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Magnus intersections of one-relator free products with small cancellation conditions

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Dedicated to Professor Avinoam Mann on the occasion of his retirement

Abstract. Donald Collins initiated the study of intersections of Magnus subgroups in one-relator groups. In particular, he characterized those intersections of Magnus subgroups that are not Magnus subgroups. In the present work we show that Collins' results extend to one-relator quotients of free products of groups with a small cancellation condition and give a complete list of those defining relators for which Magnus subgroups do not intersect in a Magnus subgroup. We use van Kampen diagrams and word combinatorics.

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1. Introduction

Let $\mathcal{P} = \langle X \mid \mathcal{R} \rangle$ be a presentation of a group. $\mathcal{P}$ is termed *one-relator presentation* if $\mathcal{R}$ consists of a single relator R. We say that a group is a *one-relator group* if it has a one-relator presentation. Let G be an one-relator group given by the one-relator presentation $\mathcal{P} = \langle X \mid \mathcal{R} \rangle$ and let F be the free group, freely generated by X.

The study of one-relator groups started with the pioneering works of Max Dehn and Wilhelm Magnus, and this was one of the central subjects of classical combinatorial group theory (see [MKS]). Among the most important achievements of this theory was the solution of the word problem for one-relator groups by W. Magnus. Magnus and his successors developed a whole (algebraic) theory of one-relator groups. The main ingredients of this theory are the subgroups of G which are generated by the images of proper subsets of X under the natural map $\pi \colon F \to G$. These groups where termed by his successors *Magnus subgroups*. Magnus proved that these groups are free, freely generated by the corresponding subset of $\pi(X)$ (the Freiheitssatz).

Groups in which subgroups generated by proper subsets of a canonical set of generators play a central role, are not exceptional in group theory; for example, if H is a Coxeter group generated by a finite set S of reflections, then the so-called parabolic subgroups – which are the subgroups generated by proper subsets of S – enter in considerations of fundamental importance in the theory of Coxeter groups and in representation theory. It is an important basic result that the intersection of two parabolic subgroups is again parabolic.

Coming back to one-relator groups, it is easy to see that intersection of Magnus subgroups is not necessarily a Magnus subgroup: let $X = \{a, b\}$ and let $R = a^2b^3$. Then $\langle a\rangle$ and $\langle b\rangle$ are Magnus subgroups with non-trivial intersection $\langle a^2\rangle$. However, $\langle a^2\rangle$ is not a Magnus subgroup. This example rises naturally the following questions.

Let X_1 and X_2 be proper subsets of X, let $Y_1 = \pi(X_1)$, $Y_2 = \pi(X_2)$, and let $H_1 = \langle Y_1\rangle$ and $H_2 = \langle Y_2\rangle$.

(1) Under what conditions on R, X_1 and X_2 is $H_1 \cap H_2$ a Magnus subgroup?

(2) If $H_1 \cap H_2$ is not a Magnus subgroup then how its structure looks like? In particular, how $H_1 \cap H_2$ is related to $\langle Y_1 \cap Y_2\rangle$?

These questions are interesting in their own, but they are also crucial in certain aspects of solutions of equations and also for cyclic presentations. (See [Ju2], [Ju3] and [Ju4], and independently, [E-H].)

The study of Magnus intersections was initiated by Donald Collins in [Co], where among other things he gave a complete answer to the second question, by showing that

$$\text{if } H_1 \cap H_2 \neq \langle Y_1 \cap Y_2\rangle \text{ then } H_1 \cap H_2 = \langle Y_1 \cap Y_2\rangle * \langle c_1\rangle = \langle Y_1 \cap Y_2\rangle * \langle c_2\rangle, \\ \text{where } c_1 \in H_1,\ c_2 \in H_2 \text{ and } c_1, c_2 \notin \langle Y_1 \cap Y_2\rangle\,. \qquad (\star)$$

Jim Howie in [Ho2], based an a conjecture of Don Collins, gave an algorithm to check whether $H_1 \cap H_2 = \langle Y_1 \cap Y_2\rangle$, or not.

Now, one-relator free products are natural generalisations of one-relator groups: we consider the free group freely generated by X as the free product of infinite cyclic groups and then replace them by arbitrary groups G_i, $G_i \neq 1$ for $i = 1, \dots, n$, $n \geq 2$ and take a one-relator quotient (see [Ho1] for more motivation). Such groups G have a free product presentations $\mathcal{P} = \langle G_1 * \cdots * G_n \mid R\rangle$, where R is a cyclically reduced word in $G_1 * \cdots * G_n$ of length at least two. We can naturally extend the notion of Magnus subgroups to one-relator free products, namely a *Magnus subgroup of* G is a subgroup generated by the image of a proper subset of $\{G_i\}$, $i = 1, \dots, n$.

In contrast with one-relator groups, very little is known on one-relator free products. Even the most fundamental problem, the word problem, is widely open. Nevertheless, under suitable conditions on the components of the free product or on the defining relator, or on both, large parts of the theory of one-relator groups can be extended to one-relator free products.

In the present work we make assumptions on R and consider questions 1 and 2 above. More precisely, we assume the small cancellation condition $C(6)$ & $T(4)$ and in Theorem A, with a mild restriction on R we give a complete classification of those words R for which $H_1 \cap H_2$ is not a Magnus subgroup, where H_1 and H_2 are Magnus subgroups of G. In Theorem B we show that the corresponding version of the theorem of D. Collins (see [Co]) mentioned above in ($\star$) holds true. We also show how to get from the defining relator R elements c_1 and c_2 in ($\star$). Finally, in Theorem C we show that Magnus subgroups are free products. (The Freiheitssatz.) We work under the following assumptions

Notation and assumptions of the main theorems. Let G be a group with a one-relator free product presentation $\mathcal{P}$, $\mathcal{P} = \langle F \mid \mathcal{R} \rangle$, where $F = G_1 * \cdots * G_n$, $n \geq 2$, G_i, $i = 1, \ldots, n$, are non-trivial groups, $\mathcal{R}$ is the symmetric closure of a cyclically reduced word R of length at least two in F such that $\mathcal{P}$ satisfies the small cancellation condition $C(6)$ & $T(4)$. (See [L-S, Ch. V] for definition.) Suppose that no letter in R has order two and if $g \in G_i$ occurs in R and g with finite order, then there is at least one more occurrence of a letter in R from G_i. Let $\nu \colon F \to G$ be the natural homomorphism which sends each element of F to its coset modulo the normal closure of R in F. For a subset Q of $\{1, \ldots, n\}$ let $G_Q = \underset{i \in Q}{*} G_i$. Let $I, J \subsetneqq \{1, \ldots, n\}$ such that $I \not\subseteq J$ and $J \not\subseteq I$ and let $D = I \cap J$. Finally, let H_Q be the image of G_Q by ν. Our main results are the following.

Theorem A. *Let notation and assumptions be as above. If $H_I \cap H_J \neq H_D$ then R has a cyclic conjugate R^* which satisfies one of the following:*

(i) *$R^* = UaU^{-1}W^{-1}$ reduced as written, where $a \in G_i$ for some i and (U, W) is inadequate* (*see Definition* 2.4 (c))*;*

(ii) *R^* is exceptional in the sense of Definition* 5.5*;*

(iii) *$R^* = AB$ reduced as written with $A \in G_I$ and $B \in G_J$.*

Moreover, if R has no cyclic conjugate R^* as in (i)*, then $H_I \cap H_J \neq H_D$ if and only if R^* is exceptional or $R^* = AB$, $A \in G_I$ and $B \in G_J$.*

The result of Theorem A is quite surprising: clearly, (iii) is an obvious case for $H_I \cap H_J \neq H_D$ and as usual in small cancellation theory we would expect that this is the only case. However, Theorem A tells us that there are also rather unexpected cases (case ii) and moreover, if (i) does not hold then these are all the additional cases. Observe that by Theorem A, the exceptional words in (ii) are precisely those R which have arbitrary length as words in $G_I *_{G_D} G_J$, yet they have a consequence of length two in it.

Theorem B. *Let notation and assumptions be as above and suppose that R has no cyclic conjugate R^* which satisfies condition* (i) *of Theorem A. If $H_I \cap H_J \neq H_D$*

then $H_I \cap H_J = H_D * \langle u \rangle = H_D * \langle v \rangle$, *where* $u \in H_I \setminus H_D$ *and* $v \in H_J \setminus H_D$. *Moreover,* R^* *contains a unique subword* $U \in G_I$ *which starts and terminates with a letter in* $I \setminus D$ *and which is maximal relative to this property and there is a unique subword* $V \in G_J$ *which starts and terminates with a letter in* $J \setminus D$ *and which is maximal relative to this property such that if* $\nu(U) = u$ *and* $\nu(V) = v$ *then* $H_D * \langle u \rangle = H_D * \langle v \rangle = H_I \cap H_J$.

Theorem C. *Let notation and assumptions be as above and suppose that no cyclic conjugate* R^* *of* R *satisfies condition* (i) *of Theorem A. Then* $H_J \cong G_J$. *In particular,* H_J *is a free product.*

We mention that J. Howie in [Ho2], independently, considered problems (1) and (2) above in one-relator free products with arbitrary defining relators, however, with the assumptions that every component G_i is locally indicable (i.e. every finitely generated non-trivial subgroup maps onto the infinite cyclic group).

Our main tools are small cancellation theory and van Kampen diagrams with word combinatorics. We prove first Theorem C. The idea is to show that under the assumptions of the theorem,

$$\text{every consequence of the defining relator } R \text{ contains at least one letter from each } G_i. \qquad (\star\star)$$

(This is one of the equivalent formulations of the Freiheitssatz by Magnus. See [L-S].)

A central ingredient in small cancellation theory is Greendlinger's Lemma, which guaranties the existence of at least two Greendlinger regions in every van Kampen diagram M, which has at least two regions. These are regions with the property that their boundary has a large common portion with the boundary of M. (For definitions of van Kampen diagrams and regions see Section 2.2.)

However for our problem, showing $(\star\star)$, a more precise information than just knowing that a large portion of the defining relator is present on the boundary of M, is needed.

Recently we developed an improved version of Greendlinger's Lemma for one-relator groups and one-relator free products with the small cancellation condition $C(6)\,\&\,T(4)$, which implies $(\star\star)$, and hence proves Theorem C. We remark that it also implies several results of different nature. In [Ju5] we solved the membership problem for Magnus subgroups of one-relator free products with small cancellation. In [Ju6] we proved the appropriate version of Magnus's Freiheitssatz for Magnus subsemigroups of one-relator groups with small cancellation. In [Ju7] we classify non-malnormal Magnus subgroups in one-relator groups and free products with small cancellation. We also plan to use it in complexes of certain types of groups to produce a lower bound on the angles between the local groups.

Theorem B follows easily from the proof of Theorem A, so we concentrate on the proof of Theorem A. The proof of Theorem A is much more demanding then the proof of Theorem C; while the improved Greendlinger's Lemma was enough for the proof of Theorem C, we need the extension of a further result from small cancellation theory. If $H_I \cap H_J \neq H_D$, as in Theorem A, then there are non-empty words A and B in G_I and G_J respectively, such that $\nu(A) = \nu(B)$ in G. Thus AB^{-1} is a consequence of R and hence there is a van Kampen diagram M with boundary label AB^{-1}. In Theorem A we recover the combinatorial structure of the word R from the combinatorial structure of its consequence AB^{-1}. In a sense, we deal with an inverse problem to the word problem. In the word problem we are *given* R and we want to check whether (another) given word W is a consequence of R; in Theorem A a word $W(= AB^{-1})$ is given and it is also given that W is a consequence of an unknown relation R, and we would like to find the combinatorial structure of all such relations, in term of the combinatorial structure of W. We are not aware of results in this direction in the literature. This is a difficult problem in general, because boundary regions of M contribute only parts of R^* to the boundary label of M and in general, it is difficult to recover R^* from these parts. There is however, one case when this is doable; this is the case when M is a one-layer diagram. Then we can use word combinatorics in order to determine the combinatorial structure of R. This is done in Sections 5.1 and 5.2. So it remains now to show that M is a one-layer diagram. We show this in Section 4.

