N-k)}|x|^{2-k} + (x, \nabla\tilde{g}) + \frac{N}{q+1}\tilde{g},$$
$$(x, \nabla G) + \frac{N}{p+1} G = \left(\frac{N}{p+1} - (N-2)\right)\frac{1}{\Gamma}|x|^{2-N} + (x, \nabla g) + \frac{N}{p+1} g,$$
$$\big(\nabla\widetilde{G}, \nabla G\big) = \frac{|x|^{-k-N+2}}{\sigma_N \Gamma^p(N-k)} - \frac{(\nabla g, x)}{\Gamma^p(N-k)}|x|^{(2-N)p} - \frac{(\nabla\tilde{g}, x)}{\sigma_N}|x|^{-N} + (\nabla\tilde{g}, \nabla g)$$

and

$$\frac{1}{p+1} G^{p+1} = \frac{1}{p+1}\left[\frac{1}{\Gamma^p}|x|^{-k} - \Delta\tilde{g}\right]\left[\frac{1}{\Gamma}|x|^{2-N} + g\right].$$

From here, we check that the terms with $|x|^{3-N-k}$ cancel out, other integrals tend to 0 since the integrands are $o(|x|^{1-N})$, and only one term of order $|x|^{1-N}$ remain, giving

$$\int_{\partial\Omega} \frac{\partial\widetilde{G}}{\partial n}\frac{\partial G}{\partial n}(x,n)\,\mathrm{d}s = -\lim_{r\to 0}\frac{1}{\sigma_N r^{N-1}}\int_{\partial B_r}\frac{N}{q+1}\tilde{g}\,\mathrm{d}s = -\frac{N}{q+1}\tilde{g}(0,0). \qquad \square$$

Proof of Theorem 1.1. (a) The part (ii) follows from Lemma 5.1,

$$\big\| |\Delta u_\varepsilon|^{1/p} \big\|_{C^{1,\alpha}(\omega)} \leqslant \|u_\varepsilon^{q_\varepsilon}\|_{L^1(\Omega)} + \|u_\varepsilon^{q_\varepsilon}\|_{L^\infty(\omega)}$$

and estimates (2.19), (2.22), and (2.23). Part (i) follows from

$$\|u_\varepsilon\|_{C^{1,\alpha}(\omega)} \leqslant \big\|v_\varepsilon^p\big\|_{L^1(\Omega)} + \big\|v_\varepsilon^p\big\|_{L^\infty(\omega)}$$

and estimate (2.18). Finally (iii) follows combining (ii) with the convergence

$$\int_{\mathbb{R}^N} |\Delta u_\varepsilon|^{(p+1)/p}\,\mathrm{d}x = \int_{\mathbb{R}^N} v_\varepsilon^{p+1}\,\mathrm{d}x \to \|V\|^{p+1}_{L^{p+1}(\mathbb{R}^N)}.$$

as $\varepsilon \to 0$. This completes part (a).

For part (b), note that from (2.6), we have the vectorial equality $\int_{\partial\Omega}(\nabla u_\varepsilon, \nabla v_\varepsilon)n\,\mathrm{d}s = 0$. In the limit for $p \geqslant N/(N-2)$, we get

$$\int_{\partial\Omega} \big(\nabla G(x,x_0), \nabla G(x,x_0)\big)n\,\mathrm{d}s = 0 \tag{3.11}$$

and similarly for $p < N/(N-2)$, we obtain

$$\int_{\partial\Omega} \big(\nabla \widetilde{G}(x,x_0), \nabla G(x,x_0)\big)n\,\mathrm{d}s = 0. \tag{3.12}$$

But we have the following result.

Lemma 3.2. *For every $x_0 \in \Omega$*

$$\int_{\partial\Omega} \big(\nabla G(x,x_0), n\big)\big(\nabla G(x,x_0), n\big)n\,\mathrm{d}s = -\nabla\phi(x_0) \tag{3.13}$$

and

$$\int_{\partial\Omega} \big(\nabla \widetilde{G}(x,x_0), n\big)\big(\nabla\big(\Delta\widetilde{G}(x,x_0)\big)^{1/p}, n\big)n\,\mathrm{d}s = -\nabla\tilde{\phi}(x_0). \tag{3.14}$$

Hence combining (3.11) with (3.13), and (3.12) with (3.14), we complete the proof of part (b) and the theorem is proven. □

