Finiteness of the set of solutions of Plateau's problem for polygonal boundary curves

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Abstract

It is proved that for a simple, closed, extreme *polygon* $\Gamma \subset \mathbb{R}^3$ every immersed, stable minimal surface spanning Γ is an isolated point of the set of all minimal surfaces spanning Γ w.r.t. the C^0-topology. Since the subset of immersed, stable minimal surfaces spanning Γ is shown to be closed in the compact set of all minimal surfaces spanning Γ, this proves in particular that Γ can bound only *finitely* many immersed, stable minimal surfaces.
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1. Introduction and main results

In 1978 Nitsche formulated the following conjecture (see [23]): *A "reasonably well behaved" simple, closed contour can bound only finitely many solutions of Plateau's problem.* The aim of the present article is a proof of the following partial result:

Theorem 1. *Let $\Gamma \subset \mathbb{R}^3$ be a simple, closed, extreme polygon. Then every immersed, stable minimal surface spanning Γ is an isolated point of the set of all minimal surfaces spanning Γ w.r.t. the C^0-topology. In particular, Γ can bound only finitely many immersed, stable minimal surfaces.*

A polygon is termed *extreme* if it is contained in the boundary of a compact, convex subset of $\mathbb{R}^3$. Furthermore a disc-type minimal surface X is called *immersed* if there holds $\inf_B |DX| > 0$, where we denote by B the open unit disc $\{w = (u, v) \in \mathbb{R}^2 \mid |w| < 1\}$. It is said to be *stable* if the second variation of the area functional $\mathcal{A}$ in X in normal direction $\xi := (X_u \wedge X_v)/|X_u \wedge X_v|$

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$$J^X(\varphi) := \int_B |\nabla\varphi|^2 + 2KE\varphi^2 \, \mathrm{d}w = \frac{\mathrm{d}^2}{\mathrm{d}\varepsilon^2} \mathcal{A}(X + \varepsilon\varphi\xi)|_{\varepsilon=0} \tag{1}$$

satisfies $J^X(\varphi) \geqslant 0$ for any $\varphi \in C_c^\infty(B)$, where E denotes $|X_u|^2$ and $K \leqslant 0$ the Gauss curvature of X.

Let Γ be a closed Jordan curve in $\mathbb{R}^3$, and denote by $\mathcal{C}^*(\Gamma)$ the *Plateau class* of surfaces $X \in H^{1,2}(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3)$ that span Γ satisfying a fixed three-point condition. We endow $\mathcal{C}^*(\Gamma)$ with the norm $\|\cdot\|_{C^0(\overline{B})}$ and denote by $\mathcal{M}(\Gamma)$ its subspace $\{X \in \mathcal{C}^*(\Gamma) \mid \Delta X = 0, \ |X_u| = |X_v|, \ \langle X_u, X_v \rangle = 0 \text{ on } B\}$ of disc-type minimal surfaces. Furthermore let $\mathcal{M}_s(\Gamma)$ be the subspace of $\mathcal{M}(\Gamma)$ consisting of those elements which are immersed and stable.

The first deep "finiteness result" was achieved by Tomi (1973) in [27]: If Γ is a regular Jordan curve of class $C^{4,\alpha}$ with the property that all minimal surfaces $X \in \mathcal{M}(\Gamma)$, which yield global minimizers of the area functional $\mathcal{A}$ on $\mathcal{C}^*(\Gamma)$, are immersed, then there are only finitely many of them. In combination with the papers [7] resp. [15] Tomi's theorem guarantees that regular Jordan curves Γ which are either of class $C^{4,\alpha}$ with total curvature less than 4π or analytic can bound only finitely many global minimizers of the area functional $\mathcal{A}$ on $\mathcal{C}^*(\Gamma)$. Four years later Tomi proved in [28]: If Γ is a regular Jordan curve of class $C^{4,\alpha}$, which bounds only minimal surfaces without boundary branch points and with interior branch points of at most first order (see Def. 1 below), then $\mathcal{M}_s(\Gamma)$ is finite. One year later Nitsche finally achieved his "6π-Theorem" in [23]: If Γ is a regular Jordan curve of class $C^{4,\alpha}$, which bounds only minimal surfaces without any branch points and whose total curvature does not exceed the value 6π, then the *entire* set $\mathcal{M}(\Gamma)$ is finite.

In 1990, Sauvigny [24] proved a finiteness result for "small" H-surfaces which is very similar to Theorem 1. If Γ is an extreme, regular Jordan curve of class $C^{4,\alpha}$ contained in $B_1^3(0)$ and $H \in [0, 1)$, then it can bound only finitely many immersed small H-surfaces X which are stable in the sense that $J^X(\varphi) - 4\int_B H^2 E\varphi^2 \, \mathrm{d}w \geqslant 0 \ \forall \varphi \in C_c^\infty(B)$.[2]

The proofs of the above quoted results depend on boundary regularity results for minimal surfaces resp. H-surfaces spanning $C^{4,\alpha}$-boundary curves due to Hildebrandt [16] resp. Heinz [8]. Analogs of these results are not available for polygons. Thus the author had to follow a completely different approach to prove Theorem 1 which uses fundamental results of Courant [1] and Heinz [10–13] in combination with ideas of Tomi [28] and Sauvigny [24–26]. Of particular importance are the asymptotic expansions for minimal surfaces in the corners of the bounding polygon due to Heinz [11] and Heinz' discovery of a deep connection between the total branch point orders of such minimal surfaces, the defects of their assigned Schwarz operators and the number of vertices of the bounding polygon, expressed in his formula (12) below, the main result of his paper [14].

2. Courant's and Heinz' results

A polygon Γ is a closed piecewise linear Jordan curve in $\mathbb{R}^3$ with $N + 3$ vertices ($N \in \mathbb{N}$) $(P_1, \dots, P_{N+3})$, where we require the pairs of vectors $(P_{j+1} - P_j, P_j - P_{j-1})$ to be linear independent for $j = 1, \dots, N + 3$, with $P_0 := P_{N+3}$ and $P_{N+4} := P_1$. We consider the "Plateau class" $\mathcal{C}^*(\Gamma)$ of surfaces $X \in H^{1,2}(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3)$ that are spanned into Γ, i.e. whose boundary values $X|_{\partial B} : \mathbb{S}^1 \to \Gamma$ are weakly monotonic mappings with degree equal to 1, satisfying a three-point condition: $X(e^{i\tau_{N+k}}) = P_{N+k}$ for $\tau_{N+k} := \frac{\pi}{2}(1 + k)$, $k = 1, 2, 3$. Our fundamental tools are Courant's [1] and Heinz' [10], [11] maps

$$\psi : T \longrightarrow \big(\mathcal{C}^*(\Gamma), \|\cdot\|_{C^0(\overline{B})}\big), \qquad \tilde{\psi} : T \longrightarrow C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3), \tag{2}$$

which are assigned to our arbitrarily fixed closed polygon Γ. Here T is an open, bounded, convex set of N-tuples $(\tau_1, \tau_2, \dots, \tau_N) =: \tau \in (0, \pi)^N$, which meet the following chain of inequalities $0 < \tau_1 < \cdots < \tau_N < \pi = \tau_{N+1}$, where $N + 3$ was the number of vertices of the considered polygon. Moreover to any $\tau \in T$ we assign the sets of surfaces

$$\mathcal{U}(\tau) := \big\{X \in \mathcal{C}^*(\Gamma) \mid X|_{\partial B}(e^{i\tau_j}) = P_j \text{ for } j = 1, \dots, N\big\} \quad \text{and}$$

$$\widetilde{\mathcal{U}}(\tau) := \big\{X \in C^0(\overline{B}, \mathbb{R}^3) \cap C^2(B, \mathbb{R}^3) \mid X(e^{i\theta}) \in \Gamma_j \text{ for } \theta \in [\tau_j, \tau_{j+1}]\big\},$$

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[2] After completion of this manuscript, Prof. Frank Morgan kindly communicated one of his results to the author [see Indiana Univ. Math. J. 35 (4) (1986) 813, Theorem 5.8] which generalizes Sauvigny's theorem especially to systems of $C^{2,\alpha}$-Jordan curves lying on the boundary of an arbitrary strictly convex set in $\mathbb{R}^3$, but only for $H = 0$. The smoothness of the boundary curves are crucial in his proof, as in the results of Tomi, Nitsche, and Sauvigny.

for $1 \leqslant j \leqslant N + 3$, where we set $\Gamma_j := \{P_j + t(P_{j+1} - P_j) \mid t \in \mathbb{R}\}$, $P_{N+4} := P_1$ and $\tau_{N+4} := \tau_1$. On account of two uniqueness results in [1] resp. [10] one can define the maps

$$\psi(\tau) := \text{unique minimizer of } \mathcal{D} \text{ within } \mathcal{U}(\tau) \quad \text{and} \quad \tilde{\psi}(\tau) := \text{unique minimizer of } \mathcal{D} \text{ within } \tilde{\mathcal{U}}(\tau),$$

where $\mathcal{D}$ denotes Dirichlet's integral. We will also use the notation $X(\cdot, \tau)$ for $\tilde{\psi}(\tau)$. Now by the result of [1] (see also [18], p. 558) Satz 1 and 2 in [10] and the main theorem of [11], resp. Satz 1 in [13], these maps have the following properties:

Theorem 2.

(i) ψ *is continuous on* T.
(ii) $f := \mathcal{D} \circ \psi$ *is of class* $C^1(T)$ *and* $\tilde{f} := \mathcal{D} \circ \tilde{\psi}$ *even of class* $C^\omega(T)$.
(iii) *There holds* $\tilde{f} \leqslant f$ *on* T *and* $\tilde{f}(\tau) = f(\tau)$ *if and only if* $\tilde{\psi}(\tau) = \psi(\tau)$, *which is again equivalent to* $\tilde{\psi}(\tau) \in \mathcal{C}^*(\Gamma)$.
(iv) $\tilde{\psi}(\tau)$ *and* $\psi(\tau)$ *are harmonic on* B $\forall \tau \in T$.
(v) *The restriction*

$$\psi|_{K(f)} : K(f) \xrightarrow{\cong} \mathcal{M}(\Gamma) \tag{3}$$

yields a homeomorphism *between the compact set of critical points of* f *and* $(\mathcal{M}(\Gamma), \|\cdot\|_{C^0(\overline{B})})$ *and a surface* $\tilde{\psi}(\tau)$ *is conformally parametrized on* B, *thus a minimal surface in* $\tilde{\mathcal{U}}(\tau)$, *if and only if* $\tau \in K(\tilde{f})$.
(vi) *Let* $\bar{\tau} \in T$ *be arbitrarily fixed and* $D \subset \mathbb{C}$ *some simply connected domain whose intersection* $D \cap B$ *with* B *is nonvoid and connected and such that* $\bar{D} \cap \{e^{i\bar{\tau}_k}\} = \emptyset$. *Then there exists some neighborhood* $U_D(\bar{\tau})$ *of* $\bar{\tau}$ *in* $\mathbb{C}^N$ *and some holomorphic continuation of* $X_w(\cdot, \cdot)$ *onto* $D \times U_D(\bar{\tau})$.
(vii) *Furthermore for any* $\bar{\tau} \in T$ *and* $k \in \{1, \ldots, N+3\}$ *there exists some neighborhood* $B_\delta(e^{i\bar{\tau}_k}) \times B_\delta^N(\bar{\tau})$ *in* $\mathbb{C} \times \mathbb{C}^N$ *about* $(e^{i\bar{\tau}_k}, \bar{\tau})$ *such that there holds the representation*

$$X_w(w, \tau) = \sum_{j=1}^{p_k} f_j^k(w, \tau)\left(w - e^{i\tau_k}\right)^{\rho_j^k} \tag{4}$$

for $(w, \tau) \in (B_\delta(e^{i\bar{\tau}_k}) \cap B) \times (B_\delta^N(\bar{\tau}) \cap \mathbb{R}^N)$, *where the functions* f_j^k *are holomorphic on* $B_\delta(e^{i\bar{\tau}_k}) \times B_\delta^N(\bar{\tau})$ *and the exponents* ρ_j^k *satisfy*

$$-1 < \rho_1^k < \cdots < \rho_{p_k}^k = 0, \quad p_k \in \{2, 3\}, \tag{5}$$

and do not depend on $\tau \in B_\delta^N(\bar{\tau}) \cap \mathbb{R}^N$.

