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Ann. I. H. Poincaré – AN 24 (2007) 897–906

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Remarks on global controllability for the Burgers equation with two control forces

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Received 24 January 2006; received in revised form 8 March 2006; accepted 7 June 2006

Available online 6 February 2007

Abstract

In this paper we deal with the viscous Burgers equation. We study the exact controllability properties of this equation with general initial condition when the boundary control is acting at both endpoints of the interval. In a first result, we prove that the global exact null controllability does not hold for small time. In a second one, we prove that the exact controllability result does not hold even for large time.

Résumé

Ce papier concerne l'équation de Burgers visqueuse. On étudie les propriétés de contrôlabilité exacte de cette équation quand on contrôle les deux extrémités de l'intervalle. Dans un premier résultat, on démontre que la contrôlabilité globale à zéro n'est pas vraie pour un temps petit. Enfin, dans un deuxième résultat on démontre que la contrôlabilité exacte n'est pas vraie pour un temps long.

Keywords: Controllability; Burgers equation

1. Introduction

We consider the following control system associated to the one-dimensional Burgers equation:

$$\begin{cases} y_t - y_{xx} + yy_x = 0, & (t,x) \in Q := (0,T) \times (0,1), \\ y(t,0) = v_1(t), \quad y(t,1) = v_2(t), & t \in (0,T), \\ y(0,x) = y_0(x), & x \in (0,1). \end{cases} \tag{1}$$

Here, $T > 0$ is a given final time and $v_1(t)$ and $v_2(t)$ are control functions which are acting over our system at both endpoints of the segment $(0,1)$. Furthermore, y_0 is the initial condition which is supposed to be in $H^1(0,1)$. In the

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[1] Partially supported by D.G.E.S. (Spain), Grant BFM2003-06446.
[2] This work is partially supported by NSF Grant DMS 0205148.

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2006.06.010

sequel, we will suppose that our control functions v_1 and v_2 belong to the space $H^{3/4}(0,T)$ and they satisfy the compatibility conditions

$$v_1(0)=y_0(0) \quad \text{and} \quad v_2(0)=y_0(1). \tag{2}$$

Under these assumptions, it is classical to see that there exists a solution y of system (1) which belongs to the space $X:=L^2(0,T;H^2(0,1))\cap H^1(0,T;L^2(0,1))$ and a continuous function $K_0>0$ such that

$$\|y\|_X \leqslant K_0\big(\|y_0\|_{H^1(0,1)}+\|v_1\|_{H^{3/4}(0,T)}+\|v_2\|_{H^{3/4}(0,T)}\big) \tag{3}$$

(see, for instance, [11]).

In general, an *exact controllability* property for system (1) reads as follows: given $y_0\in H^1(0,1)$ and $y_1\in H^1(0,1)$, do there exist controls $v_1\in H^{3/4}(0,T)$ and $v_2\in H^{3/4}(0,T)$ such that the corresponding solution of (1) satisfies $y(T,x)=y_1(x)$ in $(0,1)$? When $y_1\equiv 0$, we will refer to this problem as the *exact null controllability*.

Let us now mention the previous works which one can find in the literature concerning the exact controllability for the Burgers equation.

• We start talking about the non-viscous Burgers equation. As far as we know, only two works have been dedicated to this issue.

In [10], the author describes the attainable set of the inviscid one-dimensional Burgers equation. In particular, he proves that by means of a boundary control, the Burgers equation can be driven from the null initial condition to a constant final state M in a time $T\geqslant 1/M$. The main tool the author uses is the so-called *return method*, which was introduced in [5].

More general results are obtained in [1]. In this reference, the authors precisely describe the attainable set for general scalar non-linear conservation laws with C^2 strictly convex flux functions when starting from a null initial data. In particular, they deduce that as long as the controllability time is lower than $1/M$ ($M>0$ constant), we do not have exact controllability to the state M.

• As long as the controllability of the viscous Burgers equation is concerned, very few works have been done too. Most of the papers in the literature deal with the controlled Burgers equation with one control force, which can act over our system at one endpoint or a small interior open set.

First, in [6], the author proves that for the solutions of the viscous Burgers equation with $y(t,0)=0$ in $(0,T)$ and initial data $y_0\in L^\infty(0,1)$, there exists a constant $C_0>0$ such that, for every $T>0$ one has

$$y(t,x)\leqslant \frac{C_0}{1-x} \quad \forall (t,x)\in(0,T)\times(0,1).$$

From this a priori estimate, the author deduces that the approximate controllability to some target states does not hold.