Proof of Lemma 3.2. Equality (3.13) was proved in [3] and [15]. To prove (3.14), by (3.10) we have

$$\int_{\partial\Omega} \frac{\partial\widetilde{G}}{\partial n}\frac{\partial G}{\partial n} n\,\mathrm{d}s = \int_{\partial B_r} \left\{\frac{1}{p+1}G^{p+1}n + \frac{\partial\widetilde{G}}{\partial n}\nabla G + \frac{\partial G}{\partial n}\nabla\widetilde{G} - (\nabla\widetilde{G}, \nabla G)n\right\}\mathrm{d}s.$$

Using $\int_{\partial B_r} n = 0$, we get

$$\begin{aligned}
\int_{\partial\Omega} \frac{\partial\widetilde{G}}{\partial n}\frac{\partial G}{\partial n} n\,\mathrm{d}s &= \frac{1}{(p+1)r^{N-1}}\int_{\partial B_r}\left\{\frac{1}{\Gamma^p}r^{N-k-1}g - \Delta\tilde{g}\frac{1}{\Gamma}r - \Delta\tilde{g}gr^{N-1}\right\}n\,\mathrm{d}s \\
&\quad + \frac{1}{r^{N-1}}\int_{\partial B_r}\big\{(\nabla\tilde{g}, n)\nabla g + (\nabla g, n)\nabla\tilde{g} - (\nabla\tilde{g}, \nabla g)n\big\}r^{N-1}\,\mathrm{d}s \\
&\quad - \frac{1}{r^{N-1}}\int_{\partial B_r}\left\{\frac{1}{\sigma_N}\nabla\tilde{g} + \frac{r^{N-k}}{\Gamma^p(N-k)}\nabla g\right\}\mathrm{d}s.
\end{aligned} \tag{3.15}$$

We use the regular $\hat{g}(x,0)$ instead of $\tilde{g}(x,0)$. Thus

$$\nabla\hat{g}(x,0) = \nabla\tilde{g}(x,0) + \frac{pg(0,0)}{\Gamma^{p-1}(2N-k-2)}|x|^{N-k-2}x, \tag{3.16}$$

$$\Delta\hat{g}(x,0) = \Delta\tilde{g}(x,0) + \frac{pg(0,0)}{\Gamma^{p-1}}|x|^{N-k-2}. \tag{3.17}$$

But $g(x,0) = g(0,0) + (\nabla g(0,0), x) + o(|x|^2)$ and

$$\int_{\partial B_r} r^{-k} g(x,0) n \, ds = \int_{\partial B_1} r^{N-k-1} g(0,0) n \, ds + \int_{\partial B_1} r^{N-k} \big(\nabla g(0,0), y\big) n \, ds + o\big(r^{N-k+1}\big),$$

where $y = x/r$. Clearly the first integral in the r.h.s is zero and the other terms tends to zero as $r \to 0$. Hence

$$\lim_{r\to 0} \frac{1}{r^{N-1}} \int_{\partial B_r} r^{N-k-1} g(x,0) n \, ds = 0. \tag{3.18}$$

We replace (3.16) and (3.17) in (3.15), to obtain an identity without $\tilde{g}$. Using the limit (3.18) and that $\hat{g}$ and g are regular, we obtain

$$\int_{\partial\Omega} \frac{\partial \widetilde{G}}{\partial n} \frac{\partial G}{\partial n} n \, ds = \lim_{r\to 0} \frac{1}{r^{N-1}} \int_{\partial B_r} \frac{1}{\sigma_N} \nabla \hat{g} \, ds = \nabla \hat{g}(0,0) = \nabla \tilde{\phi}(0),$$

where the last equality follows by the observation after Theorem 1.1. □

4. Proof of Lemma 2.3

Let us recall the problem (2.10)–(2.12),

$$-\Delta u_{\varepsilon,\mu} = v_{\varepsilon,\mu}^{p} \quad \text{in } \Omega_\varepsilon, \tag{4.1}$$
$$-\Delta v_{\varepsilon,\mu} = u_{\varepsilon,\mu}^{q_\varepsilon} \quad \text{in } \Omega_\varepsilon, \tag{4.2}$$
$$u_{\varepsilon,\mu} = v_{\varepsilon,\mu} = 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{4.3}$$

where $\Omega_\varepsilon = (\Omega - x_\varepsilon)/\mu^{1-\varepsilon/2}$. Let $\overline{R} > 0$. We define $\sigma(p) := 2 + N - p(N-2)$, and the scalar function

$$J\big(|y|\big) := \begin{cases} 1 & \text{if } \sigma(p) < 2, \\ \big|\log\big(|y|/\overline{R}\big)\big| & \text{if } \sigma(p) = 2, \\ |y|^{2-\sigma(p)} & \text{if } \sigma(p) > 2. \end{cases}$$

Note that $\sigma(p) \in [0, N)$ for $p \in (2/(N-2), (N+2)/(N-2)]$ and $\sigma(q) \leqslant 0$. We consider the transformations

$$z_\varepsilon(y) = |y|^{2-N} v_{\varepsilon,\mu}\left(\frac{y}{|y|^2}\right) \quad \text{and} \quad w_\varepsilon(y) = \frac{|y|^{2-N}}{J(|y|)} u_{\varepsilon,\mu}\left(\frac{y}{|y|^2}\right)$$

in Ω_ε^*, the image of Ω_ε under $x \mapsto x/|x|^2$.