The last assertion about the independence of the exponents ρ_j^k of $\tau \in B_\delta^N(\bar{\tau}) \cap \mathbb{R}^N$ follows immediately from [10], (2.20) and (3.28), as we point out now. We set $v_k := (P_{k+1} - P_k)/|P_{k+1} - P_k|$ and consider as in (2.20) of [10] the reflections S_k at the lines $\Gamma_k - P_k = \text{Span}(v_k)$ for $k \in \{1, \ldots, N+3\}$ (with $P_{N+4} := P_1$), explicitly given by

$$S_k(x) := -x + 2\langle v_k, x \rangle v_k \quad \forall x \in \mathbb{R}^3.$$

The composed reflections $S_{k-1} \circ S_k \in SO(3)$ are diagonalizable by conjugation with unitary matrices and have eigenvalues on the $\mathbb{S}^1$. Now the ρ_j^k appear in (3.28) of [10] as pairwise different (negative) angles of these eigenvalues, precisely:

$$\text{Spec}(S_{k-1} \circ S_k) = \left\{e^{-2\pi i \rho_j^k}\right\}, \quad 1 \leqslant j \leqslant p_k,$$

ordered as in (5) with $p_k \in \{2, 3\}$, which proves the claimed independence of the exponents ρ_j^k of $\tau \in B_\delta^N(\bar{\tau}) \cap \mathbb{R}^N$. Moreover we shall note that $S_{k-1} \circ S_k \neq \mathbf{1}$ and thus $p_k > 1$ by our requirement that the vectors $P_{k-1} - P_k$ and $P_{k+1} - P_k$ have to be linearly independent. Moreover we see that $p_k = 2$ if and only if $\rho_1^k = -\frac{1}{2}$, i.e. if the spectrum of $S_{k-1} \circ S_k$ is $\{-1, -1, 1\}$, which can arise if and only if the smaller angle β_k between the vectors $P_{k-1} - P_k$ and $P_{k+1} - P_k$ is $\frac{\pi}{2}$. If in general $\beta_k \notin \{\frac{\pi}{2}, 0, \pi\}$, then the spectrum of $S_{k-1} \circ S_k$ is $\{\lambda, \bar{\lambda}, 1\}$ for some $\lambda \in \mathbb{S}^1$ with

$\Im(\lambda) \neq 0$, i.e. $\rho_1^k + \rho_2^k = -1$. One can easily see that there holds either $-\rho_1^k \pi = \beta_k$ or $(\rho_1^k + 1)\pi = \beta_k$, which is by $\rho_1^k + \rho_2^k = -1$ equivalent to the pair of possibilities $(\rho_2^k + 1)\pi = \beta_k$ or $-\rho_2^k \pi = \beta_k$. Moreover we can expand the holomorphic functions f_j^k w.r.t. w about the point $e^{i\tau_k}$ and obtain by (4) for any $k \in \{1, \ldots, N + 3\}$:

$$X_w(w, \tau) = \sum_{j=1}^{p_k} \sum_{n=0}^{\infty} f_{j,n}^k(\tau)(w - e^{i\tau_k})^{\rho_j^k + n} \tag{6}$$

$\forall w \in B_\delta(e^{i\tau_k}) \cap B$ and $\forall \tau \in B_\delta^N(\bar{\tau}) \cap \mathbb{R}^N$. Now we fix some $\bar{\tau} \in T$ and choose that pair (j, n) for which $f_{j,n}^k(\bar{\tau}) \neq 0$ in (6) and $\rho_j^k + n$ is minimal and term this pair (j^*, m), i.e. we assign this pair to the point $\bar{\tau} \in T$. Since we know that either $(\rho_{j^*}^k + 1)\pi$ or $-\rho_{j^*}^k \pi$ equals the smaller angle $\beta_k \neq 0, \pi$ between the linear independent vectors $P_{k-1} - P_k$ and $P_{k+1} - P_k$ we conclude due to $\rho_{p_k}^k = 0$ that there has to hold $j^* < p_k$. Now using these terms the author derived in [20], Corollary 2.1:

Corollary 1. *For any fixed $\bar{\tau} \in T$ and $k \in \{1, \ldots, N + 3\}$ there holds*

$$X_w(w, \bar{\tau}) = f_{j^*,m}^k(\bar{\tau})(w - e^{i\tau_k})^{\rho_{j^*}^k + m} + O\big(|w - e^{i\tau_k}|^{\rho_{j^*}^k + m + \varepsilon}\big) \tag{7}$$

for $B \ni w \to e^{i\tau_k}$, where $\varepsilon := \rho_{j^*+1}^k - \rho_{j^*}^k \in (0, 1)$.

Furthermore we derive from part (vi) of Theorem 2 that there is a Taylor expansion of $X_w(\cdot, \bar{\tau})$ about any point $w_0 \in \overline{B} \setminus \{e^{i\tau_k}\}$:

$$X_w(w, \bar{\tau}) = a_m(\bar{\tau})(w - w_0)^m + a_{m+1}(\bar{\tau})(w - w_0)^{m+1} + \cdots, \tag{8}$$

where the coefficients $\{a_j\}_{j \geqslant m}$ are holomorphic about the point $\bar{\tau}$ and $a_m(\bar{\tau}) \in \mathbb{C}^3 \setminus \{0\}$.

Definition 1.

(i) We term the exponent $m \equiv m^{\bar{\tau}}$ in (7) resp. (8) the branch point order of the surface $X(\cdot, \bar{\tau})$ at the point $e^{i\bar{\tau}_k}$, $k = 1, \ldots, N + 3$, resp. $w_0 \in \overline{B} \setminus \{e^{i\bar{\tau}_k}\}$.
(ii) A point $\overline{w} \in \overline{B}$ is termed a branch point of the minimal surface $X(\cdot, \bar{\tau}) \in \widetilde{\mathcal{M}}(\Gamma)$ if its order $m^{\bar{\tau}}(\overline{w})$ is positive.

Hence, we see that there holds $m^{\bar{\tau}}(w) = 0$ in any point $w \in \overline{B}$ if and only if

$$\inf_B |DX(\cdot, \bar{\tau})| > 0. \tag{9}$$

Furthermore one obtains easily by (7) and (8) that $X(\cdot, \bar{\tau})$ can have only finitely many branch points on $\overline{B}$. Hence, we may define its total branch point order

$$\kappa(\bar{\tau}) := \sum_{w \in B} m^{\bar{\tau}}(w) + \frac{1}{2} \sum_{w \in \partial B} m^{\bar{\tau}}(w). \tag{10}$$

Now we assign to every point $\tau \in K(\tilde{f})$, i.e. to every minimal surface $X(\cdot, \tau)$, its Schwarz operator

$$A^\tau \equiv A^{X(\cdot, \tau)} := -\Delta + 2(KE)^\tau,$$

where $(KE)^\tau(w) := (KE)(w, \tau)$ is defined as in (1), on the domain

$$\text{Dom}(A^\tau) := \big\{ \varphi \in C^2(B) \cap \mathring{H}^{1,2}(B) \mid A^\tau(\varphi) \in L^2(B) \big\} \tag{11}$$

and formulate as a central tool of our paper the "Heinz' formula" from [14]: For an arbitrary $\tau \in K(\tilde{f})$ one has

$$\dim \text{Ker } A^\tau + \text{rank}\big(D^2 \tilde{f}(\tau)\big) + 2\kappa(\tau) = N. \tag{12}$$

Furthermore by differentiation of (6) w.r.t. w we obtain exactly as in the proof of Corollary 1 on account of the holomorphy of the components of $X_w(\cdot, \bar{\tau})$ on B:

$$X_{ww}(w, \bar{\tau}) = f_{j^*,m}^k(\bar{\tau})\big(m + \rho_{j^*}^k\big)(w - e^{i\tau_k})^{\rho_{j^*}^k + m - 1} + O\big(|w - e^{i\tau_k}|^{\rho_{j^*}^k + m + \varepsilon - 1}\big), \tag{13}$$

for $B \ni w \to e^{i\bar{\tau}_k}$. Furthermore (cf. formula (2.8) in [14]) one proves by integration of $X_w(\cdot, \bar{\tau})$ in (6) w.r.t. w that for any fixed $\bar{\tau} \in T$ and $k \in \{1, \ldots, N+3\}$ there exists some $\delta > 0$ such that

$$X(w, \tau) = 2\Re\left(\int_{e^{i\tau_k}}^{w} X_z(z, \tau)\,dz\right) + X\left(e^{i\tau_k}, \tau\right)$$

$$= \Re\left(\sum_{j=1}^{p_k} g_j^k(w, \tau)\left(w - e^{i\tau_k}\right)^{\rho_j^k + 1}\right) + P_k \tag{14}$$

for $w \in B_\delta(e^{i\bar{\tau}_k}) \cap B$ and $\tau \in B_\delta^N(\bar{\tau}) \cap \mathbb{R}^N$, where the functions

$$g_j^k(w, \tau) := \sum_{n=0}^{\infty} \frac{2f_{j,n}^k(\tau)}{\rho_j^k + n + 1}\left(w - e^{i\tau_k}\right)^n$$

are holomorphic on $B_\delta(e^{i\tau_k}) \times B_\delta^N(\bar{\tau})$ and satisfy in particular $g_j^k(e^{i\tau_k}, \tau) = 2f_{j,0}^k(\tau)/\rho_j^k + 1$. Next, as stated in formula (2.9) in [14], one achieves by differentiation of (14) w.r.t. τ_l for $l \in \{1, \ldots, \hat{k}, \ldots, N\}$:

$$X_{\tau_l}(w, \tau) = \Re\left(\sum_{j=1}^{p_k} \frac{\partial g_j^k}{\partial \tau_l}(w, \tau)\left(w - e^{i\tau_k}\right)^{\rho_j^k + 1}\right), \tag{15}$$

for $w \in B_\delta(e^{i\bar{\tau}_k}) \cap B$ and $\tau \in B_\delta^N(\bar{\tau}) \cap \mathbb{R}^N$ and for $l = k$:

$$X_{\tau_l}(w, \tau) = \Re\left(\sum_{j=1}^{p_l} \frac{\partial g_j^l}{\partial \tau_l}(w, \tau)\left(w - e^{i\tau_l}\right)^{\rho_j^l + 1} - ie^{i\tau_l}\left(\rho_j^l + 1\right)g_j^l(w, \tau)\left(w - e^{i\tau_l}\right)^{\rho_j^l}\right). \tag{16}$$

To verify now formula (2.10) in [14] we set $\frac{\partial}{\partial \varphi} := u\frac{\partial}{\partial v} - v\frac{\partial}{\partial u}$ and compute for some $l = k \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, p_l\}$

$$\frac{\partial}{\partial \varphi}\left(w - e^{i\tau_l}\right)^{\rho_j^l + 1} = \left(\rho_j^l + 1\right)\left(w - e^{i\tau_l}\right)^{\rho_j^l} iw, \tag{17}$$

thus obtaining for $l = k$:

$$X_{\tau_l}(w, \tau) + X_\varphi(w, \tau) = \Re\left(\sum_{j=1}^{p_l}\left(\frac{\partial g_j^l}{\partial \tau_l}(w, \tau) + \frac{\partial g_j^l}{\partial \varphi}(w, \tau) + i\left(\rho_j^l + 1\right)g_j^l(w, \tau)\right)\left(w - e^{i\tau_l}\right)^{\rho_j^l + 1}\right) \tag{18}$$

for $w \in B_\delta(e^{i\bar{\tau}_l}) \cap B$ and $\tau \in B_\delta^N(\bar{\tau}) \cap \mathbb{R}^N$. Combining this with (15) we achieve

Corollary 2. *Let* $\bar{\tau} \in K(\tilde{f})$ *be arbitrarily chosen. If there holds* $m^{\bar{\tau}}(e^{i\bar{\tau}_l}) = 0$ *for each* $l \in \{1, \ldots, N\}$, *then the functions* $X_{\tau_l}(\cdot, \bar{\tau})$ *are linearly independent on* B.

Proof. Otherwise there would exist some linear relation

$$\sum_{l=1}^{N} \alpha_l X_{\tau_l}(\cdot, \bar{\tau}) \equiv 0 \quad \text{on } B, \tag{19}$$

where there is at least one index $k \in \{1, \ldots, N\}$ with $\alpha_k \neq 0$. By (15) we see that $X_{\tau_l}(w, \bar{\tau}) \to 0$ for $w \to e^{i\bar{\tau}_k}$ and $l \neq k$. Hence, inserting this into (19) we obtain due to $\alpha_k \neq 0$ that $X_{\tau_k}(w, \bar{\tau}) \to 0$ for $w \to e^{i\bar{\tau}_k}$. Now together with (18) we conclude that there holds also $X_\varphi(w, \bar{\tau}) \to 0$ for $w \to e^{i\bar{\tau}_k}$. Moreover as we require $\bar{\tau} \in K(\tilde{f})$, thus that $X(\cdot, \bar{\tau})$ has to be conformally parametrized on B, we have $|X_\varphi(w, \bar{\tau})|^2 = 2|w|^2|X_w(w, \bar{\tau})|^2, \forall w \in B$. Hence, we would finally obtain $X_w(w, \bar{\tau}) \to 0$ for $w \to e^{i\bar{\tau}_k}$, which would imply $m^{\bar{\tau}}(e^{i\bar{\tau}_k}) > 0$ by (7), contradicting the requirement of the corollary. $\quad\square$

Now we set

$$\rho := \min_{k=1,\dots,N+3} \rho_1^k > -1. \tag{20}$$

By the Courant–Lebesgue Lemma and point (iv) of Theorem 2 the author proved in Chapter 2 in [20] the following important

Lemma 1. *There holds*

$$\widetilde{\mathcal{M}}(\Gamma) := \{\text{set of minimal surfaces on } B\} \cap \bigcup_{\tau \in T} \widetilde{\mathcal{U}}(\tau) \cap H^{1,2}(B, \mathbb{R}^3)$$

$$= \big\{ X \in \text{image } (\tilde{\psi}) \mid X \text{ is also conformally parametrized on } B \big\}.$$

Combining this result with (8) and point (v) of Theorem 2 the author proved in Chapter 2 in [20]

Corollary 3. *There holds* $\mathcal{M}(\Gamma) \subset \widetilde{\mathcal{M}}(\Gamma)$ *and also* $K(f) \subset K(\tilde{f})$. *In particular* $X(\cdot, \tau) \equiv \tilde{\psi}(\tau)$ *coincides with* $\psi(\tau)$ *for any* $\tau \in K(f)$.