The work is organised as follows:

In Section 2 we introduce preliminary results on words and van Kampen diagrams as well as the improved version of Greendlinger's Lemma. In Section 3 we prove Theorem C while in Section 4 we prove that intersection diagrams are one-layer diagrams. In Section 5 we prove Theorems A and B.

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2. Preliminary results on words and diagrams

2.1. Words. Basic reference for this subsection is [L-S, Ch. V]. We recall here only a few basic notions and results which we need.

Let $F = G_1 * \cdots * G_n$, $n \geq 2$, be the free product of non-trivial groups G_i, $i = 1, \ldots, n$. We call the G_is the *components of* F. Let G be a group. A *free presentation for* G is a presentation of G as a homomorphic image of free group F. A *free product presentation for* G is a presentation of G as a homomorphic image of free product F. If F is a free group, freely generated by a set X then, as usual, we denote *free presentation* of G by $\langle X \mid \mathcal{R} \rangle$, where $\mathcal{R}$ is a set of defining relations for G, and if F is a free product, $F = G_1 * \cdots * G_n$, $n \geq 2$, G_i non-trivial, then we denote *free product presentation* of G by $\langle G_1 * \cdots * G_n | \mathcal{R} \rangle$, where again $\mathcal{R}$ is a set of defining relations for G, $\mathcal{R} \subseteq G_1 * \cdots * G_n$. The elements of $F \setminus \{1\}$ can be uniquely presented by finite sequences of non-trivial elements of the components, such that adjacent elements in a sequence come from different components. We call the elements of G_i, $i = 1, \ldots, n$, *letters* and the sequences of elements, *words*. For $g \in G_i$, $g \neq 1$, denote $\alpha(g) = i$. Thus, if $1 \neq W \in F$ then W can be uniquely expressed as a word: $W = b_{i_1} \ldots b_{i_k}$, where $k \geq 1$, $1 \neq b_{i_j} \in G_{i_j}$ and $\alpha(b_{i_j}) \neq \alpha(b_{i_{j+1}})$ for $j = 1, \ldots, k-1$. We call this presentation of W its *normal form*, call k its *length* and denote it by $|W|$.

Let U and V be reduced words in F. We say that the product UV is *reduced as written* if either the last letter of U and the first letter of V are in different components G_i, or if there is no cancellation between U and V, however the last letter of U and the first letter of V may come from the same component (consolidation).

Denote by $\mathcal{H}(W)$ the set of initial subwords of W and by $\mathcal{T}(W)$ the set of terminal subwords of W. Also, for a reduced non-empty word W we denote by $h(W)$ the first letter of W and by $t(W)$ the last letter of W. We start with the following well-known results on word equations over F.

Lemma 2.1. (a) *Let A, B and C be reduced words which contains no letters of order two, such that AB and BC are reduced as written. If $|AB| \geq 2$ and $AB = BC$ then $A = KL$, $C = LK$ and $B = (KL)^\beta K$, $\beta \geq 0$.*

(b) *Let A be a cyclically reduced word, $|A| \geq 2$. If $AA = UA^\varepsilon V$, reduced as written, $U \neq 1$, $V \neq 1$, and $\varepsilon \in \{1, -1\}$, then $\varepsilon = 1$ and $A = B^k$, $k \geq 2$, for some cyclically reduced word B.*

(c) *Let Z be a reduced word which contains no letters of order two.*

(i) *If for some reduced words V and U, such that ZU and $Z^{-1}V$ are reduced as written, we have $ZU = Z^{-1}V$, then $|Z| = 1$ and $V = Z^2U$. Moreover, Z and the first letters of V and U are in the same component G_i.*

(ii) *If for some reduced words U and V such that UZ and $Z^{-1}V$ are reduced as written, we have $UZ = Z^{-1}V$, then $U = Z^{-1}a$ and $V = aZ$, where a is a letter and the first letter of Z and a are in the same component G_i.*

We introduce below the key notion of the work.

Definitions and notation. (a) Let $W \in F$, $W = a_{i_1} \ldots a_{i_k}$, $a_{i_j} \in G_{i_j}$ reduced as written. Define

$$\mathrm{Supp}(W) = \{i_1, \ldots, i_k\} \subseteq \{1, \ldots, n\}.$$

(b) Let W_1 and W_2 be reduced words in F. W_2 *majorises* W_1 if $\mathrm{Supp}(W_2) \supseteq \mathrm{Supp}(W_1)$. In this case write $W_2 \succ W_1$. If $W_1 \succ W_2$ and $W_1 \succ W_3$ we shall write $W_1 \succ W_2, W_3$.

(c) For W_1 and W_2 in part (b) define $W_1 \sim W_2$ if $W_1 \prec W_2$ and $W_2 \prec W_1$. Thus $W_1 \sim W_2$ if and only if $\mathrm{Supp}(W_1) = \mathrm{Supp}(W_2)$.

Clearly "$\sim$" is an equivalence relation, which contains the equality of elements in F.

The following lemma is immediate from the definition, hence its proof is omitted.

Lemma 2.2. (a) *If A is a subword of B then $A \prec B$.*

(b) *If $A \prec B$ then $A^{\pm 1} \prec B^{\pm 1}$.*

(c) *If $A \sim B$ and $A \prec C$ then $B \prec C$.*

(d) *If $A = P_1 \dots P_m$, reduced as written and $P_i \sim Q$ for $i = 1, \dots, m$ and a reduced word Q, then $A \sim Q$.*

(e) *If $A \succ P_1, \dots, P_m$ then $A \succ W(P_1, \dots, P_m)$, for every word W on $P_1, \dots, P_m$.*

Parts (a) and (b) of the following lemma are immediate corollaries of Lemma 2.1 and Lemma 2.2. Also, the remaining parts are routine case by case checking. Hence we omit their proofs.

Lemma 2.3. (a) *Let A, B and C be as in Lemma* 2.1 (a). *Then $B \prec A \sim C \sim AB \sim BC$. If $\beta \geq 1$ then $B \sim A$.*

(b) *If $AB = KAC$ with $|B| \geq 2$ and $|KA| \geq 2$, reduced as written then $B \succ A$, $B \succ C$ and $K \succ A$.*

(c) *Let K, Q, U, V and S be non-empty words such that KQ, UV, VU and KS are reduced as written, of length at least two. If $KQ = UV$ and $KS = VU$ then either $Q \sim S \succ K, U, V$, or $U = D^{\alpha}$, $V = D^{\beta}$, $\alpha, \beta \geq 1$ and $D \succ K, Q, S$.*

(d) *Let B, Q, L, U and V be non-empty words such that BQ, UV, LB and VU have length at least two and are reduced as written. If $BQ = UV$ and $LB = VU$ then one of the following holds:*

(i) *$B = U$, $Q = L = V$; or*

(ii) *$Q \succ B, U, V, L$ and $L \succ B, U, V, Q$ (hence $L \sim Q \sim UV$).*

(e) *Let L, K, Q_1, M and N be non-empty reduced words, such that KQ_1, MN, Q_1M and LK are reduced as written with length at least two. If $KQ_1 = MN$ and $Q_1M = LK$, then one of the following holds:*

(i) *$Q_1 = N = L$ and $K = M$; or*

(ii) *$Q_1 \succ K, L, M, N$.*

Notice that if one of the products in parts (a)–(e), like BQ in part (d), has length one then the statements of Lemma 2.3 trivially hold true.

The following basic notions are crucial for the paper.

Definition 2.4. (a) Let R be a weakly cyclically reduced word in F and let P be a subword of a cyclic conjugate of R. P is a *piece* in R (or a piece relative to the symmetric closure $\mathcal{R}$ of R) if R has distinct cyclic conjugates R_1 and R_2 such that $R_1 = PR_1'$, $R_2^\varepsilon = PR_2'$, reduced as written, for some $\varepsilon \in \{1, -1\}$. Equivalently, $P^{\pm 1}$ has at least two occurrences in the cyclic word $\widehat{R}$, corresponding to the linear word R. We call the two occurrences of P in R_1 and R_2^ε, respectively, a *piece pair* and denote it by (P, P'), where $P'(= P^\varepsilon)$ is the occurrence of P^ε in R_2.

(b) A piece pair (P, P') as in part (a) of the definition is *right normalized* if $(R_1')^{-1} R_2'$ is reduced as written.

(c) Let $R = UaU^{-1}W^{-1}$, reduced as written, $a \in G_i$ for some i, $i = 1, \dots, n$. The pair (U, W) is *inadequate* if

(i) W is the product of at least four pieces over the symmetric closure of R and

(ii) $\operatorname{Supp} W \supsetneqq \operatorname{Supp} U$.

2.2. Diagrams. For basic results on diagrams see [L-S, Ch. V]. We recall here some of the basic definitions from [L-S, p. 236 and pp. 274–276] for convenience.

A *diagram over a group* F is an oriented map M and a function Φ assigning to each oriented edge e of M as a *label* an element $\Phi(e)$ of F such that if e is an oriented edge of M and e^{-1} the oppositely oriented edge, then $\Phi(e^{-1}) = \Phi(e)^{-1}$, and if $\mu = e_1 v_1 e_2 v_2 \dots e_k$ is a path in M then $\Phi(\mu) = \Phi(e_1)\Phi(e_2)\dots\Phi(e_k)$. We denote by Φ_M the labelling function of M over F. If M is fixed we shall write Φ for Φ_M.

If M is planar, connected and simply connected then it is called *a van Kampen diagram*. In the case of diagrams M over free products the vertices are divided into two types, *primary* and *secondary*. The label on every edge of M will belong to a factor G_i of F with the labels on successive edges meeting at primary vertices belonging to different factors G_j, while the labels on the edges at a secondary vertex all belong to the same factor of F. For a region D in M denote by ∂D its boundary and by ∂M the boundary of M.

Definitions 2.5. Let M be a diagram over F.

(a) Two regions D_1 and D_2 in M are *neighbours* if $\partial D_1 \cap \partial D_2 \neq \emptyset$. They are *proper neighbours* if $\partial D_1 \cap \partial D_2$ contains a non-empty edge.

(b) A region D is a *boundary region* if $\partial D \cap \partial M \neq \emptyset$. A region D is a *proper boundary region* if $\partial D \cap \partial M$ contains a non-empty edge. A region of M which is not a boundary region is an *inner region*.

(c) Let M be a connected, simply connected map. M is a *simple one-layer* map, if the dual map M^*, obtained from M by putting in each region D a vertex D^* and connecting two vertices D_1^* and D_2^* by an edge if D_1 and D_2 are proper neighbours, is a tree in which each vertex has valency at most two. (See

Figure 1 (b).) In particular, M has connected interior, every region is a boundary region, each region has at most two proper neighbours and if M contains more than one region then M contains exactly two regions, see D_1, D_r in Figure 1 (b) and D_1, D_2 in Figure 1 (c), which have exactly one neighbour each. M is a *one-layer map* if it is composed from simple one-layer maps and paths in the way shown in Figure 1 (a).