The next lemma is equivalent to Lemma 2.3, using the asymptotic behaviors (1.17).

Lemma 4.1. *Let $(w_\varepsilon, z_\varepsilon)$ solving*

$$-\Delta J\big(|y|\big) w_\varepsilon = |y|^{-\sigma(p)} z_\varepsilon^p \quad \textit{in } \Omega_\varepsilon^*, \tag{4.4}$$
$$-\Delta z_\varepsilon = |y|^{-\sigma(q)+(q_\varepsilon - q)(N-2)} \big[J\big(|y|\big) w_\varepsilon\big]^{q_\varepsilon} \quad \textit{in } \Omega_\varepsilon^*, \tag{4.5}$$
$$w_\varepsilon = z_\varepsilon = 0 \quad \textit{on } \partial\Omega_\varepsilon^*. \tag{4.6}$$

Then for any fixed $R \in (0, \overline{R})$, we have

$$\|w_\varepsilon\|_{L^\infty(\Omega_\varepsilon^R)} + \|z_\varepsilon\|_{L^\infty(\Omega_\varepsilon^R)} \leqslant C,$$

where $\Omega_\varepsilon^R = \Omega_\varepsilon^* \cap B_R$, and $C = C(R)$ independent of $\varepsilon > 0$ provided ε is sufficiently small.

Proof. Given $R > 0$, let w_0 and z_0 be solutions of

$$\Delta J\big(|y|\big) w_0 = 0 \quad \text{in } \Omega_\varepsilon^R \quad \text{and} \quad w_0 = 0 \quad \text{on } \partial\Omega_\varepsilon^*, \quad w_0 = w_\varepsilon \quad \text{on } \partial B_R,$$

and

$$\Delta z_0 = 0 \quad \text{in } \Omega_\varepsilon^R \quad \text{and} \quad z_0 = 0 \quad \text{on } \partial\Omega_\varepsilon^*, \quad z_0 = z_\varepsilon \quad \text{on } \partial B_R.$$

By the convergence in compact sets of w_ε and z_ε, see (2.13), we have $|z_\varepsilon| + |\nabla z_\varepsilon| + |w_\varepsilon| + |\nabla w_\varepsilon| \leqslant C$ in $|y| = R$ for C independent of ε. Therefore by the maximum principle, we get

$$|Jw_0| + \left|\nabla(Jw_0)\right| + |z_0| + |\nabla z_0| \leqslant C \quad \text{in } \Omega_\varepsilon^R.$$

Define $\tilde{w} = w_\varepsilon - w_0$ and $\tilde{z} = z_\varepsilon - z_0$. We now write

$$-\Delta J\big(|y|\big)\tilde{w} = a(y) z_\varepsilon \quad \text{in } \Omega_\varepsilon^R, \tag{4.7}$$

$$-\Delta \tilde{z} = b(y) J\big(|y|\big) w_\varepsilon \quad \text{in } \Omega_\varepsilon^R, \tag{4.8}$$

$$\tilde{w} = \tilde{z} = 0 \quad \text{on } \partial\Omega_\varepsilon^R, \tag{4.9}$$

where $a(y) = |y|^{-\sigma(p)} z_\varepsilon^{p-1}$ and $b(y) = |y|^{-\sigma(q)+(q_\varepsilon - q)(N-2)}[J(|y|)w_\varepsilon]^{q_\varepsilon - 1}$. Clearly by the maximum principle $\tilde{w} \geqslant 0$ and $\tilde{z} \geqslant 0$.