Moreover let

$$\xi(\cdot, \tau) := \frac{X_u \wedge X_v}{|X_u \wedge X_v|}(\cdot, \tau) = \frac{X_w \wedge X_{\overline{w}}}{\mathrm{i}|X_w|^2}(\cdot, \tau) \tag{21}$$

denote the unit normal field of some minimal surface $X(\cdot, \tau) \in \widetilde{\mathcal{M}}(\Gamma)$, i.e. for some $\tau \in K(\tilde{f})$. By (7) and (8) one achieves that $\xi(\cdot, \tau)$ can be continued continuously onto $\overline{B}$ and even analytically onto $\overline{B} \setminus \{e^{\mathrm{i}\tau_l}\}_{l=1,\dots,N+3}$, although it is not defined in the branch points of $X(\cdot, \tau)$, and that at some point $e^{\mathrm{i}\tau_k}$ it behaves asymptotically like

$$\xi(w, \tau) = \frac{f_{j^*,m}^k(\tau) \wedge \overline{f_{j^*,m}^k(\tau)}}{\mathrm{i}|f_{j^*,m}^k(\tau)|^2} + O\big(|w - e^{\mathrm{i}\tau_k}|^\varepsilon\big) \quad \text{for } w \longrightarrow e^{\mathrm{i}\tau_k} \tag{22}$$

and $k = 1, \dots, N + 3$. Together with (13) one obtains moreover:

$$\frac{|\langle \xi(w, \tau), X_{ww}(w, \tau)\rangle|}{|X_w(w, \tau)|} = O\big(|w - e^{\mathrm{i}\tau_k}|^{\varepsilon-1}\big) \quad \text{for } w \longrightarrow e^{\mathrm{i}\tau_k} \tag{23}$$

and $k = 1, \dots, N + 3$. Heinz used this and identity (3.2) in [14],

$$(KE)^\tau(w) \equiv (KE)(\tau, w) = -\frac{8|\langle \xi(w, \tau), X_{ww}(w, \tau)\rangle|^2}{|X_w(w, \tau)|^2} \tag{24}$$

for any $\tau \in K(\tilde{f})$, in order to prove: There is some constant $const.(\tau)$, depending on τ and Γ only, such that:

$$\big|(KE)^\tau(w)\big| \leqslant const.(\tau) \sum_{k=1}^{N+3} \big|w - e^{\mathrm{i}\tau_k}\big|^{-2+\alpha} \quad \forall w \in B, \tag{25}$$

for

$$\alpha := 2\min\big\{\rho_j^k - \rho_{j-1}^k \mid j = 1, \dots, p_k, \ k = 1, \dots, N+3\big\} > 0 \tag{26}$$

where we set $\rho_0^k := -1$ for each k. Thus α only depends on Γ but not on $\tau \in K(\tilde{f})$, as the ρ_j^k do not, whence $(KE)^\tau \in L^{p^*}(B)$ for any $p^* \in (1, \frac{2}{2-\alpha})$ and any $\tau \in K(\tilde{f})$. Moreover one can insure by (24) and (8) that the function $(KE)^\tau$, which is not defined in the branch points of $X(\cdot, \tau)$, is of class $L_{\mathrm{loc}}^\infty(\overline{B} \setminus \{e^{\mathrm{i}\tau_l}\}_{l=1,\dots,N+3})$ and can in fact be continued analytically onto $\overline{B} \setminus \{e^{\mathrm{i}\tau_l}\}_{l=1,\dots,N+3}$.

3. Compactness of $\mathcal{M}_s(\Gamma)$

Theorem 3. *If Γ is a rectifiable, closed Jordan curve in $\mathbb{R}^3$ which bounds only minimal surfaces without boundary branch points, then $\mathcal{M}_s(\Gamma)$ is a closed subset of $\mathcal{M}(\Gamma)$, thus compact, w.r.t. the $C^0(\overline{B})$-topology.*

Proof. Let $\{X^n\}$ be some sequence in $\mathcal{M}_s(\Gamma)$ converging to some $X \in \mathcal{M}(\Gamma)$ in $C^0(\overline{B}, \mathbb{R}^3)$. Then we infer $X^n \to X$ in $C^1_{\mathrm{loc}}(B, \mathbb{R}^3)$ from Cauchy's estimates and thus that X also cannot have any interior branch points by Theorem 1 in [24], where one has to use that X cannot be a constant map on account of the imposed three-point condition. Moreover, since X is bounded by Γ it must be free of boundary branch points, and thus immersed on $\overline{B}$. Secondly we show the stability of X. To this end we fix some $\varphi \in C^\infty_c(B)$ arbitrarily. On account of the requirement $|X^n_w| > 0$ on B we can use the identity $(KE)^n = -8|\langle \xi^n, X^n_{ww} \rangle|^2 / |X^n_w|^2$ in order to conclude by Cauchy's estimates that

$$(KE)^n \varphi^2(w) \longrightarrow KE\varphi^2(w) \quad \text{pointwise for a.e. } w \in B$$

and for $n \to \infty$. Now together with $-(KE)^n \geqslant 0$ on B and $J^{X^n}(\varphi) \geqslant 0$ for any $n \in \mathbb{N}$ we achieve by Fatou's lemma:

$$\int_B -2KE\varphi^2 \, \mathrm{d}w \leqslant \liminf_{n\to\infty} \int_B -2(KE)^n \varphi^2 \, \mathrm{d}w \leqslant \int_B |\nabla\varphi|^2 \, \mathrm{d}w,$$

i.e. $J^X(\varphi) \geqslant 0$ for any $\varphi \in C^\infty_c(B)$. Hence, $\mathcal{M}_s(\Gamma)$ is a closed subset of $\mathcal{M}(\Gamma)$ and thus compact on account of the well-known compactness of $\mathcal{M}(\Gamma)$. $\square$

Now we fix again some simple, closed polygon Γ and prove the following approximation result to be used below in Sections 6 and 7:

Lemma 2. *If X is some stable minimal surface in $\widetilde{\mathcal{M}}(\Gamma)$, then there holds $J^X(\varphi) \geqslant 0$ even for all functions $\varphi \in \mathring{H}^{1,2}(B)$.*

Proof. By Lemma 1 and point (v) of Theorem 2 we know that there exists some $\tau \in K(\tilde{f})$ such that $X = \tilde{\psi}(\tau)$. Thus we conclude by (26) that there exists some $p^* > 1$ such that $KE \in L^{p^*}(B)$. Now let $\varphi \in \mathring{H}^{1,2}(B)$ be chosen arbitrarily and $\{\varphi_j\} \subset C^\infty_c(B)$ some sequence with $\varphi_j \to \varphi$ in $\mathring{H}^{1,2}(B)$. By Sobolev's embedding theorem we have $\varphi_j \to \varphi$ in $L^q(B)$, for any $q \in [1, \infty)$, and therefore together with Hölder's inequality: $\|KE(\varphi_j^2 - \varphi^2)\|_{L^1(B)} \to 0$, for $j \to \infty$. Thus together with the required stability of X we obtain $0 \leqslant J^X(\varphi_j) \to J^X(\varphi)$ for $j \to \infty$, hence $J^X(\varphi) \geqslant 0$. $\square$

4. Extreme polygons prevent boundary branch points

In this section we collect the author's results of Chapter 4 of [20] which guarantee that Theorem 3 especially applies to *extreme* polygons. In Chapter 4 of [20] the author proved the following generalization of Hopf's "boundary point lemma":

Lemma 3. *Let $\Phi \in C^0(\overline{B}) \cap C^2(B)$ be harmonic on B and satisfy*

$$\Phi(w_0) > \Phi(w) \quad \forall w \in B, \tag{27}$$

for some fixed point $w_0 \in \partial B$. Then there exists some constant $\sigma > 0$ such that there holds

$$\frac{\Phi(w_0) - \Phi(w)}{|w_0 - w|} > \sigma \quad \forall w \in K_{\delta, \frac{\pi}{4}}(w_0), \tag{28}$$

for some sufficiently small chosen $\delta > 0$, where $K_{\delta, \frac{\pi}{4}}(w_0) := \{w \in B_\delta(w_0) \cap B \mid |\operatorname{angle}(w - w_0, -w_0)| \in [0, \frac{\pi}{4}]\}$.

By this result the author derived in Chapter 4 of [20]:

Theorem 4. *If Γ is an extreme, simple, closed polygon which is not contained in a plane, then a minimal surface $X \in \mathcal{M}(\Gamma)$ does not possess any boundary branch points.*

Secondly, for an arbitrary simple, closed polygon Γ one has

Lemma 4. *Let $X(\cdot, \tau) \in \widetilde{\mathcal{M}}(\Gamma)$ be a minimal surface whose boundary values are not monotonic on some arc $(e^{i\tau_k}, e^{i\tau_{k+1}}) \subset \mathbb{S}^1$, for some $k = 1, \ldots, N + 3$. Then there exists some angle $\theta \in (\tau_k, \tau_{k+1})$ for which $e^{i\theta}$ is a boundary branch point of $X(\cdot, \tau)$. In particular, any $X \in \widetilde{\mathcal{M}}(\Gamma) \setminus \mathcal{M}(\Gamma)$ possesses a boundary branch point.*

Thirdly, a combination of this lemma with the proof of Theorem 4 implies

Corollary 4. *Let Γ be some extreme, simple, closed polygon. If a minimal surface $X \in \widetilde{\mathcal{M}}(\Gamma)$ satisfies $2\kappa(X) = 1$ then it is already free of branch points on $\overline{B}$.*

Combining (24) with Theorem 1 on p. 175 in [3] the author derived in Chapter 4 of [20] for any extreme, simple, closed polygon Γ:

Corollary 5. *Let $\bar{\tau} \in K(f)$ and $\delta > 0$ be fixed with the property that $X(\cdot, \tau)$ has no branch points on $\overline{B}$ for $\tau \in B_\delta(\bar{\tau}) \cap K(f)$, where we set $B_\delta(\bar{\tau}) := B_\delta^N(\bar{\tau}) \cap \mathbb{R}^N$. Then there exists some $\bar{\delta} \in (0, \delta]$ and some constant C depending on Γ, $\bar{\tau}$ and $\bar{\delta}$ only such that there holds*

$$\left|(KE)^\tau(w)\right| \leqslant C \sum_{k=1}^{N+3} \left|w - e^{i\tau_k}\right|^{-2+\alpha} \quad \forall w \in B, \tag{29}$$

for any $\tau \in K(f) \cap B_{\bar{\delta}}(\bar{\tau})$, with $\alpha := 2\min\{\rho_j^k - \rho_{j-1}^k \mid j = 1, \ldots, p_k, k = 1, \ldots, N+3\} > 0$ and $\rho_0^k := -1$ for each k.

5. The Schwarz operators A^τ and $\dot{A}^\tau$ for $\tau \in K(\tilde{f})$

We set

$$C_0^2(B) := \left\{\varphi \in C^2(B) \cap C^0(\overline{B}) \mid \varphi|_{\partial B} \equiv 0\right\}$$

and consider the Schwarz operator $A^\tau := -\Delta + 2(KE)^\tau$, for $\tau \in K(\tilde{f})$, on the domain

$$\mathrm{Dom}(A^\tau) := \left\{\varphi \in C^2(B) \cap \mathring{H}^{1,2}(B) \mid A^\tau(\varphi) \in L^2(B)\right\}, \tag{30}$$

and the minimal Schwarz operator $\dot{A}^\tau$ and minimal Laplace operator $\dot{\Delta}$ on the domain $H^{2,2}(B) \cap C_0^2(B)$. Using estimate (25) one can prove assertion (3.11) in [14] (see Proposition 2.1 in [19] or Section 7.1 in [20]) which reads:

Proposition 1. *For any $\varphi \in H^{2,2}(B) \cap C_0^2(B)$ and any $\tau \in K(\tilde{f})$ there holds*

$$\left|(KE)^\tau \varphi(w)\right| \leqslant c(\tau, \alpha) \sum_{k=1}^{N+3} \left|w - e^{i\tau_k}\right|^{-1+\alpha/2} \|\dot{\Delta}\varphi\|_{L^2(B)} \quad \forall w \in B. \tag{31}$$

Now in Proposition 2.2 in [19] the author proved that $H^{2,2}(B) \cap C_0^2(B)$ is densely contained in $H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ w.r.t. the $H^{2,2}(B)$-norm, which implies that estimate (31) extends onto $H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ for any $\tau \in K(\tilde{f})$. Using these results the author achieved in [19] (or Section 7.1 in [20]) that there holds for any $\varphi \in H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ and any $\tau \in K(\tilde{f})$:

$$\left\|2(KE)^\tau \varphi\right\|_{L^2(B)} \leqslant \frac{1}{2} \|\Delta\varphi\|_{L^2(B)} + c\|\varphi\|_{L^2(B)}, \tag{32}$$

for some constant $c = c(\tau)$ that only depends on τ. Using this estimate the author proved in [19] that $\mathrm{Dom}(\bar{\Delta}) = \mathrm{Dom}(\bar{A^\tau}) = H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$, $\forall \tau \in K(\tilde{f})$. Moreover in [19] or Section 7.2 in [20] the author showed the essential self-adjointness of $\dot{\Delta}$ w.r.t. $\langle \cdot, \cdot \rangle_{L^2(B)}$, i.e. $\bar{\Delta} = (\dot{\Delta})^*$. Together with estimate (32), for $\tau \in K(\tilde{f})$, one can infer from Theorem 4.4 in [21], p. 288, that also $\dot{A}^\tau = -\dot{\Delta} + 2(KE)^\tau$ is essentially self-adjoint w.r.t. $\langle \cdot, \cdot \rangle_{L^2(B)}$, $\forall \tau \in K(\tilde{f})$.

Furthermore it is proved in [19] that A^τ is symmetric, i.e. $A^\tau \subset (A^\tau)^*$, and thus closable in $L^2(B)$. Hence, together with the fact that $\mathrm{Dom}(A^\tau)$ is densely contained in $L^2(B)$ one can derive by twice application of Theorem 5.29 in [21], p. 168, that $(A^\tau)^*$ is densely defined in $L^2(B)$ and closed, $(A^\tau)^{**} = \bar{A}^\tau$ and $(A^\tau)^* = \overline{(A^\tau)^*} = ((A^\tau)^*)^{**}$, $\forall \tau \in K(\tilde{f})$. Combining the above results with Theorem 5.29 in [21], p. 168, the author achieved in [19] or Section 7.2 in [20]:

Theorem 5. $(\dot{A}^\tau)^* = \overline{\bar{A}^\tau} = \bar{A}^\tau = (A^\tau)^*$ *are self-adjoint operators with domain* $H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$, $\forall \tau \in K(\tilde{f})$.