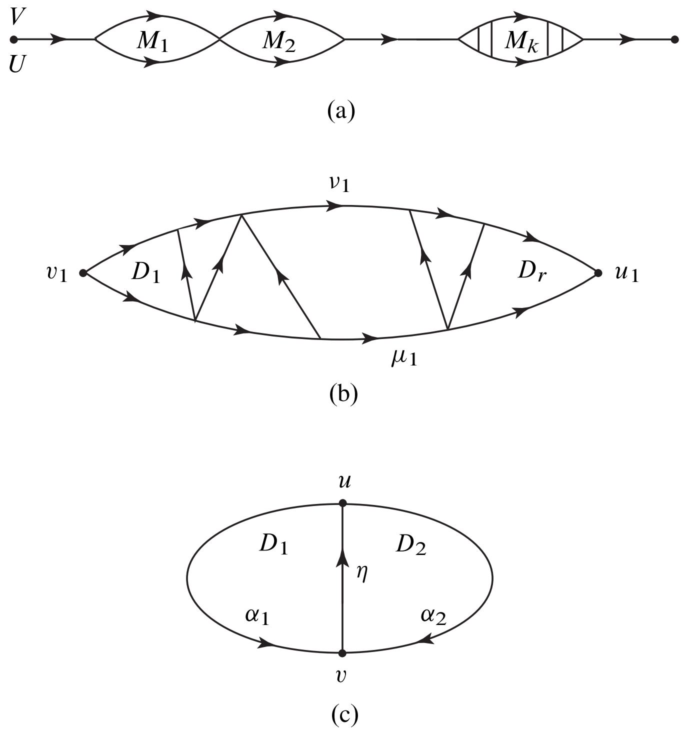


Figure 1. One-layer diagrams.

We shall need the next lemma in Section 5. As pointed out by the referee it is an immediate consequence of Lemma 2.1. We omit its proof.

Lemma 2.6. *Let $\mathcal{R}$ be the symmetric closure of a cyclically reduced word R and let M be a van Kampen diagram over $\mathcal{R}$, with a boundary label K. Let D_1 and D_2 be adjacent regions in M with boundary cycles $u\alpha_1 v\eta u$ and $u\eta^{-1}v\alpha_2^{-1}u$, respectively, where u and v are vertices.* (*See Figure* 1 (c).) *Suppose R has a cyclic conjugate A^n, $n \geq 2$ for some cyclically reduced non-empty word A. Suppose further that*

(i) *A is not a proper power* (*i.e.* $A \neq B^k$, $k \geq 2$ *for every word* B).

(ii) *M contains a minimal number of regions among all the diagrams with boundary label K.*

Then $\Phi(\eta)$ contains no cyclic conjugate of $A^{\pm 1}$.

We recall the main structure theorem from [Ju1], where it is proved in a more general setting. Observe that the condition $C(6)$ & $T(4)$ implies the condition $W(6)$ in [Ju1]. (For the definition of the standard small cancellation conditions, see [L-S, pp. 240–241]

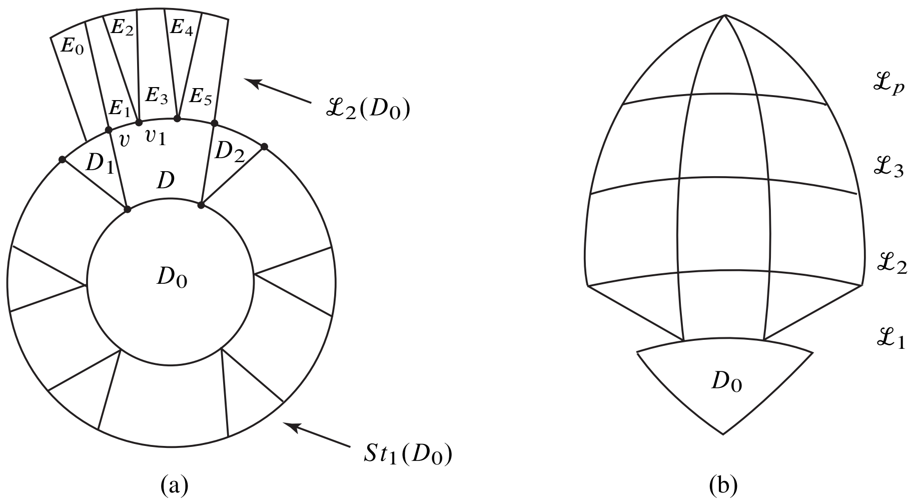


Figure 2. Layer decompositions.

Theorem 2.7 (Layer decomposition, [Ju1]). (See Figure 2.) *Let M be a simply connected map (diagram) with connected interior and let D_0 be a region of M. Assume that M satisfies the condition $C(6)$ & $T(4)$.*

Define $St_0(D_0) = D_0$ and for $i \geq 1$ let $St_i(D_0) = St_{i-1}(D) \cup \mathcal{L}_i(D_0)$, where $\mathcal{L}_i(D_0) = \langle D$ in $M \setminus St_{i-1}(D_0) \mid \partial D \cap \partial St_{i-1}(D_0) \neq \emptyset \rangle$ and $\mathcal{L}_0 = \{D_0\}$. Let p be the smallest number such that $St_p(D_0) = M$ and assume that $p > 0$ (i.e., M contains more than one region). Then each of the following holds:

(a) *Every regular submap of $St_{i+1}(D_0)$ containing $St_i(D_0)$ is simply connected for $0 \leq i \leq p$. (A submap is regular if every edge is on the boundary of a region.)*

(b) *Every connected and simply connected submap of $\mathcal{L}_i(D_0)$ is a one-layer map.*

(When D_0 is fixed, we shall abbreviate $\mathcal{L}_i(D_0)$ by $\mathcal{L}_i$ and call $\Lambda(D_0) = (\mathcal{L}_0, \ldots, \mathcal{L}_p)$ *a layer decomposition* of M. We call D_0 the *center* of the layer decomposition.)

(c) *For a region $D \in \mathcal{L}_i$, $i \geq 1$ denote by $\mathcal{A}(D)$ the set of regions E in $\mathcal{L}_{i-1}$, which have a non-trivial common edge with D, denote by $\mathcal{B}(D)$ the set of regions S in $\mathcal{L}_i$ with $\partial S \cap \partial D \neq \emptyset$ and denote by $\mathcal{C}(D)$ the set of regions K of $\mathcal{L}_{i+1}$, $(i < p)$ with $\partial K \cap \partial D \neq \emptyset$. Also, let $a(D) = |\mathcal{A}(D)|$, $b(D) = |\mathcal{B}(D)|$ and $c(D) = |\mathcal{C}(D)|$. Then $a(D) \leq 1$ and $b(D) \leq 2$. In other words, D has at most two proper neighbours in $\mathcal{L}_i$ and at most one neighbour in $\mathcal{L}_{i-1}$.*

(d) *If $v \in \partial St_i(D_0) \backslash \partial St_{i-1}(D_0)$ then v has valency at most three in $St_i(D_0)$.*

(e) *For regions D, E in M with $\partial D \cap \partial E \neq \emptyset$ we have that $\partial D \cap \partial E$ is connected.*

Remark. Let M be a connected, simply connected map (diagram) with connected interior and let D be a region in M. Let $\Lambda(D)$ be a layer decomposition of M with center D. Suppose that D is a boundary region of M with a non-empty edge on ∂M. (See Figure 3.) Then it follows from the above theorem that $\mathcal{L}_1(D)$ is not annular, hence simply connected, though not necessarily with connected interior. (See Figure 3 (a), where the interior of $\mathcal{L}_1$ is simply connected and connected and see Figure 3 (b), where the interior of $\mathcal{L}_1$ is not connected.) But then due to the simply connectedness of M, $\mathcal{L}_i$ is simply connected for every i.

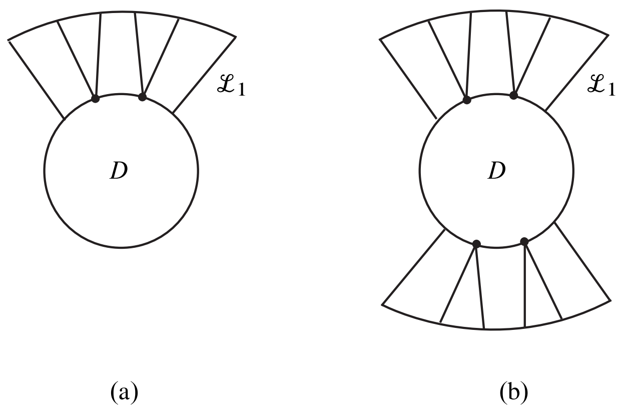


Figure 3. Simply connected and not simply connected layers.

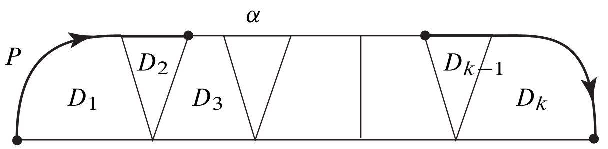
In the next definition we introduce special subdiagrams and regions, the boundaries of which share a large portion with the boundary of M.

Definition 2.8. (a) Let $\Lambda(D_0)$ be a layer decomposition of M, where D_0 is a boundary region of M with a non-empty edge on ∂M. A connected component P of the interior of $\mathcal{L}_i$ is a *peak* relative to D_0, if either $i = p$ or no region of $\mathcal{L}_{i+1}$ is a neighbour of any region in P. If $\Lambda(D_0) = (\mathcal{L}_0, \mathcal{L}_1, \ldots, \mathcal{L}_p)$ then the closure of every connected component P of the interior of $\mathcal{L}_p$ is a *peak*. (See Figure 2 (a), where $p = 2$ and $\mathcal{L}_2$ is a peak and Figure 2 (b), where $\mathcal{L}_p$ is a peak.) Related to peaks is the following notion.

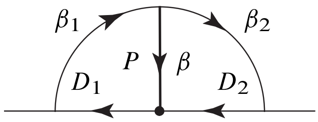
(b) A boundary region D of M is a k*-corner region* for $k = 1, 2$ if each of the following holds:

1) $\partial D \cap \partial M$ is connected and
2) D has k proper neighbours in M.

Example 2.9. Let M be a diagram of a $C(6)$ & $T(4)$ presentation. Let P be a peak, depicted in Figure 4 (a). Then its extremal regions D_1 and D_k are 2-corner regions because $a(D_1) \leq 1$ and $b(D_1) \leq 1$, due to being extremal. If P is a peak consisting of a single region E, then E is a 1-corner region due to Theorem 2.7. Also, if $a(D_{k-1}) = 0$ then D_{k-1} in Figure 4 (a) is a 2-corner region.



(a)



(b)

Figure 4. Peaks and corner regions in van Kampen diagrams.

The k-corner regions are examples of Greendlinger regions. (These are regions which satisfy the conditions of Greendlinger's Lemma. See [L-S, pp. 250–251].)

The next section is devoted to the improved version of Greendlinger's Lemma. A similar version was formulated in [Ju5] the proof of which, using Lemmas 2.3 and 2.11, easily can be adapted to the proof of Proposition 2.12 below. Therefore, we shall omit the details of the proof, which consists of case by case checking.

2.3. An improved version of Greendlinger's Lemma. The improved version of Greendlinger's Lemma is given for 1-corner regions in Lemma 2.10 and for 2-corner regions in Proposition 2.12.

In this section we assume that the conditions of Theorem A are satisfied.

2.3.1. 1-corner regions. Let D be a 1-corner region in M with proper neighbour E. Let $\alpha = \partial D \cap \partial E$, let $P = \Phi(\alpha)$, let $\eta = \partial D \cap \partial M$ and let $Q = \Phi(\eta)$. (See Figure 5 (a).) Then P is a piece and $v\alpha u\eta v$ is a boundary cycle of D with PQ a boundary label of D, where u and v are vertices.