We recall now a result from [8], where the authors prove that we cannot reach an arbitrary target function in arbitrary time with the help of one control force. Precisely, they deduced the following estimate: for each $N>5$, there exists a constant $C_1(N)>0$ such that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^b (b-x)^N y_+^4(t,x)\,\mathrm{d}x < C_1 b^{N-5},$$

where b is the lower endpoint where the control function is supported (for instance, $b=1/2$ if the control function is acting in $(1/2,1)$). Here, $y_+(t,x)=\max\{y(t,x),0\}$ is the positive part of y. From this a priori estimate, one can deduce that we can not get close to some open set of target functions in $L^2(0,1)$.

Next, in [7], the authors prove that the global null controllability does not hold with one control force. Precisely, for any initial condition $y_0\in L^2(0,1)$ with $\|y_0\|_{L^2(0,1)}=r>0$, it is proved that there exists a time $T(r)>0$ such that for any control function, the corresponding solution satisfies

$$\big|y(t,\cdot)\big|\geqslant C_2>0 \quad \forall t\in\big(0,T(r)\big) \text{ in any open interval } I\subset(0,1),$$

for some $C_2(r)>0$. Furthermore, this time $T(r)$ is proved to be sharp in the sense that there exists a constant $C_3>0$ independent of r such that, if $T>C_3T(r)$, then there exists a control function such that the corresponding solution satisfies $y(T,x)=0$ in $(0,1)$. The main tool used in this work is the comparison principle.

For the Burgers equation with two boundary controls it was shown in [8], that any steady state solution is reachable for a sufficiently large time.

Finally, in the recent paper [4], the author proved that with the help of two control forces, we can drive the solution of the Burgers equation with null initial condition to large constant states. More precisely, for any time $T > 0$, it is shown the existence of a constant $C_4 > 0$ such that for any $C \in \mathbf{R}$ satisfying $|C| \geqslant C_4$, there exist two controls $v_1(t)$ and $v_2(t)$ such that the associated solution to (1) with $y_0 \equiv 0$ satisfies $y(T, \cdot) = C$ in $(0, 1)$. The idea of the proof of this result is based on the Hopf–Cole transformation, which leads to a controllability problem for the heat equation.

Before presenting our main results, let us remark that despite the Burgers equation is viewed like a one-dimensional model for the Navier–Stokes system, its controllability properties are certainly different, since the Navier–Stokes system with control distributed over the whole boundary is globally approximately controllable. As a proof of this, see for instance [3].

In the present paper, we have two main objectives. One concerning the exact null controllability for small time and the other one concerning the exact controllability for any time $T > 0$. Both results are of negative nature.

As long as the first one is concerned, we prove that there exists a final time T and an initial condition y_0 such that the solution of (1) is far away from zero. That is to say, the global null controllability for the Burgers equation with two control forces does not hold. The precise result is given in the following theorem:

Theorem 1. *There exists $T > 0$ and $y_0 \in H^1(0, 1)$ such that, for any control functions $v_1 \in H^{3/4}(0, T)$ and $v_2 \in H^{3/4}(0, T)$ satisfying the compatibility conditions* (2)*, the associated solution $y \in X$ to* (1) *satisfies*

$$\left\| y(T, \cdot) \right\|_{H^1(0,1)} \geqslant C_5 > 0, \tag{4}$$

for some positive constant $C_5(T, y^0)$.

The second main result is a negative exact controllability result:

Theorem 2. *For any $T > 0$, there exists an initial condition $y_0 \in H^1(0, 1)$ and a target function $y_1 \in H^1(0, 1)$ such that, for any $v_1 \in H^{3/4}(0, T)$ and $v_2 \in H^{3/4}(0, T)$ satisfying* (2)*, the associated solution $y \in X$ to* (1) *satisfies*

$$\left\| y(T, \cdot) - y_1(\cdot) \right\|_{H^1(0,1)} \geqslant C_6 > 0, \tag{5}$$

for some positive constant $C_6(T, y^0, y^1)$.

Remark 1. We recall that all global controllability properties obtained up to now for the Navier–Stokes system come essentially from the same property for the Euler equation (see [9]). As we pointed out above, the global controllability for the non-viscous Burgers equation does not hold. From this point of view, one could expect to have the results stated in Theorems 1 and 2.