Let $P(y) = a(y)$ and

$$Q(y) = \begin{cases} \dfrac{1}{M} b(y) & \text{for } y \in B_R \setminus \overline{B}_r, \\ b(y) & \text{for } B_r, \end{cases}$$

where $r \in (0, R)$ and $M > 1$ both independent of ε and to be determined later. Then

$$b(y) J\big(|y|\big) w_\varepsilon = Q(y) J\big(|y|\big) w_\varepsilon + f(y),$$

where

$$f(y) = \big(b(y) - Q(y)\big) J\big(|y|\big) w_\varepsilon = \begin{cases} 0 & \text{for } y \in \Omega_\varepsilon \cap B_r, \\ \left(1 - \dfrac{1}{M}\right) b(y) J\big(|y|\big) w_\varepsilon & \text{for } y \in B_R \setminus \overline{B}_r. \end{cases}$$

It is clear that $f \in L^\infty(\Omega_\varepsilon^R)$, in fact $\|f\|_{L^\infty(\Omega_\varepsilon^R)} \leqslant (1 - 1/M) r^{-(2+N)}$ by using that $w_\varepsilon(y) \leqslant C r^{\sigma(p)-N}$ for $|y| \geqslant r$, when $p < N/(N-2)$, and $w_\varepsilon(y) \leqslant C r^{2-N}$ for $|y| \geqslant r$ when $p > N/(N-2)$. A similar bound is obtained for $p = N/(N-2)$. Then we transform (4.7)–(4.8) in the system

$$-\Delta J\tilde{w} = P z_\varepsilon \quad \text{in } \Omega_\varepsilon^R, \qquad -\Delta \tilde{z} = QJw_\varepsilon + f \quad \text{in } \Omega_\varepsilon^R.$$

We define $\eta_2(y) = \chi_{w_\varepsilon \leqslant 2\tilde{w}}(y)$ and $\eta_1(y) = \chi_{z_\varepsilon \leqslant 2\tilde{z}}(y)$ for $y \in \Omega_\varepsilon^R$, we find

$$-\Delta J\tilde{w} \leqslant 2\eta_1 P\tilde{z} + f_1 \quad \text{in } \Omega_\varepsilon^R, \qquad -\Delta\tilde{z} \leqslant 2\eta_2 QJ\tilde{w} + f_2 \quad \text{in } \Omega_\varepsilon^R.$$

Here $f_1 = (1 - \eta_1) P z_\varepsilon = \chi_{z_\varepsilon \leqslant 2z_0} P z_\varepsilon \leqslant 2Pz_0$ and $f_2 = f + (1 - \eta_2) QJw_\varepsilon$ where $(1 - \eta_2) QJw_\varepsilon \leqslant 2QJw_0$. We write the system in the form

$$-\Delta J\tilde{w} \leqslant 2\eta_1 P|y|^\gamma |y|^{-\gamma}\tilde{z} + f_1 \quad \text{in } \Omega_\varepsilon^R, \tag{4.10}$$

$$-|y|^{-\gamma}\Delta\tilde{z} \leqslant 2\eta_2 Q|y|^{-\gamma} J\tilde{w} + f_2|y|^{-\gamma} \quad \text{in } \Omega_\varepsilon^R, \tag{4.11}$$

$$\tilde{w} = \tilde{z} = 0 \quad \text{on } \partial\Omega_\varepsilon^R. \tag{4.12}$$

Let $u(y) \mapsto 2\eta_2 Q|y|^{-\gamma} u(y)$ and $u(y) \mapsto 2\eta_1 P|y|^\gamma u(y)$ be the multiplication operators $\mathcal{P}$ and $\mathcal{Q}$ respectively. Note that a multiplication operator $\mathcal{C}$ with corresponding function $c(y) \in L^s(\Omega_\varepsilon^R)$ is bounded from $L^{s_1}(\Omega_\varepsilon^R)$ to $L^{s_2}(\Omega_\varepsilon^R)$ with $1/s_2 = 1/s_1 + 1/s$.

Formally we define $-L$ as the operator $u(y) \mapsto -|y|^{-\gamma}\Delta(|y|^\gamma u(y))$. More precisely, in the appendix, we define $(-\Delta)^{-1}$ and $(-L)^{-1}$, which by the Hardy–Littlewood–Sobolev inequality are bounded, independently of ε, from $L^{m_1}(\Omega_\varepsilon^R)$ to $L^{m_2}(\Omega_\varepsilon^R)$ with $1/m_1 = 1/m_2 + 2/N$. Note that the image of these operators is a function with zero-Dirichlet boundary condition, so they are positive. Then we can write

$$J\tilde{w} \leqslant (-\Delta)^{-1}\mathcal{P}(-L)^{-1}\big(\mathcal{Q}(J\tilde{w}) + |y|^{-\gamma} f_2\big) + (-\Delta)^{-1} f_1.$$

Denoting by $K = (-\Delta)^{-1}\mathcal{P}(-L)^{-1}\mathcal{Q}$ and $h = K|y|^{-\gamma} f_2 + (-\Delta)^{-1} f_1$ we have

$$(I-K)J\tilde{w} \leqslant h.$$

The proof is complete finding m large enough such that $h \in L^m(\Omega_\varepsilon^R)$ and $(I-K)$ is invertible from $L^m(\Omega_\varepsilon^R)$ to $L^m(\Omega_\varepsilon^R)$.