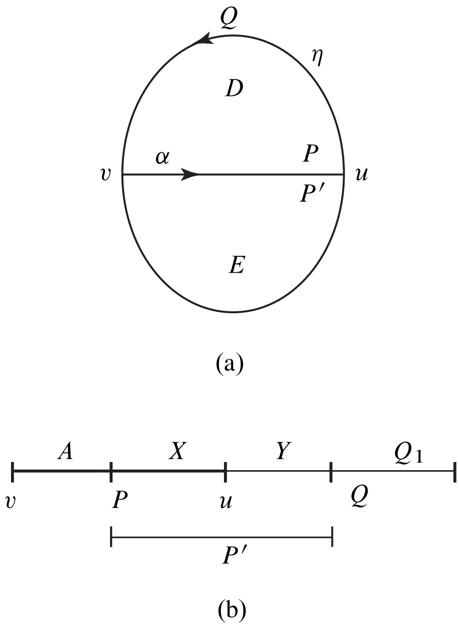


Figure 5. 1-corner regions and corresponding word equation.

Lemma 2.10. *Let notation be as above. Then $Q \succ P$.*

Proof. Let (P, P') be the corresponding piece pair. Then one of the following holds:

1) P' is a subword of Q;

2) P' overlaps with P.

In case 1) $Q \succ P$, by Lemma 2.2 (a). Also, in case 2) , if $|P| = 1$ then $Q \succ P$. Hence assume that $|P| \geq 2$. In case 2) we have $P = AX$, $P' = XY$, $Q = YQ_1$, reduced as written, $Q_1 \in \mathcal{T}(Q)$. See Figure 5 (b). Applying Lemma 2.3 (a) to the first two of these equations and remembering that P^{-1} cannot overlap P in more than one letter (see Lemma 2.1 (c)), we get $A \sim Y \succ X$ and hence, by Lemma 2.2, $P \sim Y$. Applying Lemma 2.2 to the last equation implies $Q \succ P$.

The lemma is proved. □

2.3.2. 2-corner regions. Let D be a 2-corner region in M with neighbours E_r and E_ℓ. See Figure 6. Denote $\alpha_1 = \partial D \cap \partial E_r$ and denote $\alpha_2 = \partial D \cap \partial E_\ell$. Let $v_0 = \alpha_1 \cap \partial M$, let $v_2 = \alpha_2 \cap \partial M$ and let $v_1 = \alpha_1 \cap \alpha_2$. Denote $P_1 = \Phi(\alpha_1)$, $P_2 = \Phi(\alpha_2)$ and $Q = \Phi(\partial D \cap \partial M)$. Let (P_1, P_1') and (P_2, P_2') be the piece pairs obtained from α_1 and α_2, being common edges of ∂E_ℓ and ∂D and of ∂E_r and ∂D, respectively. Thus, P_1' and P_2' are subwords of $\Phi(\partial E_r)$ and $\Phi(\partial E_\ell)$, respectively, which are equal to P_1 and P_2, respectively, as words, and since all the regions of M have the same boundary labels, up to sign, (P_1, P_1') and (P_2, P_2') are piece pairs. It is convenient and harmless to identify P_i with α_i and, similarly, P_i' with α_i', $i = 1, 2$. Then $v_2 Q v_0 P_1 v_1 P_2 v_2$ is a boundary label of D, which we may assume to coincide with R, without loss of generality, where P_1 and P_2 are pieces.

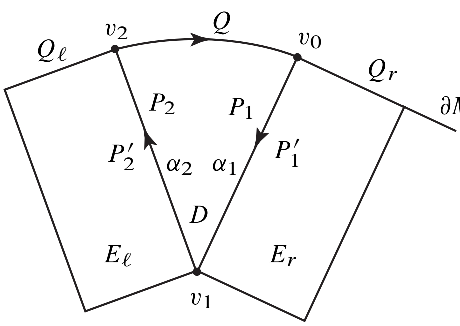


Figure 6. 2-corner regions.

The proof of the following lemma is a routine case by case checking, hence we omit it.

Lemma 2.11. *Let notation be as above and make the assumptions of the main theorems. Assume that condition* (i) *of Theorem A is not satisfied by any cyclic conjugate of R.*

(a) *If* $|P_1| = 1$, v_0 *has valency three in* M *and* v_1 *is an inner vertex of* M, *then* $Q \succ P_1, P_2$. *Similarly, if* $|P_2| = 1$ *and* v_1 *is an inner vertex of* M, *then* $Q \succ P_1, P_2$.

(b) *Suppose* $|P_1| \geq 2$ *and* P_1 *overlaps with* P_1'. *If* $\varepsilon_1 = -1$ *then either* $P_1 \prec Q$ *or* $P_1 \prec ZQ$, *where* $P_2 = cP_1^{-1}Z$ *and where* c *is a letter with* $\alpha(c) = \alpha(h(P_1^{-1}))$ *and* P_1' *contains* v_1. *The analogous result holds for* P_2.

The following is our version to Greendlinger's Lemma for 2-corner regions.

Proposition 2.12. *Let notation be as above and assume that R satisfies the assumptions of the main theorems, and no cyclic conjugate of R satisfies condition* (i) *of Theorem A. Assume that the piece pairs (P_1, P_1') and (P_2, P_2') are right normalised. Let $Q_r = \partial E_r \cap \partial M$ and let $Q_\ell = \partial E_\ell \cap \partial M$. Then the following holds:*

If $d_M(v_0) = 3$ and Q_r is not a piece, then Q_r has a head Q_ρ which is a piece over $\mathcal{R}$ such that $QQ_\rho \succ P_1P_2$ and dually, if $d_M(v_2) = 3$ and Q_ℓ is not a piece, then Q_ℓ has a tail Q_λ which is a piece over $\mathcal{R}$ such that $Q_\lambda Q \succ P_1P_2$. In particular, if both v_0 and v_2 have valency three and both Q_r and Q_ℓ are not pieces (*i.e. the products of at least two pieces*) *then both $Q_\lambda Q \succ R$ and $QQ_\rho \succ R$ hold true.*

We close this section with the following consequence of the proposition.

Proposition 2.13. *Let M be an $\mathcal{R}$-diagram. Let assumptions be as in Proposition* 2.12. *Let P be a peak relative to a layer decomposition Λ. Let $\alpha = \partial P \cap \partial M$. Then $\Phi(\alpha)$ contains a letter from each component.*

Proof. Let $P = \langle D_1, \dots, D_k\rangle$. If $k = 1$ then the result follows from Lemma 2.10. If $k \geq 3$ then it follows from Theorem 2.7 (d) and the $T(4)$ condition that either P contains a 1-corner region or contains a 2-corner region D with two neighbours E_r and E_ℓ such that $\partial D \cap \partial E_r \cap \partial M$ and $\partial D \cap \partial E_\ell \cap \partial M$ are vertices with valency three and $\partial E_r \cap \partial M$ and $\partial E_\ell \cap \partial M$ are not pieces (due to the $C(6)$ condition). In both cases the result follows by Proposition 2.12, where D_1 is E_ℓ, D_2 is D and D_3 is E_r. See Figure 4 (a).

Finally, assume $k = 2$. See Figure 4 (b). Let $P = \langle D_1, D_2\rangle$, let $\beta_1 = \partial D_1 \cap \partial M$ and let $\beta_2 = \partial D_2 \cap \partial M$. Both D_1 and D_2 are 2-corner regions, and β_i is the product of at least four ($4 = 6-2$) pieces, for $i = 1, 2$. Also, by Theorem 2.7 (d), $\beta_1 \cap \beta_2$ is a vertex with valency three. Therefore, by Proposition 2.12, $\beta_1 \cup \beta_2$ contains a letter from each component G_j and the proposition is proved. □

3. The proof of Theorem C

In proving Theorem C we may assume without loss of generality that $\mathrm{Supp}(R) = I \cup J = \{1, 2, \dots, n\}$ and we shall do so.

Suppose H_J is not a free product. Then there exists a non-empty word W in G_J such that $\nu(W) = 1$ in G. Therefore, by [L-S, Theorem 1.1, p. 237] there exists a connected, simply connected diagram M with boundary label W. Let Δ be a connected component of the interior of M. By Proposition 2.13 $\partial\Delta$ contains a letter from every component G_i for $i = 1, \dots, n$. Since, as sets, $\partial\Delta \subseteq \partial M$, ∂M also contains a letter from every component. This, however, violates $W \in G_J$, $J \subsetneqq \{1, \dots, n\}$. Therefore, H_J is a free product, $H_J \cong G_J$.

The theorem is proved.

4. The structure of intersection diagrams

For the proof of Theorems A and B we may assume without loss of generality that $\mathrm{Supp}(R) = I \cup J = \{1, \ldots, n\}$ and we shall do so. In this section we shall assume the notation of the main theorems, and, moreover, that R has no cyclic conjugate R^* which satisfies condition (i) of Theorem A. Let $W \neq 1$ be an element of $H_I \cap H_J$. Then there are non-empty words U in G_I and V in G_J such that $\nu(U) = \nu(V)$ in G. Hence, by [L-S, Theorem 1.1, p. 237] there is a van Kampen $\mathcal{R}$-diagram M with boundary label UV^{-1}. We call this diagram an *Intersection Diagram*.

Definition 4.1. Let $D = I \cap J$ and let $F_{I,J} = \langle G_{I \cup J} \rangle$. We can consider $F_{I,J}$ as the amalgamated free product $F_{I,J} := G_I *_{G_D} G_J$. We shall denote the length of a word W in F, considered as a word in $F_{I,J}$ by $\|W\|$.

Proposition 4.2. *Let assumptions be as above. Let M be an intersection diagram with boundary label UV^{-1}, where $U \in G_I$ and $V \in G_J$. If $\|R\| \geq 4$ and $\|UV^{-1}\| = 2$ then M is a one-layer diagram.*

We need Lemmas 4.4 and 4.5 for the proof of Proposition 4.2. In what follows we shall use the notation and rely on the assumptions of Proposition 4.2. Also, we shall use the following easy lemma, the proof of which we omit.

Lemma 4.3. *Let α_1, α_2 and α_3 be disjoint boundary paths of M. Let ω be a boundary cycle of M. Then $\|\Phi(\omega)\| \geq 4$ in each of the following cases:*

(a) $\|\Phi(\alpha_1)\| \geq 2$ *and* $\|\Phi(\alpha_2)\| \geq 3$*;*

(b) $\|\Phi(\alpha_i)\| \geq 2$ *for* $i = 1, 2, 3$.

Lemma 4.4. *Let P be a peak of M in $\mathcal{L}_i(D)$ and suppose $\|(\partial P \cup \partial\mathcal{L}_{i-1}(D)) \cap \partial M)\| = 2$. If $|P| > 1$ then $P = \langle D_1, D_2 \rangle$ such that $a(D_1) + a(D_2) = 1$.*

Proof. Suppose $|P| \geq 3$, $P = \langle D_1, \ldots, D_k \rangle$, $k \geq 3$. Consider the extremal regions D_1 and D_k. Start with D_1.