In order to prove these results, we first show the equivalence of the controllability problem for the Burgers equation (1) and some controllability problem for a one-dimensional linear heat equation with positive boundary controls. This is carried out in several steps, by applying Hopf–Cole type transformations.

Then, our controllability results for the Burgers equation (stated in Theorems 1 and 2 above) will be deduced from both results for the corresponding heat equation.

As a consequence of these two theorems, one can easily deduce the following corollaries:

Corollary 3. *Let us consider the following control system associated to a semilinear parabolic equation*:

$$\begin{cases} w_t - w_{xx} + \dfrac{1}{2}|w_x|^2 = \hat{v}_1(t) & \text{in } Q, \\ w(t, 0) = 0, \quad w(t, 1) = \hat{v}_2(t) & \text{in } (0, T), \\ w(0, x) = w_0(x) & \text{in } (0, 1), \end{cases} \tag{6}$$

with $\hat{v}_1 \in L^2(0, T)$ and $\hat{v}_2 \in H^1(0, T)$. Then, the exact controllability of system (6) *with H^2-data does not hold. That is to say, for any $T > 0$, there exist $w_0 \in H^2(0, 1)$ and $w_1 \in H^2(0, 1)$ satisfying $w_0(0) = w_1(0) = 0$ such that*

$$\left\| w(T, \cdot) - w_1(\cdot) \right\|_{H^2(0,1)} \geqslant C_7,$$

for some $C_7(T, w_0, w_1) > 0$. *Furthermore, the exact null controllability does not hold either. That is to say, there exists a time* $T > 0$ *and an initial condition* $w_0 \in H^2(0,1)$ *satisfying* $w_0(0) = 0$ *such that*

$$\|w(T,\cdot)\|_{H^2(0,1)} \geqslant C_8$$

for some $C_8(T, w_0) > 0$.

Corollary 4. *Let us consider the following bilinear-control system associated to a heat equation*:

$$\begin{cases} z_t - z_{xx} = \dfrac{\hat{v}_3(t) - 1}{2} z & \text{in } Q, \\ z(t,0) = 0, \quad z(t,1) = \hat{v}_4(t) & \text{in } (0,T), \\ z(0,x) = z_0(x) & \text{in } (0,1), \end{cases} \tag{7}$$

with $\hat{v}_3 \in L^2(0,T)$ *and* $\hat{v}_4 \in H^1(0,T)$. *Then, equivalently as in the previous corollary, the exact controllability* (*for large time*) *and the exact null controllability* (*for small time*) *of system* (7) *with* H^2*-data do not hold.*

The paper is organized as follows. In a first section, we reduce the control system (1) to another one concerning the heat equation. Finally, in the second section we prove both Theorems 1 and 2.

2. Reduction to a heat controllability problem

In this section, we prove that the exact controllability properties for the Burgers equation are equivalent to some others for the heat system. All through this section, we consider $T > 0$ a fixed final time.

Let us start remembering the control system associated to the Burgers equation we are working with:

$$\begin{cases} y_t - y_{xx} + y y_x = 0 & \text{in } Q = (0,T) \times (0,1), \\ y(t,0) = v_1(t), \quad y(t,1) = v_2(t) & \text{in } (0,T), \\ y(0,x) = y_0(x) & \text{in } (0,1). \end{cases} \tag{8}$$

Next, we formulate the exact controllability problem we are interested in:

$$\begin{gathered} \text{For any } y_0 \in H^1(0,1) \text{ and any } y_1 \in H^1(0,1), \text{ there exist controls} \\ v_1, v_2 \in H^{3/4}(0,T) \text{ satisfying (2) such that the solution} \\ \text{of (8) satisfies } y(T,x) = y_1(x) \text{ in } (0,1). \end{gathered} \tag{9}$$

In a first lemma, we prove that this controllability problem is equivalent to a controllability problem for a semilinear parabolic equation with time-dependent controls, one acting in the right-hand side of our equation and the other one acting at $x = 1$. Let us thus consider the following control system:

$$\begin{cases} w_t - w_{xx} + \dfrac{1}{2}|w_x|^2 = v_3(t) & \text{in } Q, \\ w(t,0) = 0, \quad w(t,1) = v_4(t) & \text{in } (0,T), \\ w(0,x) = w_0(x) & \text{in } (0,1). \end{cases} \tag{10}$$

We have:

Lemma 1. *There exists a solution of problem* (9) *if and only if there exists a solution to the following controllability problem*:

$$\begin{gathered} \textit{For any } w_0, w_1 \in H^2(0,1) \textit{ with } w_0(0) = w_1(0) = 0, \textit{ there exist} \\ \textit{controls } v_3 \in L^2(0,T) \textit{ and } v_4 \in H^1(0,T) \textit{ and a solution} \\ w \in X_1 \textit{ of } (10) \textit{ such that } w(T,x) = w_1(x) \textit{ in } (0,1). \end{gathered} \tag{11}$$

Here, we have denoted $X_1 = L^2(0,T; H^3(0,1)) \cap H^1(0,T; H^1(0,1))$.

Proof. Let us first suppose that we have a solution to problem (9). Then, we denote

$$w(t,x)=\int_0^x y(t,s)\,\mathrm{d}s \quad \forall (t,x)\in Q.$$

Then, it is very easy to check that this function $w=w(t,x)$ solves the controllability problem (11) with

$$w_0(x)=\int_0^x y_0(s)\,\mathrm{d}x\in H^2(0,1),\qquad w_1(x)=\int_0^x y_1(s)\,\mathrm{d}s\in H^2(0,1)$$

and

$$v_3(t)=\frac{v_1^2(t)}{2}-y_x(t,0),\quad v_4(t)=\int_0^1 y(t,s)\,\mathrm{d}s \quad \forall t\in(0,T).$$

From (3) and $v_1\in H^{3/4}(0,T)$, we obtain in particular that $v_3\in L^2(0,T)$ and $v_4\in H^1(0,T)$.

On the other hand, suppose that there exists a solution w to (11). In this situation, we have that $w\in X_1$ and there exists a continuous function $K_1>0$ such that

$$\|w\|_{X_1}\leqslant K_1\big(\|w_0\|_{H^2(0,1)}+\|v_3\|_{L^2(0,T)}+\|v_4\|_{H^1(0,T)}\big). \tag{12}$$

Then, the function

$$y(t,x)=w_x(t,x)\in X$$

solves problem (9) with

$$y_0(x)=w_{0,x}(x)\in H^1(0,1),\qquad y_1(x)=w_{1,x}(x)\in H^1(0,1)$$

and

$$v_1(t)=w_x(t,0),\quad v_2(t)=w_x(t,1)\quad \forall t\in(0,T).$$

From (12), $v_1,\ v_2\in H^{3/4}(0,T)$. The proof of Lemma 1 is completed. □

The next step is to prove that the previous controllability result is equivalent to a controllability problem for a linear heat equation with two time-dependent controls, one of bilinear nature (multiplying the state function) and the other one acting at $x=1$. Let us thus consider the following control system:

$$\begin{cases} z_t-z_{xx}+\dfrac{v_5(t)}{2}z=0 & \text{in } Q,\\ z(t,0)=1,\quad z(t,1)=v_6(t) & \text{in } (0,T),\\ z(0,x)=z_0(x) & \text{in } (0,1). \end{cases} \tag{13}$$

Analogously than for system (10), we have that for any $z_0\in H^2(0,1)$ with $z_0(0)=1$, $v_5\in L^2(0,T)$ and $v_6\in H^1(0,T)$ with $v_6(0)=z_0(1)$, the solution z belongs to X_1 and there exists a continuous function $K_2>0$ such that

$$\|z\|_{X_1}\leqslant K_2\big(\|z_0\|_{H^2(0,1)}+\|v_5\|_{L^2(0,T)}+\|v_6\|_{H^1(0,T)}\big). \tag{14}$$

Precisely, we have:

Lemma 2. *There exists a solution to the problem* (11) *if and only if there exists a solution to the following controllability problem*:

For any $z_0, z_1\in H^2(0,1)$ with $z_0(x), z_1(x)>0$ in $(0,1)$ and
$z_0(0)=z_1(0)=1$, there exist two controls $v_5\in L^2(0,T)$ and
a strictly positive function $v_6\in H^1(0,T)$ in $[0,T]$ such that
the solution of (13) *$z\in X_1$ satisfies $z(T,x)=z_1(x)$ in $(0,1)$.* (15)