We can estimate $Q(y)|y|^{-\gamma}$ in $L^{\frac{q+1}{q-1}}(\Omega_\varepsilon^R)$, for $\gamma = 2\sigma(p)/(p+1) \geqslant 0$, and note that $\gamma = -\sigma(q)/(q+1)$ using the Sobolev Hyperbola. Since $v_{\varepsilon,\mu} \to V$ in $L^{q+1}(\mathbb{R}^N)$, we have

$$\int_{\Omega_\varepsilon^*} \bigl[J\bigl(|y|\bigr)w_\varepsilon(y) - V\bigl(y/|y|^2\bigr)|y|^{2-N}\bigr]^{q+1}|y|^{-\sigma(q)}\,\mathrm{d}y \to 0 \quad \text{as } \varepsilon \to 0.$$

Therefore for any λ, we can take r small such that

$$\int_{\Omega_\varepsilon^r} [Jw_\varepsilon]^{(q+1)\frac{q_\varepsilon-1}{q-1}}(y)|y|^{-\sigma(q)}\,\mathrm{d}y \leqslant \int_{\Omega_\varepsilon^r} [Jw_\varepsilon]^{(q+1)}(y)|y|^{-\sigma(q)}\,\mathrm{d}y \leqslant \frac{\lambda}{2C(\delta)}$$

and M large such that for all $\varepsilon \leqslant \varepsilon_0$ we have

$$\begin{aligned}\int_{\Omega_\varepsilon^R} \bigl[Q(s)|y|^{-\gamma}\bigr]^{\frac{q+1}{q-1}}\,\mathrm{d}y &\leqslant C(\delta)\int_{\Omega_\varepsilon^r} [Jw_\varepsilon]^{(q+1)\frac{q_\varepsilon-1}{q-1}}|y|^{-\sigma(q)}\,\mathrm{d}y + \frac{C(\delta)}{M^{\frac{q+1}{q-1}}}\int_{B_R\setminus B_r} [Jw_\varepsilon]^{(q+1)\frac{q_\varepsilon-1}{q-1}}|y|^{-\sigma(q)}\,\mathrm{d}y \\ &\leqslant \lambda, \end{aligned} \tag{4.13}$$

where we have used $b(y) \leqslant C(\delta)[Jw_\varepsilon]^{q_\varepsilon-1}$ with δ given by Lemma 2.2.

Now we show that K is bounded from $L^m(\Omega_\varepsilon^R)$ to $L^m(\Omega_\varepsilon^R)$.

$$\begin{aligned}\|KJ\tilde{w}\|_{L^m(\Omega_\varepsilon^R)} &\leqslant C_1\bigl\|\mathcal{P}(-L)^{-1}\mathcal{Q}J\tilde{w}\bigr\|_{L^r(\Omega_\varepsilon^R)} \\ &\leqslant C_1\bigl\||y|^\gamma 2\eta_1 P\bigr\|_{L^{\frac{p+1}{p-1}}(\Omega_\varepsilon^R)}\bigl\|(-L)^{-1}\mathcal{Q}J\tilde{w}\bigr\|_{L^{r'}(\Omega_\varepsilon^R)} \\ &\leqslant C_1\bigl\||y|^\gamma 2\eta_1 P\bigr\|_{L^{\frac{p+1}{p-1}}(\Omega_\varepsilon^R)} C_2\|\mathcal{Q}J\tilde{w}\|_{L^{s'}(\Omega_\varepsilon^R)} \\ &\leqslant C_1C_2\bigl\||y|^\gamma 2\eta_1 P\bigr\|_{L^{\frac{p+1}{p-1}}(\Omega_\varepsilon^R)}\bigl\||y|^{-\gamma}2\eta_2 Q\bigr\|_{L^{\frac{q+1}{q-1}}(\Omega_\varepsilon^R)}\|J\tilde{w}\|_{L^{m'}(\Omega_\varepsilon^R)} \\ &\leqslant \overline{C}\bigl\||y|^\gamma P\bigr\|_{L^{\frac{p+1}{p-1}}(\Omega_\varepsilon^R)}\bigl\||y|^{-\gamma} Q\bigr\|_{L^{\frac{q+1}{q-1}}(\Omega_\varepsilon^R)}\|J\tilde{w}\|_{L^{m'}(\Omega_\varepsilon^R)}\end{aligned}$$

with $1/r = 1/m + 2/N$, so $r' > 1$ implies $m > N/(N-2)$. $1/r = (p-1)/(p+1) + 1/r'$ and $1/s' = 1/r' + 2/N$, so condition (b) in (5.1) implies $N - 2 + N/m > 2N/(p+1)$ and $s' > 1$ implies $m > (q+1)/2$ so (a) in (5.1) holds since $\gamma > 0$ and $1/s' = (q-1)/(q+1) + 1/m'$. Since

$$\frac{q-1}{q+1} + \frac{p-1}{p+1} = \frac{4}{N}, \quad \text{we have } m' = m.$$