If $a(D_1) = a(D_2) = 1$ let $\{E_1\} = \mathcal{A}(D_1)$ and $\{E_2\} = \mathcal{A}(D_2)$. If $E_1 = E_2$ then $v := \partial D_1 \cap \partial D_2 \cap \partial E_1$ is an inner vertex with valency three, violating the condition $T(4)$. Hence $E_1 \neq E_2$ and since D_1 is extremal in P and D_2 is the only region of P adjacent to D_1, hence $\mathcal{C}(E_1) = \{D_1\}$ and $d_M(E_1) = a(E_1) + b(E_1) + c(E_1) \leq 2 + 1 + 1 = 4$. Consequently, due to the $C(6)$ condition $\partial E_1 \cap \partial M$ is the product of at least two pieces, hence if $u := \partial D_1 \cap \partial E_1 \cap \partial M$ then u is a vertex with valency three and every piece on ∂E_1 starting at u and read anticlockwise is contained in $\partial E_1 \cap \partial M$. Therefore noticing that $d_M(D_1) = 2$, we may apply Proposition 2.12 to D_1 to get that

(i) If $a(D_1) = a(D_2) = 1$, then

$$\|(\partial E_1 \cap \partial M) \cup (\partial D_1 \cap \partial M)\| \geq 2. \tag{4.1}$$

(ii) If $a(D_1) = 0$ and $a(D_2) = 1$, then $d(D_1) = 1$, hence by Lemma 2.10

$$\|(\partial D_1 \cap \partial M)\| \geq 2. \tag{4.2}$$

(iii) If $a(D_1) = 1$ and $a(D_2) = 0$, then $d(D_1) = 2$ and $d(D_2) \leq 2$, hence by Proposition 2.12

$$\|(\partial D_1 \cap \partial M) \cup (\partial D_2 \cap \partial M))\| \geq 2. \tag{4.3}$$

It follows from (4.1), (4.2) and (4.3) that if we define $L = \langle E_1, D_1, D_2\rangle$ if $a(D_1) = 1$ and define $L = \langle D_1, D_2\rangle$ if $a(D_1) = 0$, then $\|\partial L \cap \partial M\| \geq 2$. A similar analysis shows that if $K = \langle E_k, D_k, D_{k-1}\rangle$ if $\mathcal{A}(D_k) = \{E_k\}$ and $K = \langle D_k, D_{k-1}\rangle$ if $a(D_k) = 0$, then $\|\partial K \cap \partial M\| \geq 2$. Consequently, if $k \geq 4$ then

$$\|(\partial P \cup \partial\mathcal{L}_{i-1}(D)) \cap \partial M)\| \geq 3, \tag{4.4}$$

violating our supposition. Hence $k \leq 3$. Since by assumption $k \geq 3$, we get $k = 3$. If one of cases (i) or (ii) above hold for D_1 (or for D_3) then (4.4) holds true. Assume therefore that case (iii) holds for both D_1 and D_3. Then $a(D_1) = 1$, $a(D_2) = 0$ and $a(D_3) = 1$. Now, $d(D_2) = a(D_2) + b(D_2) + c(D_2) = 2 + 1 + 0 = 3$, hence due to the $C(6)$ condition:

$$\partial D_2 \cap \partial M \textit{ is the product of at least three } (6 - 3 = 3) \textit{ pieces} \tag{4.5}$$

Since $d(D_1) = d(D_3) = 2$, we may apply Proposition 2.12 to the pairs (D_1, D_2) and (D_2, D_3), where in the notation of Proposition 2.12 in the first pair $D = D_1$ and $E_r = D_2$ while in the second pair $D = D_3$ and $E_\ell = D_2$. By their definition Q_ρ and Q_λ are pieces. Since $E_\ell = E_r = D_2$, Q_ρ is an initial subword of $\Phi(\partial D_2 \cap \partial M)$, which is a piece and Q_λ is a terminal subword of $\Phi(\partial D_2 \cap \partial M)$, which is a piece. Since $(\partial D_2 \cap \partial M)$ is the product of at least three pieces by (4.5), Q_ρ and Q_λ do not overlap and hence $\|\partial P \cap \partial M\| \geq 3$ violating our supposition. Therefore $|P| = 2$ and if $a(D_1) = a(D_2) = 0$ or $a(D_1) = a(D_2) = 1$ then Lemma 2.10 in the first case and Proposition 2.12 in the second case with the arguments in (i) above imply that $\|(\partial\mathcal{L}_{i-1} \cup \partial\mathcal{L}_i) \cap \partial M\| \geq 3$. Therefore, $a(D_1) + a(D_2) = 1$.

The lemma is proved. □

Lemma 4.5. *Let Λ be a layer decomposition for M and let P_1 be a peak of M relative to Λ. If P_1 is an extremal component of $\mathcal{L}_i$ then*

(a) $\|\partial P_1 \cap \partial M\| \geq 2$.

(b) *Either P_1 contains a region D with $\|\partial D \cap \partial M\| \geq 2$ or P_1 contains adjacent regions D_1 and D_2 such that $\|(\partial D_1 \cup \partial D_2) \cap \partial M\| \geq 2$.*

Proof. (a) If $|P_1| = 1$ this follows from Lemma 2.10. Assume $|P_1| \geq 2$. Let $P_1 = \langle D_1, \ldots, D_k \rangle$, $k \geq 2$ and assume P_1 is left-extremal. Then $b(D_1) = 1$ and $c(D_1) = 0$. By Theorem 2.7 (c) $a(D_1) \leq 1$. Consequently, $d(D_1) \leq 1 + 0 + 1 = 2$, hence D_1 is a 2-corner region of M. Let $v = \partial D_1 \cap \partial D_2 \cap \partial M$. Then by Theorem 2.7 (d) $d_{\mathcal{L}_i}(v) = 3$ and since $c(D_1) = c(D_2) = 0$ hence $d_{\mathcal{L}_i}(v) = d_M(v)$. Thus $d_M(v) = 3$ and Proposition 2.12 applies to D_1. Now, in the notation of Proposition 2.12, $D_1 = D$ and $D_2 = E_r$ and $d_M(D_2) = a(D_2) + b(D_2) + c(D_2) \leq 1 + 2 + 0 = 3$, hence Q_r is the product of at least three $(6 - 3 = 3)$ pieces. (Here, as in Proposition 2.12, Q_r is the label of $\partial E_r \cap \partial M$.) Therefore, it follows from Proposition 2.12 that $\|(\partial D_1 \cap \partial M) \cup (\partial D_2 \cap \partial M)\| \geq 2$, as required. Similarly, if P_1 is right-extremal then the above argument applies to D_k.

(b) follows immediate from the proof of part (a).

The lemma is proved. □

Now, it follows from Greendlinger's Lemma (see [L-S, p. 250]) that due to the $C(4)$ & $T(4)$ condition (which is implied by the $C(6)$ & $T(4)$ condition) M contains at least two k-corner regions with $k \leq 2$. Consider the layer structure of M with center D_0, where D_0 is a k-corner region of M, $k \leq 2$. Since D_0 is a boundary region of M, hence the layer structure of M with center D_0 has a peak P_0 in its last layer. Hence by Lemma 4.5 (b) either P_0 contains a boundary region D such that $\|\partial D \cap \partial M\| \geq 2$ or contains adjacent regions D and D_1 such that $\|(\partial D \cup \partial D_1) \cap \partial M\| \geq 2$. Consider the layer structure Λ of M with center D. Since $d(D) \leq 3$ all the layers of Λ are simply connected (i.e. not annular) and in particular its last layer $\mathcal{L}_p$ is. If $\mathcal{L}_p$ has more than one component then it follows from Lemma 4.5 (a) that $\|\partial \mathcal{L}_p \cap \partial M\| \geq 3$, hence by Lemma 4.3 $\|\partial M\| \geq 4$, since $\|\partial D \cap \partial M\| \geq 2$ or $\|(\partial D \cup \partial D_1) \cap \partial M\| \geq 2$, and may assume that $D_1, D \nsubseteq \mathcal{L}_p$. (If $D_1 \subseteq \mathcal{L}_p$ or $D \subseteq \mathcal{L}_p$ then $p \leq 1$ and in this case $\|\partial M\| \geq 4$ easily follows.) Similarly, it follows that

if the interior of $\mathcal{L}_i$ contains more than one component then $\|\partial M\| \geq 4$. (4.6)

Now we turn to the proof of Proposition 4.2.

Proof. First observe that $\|\partial M\| \geq 2$ due to Lemma 4.5 (a) (or Theorem C), and if $\|\partial M\| > 2$ then $\|\partial M\| \geq 4$. Suppose by way of contradiction that M is not a one-layer diagram and show that $\|\partial M\| \geq 4$. Let D, Λ and $\mathcal{L}_p$ be as above. Then due to (4.6) we may assume that $\mathcal{L}_p$ has connected interior. It follows that all layers of Λ have connected interior.

Let $P = \mathcal{L}_p$. Let $\partial M = \alpha\beta$, where $\alpha = \partial D \cap \partial M$ if $\|\partial D \cap \partial M\| \geq 2$ and $\alpha = (\partial D \cup \partial D_1) \cap \partial M$ if $\|(\partial D \cup \partial D_1) \cap \partial M\| \geq 2$. Then due to Lemmas 4.5

and 4.3, it is enough to show that $\|\beta\| \geq 3$. Clearly, $(\partial P \cup \partial \mathcal{L}_{p-1}) \cap \partial M \subseteq \beta$, hence if $\|(\partial P \cup \partial \mathcal{L}_{p-1}) \cap \partial M\| \geq 3$ then $\|\partial M\| \geq 4$. Assume therefore that $\|\beta\| \leq 2$ and $\|(\partial P \cup \partial \mathcal{L}_{p-1}) \cap \partial M\| = 2$. Then by Lemma 4.4 either $|P| = 1$ or $P = \langle D_1, D_2\rangle$ such that $a(D_1) + a(D_2) = 1$.

Claim. *Consider the following statement:*

$$\text{either } |\mathcal{L}_i| = 1 \text{ or } \mathcal{L}_i = \langle D_1, D_2\rangle \text{ such that } a(D_1) + a(D_2) = 1. \qquad (*)$$

Then $(*)$ *holds for every* $i,\ i = 1, \ldots, p$.

Proof of the Claim. By the last argument the Claim holds true for $i = p$. Suppose the Claim holds true for $\mathcal{L}_p, \ldots, \mathcal{L}_i$ and prove for $\mathcal{L}_{i-1}$. Suppose $|\mathcal{L}_{i-1}| \geq 2$ and let $\mathcal{L}_{i-1} = \langle E_1, \ldots, E_k\rangle$. Let $D_1 \in \mathcal{L}_i$ with $a(D_1) = 1$ and let $\mathcal{A}(D_1) = \{E_j\}$ for some $j,\ j = 1, \ldots, k$. Assume that either $j \neq 1$ or $j \neq k$. Suppose first $j \neq 1$. If $a(E_1) = 0$ then $\|\partial E_1 \cap \partial M\| \geq 2$ by Lemma 2.10, hence $\|\beta\| \geq 3$ by Lemma 4.3, since $\|\partial P \cap \partial M\| \geq 2$ by Proposition 2.13 and $E_1 \notin P$. This contradicts our assumption that $\|\beta\| \leq 2$. If $a(E_1) = 1$ and $a(E_2) = 1$ with $\mathcal{A}(E_1) = \{F_1\}$ and $\mathcal{A}(E_2) = \{F_2\}$ then $F_1 \neq F_2$ and $\|(\partial E_1 \cup \partial F_1) \cap \partial M\| \geq 2$ hence $\|\beta\| \geq 3$ by Lemma 4.3 (a), contradiction. (See proof of part (i) in Lemma 4.4). Therefore,

(i) *if* $j \neq 1$ *then* $a(E_1) = 1$ *and* $a(E_2) = 0$.

Suppose now that $j \neq k$. Then the arguments of the case $j \neq 1$ for E_k apply and yield

(ii) *if* $j \neq k$ *then* $a(E_k) = 1$ *and* $a(E_{k-1}) = 0$.

Assume now that $k \geq 3$. If $j \neq 2$ and $j \neq 1$ then it follows from (i) and Proposition 2.12 that $\|(\partial E_1 \cup \partial E_2) \cap \partial M\| \geq 2$ and hence $|\beta\| \geq 3$, violating our assumption. Thus

(iii) *if* $k \geq 3$ *then either* $j = 1$ *or* $j = 2$.