Proof. Let us first suppose that we have a solution $w \in X_1$ to the controllability problem (11). Then, we set $z(t,x) = \mathrm{e}^{-w(t,x)/2}$ for $(t,x) \in Q$ which satisfies $z \in X_1$ and solves problem (15) with

$$z_0(x) = \exp\{-w_0(x)/2\}, \quad z_1(x) = \exp\{-w_1(x)/2\} \quad \forall x \in (0,1)$$

and

$$v_5(t) = v_3(t), \quad v_6(t) = \exp\{-v_4(t)/2\} \quad \forall t \in (0,T).$$

Now, let us consider $z = z(t,x)$ a solution of problem (15). We define the function

$$w(t,x) = -2\ln z(t,x) \quad \forall (t,x) \in Q.$$

Observe that this is a good definition since $z(t,x) > 0$ in Q. This can readily be checked in a classical way by just proving that the negative part of the function z is identically zero. Again, immediate computations tell us that w fulfills the controllability problem (11) with

$$w_0(x) = -2\ln z_0(x), \quad w(T,x) = -2\ln z_1(x) \quad \forall x \in (0,1)$$

and

$$v_3(t) = v_5(t), \quad v_4(t) = -\ln v_6(t) \quad \forall t \in (0,T).$$

The proof of Lemma 2 is complete. □

In the last step, we will prove that the previous exact controllability problem is equivalent to an exact controllability problem for the heat equation with positive controls acting at both ends $x = 0$ and $x = 1$. Let us, thus, introduce the following control system:

$$\begin{cases} h_t - h_{xx} = 0 & \text{in } Q, \\ h(t,0) = v_7(t), \quad h(t,1) = v_8(t) & \text{in } (0,T), \\ h(0,x) = h_0(x) & \text{in } (0,1). \end{cases} \tag{16}$$

For the solution of this heat equation with initial data $h_0 \in H^2(0,1)$ and $v_7, v_8 \in H^1(0,T)$ with $v_7(0) = h_0(0)$ and $v_8(0) = h_0(1)$, we have again that $h \in X_1$ and there exists a continuous function K_3 such that

$$\|h\|_{X_1} \leqslant K_3\big(\|h_0\|_{H^2(0,1)} + \|v_7\|_{L^2(0,T)} + \|v_8\|_{H^1(0,T)}\big). \tag{17}$$

We have:

Lemma 3. *There exists a solution to the exact controllability problem* (15) *if and only if there exists a solution to the following one*:

For any $h_0, h_1 \in H^2(0,1)$ *with* $h_0(x), h_1(x) > 0$ *in* $(0,1)$ *and*
$h_0(0) = h_1(0) = 1$, *there exists a constant* $K > 0$ *and two controls*
$v_7(t), v_8(t) \in H^1(0,T)$ *which are strictly positive in* $[0,T]$ *such that*
the solution of (16) *satisfies* $h(T,x) = Kh_1(x)$ *in* $(0,1)$. (18)

Proof. We start by assuming that there exists a solution to the controllability problem (15). Then, we define

$$h(t,x) = \exp\left\{\int_0^t \frac{v_5(s)}{2}\,\mathrm{d}s\right\} z(t,x) \quad \forall (t,x) \in Q.$$

Then, it is very easy to check that h solves problem (18) with

$$h_0(x) = z_0(x), \quad h_1(x) = z_1(x) \quad \forall x \in (0,1)$$

$$v_7(t) = \exp\left\{\int_0^t \frac{v_5(s)}{2}\,\mathrm{d}s\right\}, \quad v_8(t) = \exp\left\{\int_0^t \frac{v_5(s)}{2}\,\mathrm{d}s\right\} v_6(t) \quad \forall t \in (0,T)$$

and

$$K=\exp\left\{\int_0^T \frac{v_5(s)}{2}\,\mathrm{d}s\right\}.$$

Assume now that there exists a solution h of problem (18) for some constant $K>0$. Then, the function

$$z(t,x)=\frac{1}{v_7(t)}h(t,x)\quad \forall (t,x)\in Q$$

solves problem (15) with

$$z_0(x)=\frac{1}{v_7(0)}h_0(x),\quad z_1(x)=\frac{K}{v_7(T)}h_1(x)\quad \forall x\in(0,1)$$

and

$$v_5(t)=2\frac{v_{7,t}(t)}{v_7(t)},\quad v_6(t)=\frac{v_8(t)}{v_7(t)}\quad \forall t\in(0,T).$$