Now we will denote $S\mathring{H}^{1,2}(B) := \{\varphi \in \mathring{H}^{1,2}(B) \mid \|\varphi\|_{L^2(B)} = 1\}$, and analogously $S(H^{2,2}(B) \cap \mathring{H}^{1,2}(B))$ and $S\,\mathrm{Dom}(A^\tau)$. Moreover we consider the bilinear form

$$\mathcal{L}^\tau(\varphi, \psi) := \int_B \nabla\varphi \cdot \nabla\psi + 2(KE)^\tau \varphi\psi \, \mathrm{d}w,$$

for $\varphi, \psi \in \mathring{H}^{1,2}(B)$, and denote $J^\tau(\varphi) := \mathcal{L}^\tau(\varphi, \varphi)$. We fix some $\tau \in K(\tilde{f})$ and $p^* \in (1, \frac{2}{2-\alpha})$ arbitrarily and abbreviate $A := A^\tau$, $\mathcal{L} := \mathcal{L}^\tau$ and $J := J^\tau$. In Section 4 of [19] or Chapter 8 in [20] the author proved by Ehrling's interpolation lemma that there exists some constant $C(p^*)$ such that

$$J(\varphi) \geqslant \frac{1}{2} \int_B |\nabla\varphi|^2 \, \mathrm{d}w - C(p^*)\|KE\|_{L^{p^*}(B)} \quad \forall \varphi \in S\mathring{H}^{1,2}(B). \tag{33}$$

Combining this result with L^2-regularity theory, Theorem 8.13 in [5], Theorem 5 and well known ideas of spectral theory the author achieved in [19] or Chapter 8 in [20]:

Theorem 6. *The spectra of A and $\bar{A}$ coincide, are discrete and accumulate only at ∞, thus their eigenspaces are finite dimensional. Furthermore there holds for their common smallest eigenvalue*:

$$\lambda_{\min}(A) = \inf_{S\,\mathrm{Dom}(A)} J = \inf_{S\mathring{H}^{1,2}(B)} J = \inf_{S(H^{2,2}(B)\cap\mathring{H}^{1,2}(B))} J = \lambda_{\min}(\bar{A}). \tag{34}$$

Thus we may abbreviate $\lambda_{\min} := \lambda_{\min}(A) = \lambda_{\min}(\bar{A})$. Moreover, applying Harnack's inequality, Theorem 8.20 in [5], to the modulus $|\varphi^*|$ of some eigenfunction $\varphi^* \in ES_{\lambda_{\min}}(\bar{A}) \subset H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$, with $\|\varphi^*\|_{L^2(B)} = 1$, the author finally derived from the above theorem (see Theorem 1.2 in [19]):

Theorem 7.

(i) *For an eigenfunction $\varphi^* \in ES_{\lambda_{\min}}(\bar{A})$ there holds $|\varphi^*| > 0$ on B and therefore*:

$$\dim ES_{\lambda_{\min}}(\bar{A}) = \dim ES_{\lambda_{\min}}(A) = 1. \tag{35}$$

(ii) *Especially an eigenfunction $\varphi^* \in ES_{\lambda_{\min}}(A)$ satisfies $|\varphi^*| > 0$ on B.*

6. The component $K(\tilde{f})^1_{\tau*}$ of $K(\tilde{f})$ is a closed C^ω-curve

From this section on we assume that Γ is an extreme, simple, closed polygon which is not contained in a plane. We shall prove the main result, Theorem 1, by contradiction. Thus we assume the existence of some $X^* \in \mathcal{M}_s(\Gamma)$ and some sequence $\{X^n\} \subset \mathcal{M}(\Gamma)$ with

$$X^n \longrightarrow X^* \quad \text{in } C^0(\overline{B}, \mathbb{R}^3). \tag{36}$$

Thus by means of (3) the points $\tau^* := \psi^{-1}(X^*) \in K_s(f)$ and $\tau^n := \psi^{-1}(X^n) \in K(f)$ satisfy

$$\tau^n \longrightarrow \tau^* \quad \text{in } K(f), \tag{37}$$

where we introduced the notation

$$K_s(f) := \psi^{-1}(\mathcal{M}_s(\Gamma)).$$

By $K(f) \subset K(\tilde{f})$ τ^* would be a non-isolated critical point of $\tilde{f}$ and therefore $\mathrm{rank}(D^2 \tilde{f}(\tau^*)) \leqslant N - 1$. Moreover we know that $X(\cdot, \tau^*) \equiv \tilde{\psi}(\tau^*)$ coincides with $\psi(\tau^*) = X^* \in \mathcal{M}_s(\Gamma)$ by Corollary 3. Hence, we have $\kappa(\tau^*) = 0$ and could conclude now by Heinz' formula (12) $\dim \mathrm{Ker}\, A^{X(\cdot, \tau^*)} \geqslant 1$, thus 0 would be an eigenvalue of $A^{\tau^*} := A^{X(\cdot, \tau^*)}$. Moreover we know together with Lemma 2 that there holds

$$J^{\tau^*} := J^{X(\cdot, \tau^*)} \geqslant 0 \quad \text{on } \mathring{H}^{1,2}(B), \tag{38}$$

thus in particular on $\mathrm{Dom}(A^{\tau^*})$. Therefore 0 would even be the smallest eigenvalue of A^{τ^*}. For if there were some negative eigenvalue $\lambda^* < 0$ of A^{τ^*} with some eigenfunction φ^*, we would obtain by the proof of the symmetry of A^{τ} in [19] that

$$J^{\tau^*}(\varphi^*) = \mathcal{L}^{\tau^*}(\varphi^*, \varphi^*) = \langle A^{\tau^*}(\varphi^*), \varphi^* \rangle_{L^2(B)} = \lambda^* \langle \varphi^*, \varphi^* \rangle_{L^2(B)} < 0, \tag{39}$$

which is a contradiction. Hence, together with (35) we would arrive at

$$\dim \mathrm{Ker}\big(A^{\tau^*}\big) \equiv \dim ES_{\lambda_{\min}=0}\big(A^{\tau^*}\big) = 1. \tag{40}$$

In combination with $\kappa(\tau^*) = 0$ we could therefore derive from formula (12) exactly $\mathrm{rank}(D^2 \tilde{f}(\tau^*)) = N - 1$. Hence, there would be a $\delta > 0$ such that

$$\mathrm{rank}\big(D^2 \tilde{f}\big) \geqslant N - 1 \quad \text{on } B_\delta(\tau^*), \tag{41}$$

where we abbreviate $B_\delta(\tau^*)$ for $B_\delta^N(\tau^*) \cap \mathbb{R}^N \subset\subset T$. Due to $\tilde{f} \in C^\omega(T)$ we know that $K(\tilde{f})$ is an analytic set which possesses therefore a locally finite analytic triangulation due to [22], p. 463, and is therefore especially locally connected, such that a combination of (41) and (37) shows that the sequence $\{\tau^n\}$ would approach τ^* along the 1-skeleton $K(\tilde{f})^1$ of the simplicial complex $K(\tilde{f})$. Hence, we would arrive at the

Contradiction hypothesis. If the assertion of the main result, Theorem 1, were wrong, then there would have to exist some point $\tau^* \in K_s(f)$ which is also contained in the closure of the union of 1-simplices of the simplicial complex $K(\tilde{f})$, i.e. $\tau^* \in K_s(f) \cap \overline{(K(\tilde{f})^1 \setminus K(\tilde{f})^0)} \neq \emptyset$.

Now this gives rise to the idea to analyze the following subset Z of the connected component $K(\tilde{f})^1_{\tau^*}$ of the 1-skeleton $K(\tilde{f})^1$ (of $K(\tilde{f})$) that contains τ^*:

$$Z := \big\{ \tau \in K(\tilde{f})^1_{\tau^*} \mid \kappa(\tau) = 0, \mathrm{rank}\big(D^2 \tilde{f}(\tau)\big) = N - 1, J^\tau \geqslant 0 \text{ on } C_c^\infty(B) \big\}.$$

Now we prove the following crucial

Theorem 8. *The set Z ($\neq \emptyset$) is an open and closed subset of $K(\tilde{f})^1_{\tau^*}$, thus $Z = K(\tilde{f})^1_{\tau^*}$.*

Proof. (a) By $\tau^* \in Z$ we know that Z is not empty.

(b) Secondly we derive the "openness" of the condition $\kappa(\tau) = 0$, i.e. we prove: Let $\bar{\tau} \in Z$ be some arbitrarily fixed point, then there exists some $\delta > 0$ such that $\kappa \equiv 0$ on $B_\delta(\bar{\tau}) \cap K(\tilde{f})$. In fact, since we have $\mathrm{rank}\, D^2(\tilde{f})(\bar{\tau}) = N - 1$ there exists some $\delta > 0$ such that $\mathrm{rank}\, D^2(\tilde{f})(\tau) \geqslant N - 1$ for any $\tau \in B_\delta(\bar{\tau})$. Hence, together with Heinz' formula (12) we can conclude that $2\kappa(\tau) \leqslant 1$ for any $\tau \in B_\delta(\bar{\tau}) \cap K(\tilde{f})$. Thus recalling Corollary 4 we achieve in fact $\kappa(\tau) = 0$ for any $\tau \in B_\delta(\bar{\tau}) \cap K(\tilde{f})$.

(c) Next we show that $\kappa(\tau) = 0$ and $J^\tau \geqslant 0$ on $C_c^\infty(B)$ are "closed conditions". To this end we combine the above reasoning with Corollary 4 in order to achieve: Let $\bar{\tau} \in Z$ be some arbitrarily fixed point, then there exists some $\delta > 0$ such that there holds $K(\tilde{f}) \cap B_\delta(\bar{\tau}) = K(f) \cap B_\delta(\bar{\tau})$. In particular, this shows $Z \subset K(f)$. This can be seen as follows: The inclusion "$\supset$" follows from Corollary 3. "$\subset$": By part (b) of the proof we know that there exists some $\delta > 0$ such that there holds $\kappa \equiv 0$ on $K(\tilde{f}) \cap B_\delta(\bar{\tau})$. Now if the assertion were wrong there would have to exist some point $\tau^* \in K(\tilde{f}) \cap B_\delta(\bar{\tau})$ which is contained in $K(\tilde{f}) \setminus K(f)$ and therefore $X(\cdot, \tau^*) \in \widetilde{\mathcal{M}}(\Gamma) \setminus \mathcal{M}(\Gamma)$ on account of Lemma 1 and points (v) and (iii) of Theorem 2. By Corollary 4 this implies that $X(\cdot, \tau^*)$ would have to possess a boundary branch point in contradiction to $\kappa(\tau^*) = 0$. Now the second assertion follows from the first one due to $Z \subset K(\tilde{f})$ by its definition. Now we consider a sequence $\{\tau^n\} \subset Z$ that converges to some point $\hat{\tau} \in K(\tilde{f})^1_{\tau^*}$. On

account of $Z \subset K(f)$ and the closedness of $K(f)$ we have $\tau^n \to \hat{\tau}$ in $K(f)$. Thus by the properties of the points of Z, Corollary 3, (3) and Theorems 3 and 4, i.e. by the closedness of $\mathcal{M}_s(\Gamma)$, we see:

$$X(\cdot, \tau^n) = \psi(\tau^n) \longrightarrow \psi(\hat{\tau}) = X(\cdot, \hat{\tau}) \quad \text{in } \left(\mathcal{M}_s(\Gamma), \|\cdot\|_{C^0(\overline{B})}\right), \tag{42}$$

which means that $\kappa(\hat{\tau}) = 0$ and $J^{\hat{\tau}} \geqslant 0$ on $C_c^\infty(B)$, proving the closedness of the conditions $\kappa(\tau) = 0$ and $J^\tau \geqslant 0$ on $C_c^\infty(B)$.

(d) Now we show the openness of the condition $\text{rank}(D^2 \tilde{f}) = N - 1$. As already used in (41) we achieve for any fixed point $\bar{\tau} \in Z$ the existence of some $\delta > 0$ such that

$$\text{rank}(D^2 \tilde{f}) \geqslant N - 1 \quad \text{on } B_\delta(\bar{\tau}). \tag{43}$$

Now since $K(\tilde{f})_{\tau*}^1$ is a piecewise analytic curve none of its points can be an isolated critical point of $\tilde{f}$, which implies in fact by (43):

$$\text{rank}(D^2 \tilde{f}) \equiv N - 1 \quad \text{on } B_\delta(\bar{\tau}) \cap K(\tilde{f})_{\tau*}^1. \tag{44}$$

Moreover the condition $\text{rank}(D^2 \tilde{f}) = N - 1$ is also closed, for let $\{\tau^n\} \subset Z$ be some sequence converging to some point $\hat{\tau} \in K(\tilde{f})_{\tau*}^1$, then $\text{rank}(D^2 \tilde{f}(\hat{\tau})) \leqslant N - 1$, since otherwise $\hat{\tau}$ would be an isolated critical point of $\tilde{f}$. Inserting this and $\kappa(\hat{\tau}) = 0$ (by (c)) into formula (12) we see that 0 is an eigenvalue of $A^{\hat{\tau}}$. Thus as we also know $J^{\hat{\tau}} \geqslant 0$ on $\text{Dom}(A^{\hat{\tau}})$ by (c) and Lemma 2 we gain as in (39) that 0 is even the smallest eigenvalue of $A^{\hat{\tau}}$ and therefore as in (40):

$$\dim ES_{\lambda_{\min}=0}(A^{\hat{\tau}}) = \dim \text{Ker}(A^{\hat{\tau}}) = 1.$$

Hence, inserting this and $\kappa(\hat{\tau}) = 0$ into Heinz' formula (12) again we achieve exactly $\text{rank}(D^2(\tilde{f})(\hat{\tau})) = N - 1$.