By

$$\int_{\Omega_\varepsilon^*} \bigl[z_\varepsilon(y) - U\bigl(y/|y|^2\bigr)|y|^{2-N}\bigr]^{p+1}|y|^{-\sigma(p)}\,\mathrm{d}y \to 0 \quad \text{as } \varepsilon \to 0,$$

we deduce that $\||y|^{\gamma-\sigma(p)}z_\varepsilon^{p-1}\|_{L^{\frac{p+1}{p-1}}(\Omega_\varepsilon^R)} = \||y|^\gamma P\|_{L^{\frac{p+1}{p-1}}(\Omega_\varepsilon^R)} \leqslant C(\varepsilon_0)$ with $C(\varepsilon_0) > 0$ and for all $\varepsilon \in (0, \varepsilon_0)$. Since λ in (4.13) can be arbitrarily small then the norm of K is small and so $I - K$: $L^m(\Omega_\varepsilon^R) \to L^m(\Omega_\varepsilon^R)$ invertible for m large. We have that

$$\bigl\||y|^{-\gamma} f_2\bigr\|_{L^m(\Omega_\varepsilon^R)} \leqslant r^{-\gamma}\|f_2\|_{L^\infty(\Omega_r^R)}\bigl(\mathrm{meas}\bigl(\Omega_r^R\bigr)\bigr)^{1/m}$$

is bounded, since f_2 is zero outside Ω_r^R and

$$\bigl\|\Delta^{-1} f_1\bigr\|_{L^m(\Omega_\varepsilon^R)} \leqslant C_1\|f_2\|_{L^r(\Omega_\varepsilon^R)} \leqslant \|f_1\|_{L^\infty(\Omega_r^R)}\bigl(\mathrm{meas}\bigl(\Omega_\mu^R\bigr)\bigr)^{1/r} \leqslant C(z_0)\bigl(\mathrm{meas}(B_R)\bigr)^{1/r}.$$

This implies $\|Jw\|_{L^m(\Omega_\varepsilon^R)} \leqslant M$ for every m large, and consequently for every $m \geqslant 1$. (Use the w_0 to get that $\|Jw_\varepsilon\|_{L^m(\Omega_\varepsilon^R)} \leqslant M$.) Now we have that

$$-\Delta\tilde{z} = b(y)Jw_\varepsilon = |y|^{-\sigma(q)+(q-q_\varepsilon)(N-2)}[Jw_\varepsilon]^{q_\varepsilon}.$$

Since $\sigma(q) \leqslant 0$, if we take m large such that $mq_\varepsilon > N/2$ then

$$\|\tilde{z}\|_{L^\infty(\Omega_\varepsilon^R)} \leqslant \widetilde{M} \quad \text{and therefore} \quad \|z_\varepsilon\|_{L^\infty(\Omega_\varepsilon^R)} \leqslant M \tag{4.14}$$

for some M independent of ε. We study now each case of J separately. We have

$$-\Delta Jw_\varepsilon = |y|^{-\sigma(p)}z_\varepsilon^p \quad \text{in } \Omega_\varepsilon^*. \tag{4.15}$$

(a) In the case $J = 1$, since $\sigma(p) < 2$, using (4.14), we have $-\Delta\tilde{w}_\varepsilon \in L^q(\Omega)$ for any $q \in (N/2, N/\sigma(p))$. By regularity, we get

$$\|w_\varepsilon\|_{L^\infty(\Omega_\varepsilon^R)} \leqslant M.$$

(b) For $J(|y|) = -\log(|y|/\overline{R}) > \log(\overline{R}/R)$, we have

$$-\Delta\tilde{w} - \frac{\nabla J}{J}\nabla\tilde{w} - \frac{\Delta J}{J}\tilde{w} = \frac{1}{J|y|^2}z_\varepsilon^p \quad \text{in } \Omega_\varepsilon^R$$

or equivalently

$$-\Delta\tilde{w} + \frac{1}{J|y|^2}(y, \nabla\tilde{w}) + \frac{1}{J|y|^2}(N-2)\tilde{w} = \frac{1}{J|y|^2}z_\varepsilon^p \quad \text{in } \Omega_\varepsilon^R.$$

Using (4.14), we can take $u = \tilde{w} - M$ with $M = \sup_{\varepsilon>0}\sup_{y\in\Omega_\varepsilon^R} z_\varepsilon^p(y)/(N-2)$, and we get

$$-J|y|^2\Delta u + (y, \nabla u) + (N-2)u \leqslant 0 \quad \text{in } \Omega_\varepsilon^R.$$

Since $u = -M < 0$ on the boundary, $u \leqslant 0$ in Ω_ε^R. This gives $w_\varepsilon \leqslant M$ in Ω_ε^R.