The proof of this lemma is finished. □

Remark 2. Equivalently to the exact controllability problem stated in (9), one can formulate the null controllability problem taking $y_1\equiv 0$ and so the null controllability property

$$\begin{aligned}&\text{For any } y_0\in H^1(0,1)\text{, there exist controls } v_1,v_2\in H^{3/4}(0,T)\text{ satisfying}\\&\text{(2) and such that the solution of (8) satisfies } y(T,x)=0 \text{ in } (0,1)\end{aligned}\tag{19}$$

is equivalent (in the sense of the previous lemmas) to

$$\begin{aligned}&\text{For any } h_0\in H^2(0,1)\text{ with } h_0(x)>0 \text{ in } (0,1)\text{ and } h_0(0)=1\text{, there}\\&\text{exists a constant } K>0\text{ and controls } 0<v_7(t),\ v_8(t)\in H^1(0,T)\text{ such}\\&\text{that the solution } h\in X_1 \text{ of (16) satisfies } h(T,x)=K \text{ in } (0,1).\end{aligned}\tag{20}$$

3. Proofs of main results

In this section, we will prove both Theorems 1 and 2.

We first state a technical result which expresses the local results for solutions of heat equations:

Lemma 4. *Let $0<\xi_0<\xi_1<\xi_2<1$. Then, for each $\theta>0$ there exists a time $T^*=T^*(\theta)>0$ such that the solution of the backwards heat equation*

$$\begin{cases}-U_t-U_{xx}=0, & (t,x)\in(0,T^*)\times(0,1),\\ U(t,0)=U(t,1)=0, & t\in(0,T^*),\\ U(T^*,x)=\delta_{\xi_0}-\theta\delta_{\xi_1}+\delta_{\xi_2}, & x\in(0,1)\end{cases}\tag{21}$$

satisfies

$$U_x(t,0)>0\quad \text{and}\quad U_x(t,1)<0\quad \forall t\in(0,T^*).$$

In the previous lemma, we have denoted by δ_x the Dirac mass distribution at point x.

A much more intrinsic result is the one given in the following lemma which was proved in [2]:

Lemma 5. *Let $T>0$ and $0<\xi_0<\xi_1<\xi_2<1$. Then, there exists $\theta>0$ such that the solution of the backwards heat equation*

$$\begin{cases}-U_t-U_{xx}=0, & (t,x)\in Q,\\ U(t,0)=U(t,1)=0, & t\in(0,T),\\ U(T,x)=\delta_{\xi_0}-\theta\delta_{\xi_1}+\delta_{\xi_2}, & x\in(0,1)\end{cases}\tag{22}$$

satisfies

$$U_x(t,0) > 0 \quad \textit{and} \quad U_x(t,1) < 0 \quad \forall t \in (0,T).$$

3.1. No null controllability result

In this paragraph, we provide the proof of Theorem 1. We first recall that from the computations made in the previous section, our control system is reduced to

$$\begin{cases} h_t - h_{xx} = 0, & (t,x) \in Q, \\ h(t,0) = \tilde{v}_1(t), \quad h(t,1) = \tilde{v}_2(t), & t \in (0,T), \\ h(t,x) = h_0(x), & x \in (0,1), \end{cases} \tag{23}$$

where

$$h_0(x) = \exp\left\{ \int_0^x \frac{y_0(s)}{2}\, \mathrm{d}s \right\} \quad \forall x \in (0,1)$$

and condition $y(T,x) = 0$ in $(0,1)$ now reads $h(T,x) = K$ in $(0,1)$.

More precisely, we have proved (see Lemmas 1–3) that the null controllability property for system (1) is equivalent to the existence of a positive constant K and positive controls $\tilde{v}_1, \tilde{v}_2 \in H^1(0,T)$ such that the solution $h \in X_1$ of (23) satisfies $h(T,x) = K$ in $(0,1)$.

Proof of Theorem 1. We prove Theorem 1 by contradiction. Thus, suppose that for any $T > 0$ and any $h_0 \in H^2(0,1)$ with $h_0 > 0$ and $h_0(0) = 1$, there exists a constant $K > 0$ and two controls $0 < \tilde{v}_1 \in H^1(0,T)$ and $0 < \tilde{v}_2(t) \in H^1(0,T)$ such that the solution of (23) satisfies

$$h(T,x) = K \quad \text{in } (0,1).$$

Then, let us consider the function U given by Lemma 4 for some $\theta \geqslant 2$, which is thus defined up to a time $t = T^*$. Multiplying the equation of h by U and integrating in $(0,T^*) \times (0,1)$, we obtain

$$-\int_0^{T^*} \big(U_x(t,0)\tilde{v}_1(t) - U_x(t,1)\tilde{v}_2(t)\big)\,\mathrm{d}t + K(2-\theta) - \int_0^1 U(0,x)\,h_0(x)\,\mathrm{d}x = 0 \tag{24}$$

for any $h_0 \in H^2(0,1)$. From the facts that the normal derivative of U is negative and $\theta \geqslant 2$, the two first terms in the previous identity are non-positive.