Similarly, if $j \neq k - 1$ and $j \neq k$ then it follows from (ii) and Proposition 2.12 that $\|(\partial E_k \cup \partial E_{k-1}) \cap \partial M\| \geq 2$ and hence $|\beta\| \geq 3$, violating our assumption. Thus

(iv) *If* $k \geq 3$ *then either* $j = k - 1$ *or* $j = k$.

Therefore, by (iii) and (iv), if $k \geq 3$ then $j \in \{1, 2\} \cap \{k - 1, k\}$. In particular, $\{1, 2\} \cap \{k - 1, k\} \neq \emptyset$. It follows that if $k \geq 3$ then $j = k - 1 = 2$, hence $k = 3$ and $j = 2$. Since $d(E_2) = 2$ and $\partial E_2 \cap \partial D_1$ is a piece, either $\partial E_2 \cap \partial M$ is connected and is the product of at least three $\big(6 - (2 + 1) = 3\big)$ pieces or $\partial E_2 \cap \partial M$ has two connected components γ_1 and γ_3 such that $\partial E_2 \cap \partial St_{i-1} = \gamma_1\gamma_2\gamma_3$ with $\gamma_2 = \partial E_2 \cap \partial D_1$ and either γ_1 is the product of at least two pieces or γ_3 is the product of at least two pieces. Therefore we may apply Proposition 2.12 for E_1 or for E_3 to give $\|(\partial \mathcal{L}_{i-1} \cup \partial \mathcal{L}_i) \cap \partial M\| \geq 3$, a contradiction. Consequently, $k \leq 2$, i.e. $|\mathcal{L}_{i-1}| \leq 2$.

Now it easily follows by arguments we made several times above that if $|\mathcal{L}_{i-1}| = 2$, then $a(E_1) = 1$ and $a(E_2) = 1$ would imply that either $\|(\partial E_1 \cup \partial F_1) \cap \partial M\| \geq 2$ or $\|(\partial E_2 \cup \partial F_2) \cap \partial M\| \geq 2$. This would imply $\|\beta\| \geq 3$, violating our assumption, proving the claim.

We show that $(*)$ implies M is a one-layer diagram. Let K be a region of M. Suppose K is in $\mathcal{L}_i$, $0 < i < p$ and $\mathcal{L}_i$ consists of two regions. Then $b(K) = 1$ by condition $(*)$. Also, $a(K) \leq 1$ by Theorem 2.7 (c), and $c(K) \leq 1$ by condition $(*)$. Hence $d(K) \leq 3$. Let $\mathcal{L}_i = \langle K, L\rangle$. If $d(K) = 3$ then it follows from $(*)$ that $c(L) = 0$, $a(L) = 0$ and $b(L) = 1$, hence $d(L) = 1$ and hence $\|\partial L \cap \partial M\| \geq 2$ by Lemma 2.10, implying $\|\beta\| \geq 3$. Therefore $d(K) \leq 2$. Suppose $d(K) = 1$. Then $\|\partial K \cap \partial M\| \geq 2$ by Lemma 2.10 implying again $\|\beta\| \geq 3$. Therefore $d(K) = 2$. Thus, every region in $\mathcal{L}_i$, $1 < i < p$ has exactly two neighbours. But now, $\mathcal{L}_0 = \{D\}$ and D by $(*)$ has exactly one neighbour (in $\mathcal{L}_1$) and either $\mathcal{L}_p = \{D_1\}$ in which case $d(D_1) = 1$ due to the $T(4)$ condition and Theorem 2.7 (d), or $\mathcal{L}_p = \langle D_1, D_2\rangle$ in which case either $d(D_1) = 2$ and $d(D_2) = 1$ or $d(D_2) = 2$ and $d(D_1) = 1$, by Lemma 4.4. Consequently, M is a one-layer diagram.

The proposition is proved. □

5. The proofs of Theorems A and B

5.1. Decompositions of *R*. For the proof of Theorem A we may assume $\mathrm{Supp}(R) = I \cup J = \{1, 2, \ldots, n\}$. We start with various decompositions of words in $\langle G_I, G_J\rangle$. Consider $gp\,\langle G_I, G_J\rangle$ as the amalgamated free product $G_I *_{G_D} G_J$ and denote its length function by $\|\cdot\|$. Every F-reduced element W of $G_I *_{G_D} G_J$ with $\|W\| \geq 2$ can be written by

$$W = W_1 \ldots W_k, \quad k \geq 1, \qquad (*)$$

F-reduced as written, such that each of the following holds:

(i) $W_1 \in \mathcal{T}(A_1B_1K_1L_1)$, $W_k \in \mathcal{H}(A_kB_kK_kL_k)$, and $W_i = A_iB_iK_iL_i$ for $i = 2, \ldots, k-1$, F-reduced as written with $A_i \in G_I$, $K_i \in G_J$, $B_i \in G_D$ and $L_i \in G_D$, $i = 1, \ldots, k$;

(ii) A_i starts and terminates with an element of $G_{I\setminus D} \setminus \{1\}$;

(iii) K_i starts and terminates with an element of $G_{J\setminus D} \setminus \{1\}$.

We call this decomposition of W its $(*)$-*decomposition*. We say that the $(*)$-decomposition is *complete* if $W_1 = A_1B_1K_1L_1$ and $W_k = A_kB_kK_kL_k$. If W is cyclically reduced then it has a cyclic conjugate W^* with a complete $(*)$-decomposition.

Since $G_I = G_{I\setminus D} * G_D$ and $G_J = G_{J\setminus D} * G_D$, it follows from the normal form theorem for free products (see [L-S, p. 175]) that W has a *unique* $(*)$-decomposition. As a result, we have the following lemma, the proof of which is a routine application of the normal form theorem for free products, hence we omit it.

Lemma 5.1. *Let W and S be elements of $\langle G_I, G_J\rangle$ with $(*)$-decompositions $W = W_1 \dots W_k$, $k \geq 1$ and $S = S_1 \dots S_\ell$, $\ell \geq 1$, respectively. Let $W = HPT$ and let $S = H'P'T'$ be decomposition of W and S, as words in F, reduced as written. Assume that $\|P\| \geq 2$ and $\|P'\| \geq 2$. Then*

(a) $P = W_i'' W_{i+1} \dots W_j W_{j+1}'$, *where* $W_i'' \in \mathcal{T}(W_i)$ *and* $W_{j+1}' \in \mathcal{H}(W_{j+1})$, *reduced as written in* F, $S = S_p'' S_{p+1} \dots S_q S_{q+1}'$, *where* $S_p'' \in \mathcal{T}(S_p)$ *and* $S_{q+1}' \in \mathcal{H}(S_{q+1})$, *reduced as written in* F.

(b) *If* $P = P'$ *and* $j \geq i+1$, *then* $j - i = q - p$ *and*

  (i) $S_{p+t} = W_{i+t}$ *for* $t = 1, \dots, q-p$;

  (ii) $W_i'' = S_p''$ *and* $S_{q+1}' = W_{j+1}'$.

5.2. Word equations that define R. Assume now results (i) and (iii) of Theorem A do not hold and consider $H_I \cap H_J$. We shall prove that necessarily result (ii) holds true. Let w be an element of $H_I \cap H_J$, $w \neq 1$. Then there are reduced words U in G_I and V in G_J such that $w = \nu(U) = \nu(V)$. If $U \in G_D$ then also $V \in G_D$, otherwise UV^{-1} is a non-trivial relation in G_J, violating Theorem C. Hence if $V \notin G_D$ then $U \notin G_D$. Since by assumption $H_D \neq H_I \cap H_J$, we may assume $U \in G_I \setminus G_D$ and $V \in G_J \setminus G_D$. Now, since $\nu(U) = \nu(V)$, we have $\nu(UV^{-1}) = 1$, hence there is a van Kampen $\mathcal{R}$-diagram with a boundary label UV^{-1}. Since $V \notin G_D$ and $U \notin G_D$, hence M is not-empty. Since we assumed that result (iii) of Theorem A doesn't hold, hence $\|R^*\| \geq 4$ for every cyclic conjugate R^* of R. Hence Proposition 4.2 applies, implying that M is a one-layer diagram, which without loss of generality has connected interior. Since $\|R^*\| \geq 4$, while $\|UV^{-1}\| = 2$, we get $|M| \geq 2$.

Lemma 5.2. *Let M be a connected, simply connected $\mathcal{R}$-diagram with connected interior. Suppose that M is a one-layer diagram; $M = \langle D_0, \dots, D_t\rangle$, $t \geq 1$, with boundary cycle $u\mu v\nu^{-1}$ such that $u \in \partial D_0$ and $v \in \partial D_t$. Suppose $\Phi(\mu) \in G_I$ and $\Phi(\nu) \in G_J$. Let $\theta = \partial D_0 \cap \partial D_1$ and let $P = \Phi(\theta)$. Let $(P, P') = (P, P^\varepsilon)$ be the corresponding piece pair. If $\varepsilon = 1$ and $\|R\| \geq 4$ then R has a cyclically reduced (in F) cyclic conjugate R^* with $(*)$-decomposition $W_1 \dots W_k$, which satisfies the word equation $SW_jW_{j+1}Z = ZW_1W_2S$, where $2 \leq j \leq k-2$, $S = W_3 \dots W_{j-1}$ and $Z = W_{j+1} \dots W_k$.*

Proof. Let $\mu_i = \mu \cap \partial D_i$, $\nu_i = \nu \cap \partial D_i$, let $H_i = \Phi(\mu_i)$ and let $T_i = \Phi(\nu_i)$ for $i = 0, \dots, t$. Consider the subdiagram $\langle D_0, D_1\rangle$. (See Figure 7 (a).) Let $\mu = \mu_0 z_1 \mu_1 \dots z_t \mu_t$ and $\nu = \nu_0 w_1 \nu_1 \dots w_t \nu_t$, where z_i and w_i are vertices, $i = 1, \dots, t$. Then $u\mu_0 z_1 \theta^{-1} w_1 \nu_0^{-1}$ is a boundary cycle of D_0 with label $H_0 P T_0^{-1}$.Now, by Lemma 5.1 R has a cyclic conjugate R^* with $(*)$-decomposition $R^* = W_1 \dots W_k$ with $W_i = A_i B_i K_i L_i$ for $i = 1, 2, \dots, k$, $k \geq 2$, like in (i), (ii) and (iii), in the beginning of Section 5.1. We have $k \geq 2$ due to the assumption $\|R\| \geq 4$.

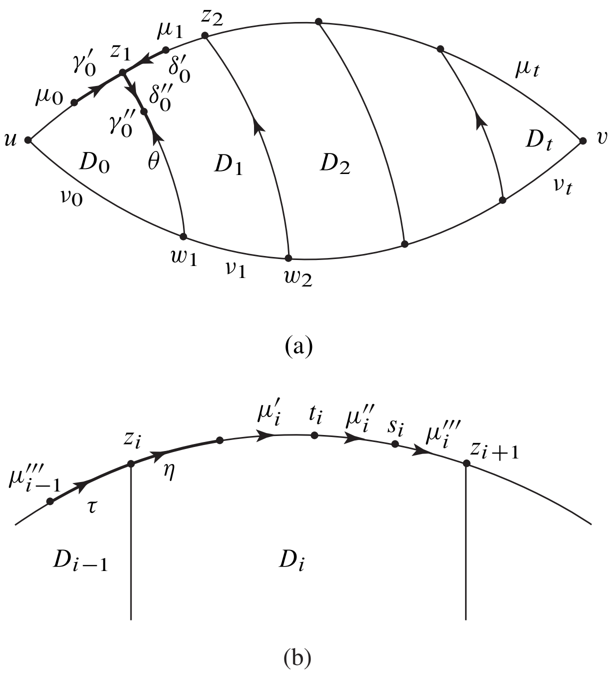


Figure 7. The diagram for $U = V$.