(e) Finally we prove the openness of the stability condition, i.e. of $J^\tau \geqslant 0$ on $C_c^\infty(B)$. To this end let

$$\iota : \left(H^{2,2}(B) \cap \mathring{H}^{1,2}(B), \|\cdot\|_{H^{2,2}(B)}\right) \hookrightarrow \left(L^2(B), \|\cdot\|_{L^2(B)}\right)$$

denote the inclusion, thus $\text{image}(\iota) = \text{Dom}(\bar{A}^\tau)$ for any $\tau \in K(\tilde{f})$,

$$\mathcal{S}\left(H^{2,2}(B) \cap \mathring{H}^{1,2}(B)\right) := \left\{\varphi \in H^{2,2}(B) \cap \mathring{H}^{1,2}(B) \mid \|\varphi\|_{H^{2,2}(B)} = 1\right\}$$

and $\|\cdot\|$ the operator norm for bounded linear operators L mapping $(H^{2,2}(B) \cap \mathring{H}^{1,2}(B), \|\cdot\|_{H^{2,2}(B)})$ into $L^2(B)$, i.e.

$$\|L\| := \sup\left\{\|L(\varphi)\|_{L^2(B)} \mid \varphi \in \mathcal{S}\left(H^{2,2}(B) \cap \mathring{H}^{1,2}(B)\right)\right\}.$$

Now we prove:

For any fixed $\bar{\tau} \in Z$ there exists some $\bar{\delta} > 0$ and some constant $C(\alpha)$ only depending on α, $\bar{\tau}$ and $\bar{\delta}$ such that there holds

$$\left|(KE)^\tau \varphi(w)\right| \leqslant C(\alpha) \sum_{k=1}^{N+3} \left|w - e^{i\tau_k}\right|^{-1+\frac{\alpha}{2}} \|\Delta\varphi\|_{L^2(B)} \qquad \forall w \in B, \tag{45}$$

for any $\tau \in B_{\bar{\delta}}(\bar{\tau}) \cap K(f)$ and any $\varphi \in H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$. In fact, we already derived the existence of some neighborhood $B_\delta(\bar{\tau})$ such that $X(\cdot, \tau)$ is free of branch points on $\overline{B}$ for any $\tau \in B_\delta(\bar{\tau}) \cap K(f)$. Thus we obtain the existence of some $\bar{\delta} \in (0, \delta]$ such that estimate (29) holds for any $\tau \in B_{\bar{\delta}}(\bar{\tau}) \cap K(f)$ with some constant C that does not depend on τ. Applying this in the ending of the proof of Proposition 1, i.e. of Proposition 2.1 in [19], we achieve estimate (45) for any $\tau \in B_{\bar{\delta}}(\bar{\tau}) \cap K(f)$ and any $\varphi \in H^{2,2}(B) \cap C_0^2(B)$ and then even for any $\varphi \in H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ by the fact that $H^{2,2}(B) \cap C_0^2(B)$ is densely contained in $H^{2,2}(B) \cap \mathring{H}^{1,2}(B)$ w.r.t. the $H^{2,2}(B)$-norm, proved in Proposition 1 in [19]. Using estimate (45) we prove now:

Let $\bar{\tau}$ be some arbitrary point of Z. Then there holds for an arbitrary sequence $\{\tau^n\} \subset K(\tilde{f})$ with $\tau^n \to \bar{\tau}$:

$$\left\|\bar{A}^{\tau^n} \circ \iota - \bar{A}^{\bar{\tau}} \circ \iota\right\| \longrightarrow 0 \quad \text{for } n \longrightarrow \infty. \tag{46}$$

Set

$$S_n := \sup\left\{\left\|\left((KE)^{\tau^n} - (KE)^{\bar{\tau}}\right)\varphi\right\|_{L^2(B)} \mid \varphi \in \mathcal{S}\left(H^{2,2}(B) \cap \mathring{H}^{1,2}(B)\right)\right\}$$
$$= \left\|\bar{A}^{\tau^n} \circ \iota - \bar{A}^{\bar{\tau}} \circ \iota\right\|,$$

and let $\{\varepsilon_n\}$ be an arbitrary null-sequence. By the definition of the supremum there exists for each $n \in \mathbb{N}$ some function $\varphi_n \in \mathcal{S}(H^{2,2}(B) \cap \mathring{H}^{1,2}(B))$ that satisfies

$$0 \leqslant S_n - \left\| \left((KE)^{\tau^n} - (KE)^{\bar{\tau}} \right) \varphi_n \right\|_{L^2(B)} < \varepsilon_n. \tag{47}$$

We set $g_n := ((KE)^{\tau^n} - (KE)^{\bar{\tau}})\varphi_n$ for each n. Firstly we infer from the requirement that $\{\tau^n\} \subset K(\tilde{f})$ converges to $\bar{\tau} \in Z$ that $\kappa(\tau^n) = \kappa(\bar{\tau}) = 0$ and that τ^n and $\bar{\tau}$ are contained in $K(f)$ for sufficiently large n on account of parts (b) and (c) of the proof. Thus we can see by Sobolev's embedding theorem due to $2 - \frac{2}{2} = 1$ and $\|\varphi_n\|_{H^{2,2}(B)} = 1$ in combination with Corollary 3, (3), Cauchy's estimates and (24):

$$\begin{aligned}
\left| g_n(w) \right| &\leqslant \|\varphi_n\|_{L^\infty(B)} \left| (KE)^{\tau^n} - (KE)^{\bar{\tau}} \right|(w) \\
&\leqslant const. \left| (KE)^{\tau^n} - (KE)^{\bar{\tau}} \right|(w) \longrightarrow 0 \quad \text{for } n \to \infty,
\end{aligned} \tag{48}$$

pointwise for any $w \in B$. Furthermore on account of the facts that τ^n and $\bar{\tau}$ are contained in $K(f)$ for large n and $\tau^n \to \bar{\tau}$ we can apply estimate (45) and obtain together with $\|\Delta\varphi_n\|_{L^2(B)} \leqslant 1$ the estimate

$$\begin{aligned}
\left| g_n(w) \right| &\leqslant \left| (KE)^{\tau^n}\varphi_n \right|(w) + \left| (KE)^{\bar{\tau}}\varphi_n \right|(w) \\
&\leqslant C(\alpha) \left(\sum_{k=1}^{N+3} \left| w - e^{i\tau_k^n} \right|^{-1+\alpha/2} + \left| w - e^{i\bar{\tau}_k} \right|^{-1+\alpha/2} \right),
\end{aligned} \tag{49}$$

for any $w \in B$ and $n > \overline{N}$, with $\overline{N}$ sufficiently large. Now we verify the requirements of Vitali's theorem applied to $\{g_n\}$. To this end let $E \subset B$ be an arbitrary measurable subset with positive $\mathcal{L}^2$-measure and define $R := \sqrt{\mathcal{L}^2(E)}$. Now $\tau^n \to \bar{\tau}$ implies in particular the existence of some number $d > 0$ such that $\text{dist}(\tau^n, \partial T) > d$, $\forall n \in \mathbb{N}$. Thus we can conclude that there has to exist some $\bar{R} > 0$ such that

$$2\bar{R} < \min_{k=1,\ldots,N+3} \left\{ \left| e^{i\tau_{k+1}^n} - e^{i\tau_k^n} \right|, \left| e^{i\bar{\tau}_{k+1}} - e^{i\bar{\tau}_k} \right| \right\},$$

with $\tau_{N+4} := \tau_1$, uniformly $\forall n \in \mathbb{N}$. Then we obtain the following estimate:

$$\begin{aligned}
\left\| \left| w - e^{i\tau_k^n} \right|^{-1+\alpha/2} \right\|_{L^2(E)}^2 &\leqslant \int\limits_{\bigcup_{j=1}^{N+3} B_R(e^{i\tau_j^n})} \left| w - e^{i\tau_k^n} \right|^{-2+\alpha} dw + \int\limits_{E \setminus \bigcup_{j=1}^{N+3} B_R(e^{i\tau_j^n})} \left| w - e^{i\tau_k^n} \right|^{-2+\alpha} dw \\
&\leqslant (N+2)R^{-2+\alpha}\pi R^2 + 2\pi \int\limits_0^R r^{-2+\alpha} r \, dr + \mathcal{L}^2(E)R^{-2+\alpha} \\
&= \left((N+2)\pi + \frac{2\pi}{\alpha} + 1 \right) R^\alpha \longrightarrow 0
\end{aligned}$$

for $\bar{R} > R \searrow 0$, i.e. for $\mathcal{L}^2(E) \searrow 0$, uniformly for any $n \in \mathbb{N}$ and for any $k = 1, \ldots, N + 3$. Thus by the same estimate for the summands in (49) involving the $\bar{\tau}_k$ and by Minkowski's inequality we achieve finally $\|g_n\|_{L^2(E)} \to 0$ if $\mathcal{L}^2(E) \searrow 0$, uniformly for any $n > \overline{N}$. Hence, together with (48) Vitali's theorem yields $\|g_n\|_{L^2(B)} \to 0$, for $n \to \infty$, and therefore together with (47):

$$0 \leqslant S_n < \|g_n\|_{L^2(B)} + \varepsilon_n \longrightarrow 0 \quad \text{for } n \to \infty,$$

which proves (46). Furthermore using that

$$\bar{A}^\tau \circ \iota : \left(H^{2,2}(B) \cap \mathring{H}^{1,2}(B), \| \cdot \|_{H^{2,2}(B)} \right) \longrightarrow \left(L^2(B), \| \cdot \|_{L^2(B)} \right)$$

are bounded operators due to estimate (32) and by $\text{image}(\iota) = \text{Dom}(\bar{A}^\tau)$ for any $\tau \in K(\tilde{f})$ we can immediately conclude from (46) and Theorem 2.29 on p. 207 in Kato's book [21] that

$$\bar{A}^{\tau^n} \longrightarrow \bar{A}^{\bar{\tau}} \quad \text{in the generalized sense,} \tag{50}$$

if $K(\tilde{f}) \ni \tau^n \to \bar{\tau}$, for an arbitrarily fixed point $\bar{\tau} \in Z$, where we used Kato's terminology in [21], p. 202. Now since $\bar{A}^\tau$ has a discrete spectrum only accumulating at ∞, for any $\tau \in K(\tilde{f})$, by Theorem 6 we can apply Theorem 3.16 on pp. 212–213 in [21] which yields due to (50):

Let $\bar{\tau}$ be some arbitrary point of Z, $\{\tau^n\} \subset K(\tilde{f})$ an arbitrary sequence with $\tau^n \to \bar{\tau}$ and $d \in \mathbb{R} \setminus \mathrm{Spec}(\bar{A}^{\bar{\tau}})$ arbitrarily fixed. Then there holds also $d \in \mathbb{R} \setminus \mathrm{Spec}(\bar{A}^{\tau^n})$ and

$$\dim \bigoplus_{\lambda < d} ES_\lambda(\bar{A}^{\tau^n}) \equiv \dim \bigoplus_{\lambda < d} ES_\lambda(\bar{A}^{\bar{\tau}}), \tag{51}$$

for sufficiently large n. Now together with parts (b) and (d) of the proof and formulas (12), (34) and (35) we can finally see:

For any $\bar{\tau} \in Z$ there is some neighborhood $B_\varepsilon(\bar{\tau})$ such that $J^\tau \geqslant 0$ on $\mathrm{Dom}(A^\tau)$ for any $\tau \in B_\varepsilon(\bar{\tau}) \cap K(\tilde{f})^1_{\tau*}$.

For, suppose the assertion were wrong, i.e. that there exists some sequence $\{\tau^n\} \subset K(\tilde{f})^1_{\tau*}$ with $\tau^n \to \bar{\tau}$ and $\inf_{\mathrm{Dom}(A^{\tau^n})} J^{\tau^n} < 0 \ \forall n \in \mathbb{N}$. Hence, by (34) we achieve:

$$\lambda_{\min}(\bar{A}^{\tau^n}) = \lambda_{\min}(A^{\tau^n}) = \inf_{S\,\mathrm{Dom}(A^{\tau^n})} J^{\tau^n} < 0 \quad \forall n \in \mathbb{N}. \tag{52}$$

Now by parts (b) and (d) we know already that $\bar{\tau}$ possesses some neighborhood $B_\delta(\bar{\tau})$ such that there hold $\kappa(\tau) = 0$ and $\mathrm{rank}(D^2(\tilde{f})(\tau)) = N - 1 \ \forall \tau \in B_\delta(\bar{\tau}) \cap K(\tilde{f})^1_{\tau*}$. Hence, in combination with Heinz' formula (12) we conclude that $\mathrm{Ker}(A^{\tau^n}) \neq \{0\}$, thus by $\mathrm{Dom}(A^{\tau^n}) \subset \mathrm{Dom}(\bar{A}^{\tau^n})$ that $\mathrm{Ker}(\bar{A}^{\tau^n}) \neq \{0\}$ for $n > \bar{n}$ and some sufficiently large $\bar{n}$. Therefore we achieve together with (52):

$$\dim \bigoplus_{\lambda \leqslant 0} ES_\lambda(\bar{A}^{\tau^n}) \geqslant 2 \quad \forall n > \bar{n}. \tag{53}$$

Now we know for $\bar{\tau} \in Z$ by the definition of Z and formula (12) that $\dim \mathrm{Ker}(A^{\bar{\tau}}) = 1$, thus especially that 0 is an eigenvalue of $A^{\bar{\tau}}$, and since Lemma 2 yields $J^{\bar{\tau}} \geqslant 0$ on $\mathrm{Dom}(A^{\bar{\tau}})$ we can conclude as in (39) that 0 is even the smallest eigenvalue of $A^{\bar{\tau}}$. Hence, we infer from formula (35) that

$$\dim ES_{\lambda_{\min}=0}(\bar{A}^{\bar{\tau}}) = \dim ES_{\lambda_{\min}=0}(A^{\bar{\tau}}) = 1. \tag{54}$$

Now on account of $K(\tilde{f}) \ni \tau^n \to \bar{\tau}$ we can apply (51) with $d := (\lambda_2(\bar{A}^{\bar{\tau}}) - \lambda_{\min}(\bar{A}^{\bar{\tau}}))/2 > 0$, which yields together with (53), $\lambda_{\min}(\bar{A}^{\bar{\tau}}) = 0$ and (54):

$$2 \leqslant \dim \bigoplus_{\lambda < d} ES_\lambda(\bar{A}^{\tau^n}) \equiv \dim \bigoplus_{\lambda < d} ES_\lambda(\bar{A}^{\bar{\tau}}) = \dim ES_0(\bar{A}^{\bar{\tau}}) = 1$$

for sufficiently large n, which is a contradiction.