Now, since the normal derivative of U is negative and U satisfies homogeneous Dirichlet boundary conditions, there exists $\delta > 0$ such that

$$U(0,x) \geqslant \delta x \quad \forall x \in (0,\delta) \quad \text{and} \quad U(0,x) \geqslant \delta(1-x) \quad \forall x \in (1-\delta,1).$$

Then, we can choose an initial condition h_0 with $h_0(0) = 1$ such that

$$-\int_0^1 U(0,x)h_0(x)\,\mathrm{d}x < 0.$$

Indeed, on the one hand, from a priori estimates for system (21), we obtain the existence of a positive constant $C^* = C^*(T^*,\theta)$ such that

$$\|U(0,\cdot)\|_{L^2(0,1)} \leqslant C^*.$$

On the other hand, if we choose $h_0 = h_0(x) \in (0,1)$ for all $x \in (0,1)$ such that

$$h_0(x) = \frac{4C^*}{\delta^3} \quad \forall x \in \left(\frac{\delta}{4}, \frac{3\delta}{4}\right) \cup \left(1 - \frac{3\delta}{4}, 1 - \frac{\delta}{4}\right),$$

we have (recall that $h_0 > 0$)

$$
\begin{aligned}
-\int_0^1 U(0,x)h_0(x)\,\mathrm{d}x &\leqslant -\int_{\delta/4}^{3\delta/4}\left(\frac{4C^*}{\delta^3}\right)U(0,x)\,\mathrm{d}x - \int_{1-3\delta/4}^{1-\delta/4}\left(\frac{4C^*}{\delta^3}\right)U(0,x)\,\mathrm{d}x - \int_{\delta}^{1-\delta} U(0,x)h_0(x)\,\mathrm{d}x \\
&\leqslant -\left(\frac{C^*}{2}+\frac{C^*}{2}\right) + (1-2\delta)C^* = -2\delta C^* < 0.
\end{aligned}
$$

This is obviously a contradiction when combined with identity (24). □

3.2. No exact controllability result

In this paragraph, we prove Theorem 2. Going back again to the previous section, we consider the following control system:

$$
\begin{cases}
h_t - h_{xx} = 0, & (t,x) \in (0,T)\times(0,1), \\
h(t,0) = \tilde{v}_1(t), \quad h(t,1) = \tilde{v}_2(t), & t \in (0,T), \\
h(t,x) = h_0(x), & x \in (0,1)
\end{cases}
\tag{25}
$$

where now the condition $y(T,x) = y_1(x)$ in $(0,1)$ reads

$$
h(T,x) = Kh_1(x) := K\exp\left\{-\int_0^x \frac{y_1(s)}{2}\,\mathrm{d}s\right\} \quad \text{in } (0,1).
$$

Proof of Theorem 2. Again, we prove Theorem 2 by contradiction. Thus, we assume the existence of a final time $T > 0$ such that for any $0 < h_0 \in H^2(0,1)$ and any $0 < h_1 \in H^2(0,1)$ with $h_0(0) = h_1(0) = 1$, there exists a constant $K > 0$ and two controls $0 < \tilde{v}_1(t) \in H^1(0,1)$ and $0 < \tilde{v}_2(t) \in H^1(0,T)$ such that

$$
h(T,x) = Kh_1(x) \quad \text{in } (0,1).
$$

Let U be the function given by Lemma 5. Analogously as we did in the previous paragraph, by a simple integration by parts we get

$$
-\int_0^T \big(U_x(t,0)\tilde{v}_1(t) - U_x(t,1)\tilde{v}_2(t)\big)\,\mathrm{d}t - \int_0^1 U(0,x)\,h_0(x)\,\mathrm{d}x + K\int_0^1 \big(h_1(\xi_0) - \theta h_1(\xi_1) + h_1(\xi_2)\big)\,\mathrm{d}x = 0 \tag{26}
$$

for any $0 < h_0 \in H^2(0,1)$ and any $0 < h_1 \in H^2(0,1)$. Recall that U is now the solution of system (22). Again, we have that the first term of identity (26) is non-positive.