Since $H_0 \in G_I$ and $T_0 \in G_J$, H_0 is a subword of $L_i A_{i+1} B_{i+1}$ and T_0^{-1} is a subword of $B_i K_i L_i$ for some i, $i = 1, 2, \ldots, k$. Since $T_0^{-1} H_0$ is a subword of R^*, hence $T_0^{-1} H_0$ is a subword of $B_i K_i L_i A_{i+1} B_{i+1}$. Therefore, L_i decomposes to $L_i = L_i' L_i''$ such that T_0^{-1} is a terminal subword of $B_i K_i L_i'$ and H_0 is an initial subword of $L_i'' A_{i+1} B_{i+1}$. Hence,

P has a $(*)$-decomposition

$$P = (P' K_{i+1} L_{i+1}) W_{i+2} \ldots W_k \ldots W_{i-1} (A_i P''), \tag{5.7}$$

where P' is a terminal subword of $L_i'' A_{i+1} B_{i+1}$ and P'' is an initial subword of $B_i K_i L_i'$.

Now, P is the label of a common boundary path of D_0 and D_1, hence P^ε occurs as a subword of $\widehat{R}$ (the cyclic word R) in different positions, for some $\varepsilon \in \{1, -1\}$. (The positions of these occurrences are different because M is a reduced diagram.) Therefore, by Lemma 5.1 either P also has a $(*)$-decomposition

$$P = (Q' K_{j+1} L_{j+1}) W_{j+2} \ldots W_{j-1} (A_j Q'') \quad \textit{for some } j, \tag{5.7'}$$

or a $(*)$-decomposition

$$P^{-1} = (Q'K_{j+1}L_{j+1})W_{j+2}\dots W_{j-1}(A_jQ'') \quad \textit{for some } j,\ 1 \le j \le n. \tag{5.8}$$

Since $\varepsilon = 1$ by the assumption of the lemma, P' is given by (5.7′). Due to Lemma 5.1 we have

$$\begin{aligned}&P' = Q',\ \ K_{i+1} = K_{j+1},\ \ L_{i+1} = L_{j+1},\ \ W_{i+t} = W_{j+t},\ \textit{for}\ t = 2,\dots,k-1,\\ &A_i = A_j\ \textit{ and }\ P'' = Q''\ \ (\text{ we count } j+t \text{ and } i+t,\ \bmod k).\end{aligned} \tag{5.9}$$

Without loss of generality we may assume $i = 1$ and $k \ge j > 1$. If $k \ge 3$ then $P = (P'K_2L_2)W_3\dots W_k(A_1P'')$, hence if $k \ge 3$ we get $W_3\dots W_k = W_{j+2}\dots W_{j-1}$ from (5.9). Denote $[P] = W_3\dots W_k$, $Z = W_{j+2}\dots W_k$ and $S = W_3\dots W_{j-1}$. Since $2 \le j \le k$, we have $4 \le j+2 \le k+2$. If $j+2 \le k$ we split $W_{j+2}\dots W_{j-1}$ into the product $(W_{j+2}\dots W_k)\cdot(W_1\dots W_{j-1})$. Thus

$$\begin{aligned}&\textit{if } j \le k-2 \textit{ then } SW_jW_{j+1}Z = ZW_1W_2S,\\ &\textit{where } S = W_3\dots W_{j-1} \textit{ and } Z = W_{j+1}\dots W_k.\end{aligned} \tag{5.10}$$

If $j+2 > k$ then either $j+2 = k+1$ or $j+2 = k+2$, i.e. either $j = k$ or $j = k-1$.

$$\textit{If } j = k \textit{ then } W_2S = SW_k. \tag{5.11}$$

Finally, suppose $j = k-1$.

$$\textit{If } j = k-1 \textit{ then } W_1W_2S = SW_{k-1}W_k. \tag{5.12}$$

We claim that cases (5.11) and (5.12) cannot occur. Due to Lemma 5.1 we have in case (5.11) $W_2 = W_3$, $W_3 = W_4, \dots, W_{k-1} = W_k$, hence $W_1 = W_2 = \dots = W_k$. But then $R^* = W_1\dots W_k = W_1^k$, a proper power, hence $\mu_0\mu_1$ is not reduced, a contradiction. In case (5.12) $W_1 = W_3$, $W_2 = W_4$, $W_3 = W_5, \dots, W_{j-2} = W_{k-1}$, $W_{j-1} = W_k$. Since $j = k-1$, if k even, we get $W_1 = W_{2\ell+1}$, $\ell = 1,\dots,\frac{k}{2}-1$, $W_2 = W_\ell$, $\ell = 2,\dots,\frac{k}{2}$. But then $R^* = (W_1W_2)^{\frac{k}{2}}$ and again $\Phi(\mu_0\mu_1)$ is not reduced as written. If k is odd then $W_1 = W_2 = \dots = W_k$, i.e. $R^* = (W_1)^k$, which leads to a contradiction, as above.

The lemma is proved. □

In order to find the explicit from of the relator, it is convenient to consider $W_1,\dots,W_k$ as symbols, not in F and consider the equation in (5.10) as a word equation in the free semigroup, freely generated by $W_1,\dots,W_k$. We can do this due to Lemma 5.1.

We have now to find out the conditions under which the equation in (5.10) is solvable and to find the solutions. To this end we introduce some types of words.

Definition 5.3 (1-solutions for the defining equations). (a) Let F_0 be the free group, freely generated by two elements X_1 and X_2. For a natural number α_0 ($\alpha_0 \geq 0$) define: $U_{\alpha_0} = (X_1X_2)^{\alpha_0}X_1$, $V_{\alpha_0} = (X_2X_1)^{\alpha_0}X_2$, $M_{\alpha_0} = X_1^{\alpha_0}X_2$, $N_{\alpha_0} = X_2^{\alpha_0}X_1$. (Observe that U_{α_0}, V_{α_0}, M_{α_0} and N_{α_0} are in the same orbit under $Aut(F_0)$.)

(b) Let k be a natural number, $k \geq 1$ and let $\alpha = (\alpha_1, \ldots, \alpha_k)$ be a sequence of natural numbers, $\alpha_i \geq 0$, $i = 1, \ldots, k$. Define: $U_\alpha = U_{\alpha_1} \ldots U_{\alpha_k}$, $V_\alpha = V_{\alpha_1} \ldots V_{\alpha_k}$, $M_\alpha = M_{\alpha_1} \ldots M_{\alpha_k}$, and $N_\alpha = N_{\alpha_1} \ldots N_{\alpha_k}$.

(c) Let F be a free group and let A and B be reduced or empty words in F. Let E be the equation $AxyB = BuvA$ over F in the indeterminates x, y, u and v. *A* 1-*solution of* E is an element $(x_0, y_0, u_0, v_0) \in F^4$ with $|x_0| = |y_0| = |u_0| = |v_0| = 1$ such that Ax_0y_0B and Bu_0v_0A are reduced as written and $Ax_0y_0B = Bu_0v_0A$ holds true in F. E is *1-solvable over* F if it has a 1-solution. Denote $W_E = W_E(u_0, v_0) := Bu_0v_0A$.

Proposition 5.4. *Let notation be as in Definition* 5.3. *Then* E *is* 1-*solvable with* 1-*solution* (x_0, y_0, u_0, v_0) *if and only if one of the following holds:*

I. (i) $W_E = u_0^a$, $a \geq 0$, $x_0 = y_0 = u_0 = v_0$*;*

(ii) $W_E = (u_0v_0)^a$, $a \geq 0$, $x_0 = u_0$ *and* $y_0 = v_0$*;*

(iii) $W_E = (Yu_0v_0)^aY$, $B = (Yu_0v_0)^bY$, *for some non-empty reduced word* Y, $a, b \geq 0$, $x_0 = u_0$, $y_0 = v_0$

II. *Let* $\alpha = (\alpha_1, \ldots, \alpha_m)$, $m \geq 1$, $0 \leq \alpha_i \in \mathbb{Z}$, $i = 1, \ldots, m$ *with the property that if* $m \geq 2$ *then there exists a natural number* k, $1 \leq k \leq m$ *such that one of the following holds:*

$$(\alpha_1 - 1, \alpha_2, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_m) = (\alpha_{k+1}, \ldots, \alpha_m, \alpha_1, \ldots, \alpha_k - 1) \quad (*)$$

or

$$(\alpha_1, \ldots, \alpha_k, \alpha_{k+1}, \ldots, \alpha_m - 1) = (\alpha_{k+1} - 1, \ldots, \alpha_m, \alpha_1, \ldots, \alpha_k) \quad (**)$$

Then one of the following holds:

(i) $W_E = U_\alpha(u_0, v_0)$, *where* α *satisfies* $(*)$ *and* $x_0 = v_0$, $y_0 = u_0$*;*

(ii) $W_E = V_\alpha(u_0, v_0)$, *where* α *satisfies* $(**)$ *and* $x_0 = v_0$, $y_0 = u_0$*;*

(iii) $W_E = M_\alpha(u_0, v_0)$, *where* α *satisfies* $(*)$ *and* $x_0 = v_0$, $y_0 = u_0$*;*

(iv) $W_E = N_\alpha(u_0, v_0)$, *where* α *satisfies* $(**)$ *and* $x_0 = v_0$, $y_0 = u_0$*;*

(v) $W_E = (Yu_0^2)^aY$, $a \geq 0$, Y *a reduced word and* $x_0 = y_0 = u_0 = v_0$.

Proof. The proof of the proposition is straightforward. We first show that for any 1-solution either $x_0 = u_0$ and $y_0 = v_0$, or $x_0 = v_0$ and $y_0 = u_0$. Next we check each of the cases: $AxyB = ByxA$ and $AxyB = BxyA$, respectively. We omit details. □

Definition 5.5 (Exceptional words). (a) Let $W \in F$ be a reduced word and let V_1, V_2 be reduced words. Call W *exceptional with respect to* V_1, V_2 if $W = W_E(V_1, V_2)$, where W_E is given by one of II(i) – II(iv) of Proposition 5.4. (By $W_E(V_1, V_2)$ we mean the word obtained from $W_E(u_0, v_0)$ by substituting V_1 in place of u_0 and V_2 in place of v_0.)

(b) Let A be a non-empty word in G_I, reduced in F, which starts and terminates with an element from $G_{I \setminus D}$ and let K be a non-empty word in G_J, reduced in F, which starts and terminates with an element from $G_{J \setminus D}$. Let L, B_1 and B_2 be elements of G_D, $B_1 \neq B_2$ such that AB_1KL and AB_2KL are reduced in F. Define

$$\widehat{W}_1 = AB_1KL \quad and \quad \widehat{W}_2 = AB_2KL \tag{5.13}$$

(c) Let $I, J \subseteq \{1, \dots, n\}$ be as in the beginning of this section. W is (I, J)-*exceptional* if it is exceptional with respect to $\widehat{W}_1$ and $\widehat{W}_2$ as given by (5.13).

Thus, W is (I, J)-exceptional if W is obtained from W_E in parts of II (i) – II (iv) of Proposition 5.4, by substituting $\widehat{W}_1$ for v_0 and $\widehat{W}_2$ for u_0, where $\widehat{W}_1$ and $\widehat{W}_2$ are given by (5.13).

5.3. The proof of Theorem A. We keep the notation and assumptions of Sections 5.1 and 5.2. To simplify notation we shall write u for u_0 and v for v_0.