Hence, in fact $Z \neq \emptyset$ is an open and closed subset of the connected set $K(\tilde{f})^1_{\tau*}$, thus $Z = K(\tilde{f})^1_{\tau*}$. $\quad\square$

Next combining this result with the implicit function theorem for real analytic functions, [4] p. 268, we finally achieve

Corollary 6. *The set* $Z = K(\tilde{f})^1_{\tau*}$ *is a* closed *analytic Jordan curve.*

Proof. Firstly we know that $K(\tilde{f})$ is a closed subset of T and therefore also its 1-skeleton $K(\tilde{f})^1$ and its connected component $K(\tilde{f})^1_{\tau*}$. Moreover we know by part (c) of the proof of the above theorem that $Z = K(\tilde{f})^1_{\tau*}$ is contained in $K(f) \subset\subset T$, which yields the closedness of the set Z w.r.t. the standard topology of $\mathbb{R}^N$ and therefore its compactness. Thus together with the fact that the analytic set $K(\tilde{f})$ possesses a locally finite analytic triangulation $Z = K(\tilde{f})^1_{\tau*}$ can only consist of a finite number of consecutive analytic arcs. Now we show that the set Z is not only a piecewise but an entirely analytic curve, i.e. it does not have any "corners". Firstly we fix some point $\bar{\tau} \in Z$ and obtain the existence of some $\delta > 0$ such that $K(\tilde{f})^1_{\tau*} \cap B_\delta(\bar{\tau})$ is connected on account of the local connectedness of $K(\tilde{f})^1_{\tau*}$. Moreover we have $\mathrm{rank}(D^2 \tilde{f}(\bar{\tau})) = N - 1$ due to $\bar{\tau} \in Z$ and derive from the symmetry of $D^2 \tilde{f}(\bar{\tau})$ the existence of a uniquely determined permutation of the coordinates $\tau_1, \ldots, \tau_N$ in T such that there holds $\det D_{\hat{\tau}}(\nabla_{\hat{\tau}}(\tilde{f}))(\hat{\bar{\tau}}, \bar{\tau}_N) \neq 0$, where we denote by $\hat{\tau} := (\tau_1, \ldots, \tau_{N-1})$ the tuple of the first $N - 1$ permuted coordinates. Hence we can choose the above δ that small, depending on $\bar{\tau}$, such that there holds

$$\det D_{\hat{\tau}}(\nabla_{\hat{\tau}}(\tilde{f}))(\hat{\tau}, \tau_N) \neq 0 \quad \forall (\hat{\tau}, \tau_N) = \tau \in B_\delta(\bar{\tau}). \tag{55}$$

Hence, we obtain by the implicit function theorem for analytic functions, [4] p. 268, applied to $\nabla_{\hat{\tau}}\tilde{f} \in C^{\omega}(T, \mathbb{R}^{N-1})$ that

$$M_{\delta}(\bar{\tau}) := \left\{(\hat{\tau}, \tau_N) = \tau \in B_{\delta}(\bar{\tau}) \mid \nabla_{\hat{\tau}}(\tilde{f})(\hat{\tau}, \tau_N) = 0\right\} \tag{56}$$

is a one-dimensional analytic submanifold of $B_{\delta}(\bar{\tau})$, containing $K(\tilde{f})^1_{\tau^*} \cap B_{\delta}(\bar{\tau})$ in particular. Thus we can conclude that $K(\tilde{f})^1_{\tau^*} \cap B_{\delta}(\bar{\tau})$ is a one-dimensional connected analytic manifold, possibly with boundary, which proves that $Z = K(\tilde{f})^1_{\tau^*}$ is a one-dimensional compact connected analytic manifold, possibly with boundary, as the point $\bar{\tau}$ was chosen arbitrarily in Z. Hence, we infer from the classification theorem of one-dimensional compact connected smooth manifolds (see the appendix in [6]) that Z is either homeomorphic to $[0, 1]$ or $\mathbb{S}^1$. Now we suppose $Z \cong [0, 1]$ and consider some boundary point $\bar{\tau} \in \partial Z$. We note that there holds $\nabla_{\hat{\tau}}(\tilde{f})(\bar{\tau}) = 0$ and (55). Now fixing some sufficiently small $\tilde{\delta} \in (0, \delta]$ the implicit function theorem for analytic functions yields the existence of some neighborhood $J := [\bar{\tau}_N - \varepsilon_1, \bar{\tau}_N + \varepsilon_2]$ of $\bar{\tau}_N$, depending on $\tilde{\delta}$, and some C^{ω}-map $g : J \to \mathbb{R}^{N-1}$ such that $g(\bar{\tau}_N) = \hat{\bar{\tau}}$ and

$$\text{graph}(g|_{\mathring{J}}) := \left\{(g(\tau_N), \tau_N) \mid \tau_N \in \mathring{J}\right\} = M_{\tilde{\delta}}(\bar{\tau}). \tag{57}$$

Hence, recalling definition (56) we conclude immediately that

$$\text{graph}(g|_{\mathring{J}}) \supset Z \cap B_{\tilde{\delta}}(\bar{\tau}). \tag{58}$$

Now the continuity and injectivity of $(g(\cdot), \cdot)$ on J implies that $(g(\cdot), \cdot) : \mathring{J} \xrightarrow{\cong} \text{graph}(g|_{\mathring{J}})$ performs a homeomorphism. Hence, since $Z \cap B_{\tilde{\delta}}(\bar{\tau})$ is connected we conclude that $(g(\cdot), \cdot)^{-1}(Z \cap B_{\tilde{\delta}}(\bar{\tau}))$ is connected as well and therefore an interval I. Moreover we infer from $\bar{\tau} \in \partial Z$ and $(g(\bar{\tau}_N), \bar{\tau}_N) = \bar{\tau}$ that $\bar{\tau}_N \in \partial I$. Thus we have either $I \subset (\bar{\tau}_N - \varepsilon_1, \bar{\tau}_N]$ or $I \subset [\bar{\tau}_N, \bar{\tau}_N + \varepsilon_2)$ and we shall assume the first case without loss of generality. Then we infer especially that

$$\frac{\partial \tilde{f}}{\partial \tau_N}\left(g(\tau_N), \tau_N\right) = 0 \quad \forall \tau_N \in I \tag{59}$$

by $Z = K(\tilde{f})^1_{\tau^*}$. Moreover since $\frac{\partial \tilde{f}}{\partial \tau_N}(g(\cdot), \cdot)$ is analytic on J we can conclude by the identity theorem for real analytic functions that (59) extends in fact onto J, i.e.

$$\frac{\partial \tilde{f}}{\partial \tau_N}\left(g(\tau_N), \tau_N\right) = 0 \quad \forall \tau_N \in J.$$

Now together with (56) and (57) this implies firstly $(g(\tau_N), \tau_N) \in K(\tilde{f}) \cap B_{\tilde{\delta}}(\bar{\tau}) \; \forall \tau_N \in \mathring{J}$. Next we know that $K(\tilde{f}) \cap B_{\tilde{\delta}}(\bar{\tau})$ is contained in the one dimensional manifold $M_{\tilde{\delta}}(\bar{\tau})$ implying $K(\tilde{f}) \cap B_{\tilde{\delta}}(\bar{\tau}) = K(\tilde{f})^1 \cap B_{\tilde{\delta}}(\bar{\tau})$, and thus we obtain

$$\text{graph}(g|_{\mathring{J}}) \subset K(\tilde{f})^1_{\tau^*} \cap B_{\tilde{\delta}}(\bar{\tau}) = Z \cap B_{\tilde{\delta}}(\bar{\tau}),$$

where we used that $(g(\bar{\tau}_N), \bar{\tau}_N) = \bar{\tau}$ is contained in the connected component $K(\tilde{f})^1_{\tau^*}$ of τ^* and thus the entire graph of $g|_{\mathring{J}}$. Together with (58) and the definition of I we obtain therefore:

$$\text{graph}(g|_{\mathring{J}}) = Z \cap B_{\tilde{\delta}}(\bar{\tau}) = \text{graph}(g|_I).$$

Hence, we can infer that for any point $\tau_N^2 \in (\bar{\tau}_N, \bar{\tau}_N + \varepsilon_2)$ there would have to exist some point $\tau_N^1 \in I \subset (\bar{\tau}_N - \varepsilon_1, \bar{\tau}_N]$ such that $(g(\tau_N^1), \tau_N^1) = (g(\tau_N^2), \tau_N^2)$, thus especially $\tau_N^1 = \tau_N^2$, which contradicts $\tau_N^1 \leqslant \bar{\tau}_N < \tau_N^2$ and proves in fact $Z \cong \mathbb{S}^1$. $\quad \square$

7. Strict monotonicity of Tomi's function $\mathcal{F}(X^{(\cdot)})$

Now the implicit function theorem (in its C^{ω}-version) yields an analytic regular parametrization $\tilde{\tau} : [0, 2\pi]/(0 \sim 2\pi) \to Z$ of the analytic closed Jordan curve $Z = K(\tilde{f})^1_{\tau^*}$, which corresponds via ψ to the closed path $X^t := X(\cdot, \tilde{\tau}(t))$ of minimal surfaces in $\mathcal{M}_s(\Gamma)$, where we recall that ψ and $\tilde{\psi}$ coincide on $K(f) \supset Z$ by Corollary 3. Following an idea due to Tomi in [27] and [28] we are going to consider the composition of the so-called volume functional $\mathcal{F}$ (up to a factor $\frac{1}{3}$) with this closed path of minimal surfaces X^t, i.e.

$$\mathcal{F}(X^t) := \int_B \langle X_u^t \wedge X_v^t, X^t \rangle \, dw, \tag{60}$$

which we shall call *Tomi's function* and whose existence is guaranteed by estimate (61) below. Just as Tomi did we aim to derive its strict monotonicity on $[0, 2\pi]$, which contradicts $\tilde{\tau}(0) = \tilde{\tau}(2\pi)$ as a result of our *contradiction hypothesis* in Section 6 and thus proves our main theorem.

We fix some $\bar{\tau} \in T$ and $l \in \{1, \ldots, N\}$ arbitrarily and infer from Theorem 2(vii), (15) and (16) that there exists some $\delta > 0$ such that there hold the estimates

$$\left| X_w(w, \tau) \right| \leqslant const.(\delta, \bar{\tau}, k) \left| w - e^{i\tau_k} \right|^\rho, \tag{61}$$

for any $k \in \{1, \ldots, N + 3\}$ and for $k \neq l$:

$$\left| X_{\tau_l}(w, \tau) \right| \leqslant const.(\delta, \bar{\tau}, l, k) \left| w - e^{i\tau_k} \right|^{\rho+1}, \tag{62}$$

$\forall w \in B_\delta(e^{i\bar{\tau}_k}) \cap B$ and $\forall \tau \in B_\delta(\bar{\tau}) := B_\delta^N(\bar{\tau}) \cap \mathbb{R}^N \subset\subset T$, but for $k = l$ only $\left| X_{\tau_l}(w, \tau) \right| \leqslant const.(\delta, \bar{\tau}, l) |w - e^{i\tau_l}|^\rho$, which we shall avoid in the sequel by using (18), thus the estimate

$$\left| X_{\tau_l}(w, \tau) + X_\varphi(w, \tau) \right| \leqslant const.(\delta, \bar{\tau}, l) \left| w - e^{i\tau_l} \right|^{\rho+1}, \tag{63}$$

$\forall w \in B_\delta(e^{i\bar{\tau}_l}) \cap B$ and $\forall \tau \in B_\delta(\bar{\tau})$, where we abbreviate $\rho := \min_{k=1,\ldots,N+3} \rho_1^k \in (-1, 0]$ for the smallest exponent of the ρ_j^k, for $j = 1, \ldots, p_k$ and $k = 1, \ldots, N + 3$. Now we consider the functions

$$q^l(w, \tau) := \left\langle X_u(w, \tau) \wedge X_v(w, \tau), X_{\tau_l}(w, \tau) \right\rangle = \left\langle X_u(w, \tau) \wedge X_v(w, \tau), X_{\tau_l}(w, \tau) + X_\varphi(w, \tau) \right\rangle$$

for $l = 1, \ldots, N$, $w \in \overline{B} \setminus \{e^{i\tau_k}\}$ and $\tau \in T$, where we used that

$$\langle X_u \wedge X_v, X_\varphi \rangle(w) = u\langle X_u \wedge X_v, X_v \rangle(w) - v\langle X_u \wedge X_v, X_u \rangle(w) = 0$$

for any $X \in C^1(B, \mathbb{R}^3)$ and $w \in B$. Moreover we infer from (61)–(63):

$$\left| q^l(w, \tau) \right| \leqslant c(\delta, \bar{\tau}, l, k) \left| w - e^{i\tau_k} \right|^{2\rho}, \tag{64}$$

$\forall w \in B_\delta(e^{i\bar{\tau}_k}) \cap B$, $k = 1, \ldots, N + 3$, and $\forall \tau \in B_\delta(\bar{\tau})$. In the sequel we will denote $\Delta_\sigma(\bar{\tau}) := B \setminus \bigcup_{k=1}^{N+3} \overline{B_\sigma(e^{i\bar{\tau}_k})}$, for $\sigma < \delta$, and prove for any fixed $l = 1, \ldots, N$:

Proposition 2. *The function* $Q^l(\cdot) := \int_B q^l(w, \cdot) \, dw$ *is continuous on* T.