For the remaining case $p < N/(N-2)$, we have

$$-\Delta\tilde{w} - \frac{\nabla J}{J}\nabla\tilde{w} - \frac{\Delta J}{J}\tilde{w} = \frac{1}{|y|^2}z_\varepsilon^p \quad \text{in } \Omega_\varepsilon^R.$$

As before, defining $u = \tilde{w} - M$ with $M = \sup_{\varepsilon>0}\sup_{y\in\Omega_\varepsilon^R} z_\varepsilon^p/[(\sigma(p)-2)(N-\sigma(p))]$ then

$$-|y|^2\Delta u - \big(2-\sigma(p)\big)(y, \nabla u) - \big(2-\sigma(p)\big)\big(N-\sigma(p)\big)u \leqslant 0 \quad \text{in } \Omega_\varepsilon^R.$$

Since $u = -M < 0$ on the boundary, $u \leqslant 0$ in Ω_ε^R. This implies $w_\varepsilon \leqslant M$ in Ω_ε^R. $\square$

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Appendix A

Let $N > 2$. Let h and v be functions in $L^{s'}(\Omega_\varepsilon^R)$. Given the Green's function G solution of $-\Delta G(x,\cdot) = \delta_x$ in Ω_ε^R, $G(x,\cdot) = 0$ on $\partial\Omega_\varepsilon^R$, we define

$$(-\Delta)^{-1}h(\xi) = \int_{\Omega_\varepsilon^R} G(x,\xi)h(x)\,\mathrm{d}x, \quad \xi \in \Omega_\varepsilon^R$$

and

$$(-L)^{-1}v(\xi) = |\xi|^{-\gamma} \int\limits_{\Omega_\varepsilon^R} G(x,\xi)|x|^\gamma v(x)\,\mathrm{d}x, \quad \xi \in \Omega_\varepsilon^R.$$

Note that G is positive, so both operators are positive. We know that $(-\Delta)^{-1}$ is bounded, independently of ε, from $L^{s'}(\Omega_\varepsilon^R)$ to $L^{r'}(\Omega_\varepsilon^R)$ with $1/r' = 1/s' - 2/N$. Next we prove the same result for $(-L)^{-1}$. By the weighted Hardy–Littlewood–Sobolev inequality [5,18], for $|\xi|^{-\gamma} f \in L^{s'}(\Omega_\varepsilon^R)$, we have that

$$\left\|\xi^{-\gamma}(-\Delta)^{-1} f\right\|_{L^{r'}(\Omega_\varepsilon^R)} \leqslant 2\left\| |\xi|^{-\gamma} \int\limits_{\Omega_\varepsilon^R} \frac{C}{|x-\xi|^{N-2}} f(x)\,\mathrm{d}x \right\|_{L^{r'}(\Omega_\varepsilon^R)} \leqslant C\left\| |\xi|^{-\gamma} f\right\|_{L^{s'}(\Omega_\varepsilon^R)}$$

for $1 < s' < r' < \infty$, with $1/r' = 1/s' - 2/N$ and

$$\text{(a)}\ -\gamma < N(1-1/s') = N-2-N/r' \quad \text{and} \quad \text{(b)}\ \gamma < N/r'. \tag{5.1}$$

In other words, for any $v \in L^{s'}(\Omega_\varepsilon^R)$, we have

$$\begin{aligned}
\left\|(-L)^{-1}v\right\|_{L^{r'}(\Omega_\varepsilon^R)} &= \left\| |\xi|^{-\gamma}(-\Delta)^{-1}|x|^\gamma v\right\|_{L^{r'}(\Omega_\varepsilon^R)} \\
&\leqslant 2\left\| |\xi|^{-\gamma} \int\limits_{\Omega_\varepsilon^R} \frac{C}{|x-\xi|^{N-2}} |x|^\gamma v(x)\,\mathrm{d}x \right\|_{L^{r'}(\Omega_\varepsilon^R)} \\
&\leqslant C\|v\|_{L^{s'}(\Omega_\varepsilon^R)}.
\end{aligned} \tag{5.2}$$

Lemma 5.1. *Let u solve*

$$\begin{cases} -\Delta u = f & \text{in } \Omega \subset \mathbb{R}^N, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Let ω be a neighborhood of $\partial\Omega$. Then

$$\|u\|_{W^{1,q}(\Omega)} + \|\nabla u\|_{C^{0,\alpha}(\omega')} \leqslant C\big(\|f\|_{L^1(\Omega)} + \|f\|_{L^\infty(\omega)}\big)$$

for $q < N/(N-1)$, $\alpha \in (0,1)$ and $\omega' \subset \omega$ is a strict subdomain of ω.