Using the same construction as in the previous paragraph, one can choose $0 < h_0 \in H^2(0,1)$ with $h_0(0) = 1$ such that

$$
-\int_0^1 U(0,x)h_0(x)\,\mathrm{d}x < 0. \tag{27}
$$

That is to say, on the one hand from a priori estimates for the heat equation, we obtain

$$
\big\|U(0,\cdot)\big\|_{L^2(0,1)} \leqslant \widehat{C}
$$

for some $\widehat{C} > 0$. On the other hand, we choose $h_0 = h_0(x) \in (0,1)\ \forall x \in (0,1)$ with the following property:

$$
h_0(x) = \frac{4C^*}{\delta^3} \quad \forall x \in \left(\frac{\delta}{4},\frac{3\delta}{4}\right) \cup \left(1-\frac{3\delta}{4}, 1-\frac{\delta}{4}\right).
$$

Then, using the fact that

$$
U(0,x) \geqslant \delta x \quad \forall x \in (0,\delta) \quad \text{and} \quad U(0,x) \geqslant \delta(1-x) \quad \forall x \in (1-\delta,1)
$$

and splitting the integral in (27) into three terms, we get

$$-\int_0^1 U(0,x)h_0(x)\,\mathrm{d}x \leqslant -2\delta\widehat{C} < 0.$$

Finally, we take a function $0 < h_1(x) \in H^2(0,1)$ such that

$$h_1(\xi_0) - \theta h_1(\xi_1) + h_1(\xi_2) < 0.$$

For instance, one can take a function h_1 whose value in ξ_1 is $3/\theta$, while $h_1(\xi_0) = h_1(\xi_2) = 1$.

This is a contradiction with identity (26). The proof of Theorem 2 is complete. □

References

[1] F. Ancona, A. Marson, On the attainable set for scalar nonlinear conservation laws with boundary control, SIAM J. Control Optim. 36 (1) (1998) 290–312.

[2] M.I. Belishev, On approximating properties of solutions of the heat equation, in: Control Theory of Partial Differential Equations, in: Lecture Notes in Pure and Appl. Math., vol. 242, Chapman and Hall/CRC, Boca Raton, FL, 2005, pp. 43–50.

[3] J.-M. Coron, On the controllability of the 2-D incompressible Navier–Stokes equations with the Navier slip boundary conditions, ESAIM Control Optim. Calc. Var. 1 (1995/96) 35–75.

[4] J.-M. Coron, Some open problems on the control of nonlinear partial differential equations, in: H. Berestycki, M. Bertsch, B. Peletier, L. Véron (Eds.), Perspectives in Nonlinear Partial Differential Equations: In Honor of Haïm Brezis, in: Contemporary Mathematics, American Mathematical Society, Providence, RI, in press.

[5] J.-M. Coron, Global asymptotic stabilization for controllable systems without drift, Math. Control Signals Systems 5 (3) (1992) 285–312.

[6] J.I. Díaz, Obstruction and some approximate controllability results for the Burgers equation and related problems, in: Control of Partial Differential Equations and Applications, in: Lecture Notes in Pure and Appl. Math., vol. 174, Dekker, New York, 1995, pp. 63–76.

[7] E. Fernández-Cara, S. Guerrero, On the controllability of Burgers system, C. R. Acad. Sci. Paris, Ser. I 341 (2005) 229–232.

[8] A. Fursikov, O.Yu. Imanuvilov, On controllability of certain systems simulating a fluid flow, in: Flow Control, Minneapolis, MN, 1992, in: IMA Vol. Math. Appl., vol. 68, Springer, New York, 1995, pp. 149–184.

[9] O. Glass, Exact boundary controllability of 3-D Euler equation, ESAIM Control Optim. Calc. Var. 5 (2000) 1–44.

[10] T. Horsin, On the controllability of the Burgers equation, ESAIM Control Optim. Calc. Var. 3 (1998) 83–95.

[11] J.-L. Lions, E. Magenes, Non-Homogeneous Boundary Value Problems and Applications, vol. I, Translated from the French by P. Kenneth. Die Grundlehren der Mathematischen Wissenschaften, Band 181, Springer-Verlag, New York–Heidelberg, 1972.