Proof. Firstly we can infer from estimate (64) that the function Q^l exists in any point $\bar{\tau}$ of T. Now we fix some $\sigma \in (0, \delta)$ and $\bar{\tau} \in T$ arbitrarily and introduce the functions $Q_\sigma^l(\tau) := \int_{\Delta_\sigma(\bar{\tau})} q^l(w, \tau) \, dw$, for $\tau \in B_\delta(\bar{\tau}) \subset\subset T$, which are well-defined again due to estimate (64). We have:

$$\left| Q_\sigma^l(\bar{\tau}) - Q_\sigma^l(\tau) \right| \leqslant \int_{\Delta_\sigma(\bar{\tau})} \left| q^l(w, \bar{\tau}) - q^l(w, \tau) \right| dw. \tag{65}$$

By Hilfssatz 1(A) in [13] $X_{\tau_l}(\cdot, \cdot)$ is uniformly continuous on $\overline{\Delta_\sigma(\bar{\tau})} \times \overline{B_{\sigma/2}(\bar{\tau})}$ due to $e^{i\tau_k} \in \overline{B_{\sigma/2}(e^{i\bar{\tau}_k})}$ for $|\bar{\tau} - \tau| \leqslant \frac{\sigma}{2}$, $\forall k \in \{1, \ldots, N + 3\}$. Hence, for every $\varepsilon > 0$ there is some $\varrho(\varepsilon) > 0$ such that $|X_{\tau_l}(w, \bar{\tau}) - X_{\tau_l}(w, \tau)| < \varepsilon$, if $|(w, \bar{\tau}) - (w, \tau)| < \varrho$, i.e. if $|\bar{\tau} - \tau| < \varrho$ uniformly for any $w \in \overline{\Delta_\sigma(\bar{\tau})}$, which means that

$$X_{\tau_l}(\cdot, \tau) \longrightarrow X_{\tau_l}(\cdot, \bar{\tau}) \quad \text{in } C^0\left(\overline{\Delta_\sigma(\bar{\tau})}\right), \tag{66}$$

for $\tau \to \bar{\tau}$. Furthermore we know by Hilfssatz 1(A) in [13] resp. point (vi) of Theorem 2 that $D_{(u,v)}X(\cdot, \cdot)$ is uniformly continuous on $\overline{\Delta_\sigma(\bar{\tau})} \times \overline{B_{\sigma/2}(\bar{\tau})}$ again due to $e^{i\tau_k} \in \overline{B_{\sigma/2}(e^{i\bar{\tau}_k})}$ for $|\bar{\tau} - \tau| \leqslant \frac{\sigma}{2}$, $\forall k \in \{1, \ldots, N + 3\}$. Hence, as above we obtain that

$$D_{(u,v)}X(\cdot, \tau) \longrightarrow D_{(u,v)}X(\cdot, \bar{\tau}) \quad \text{in } C^0(\overline{\Delta_\sigma(\bar{\tau})}), \tag{67}$$

for $\tau \to \bar{\tau}$. Thus combining (66) and (67) we infer $q^l(\cdot, \tau) \to q^l(\cdot, \bar{\tau})$ in $C^0(\overline{\Delta_\sigma(\bar{\tau})})$, which yields together with Lebesgue's convergence theorem and (65):

$$\left| Q_\sigma^l(\bar{\tau}) - Q_\sigma^l(\tau) \right| \leqslant \int_{\Delta_\sigma(\bar{\tau})} \left| q^l(w, \bar{\tau}) - q^l(w, \tau) \right| dw \longrightarrow 0, \tag{68}$$

for $\tau \to \bar\tau$ and any fixed $\sigma < \delta$. Moreover we obtain by estimate (64) for any $\sigma \in (0, \delta)$ and $\tau \in B_\delta(\bar\tau)$:

$$
\begin{aligned}
\left| Q^l(\tau) - Q_\sigma^l(\tau) \right| &= \left| \int_B q^l(w, \tau)\, \mathrm{d}w - \int_{\Delta_\sigma(\bar\tau)} q^l(w, \tau)\, \mathrm{d}w \right| \\
&\leqslant \sum_{k=1}^{N+3} \int_{B_\sigma(\mathrm{e}^{\mathrm{i}\bar\tau_k}) \cap B} \left| q^l(w, \tau) \right| \mathrm{d}w \\
&\leqslant \sum_{k=1}^{N+3} c(\delta, \bar\tau, l, k) \int_{B_\sigma(\mathrm{e}^{\mathrm{i}\bar\tau_k}) \cap B} \left| w - \mathrm{e}^{\mathrm{i}\tau_k} \right|^{2\rho} \mathrm{d}w.
\end{aligned}
\tag{69}
$$

Now we estimate for any $k = 1, \ldots, N+3$ and $\sigma \in (0, \delta)$:

$$
\begin{aligned}
\int_{B_\sigma(\mathrm{e}^{\mathrm{i}\bar\tau_k})} \left| w - \mathrm{e}^{\mathrm{i}\tau_k} \right|^{2\rho} \mathrm{d}w &\leqslant \int_{B_\sigma(\mathrm{e}^{\mathrm{i}\tau_k})} \left| w - \mathrm{e}^{\mathrm{i}\tau_k} \right|^{2\rho} \mathrm{d}w + \int_{B_\sigma(\mathrm{e}^{\mathrm{i}\bar\tau_k}) \setminus B_\sigma(\mathrm{e}^{\mathrm{i}\tau_k})} \left| w - \mathrm{e}^{\mathrm{i}\tau_k} \right|^{2\rho} \mathrm{d}w \\
&\leqslant 2\pi \int_0^\sigma r^{2\rho+1}\, \mathrm{d}r + \left| B_\sigma(\mathrm{e}^{\mathrm{i}\bar\tau_k}) \setminus B_\sigma(\mathrm{e}^{\mathrm{i}\tau_k}) \right| \sigma^{2\rho} \leqslant \frac{\rho+2}{\rho+1} \pi \sigma^{2\rho+2}.
\end{aligned}
$$

Hence, together with (69) and $2\rho + 2 > 0$ we achieve:

$$
\left| Q^l(\tau) - Q_\sigma^l(\tau) \right| \leqslant (N+3)\, const.(\delta, \bar\tau, l) \frac{\rho+2}{\rho+1} \pi \sigma^{2\rho+2} \longrightarrow 0,
\tag{70}
$$

for $\sigma \searrow 0$, uniformly in $\tau \in B_\delta(\bar\tau)$. Now we split:

$$
\left| Q^l(\bar\tau) - Q^l(\tau) \right| \leqslant \left| Q^l(\bar\tau) - Q_\sigma^l(\bar\tau) \right| + \left| Q_\sigma^l(\bar\tau) - Q_\sigma^l(\tau) \right| + \left| Q_\sigma^l(\tau) - Q^l(\tau) \right|
\tag{71}
$$

for any $\sigma \in (0, \delta)$ and $\tau \in B_\delta(\bar\tau)$. We choose some $\varepsilon > 0$ arbitrarily and obtain by (70) the existence of some $\bar\sigma(\varepsilon) \in (0, \delta)$ such that

$$
\left| Q^l(\tau) - Q_{\bar\sigma}^l(\tau) \right| < \frac{\varepsilon}{3} \quad \text{uniformly } \forall \tau \in B_\delta(\bar\tau).
\tag{72}
$$

Next we know by (68) applied to $\sigma := \bar\sigma(\varepsilon)$ that there exists some $\bar\delta(\varepsilon) \in (0, \delta)$ such that

$$
\left| Q_{\bar\sigma}^l(\bar\tau) - Q_{\bar\sigma}^l(\tau) \right| < \frac{\varepsilon}{3} \quad \forall \tau \in B_{\bar\delta}(\bar\tau).
\tag{73}
$$

Hence, combining (71) with $\sigma := \bar\sigma(\varepsilon)$, (72) and (73) we achieve that for any $\varepsilon > 0$ there exists some sufficiently small $\bar\delta(\varepsilon) \in (0, \delta)$ such that $|Q^l(\bar\tau) - Q^l(\tau)| \leqslant 3\frac{\varepsilon}{3} = \varepsilon$, if $\tau \in B_{\bar\delta}(\bar\tau)$, which proves the continuity of Q^l in $\bar\tau$, for the arbitrarily chosen point $\bar\tau \in T$, and thus its continuity on T. $\square$

Due to the analyticity of $\tilde\tau$ and $\frac{\partial}{\partial t} X^t = \sum_{l=1}^N \frac{\mathrm{d}\tilde\tau_l}{\mathrm{d}t} X_{\tau_l}^t$ the above proposition implies in particular that the integral

$$
\Phi_1(t) := \int_{B_1(0)} \left\langle X_u^t \wedge X_v^t, \frac{\partial}{\partial t} X^t \right\rangle \mathrm{d}w
\tag{74}
$$

depends continuously on $t \in [0, 2\pi]/(0 \sim 2\pi)$. Now we are going to prove

Proposition 3. *For $r \nearrow 1$ one has*

$$
\int_{\partial B_r(0)} \left\langle X_\varphi^t \wedge X^t, \frac{\partial}{\partial t} X^t \right\rangle \mathrm{d}s \longrightarrow 0 \quad in\ C^0([0, 2\pi]).
\tag{75}
$$

Proof. We consider for some fixed $l \in \{1, \ldots, N\}$ the functions

$$
\begin{aligned}
h_r^l(\varphi, \tau) &:= \left\langle X_\varphi\left(re^{i\varphi}, \tau\right) \wedge X\left(re^{i\varphi}, \tau\right), X_{\tau_l}\left(re^{i\varphi}, \tau\right)\right\rangle \\
&\equiv \left\langle X_\varphi\left(re^{i\varphi}, \tau\right) \wedge X\left(re^{i\varphi}, \tau\right), X_{\tau_l}\left(re^{i\varphi}, \tau\right) + X_\varphi\left(re^{i\varphi}, \tau\right)\right\rangle,
\end{aligned}
\tag{76}
$$

for $r \in (0, 1)$, $\tau \in T$ and $\varphi \in [0, 2\pi]/(0 \sim 2\pi)$. Again we fix some $\bar{\tau} \in T$ arbitrarily. Firstly we derive a uniform bound for $|X(\cdot, \cdot)|$ on $B \times \overline{B_\varepsilon(\bar{\tau})}$ for some $\varepsilon > 0$. We know that there exists some $\delta > 0$ such that there hold (14) and (61) on $(B_\delta(e^{i\bar{\tau}_k}) \cap B) \times B_\delta(\bar{\tau})$ for $k = 1, \ldots, N + 3$. Moreover we may apply Theorem 2 (vi) to the domain $D := \Delta_{\delta/2}(\bar{\tau})$ which yields a uniform bound $b(\delta, \bar{\tau})$ of $|X_w(\cdot, \cdot)|$ on $\overline{\Delta_\delta(\bar{\tau})} \times \overline{B_\varepsilon(\bar{\tau})}$ for any $\varepsilon \in (0, \frac{\delta}{2})$ due to $e^{i\tau_k} \in B_{\delta/2}(e^{i\bar{\tau}_k})$ for $|\bar{\tau} - \tau| \leqslant \varepsilon < \frac{\delta}{2}$. Hence, we can estimate for some arbitrarily chosen k:

$$
\begin{aligned}
\left|X(0, \tau) - P_k\right| &= \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} X\left(te^{i\tau_k}, \tau\right) \mathrm{d}t\right| \leqslant \int_0^1 \left|DX\left(te^{i\tau_k}, \tau\right)\right| \mathrm{d}t \\
&= \int_0^{1-\delta} 2\left|X_w\left(te^{i\tau_k}, \tau\right)\right| \mathrm{d}t + \int_{1-\delta}^1 2\left|X_w\left(te^{i\tau_k}, \tau\right)\right| \mathrm{d}t \\
&\leqslant 2b(\delta, \bar{\tau})(1 - \delta) + \int_{1-\delta}^1 2c(\delta, \bar{\tau}, k)\left|te^{i\tau_k} - e^{i\tau_k}\right|^\rho \mathrm{d}t \\
&= 2\left(b(\delta, \bar{\tau})(1 - \delta) + c(\delta, \bar{\tau}, k)\frac{\delta^{\rho+1}}{\rho + 1}\right),
\end{aligned}
$$

and therefore

$$
\begin{aligned}
\left|X(w, \tau)\right| &\leqslant \left|X(0, \tau)\right| + \left|\int_0^1 \frac{\mathrm{d}}{\mathrm{d}t} X(tw, \tau) \mathrm{d}t\right| \\
&\leqslant \left|X(0, \tau)\right| + \int_0^1 \left|DX(tw, \tau)\right| |w| \mathrm{d}t \\
&\leqslant |P_k| + 2\left(b(\delta, \bar{\tau})(1 - \delta) + c(\delta, \bar{\tau}, k)\frac{\delta^{\rho+1}}{\rho + 1} + b(\delta, \bar{\tau})\right),
\end{aligned}
$$

for $(w, \tau) \in \overline{\Delta_\delta(\bar{\tau})} \times \overline{B_\varepsilon(\bar{\tau})}$ and an arbitrarily chosen k. Hence, together with (14) on $(B_\delta(e^{i\bar{\tau}_k}) \cap B) \times B_\delta(\bar{\tau})$, for $k = 1, \ldots, N + 3$, and $\rho + 1 > 0$ we achieve the desired uniform bound for $|X(\cdot, \cdot)|$ on $B \times \overline{B_\varepsilon(\bar{\tau})}$ for any $\varepsilon \in (0, \frac{\delta}{2})$. Moreover we obtain by Hilfssatz 1 (A) in [13] that $X_{\tau_l}(\cdot, \cdot)$ is uniformly continuous on $\overline{\Delta_\delta(\bar{\tau})} \times \overline{B_\varepsilon(\bar{\tau})}$ for any $\varepsilon \in (0, \frac{\delta}{2})$, in particular uniformly bounded, due to $e^{i\tau_k} \in B_{\delta/2}(e^{i\bar{\tau}_k})$ for $|\bar{\tau} - \tau| \leqslant \varepsilon < \frac{\delta}{2}$, $\forall k \in \{1, \ldots, N + 3\}$. Hence, together with (62) we have proved the existence of some $\delta > 0$ such that $|X_{\tau_l}(\cdot, \cdot)|$ is uniformly bounded on $(B \setminus B_\delta(e^{i\bar{\tau}_l})) \times \overline{B_\varepsilon(\bar{\tau})}$ for any $\varepsilon \in (0, \frac{\delta}{2})$. Thus taking also (61) and (63) into account we conclude that there holds for any $\varepsilon \in (0, \frac{\delta}{2})$:

$$
\left|h_r^l(\varphi, \tau)\right| \leqslant const.(\delta, \bar{\tau}, l) \sum_{k=1}^{N+3} \left|re^{i\varphi} - e^{i\tau_k}\right|^\rho,
\tag{77}
$$

$\forall r \in (0, 1)$, $\forall \varphi \in [0, 2\pi]$ and $\forall \tau \in B_\varepsilon(\bar{\tau})$, where we abbreviate $\rho := \min_{k=1,\ldots,N+3}\{\rho_1^k\} \in (-1, 0]$ for the smallest exponent of the ρ_j^k for $j = 1, \ldots, p_k$ and $k = 1, \ldots, N + 3$. Now we estimate $|re^{i\varphi} - e^{i\tau_k}|^\rho$ independently of $r \in (0, 1)$. To this end we fix some $\tau \in B_\varepsilon(\bar{\tau})$, $k \in \{1, \ldots, N + 3\}$, $r \in (0, 1)$ and $\varphi \in [0, 2\pi] \setminus \{\tau_k\}$ and choose some $R \in (0, \frac{1}{8})$ arbitrarily. Now there are two possibilities:

(I) There holds $|re^{i\varphi} - e^{i\tau_k}| < R$ or (II) $|re^{i\varphi} - e^{i\tau_k}| \geqslant R$.