References

[1] F.V. Atkinson, L.A. Peletier, Elliptic equations with nearly critical growth, J. Differential Equations 70 (1987) 349–365.

[2] M. Ben Ayed, K. El Mehdi, On a biharmonic equation involving nearly critical exponent, preprint.

[3] H. Brézis, L.A. Peletier, Asymptotics for elliptic equations involving critical growth, in: Partial Differential Equations and the Calculus of Variations, vol. I, Birkhäuser, Boston, MA, 1989, pp. 149–192.

[4] Ph. Clément, D.G. de Figueiredo, E. Mitidieri, Positive solutions of semilinear elliptic systems, Comm. Partial Differential Equations 17 (1992) 923–940.

[5] X. Chen, C. Li, B. Ou, Classification of solutions for a system of integral equations, Comm. Partial Differential Equations 30 (2005) 59–65.

[6] K.-S. Chou, D. Geng, Asymptotics of positive solutions for a biharmonic equation involving critical exponent, Differential Integral Equations 13 (7–9) (2000) 921–940.

[7] D.G. De Figueiredo, P. Felmer, On superquadratic elliptic systems, Trans. Amer. Math. Soc. 343 (1994) 99–116.

[8] D.G. De Figueiredo, P.L. Lions, R.D. Nussbaum, A priori estimates and existence of positive solutions of semilinear elliptic equations, J. Math. Pures Appl. 61 (1982) 41–63.

[9] D.G. De Figueiredo, B. Ruf, Elliptic systems with nonlinearities of arbitrary growth, Mediterranean. J. Math. 1 (4) (2004) 417–431.

[10] M. Flucher, J. Wei, Semilinear Dirichlet problem with nearly critical exponent, asymptotic location of hot spots, Manuscripta Mat. 94 (1997) 337–346.

[11] B. Gidas, W. Ni, L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68 (1979) 209–243.

[12] D. Gilbarg, N.S. Trudinger, Elliptic Partial Differential Equations of Second Order, second ed., Springer-Verlag, Berlin, 1983.

[13] I.A. Guerra, Asymptotic behaviors of a semilinear elliptic system with a large exponent, J. Dynam. Differential Equations 19 (1) (2007) 243–263.

[14] M. Hammami, Concentration phenomena for fourth-order elliptic equations with critical exponent, Electron. J. Differential Equations 121 (2004) 22.

[15] Z.-C. Han, Asymptotic approach to singular solutions for nonlinear elliptic equations involving critical Sobolev exponent, Ann. Inst. H. Poincaré 8 (2) (1991) 159–174.
[16] J. Hulshof, R.C.A.M. van der Vorst, Differential systems with strongly indefinite variational structure, J. Funct. Anal. 114 (1993) 32–58.
[17] J. Hulshof, R.C.A.M. van der Vorst, Asymptotic behaviour of ground states, Proc. Amer. Math. Soc. 124 (8) (1996) 2423–2431.
[18] E.H. Lieb, Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities, Ann. of Math. 118 (1983) 349–374.
[19] E. Mitidieri, A Rellich type identity and applications, Comm. Partial Differential Equations 18 (1–2) (1993) 125–151.
[20] L.A. Peletier, R.C.A.M. van der Vorst, Existence and nonexistence of positive solutions of non-linear elliptic systems and the biharmonic equation, Differential Integral Equations 5 (4) (1992) 747–767.
[21] O. Rey, Proof of two conjectures of H. Brézis and L.A. Peletier, Manuscripta Mat. 65 (1) (1989) 19–37.
[22] M. Struwe, Global compactness result for elliptic boundary value problems involving limiting nonlinearities, Math. Z. 187 (1984) 511–517.
[23] R.C.A.M. van der Vorst, Variational identities and applications to differential systems, Arch. Ration. Mech. Anal. 116 (1991) 375–398.
[24] X.J. Wang, Sharp constant in a Sobolev inequality, Nonlinear Anal. TMA 20 (1993) 261–268.