We found in the proof of Lemma 5.2 that in the piece pair $(P, P') = (P, P^\varepsilon)$, P is given by (5.7), P' is given by (5.7$'$) if $\varepsilon = 1$ and P' is given by (5.8) if $\varepsilon = -1$. Assume first $\varepsilon = 1$. Then due to Lemma 5.2 we get from (5.10)

$$\begin{gathered} R^* = W_1W_2[P], \text{ where } [P] = W_3 \dots W_k = SxyZ = ZuvS \\ W_j = x, \quad W_{j+1} = y, \quad W_1 = u, \quad W_2 = v, \\ Z = W_{j+1} \dots W_k, \quad S = W_3 \dots W_{j-1} \end{gathered} \tag{5.14}$$

$$R^* = uv[P] \tag{5.3$'$}$$

We apply Proposition 5.4 to (5.3).

We claim that Main Case (I) of Proposition 5.4 and Main Case (II), case (v) cannot occur. Consider first Main Case (I). In this case $x = u$ and $v = y$, hence by (5.3), $W_j = W_1$ and $W_{j+1} = W_2$. The three cases of Main Case (I) are:

(i) $R^* = u^2u^au^2u^b = u^{a+b+4}, a, b \geq 0$.

(ii) $R^* = (uv)(uv)^a(uv)(uv)^b = (uv)^{a+b+2}, a + b \geq 1$.

(iii) $R^* = (uv)(Yuv)^aY = (uvY)^{a+1}, a \geq 1$.

Hence in all cases R^* is a proper power. Since M is reduced and u, uv and v are not proper powers and $\left|[P]\right| \geq 3\|Q\|$, ($\left|[P]\right| \geq 3\|A\|$ in the notation of Lemma 2.6), it follows by Lemma 2.6 that cases (i) and (ii) can not occur. If uvY is not a proper power and $a \geq 2$ or $uvY = Q^m$, $m \geq 2$, Q not a proper power, then by Lemma 2.6 these cases cannot occur. Assume therefore that $a = 1$ and uvY is not a proper power. Then $R^* = (uvY)^2$. Since $\|T_0^{-1}H\| = 2$, $YuvY$ is a subword of P, hence P contains $Q(= uvY)$ as a subword, violating Lemma 2.6. Hence, none of these cases may occur. By a similar argument Case (II) (v) cannot occur. Therefore, by (5.3), $W_j = W_2$, $W_{j+1} = W_1$ and R^* is one of the words given by Case (II)(i) – (iv). Consequently, R^* is an (I, J)-exceptional word, provided that we can show that $A_1 = A_2$, $K_1 = K_2$ and $L_1 = L_2$. To this end consider $z_1\theta^{-1}w_1$. (See Figure 7 (a).) Since $H_0 \in G_J$, D_0 has a boundary path γ_0 such that $\gamma_0 = \gamma_0' z_1 \gamma_0''$ with $\Phi(\gamma_0) = W_1$ and, similarly, D_1 has a boundary path δ_0 such that $\delta_0 = \delta_0' z_1 \delta_0''$ with $\Phi(\delta_0) = W_2$, and satisfy $\Phi(\gamma_0'') = \Phi(\delta_0'')$ with $K_1 L_1 \in \mathcal{T}\big(\Phi(\gamma_0'')\big)$ and $K_2 L_2 \in \mathcal{T}\big(\Phi(\delta_0'')\big)$. Consequently, $K_1 = K_2$ and $L_1 = L_2$. By a similar argument in W_1 we get $A_1 = A_2$. Consequently, $W_1 = A_1 B_1 K_1 L_1$ and $W_2 = A_1 B_2 K_1 L_1$. Now $B_1 \neq B_2$ for otherwise $W_1 = W_2$, hence R^* is a proper power, which as we saw above can not occur due to Lemma 2.6. Since the last argument applies also for the case $k = 2$, it follows that R^* is a (I, J)-exceptional word, as required.

Next, suppose $\varepsilon = -1$. Then P' is given by (5.8). First observe that $W_j^{-1} = L_j^{-1} K_j^{-1} B_j^{-1} A_j^{-1}$, hence if W_j^{-1} is a subword of $W_1 W_2$ then $B_1 = L_j^{-1}$, $K_1 = K_j^{-1}$, $L_1 = B_j^{-1}$ and $A_2 = A_j^{-1}$. Since $K_1 \neq K_1^{-1}$ and $A_1 \neq A_2^{-1}$, we have $j \notin \{1, 2\}$.

Hence, if $k = 3$ then $j = 3$ and therefore $P^{-1} = Q' K_1 L_1 (A_2 B_2 K_2 L_2) A_3 Q''$, hence $P = Q''^{-1} A_3^{-1} L_2^{-1} K_2^{-1} B_2^{-1} A_2^{-1} L_1^{-1} K_1^{-1} Q'^{-1}$. On the other hand, by (5.7) $P = P' K_2 L_2 A_3 B_3 K_3 L_3 A_1 P''$, where $P' \in \mathcal{T}(L_1'' A_2 B_2)$ and $P'' \in \mathcal{H}(B_1 K_1 L_1')$. Consequently, by Lemma 5.1 either $A_3^{-1} = A_2$, $L_2^{-1} = B_2$, $K_2^{-1} = K_2$, a contradiction since $K_2 \neq K_2^{-1}$, or $A_3^{-1} = A_3$, again a contradiction. Hence $k \neq 3$. Also, a similar argument shows that $k \neq 2$. Therefore we may assume $k \geq 4$. We have by (5.3$'$) that $R^* = W_1 W_2 [P]$. Hence, if $\varepsilon = -1$ then we have the following word equation $W_1 W_2 [P] W_1 W_2 [P] = Q_1 [P]^{-1} Q_2$ in F, for some subwords Q_1 and Q_2, which define the occurrence of $[P]^{-1}$, where $\big([P], [P]^{-1}\big)$ is a piece pair. Consequently, either Q_1 is a subword of $W_1 W_2$ in which case $[P]$ and $[P]^{-1}$ overlap and we have:

$$W_1 W_2 [P] = Q_1 [P]^{-1} Q_2', \text{ where } Q_2' \text{ is a head of } Q_2, \text{ or empty,} \tag{5.15}$$

or $W_1 W_2$ is a subword of Q_1 in which case

$$[P]^{-1} = Q_1'' W_1 W_2 Q_1''', \text{ where } Q_1'' \in \mathcal{T}([P]), \text{ or empty and } Q_1''' \in \mathcal{H}([P]), \text{ or empty.} \tag{5.16}$$

Consider equation (5.15). $[P]^{-1} = Xc$, $c \in G_i$ for some i, by Lemma 2.1 (c) and $W_1 W_2 = c' X$, $c' \in G_i$. Therefore $R^* = W_1 W_2 [P] = W_1 W_2 c W_2^{-1} W_1^{-1} c'$. But then $W_1 W_2$ is a piece and R^* is a product of at most three pieces, contradicting the $C(6)$ condition. Finally, consider equation (5.16). We have $[P] = U_1 Q_1''$ and

$[P] = Q_1'''U_2$. From equation (5.16) we also have $[P] = (Q_1''')^{-1}W_2^{-1}W_1^{-1}(Q_1'')^{-1}$. These three equations imply, due to Lemma 2.1 (c) that Q_1'', $Q_1''' \in G_i$, hence again we get $R^* = W_1W_2c_1(W_1W_2)^{-1}c_1$, contradicting the $C(6)$ condition.

Theorem A is proved.

5.4. The proof of Theorem B. Let notation be as in Figure 7. By Theorem A either $\|R\| = 2$ or $\|R\| \geq 4$ and R^* has the form

$$A_1B_{i_1}K_1L_1 \cdot A_1B_{i_2}K_1L_1 \dots A_1B_{i_{k+2}}K_1L_1,$$

where $B_{i_j} \in \{B_1, B_2\}$. Assume $\|R\| \geq 4$. Then $\Phi(\mu_i) = X_iY_iZ_i$, where $Y_i \in G_D * \langle A_1\rangle$, X_i is a terminal subword of a word in $G_D * \langle A_1\rangle$ and Z_i is an initial subword of a word in $G_D * \langle A_1\rangle$. Now, let $z_i\mu_iz_{i+1} = z_i\mu_i't_i\mu_i''s_i\mu_i'''z_{i+1}$, where z_i, t_i, s_i, z_{i+1} are vertices and $\Phi(\mu_i') = X_i$, $\Phi(\mu_i'') = Y_i$ and $\Phi(\mu_i''') = Z_i$. We claim that

$$\text{if } \Phi(\mu_i') \text{ has a head } \eta \text{ such that } \Phi(\eta) \text{ is in } G_I \text{ then it is a tail of } A_1^{\pm 1} \text{ and } \mu_{i-1}''' \text{ has a tail } \tau, \text{ such that } A_1^{\pm 1} \text{ is a head of } \Phi(\tau z_i\mu_i'). \qquad (*)$$

Since A is the unique maximal subword of R^* in G_I, which starts and ends with a letter from $G_{I\setminus D}$ this is clear if $d_M(z_i) = 3$. (See Figure 7 (b).) Now it follows by induction on $d_M(z_i)$, by the same argument, that $A_1^{\pm 1}$ is a head of $\Phi(\tau z_i\mu_i')$. This proves that $\Phi(\mu) \in G_D * \langle A\rangle$. By the same argument $\Phi(\nu) \in G_D * \langle K\rangle$. Also, observe that the above arguments clearly apply for the case $\|R\| = 2$.

Theorem B is proved.

References

[Co] D. J. Collins, Intersections of Magnus subgroups of one-relator roups. In *Groups: topological, combinatorial and arithmetic aspects*, London Math. Soc. Lecture Note Ser. 311, Cambridge Univ. Press, Cambridge 2004, 255–296. Zbl 1078.20031 MR 2073350

[E-H] M. Edjvet and J. Howie, Intersections of Magnus subgroups and embedding theorems for cyclically presented groups. Preprint 2005.

[Ho1] J. Howie, How to generalize one-relator group theory. In *Combinatorial group theory and topology* (Alta, Utah, 1984), Ann. of Math. Stud. 111, Princeton University Press, Princeton, NJ 1987, 53–78. Zbl 0632.20021 MR 0895609

[Ho2] J. Howie, Magnus intersections in one-relator products. *Michigan Math. J.* **53** (2005), 597–623. Zbl 1101.20016 MR MR2207211

[Ju1] A. Juhász, Small cancellation theory with a unified small cancellation condition. I. *J. London Math. Soc.* (2) **40** (1989), 57–80. Zbl 0647.20030 MR 1028914

[Ju2] A. Juhász, Cyclic presentations and solutions of singular equations over free groups. *Publ. Math. Debrecen* **69** (2006), 333–340. Zbl 05150122 MR 2273985

[Ju3] A. Juhász, On a Freiheitssatz for cyclic presentations. *Internat. J. Algebra Comput.* **17** (5–6) (2007), 1043–1053.

[Ju4] A. Juhász, Singular equations with solutions of finite order. Manuscript 2005.

[Ju5] A. Juhász, Solution of the membership problem for Magnus subgroups in certain one-relator free products. In *Geometric group theory*, Trends Math., Birkhäuser, Basel 2007, 169–195.

[Ju6] A. Juhász, A Freiheitssatz for subsemigroups of one-relator groups with small cancellation condition. In *Semigroups and formal languages* (Lisbon 2005), World Scientific, Singapore 2007, 111–128.

[Ju7] A. Juhász, Intersection of conjugates of Magnus subgroups in one-relator free groups and free products with small cancellation. In preparation.

[L-S] R. C. Lyndon and P.E. Schupp, *Combinatorial group theory*. Ergeb. Math. Grenzgeb. 89, Springer-Verlag, Berlin 1977. Zbl 0368.20023 MR 0577064

[MKS] W. Magnus, A. Karrass, and D. Solitar, *Combinatorial group theory*. 2nd rev. ed., Dover, New York 1976. Zbl 0362.20023 MR 0422434

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