Case (I): We consider the angle $\gamma := |\operatorname{angle}(e^{i\tau_k} - e^{i\varphi}, re^{i\varphi} - e^{i\varphi})|$ which depends on the fixed φ only and note that $\gamma \in (\frac{\pi}{4}, \frac{\pi}{2})$ by the requirement of Case (I) and $\varphi \neq \tau_k$. Now we compute

$$\left| re^{i\varphi} - e^{i\tau_k} \right|^2 = (1 - r)^2 + \left| e^{i\varphi} - e^{i\tau_k} \right|^2 - 2(1 - r)\left| e^{i\varphi} - e^{i\tau_k} \right| \cos(\gamma)$$

and consider this expression as a quadratic function of $1 - r$:

$$q(x) := x^2 + y^2 - 2xy \cos(\gamma) = \left(x - y \cos(\gamma) \right)^2 + \left(y \sin(\gamma) \right)^2$$

for $x \in [0, 1]$ and $y := |e^{i\varphi} - e^{i\tau_k}|$. Due to $q \geqslant (y \sin(\gamma))^2$ on $[0, 1]$ we thus conclude

$$\left| re^{i\varphi} - e^{i\tau_k} \right| = \sqrt{q(1 - r)} \geqslant \left| e^{i\varphi} - e^{i\tau_k} \right| \sin(\gamma),$$

and therefore by $\rho \in (-1, 0]$:

$$\left| re^{i\varphi} - e^{i\tau_k} \right|^\rho \leqslant \left| e^{i\varphi} - e^{i\tau_k} \right|^\rho \sin(\gamma)^\rho. \tag{78}$$

Moreover we recall that $|e^{i\varphi} - e^{i\tau_k}| = 2 \sin(\frac{|\varphi - \tau_k|}{2})$. Now by $\sin(\theta) \geqslant \frac{\theta}{2}$ for $\theta \in [0, \frac{\pi}{2}]$ and $\frac{\pi}{2} \geqslant \frac{|\varphi - \tau_k|}{2}$, we have

$$\sin\left(\frac{|\varphi - \tau_k|}{2} \right) \geqslant \frac{|\varphi - \tau_k|}{4}. \tag{79}$$

Furthermore we gain by $\gamma \in (\frac{\pi}{4}, \frac{\pi}{2})$ that $\sin(\gamma) > \frac{1}{\sqrt{2}} > \frac{1}{2}$ and therefore together with (78) and (79):

$$\left| re^{i\varphi} - e^{i\tau_k} \right|^\rho \leqslant \left(2 \sin\left(\frac{|\varphi - \tau_k|}{2} \right) \right)^\rho \sin(\gamma)^\rho < \left(\frac{|\varphi - \tau_k|}{4} \right)^\rho,$$

for every $\varphi \in [0, 2\pi] \setminus \{\tau_k\}$ and $k \in \{1, \ldots, N + 3\}$. On the other hand in Case (II) we have $|re^{i\varphi} - e^{i\tau_k}|^\rho \leqslant R^\rho$. Hence, we achieve in any case the estimate

$$\left| re^{i\varphi} - e^{i\tau_k} \right|^\rho \leqslant \left(\frac{|\varphi - \tau_k|}{4} \right)^\rho + R^\rho, \tag{80}$$

for any $k \in \{1, \ldots, N + 3\}$, and therefore together with (77):

$$\left| h_r^l(\varphi, \tau) \right| \leqslant c(\delta, \bar{\tau}, l) \sum_{k=1}^{N+3} \left(\left(\frac{|\varphi - \tau_k|}{4} \right)^\rho + R^\rho \right), \tag{81}$$

$\forall r \in (0, 1)$, $\forall \varphi \in [0, 2\pi] \setminus \{\tau_j\}_{j=1,\ldots,N+3}$, $\forall \tau \in B_\varepsilon(\bar{\tau})$, yielding a Lebesgue dominating term for the family $\{h_r^l(\cdot, \tau)\}_{r \in (0,1)}$ in $L^1([0, 2\pi])$ for every $\tau \in B_\varepsilon(\bar{\tau})$ on account of $\rho > -1$. Furthermore we conclude from the property $X(e^{i\varphi}, \tau) \in \Gamma_j$ for $\varphi \in [\tau_j, \tau_{j+1}]$, $j = 1, \ldots, N + 3$, $(\tau_{N+4} := \tau_1)$ $\forall \tau \in T$ that $X_\varphi(e^{i\varphi}, \tau) \in \Gamma_j - P_j \equiv \operatorname{Span}(P_{j+1} - P_j)$ and also $X_{\tau_l}(e^{i\varphi}, \tau) \in \operatorname{Span}(P_{j+1} - P_j)$ for $l = 1, \ldots, N$, $\varphi \in (\tau_j, \tau_{j+1})$ and any $\tau \in T$ (see also (4.74) in [14]). Inserting this into (76) we obtain

$$h_r^l(\varphi, \tau) \longrightarrow 0 \equiv h_1^l(\varphi, \tau) \qquad \text{for } r \nearrow 1, \tag{82}$$

pointwise for every $\varphi \in [0, 2\pi] \setminus \{\tau_j\}_{j=1,\ldots,N+3}$ and for any $\tau \in T$. Moreover,

$$\frac{\partial}{\partial r} \left(\langle X_\varphi \wedge X, X_{\tau_l} \rangle \right)(w, \tau) = \langle X_{\varphi r} \wedge X, X_{\tau_l} \rangle(w, \tau) + \langle X_\varphi \wedge X_r, X_{\tau_l} \rangle(w, \tau) + \langle X_\varphi \wedge X, X_{\tau_l r} \rangle(w, \tau),$$

$\forall w \in B$ and for any $\tau \in T$. Hence, by formulas (15)–(17), estimates (61)–(63) and again (80), with ρ replaced by $2\rho - 1$, we achieve

$$\left| \frac{\partial}{\partial r} h_r^l(\varphi, \tau) \right| \leqslant const.(\delta, \bar{\tau}, l) \sum_{k=1}^{N+3} \left| re^{i\varphi} - e^{i\tau_k} \right|^{2\rho - 1}$$

$$\leqslant c(\delta, \bar{\tau}, l) \sum_{k=1}^{N+3} \left(\left(\frac{|\varphi - \tau_k|}{4} \right)^{2\rho - 1} + R^{2\rho - 1} \right), \tag{83}$$

$\forall r \in (0,1)$, $\varphi \in [0,2\pi] \setminus \{\tau_j\}_{j=1,\ldots,N+3}$ and $\forall \tau \in B_\varepsilon(\bar\tau)$, where we fixed some $\varepsilon \in (0, \frac{\delta}{2})$ and $R \in (0, \frac{1}{8})$ arbitrarily. Now we choose some $\varepsilon > 0$ arbitrarily small. One can easily see that for an arbitrary $\tau \in T$ there holds

$$\text{dist}(\tau, \partial T) = \frac{1}{\sqrt{2}} \min_{j=1,\ldots,N+3} \{|\tau_{j+1} - \tau_j|\}.$$

Hence, the union $\bigcup_{k=1}^{N+3} B_s(\tau_k)$ of intervals is disjoint for any $\tau \in B_\varepsilon(\bar\tau)$ if we choose $s \in (0, \frac{\text{dist}(\bar\tau, \partial T) - \varepsilon}{\sqrt{2}})$, and we achieve for those s by (81):

$$\int_{\bigcup_{k=1}^{N+3} B_s(\tau_k)} |h_r^l(\varphi, \tau)| r \, d\varphi \leqslant c(\delta, \bar\tau, l)(N+3) \sum_{k=1}^{N+3} \int_{B_s(\tau_k)} \left(\frac{|\varphi - \tau_k|}{4}\right)^\rho + R^\rho \, d\varphi,$$

$$= c(\delta, \bar\tau, l) 2(N+3)^2 \left(\frac{s^{\rho+1}}{4^\rho(\rho+1)} + sR^\rho\right),$$

$\forall r \in (0,1)$ and uniformly $\forall \tau \in B_\varepsilon(\bar\tau)$. Thus due to $\rho + 1 > 0$ we achieve the existence of some $\bar s(\varepsilon) \in (0, \frac{\text{dist}(\bar\tau, \partial T) - \varepsilon}{\sqrt{2}})$ such that

$$\int_{\bigcup_{k=1}^{N+3} B_{\bar s(\varepsilon)}(\tau_k)} \left|h_r^l(\varphi, \tau)\right| r \, d\varphi < \frac{\varepsilon}{2}, \tag{84}$$

$\forall r \in (0,1)$ and uniformly $\forall \tau \in B_\varepsilon(\bar\tau)$. Moreover we obtain by the mean value theorem, $h_1^l(\varphi, \tau) \equiv 0$, for every $\varphi \in [0,2\pi] \setminus \{\tau_j\}_{j=1,\ldots,N+3}$ and any $\tau \in T$ by (82), and (83):

$$\int_{[0,2\pi]\setminus\bigcup_{k=1}^{N+3} B_{\bar s(\varepsilon)}(\tau_k)} \left|h_r^l(\varphi, \tau)\right| r \, d\varphi = \int_{[0,2\pi]\setminus\bigcup_{k=1}^{N+3} B_{\bar s(\varepsilon)}(\tau_k)} \left|h_1^l(\varphi, \tau) - h_r^l(\varphi, \tau)\right| r \, d\varphi$$

$$< 2\pi \, const.(\delta, \bar\tau, l)(N+3)\left(\left(\frac{\bar s(\varepsilon)}{4}\right)^{2\rho-1} + R^{2\rho-1}\right)(r - r^2)$$

$\forall r \in (0,1)$ and uniformly $\forall \tau \in B_\varepsilon(\bar\tau)$. Thus there exists some radius $\bar r < 1$ (near 1) depending on $\bar s(\varepsilon)$, i.e. on ε, such that

$$\int_{[0,2\pi]\setminus\bigcup_{k=1}^{N+3} B_{\bar s(\varepsilon)}(\tau_k)} \left|h_r^l(\varphi, \tau)\right| r \, d\varphi < \frac{\varepsilon}{2} \tag{85}$$

$\forall r \in (\bar r(\varepsilon), 1)$ and uniformly $\forall \tau \in B_\varepsilon(\bar\tau)$. Hence, combining (84) and (85) we achieve for any $\varepsilon > 0$ the existence of some radius $\bar r(\varepsilon) < 1$ with the property that $\int_0^{2\pi} |h_r^l(\varphi, \tau)| r \, d\varphi < \varepsilon$, for any $r \in (\bar r(\varepsilon), 1)$ and uniformly $\forall \tau \in B_\varepsilon(\bar\tau)$, thus

$$H_r^l := \int_0^{2\pi} h_r^l(\varphi, \cdot) r \, d\varphi \longrightarrow 0 \quad \text{in } C^0\big(B_\varepsilon(\bar\tau)\big), \tag{86}$$

for $r \nearrow 1$. Hence, since $\bar\tau$ was arbitrarily chosen in T we can conclude for any compactly contained subdomain $T' \subset\subset T$ with $Z \subset T'$ that $H_r^l \to 0$ in $C^0(\bar T')$, for $r \nearrow 1$ and for any $l = 1, \ldots, N$. Now noting that

$$\frac{\partial}{\partial t} X^t = \sum_{l=1}^{N} \frac{d\tilde\tau_l}{dt} X_{\tau_l}^t \quad \text{and} \quad \left|\frac{d\tilde\tau}{dt}\right| \leqslant const. \quad \text{on } [0,2\pi]$$

we infer for $r \nearrow 1$:

$$\int_{\partial B_r(0)} \left\langle X_\varphi^t \wedge X^t, \frac{\partial}{\partial t} X^t \right\rangle ds = \sum_{l=1}^{N} \frac{d\tilde\tau_l}{dt} H_r^l\big(\tilde\tau(t)\big) \longrightarrow 0 \quad \text{in } C^0\big([0,2\pi]\big). \qquad \square$$

Moreover we are going to use the integral identity (1.9) in [9] due to Heinz (see Lemma 3.3 in [17] for a proof):

Lemma 5. *For $Y^1, Y^2 \in C^0(\overline{B}, \mathbb{R}^3) \cap H^{1,2}(B, \mathbb{R}^3)$ there holds the formula*

$$\mathcal{F}_{B_r(0)}(Y^1 + Y^2) - \mathcal{F}_{B_r(0)}(Y^1) = 3 \int\limits_{B_r(0)} \langle Y_u^1 \wedge Y_v^1, Y^2 \rangle \, dw$$

$$+ \int\limits_{B_r(0)} \langle 3Y^1 + Y^2, Y_u^2 \wedge Y_v^2 \rangle \, dw + \frac{1}{r} \int\limits_{\partial B_r(0)} \langle Y^1, Y^2 \wedge (Y_\varphi^1 - Y_\varphi^2) \rangle \, ds, \qquad (87)$$

for a.e. $r \in (0, 1)$.

Using

$$\frac{\partial}{\partial t} X^t = \sum_{l=1}^{N} \frac{d\tilde{\tau}_l}{dt} X_{\tau_l}^t$$

we may infer from (66), (67), Lebesgue's convergence theorem and the analyticity of $\tilde{\tau}$ that the functions

$$\Phi_r(t) := \int\limits_{B_r(0)} \left\langle X_u^t \wedge X_v^t, \frac{\partial}{\partial t} X^t \right\rangle dw \qquad (88)$$

are continuous in $t \in [0, 2\pi]/(0 \sim 2\pi)$ for any $r \in (0, 1)$ (compare with (74) for $r = 1$). Together with Theorem 2 (i), (iv), (vi) and Cauchy's estimates we can prove the following connection between these integrals Φ_r and $\mathcal{F}_{B_r(0)}(X^{(\cdot)})$:

Proposition 4. *There holds for a.e. $r \in (0, 1)$ and any $t \in [0, 2\pi]/(0 \sim 2\pi)$:*

$$\frac{d}{dt} \mathcal{F}_{B_r(0)}(X^t) = 3\Phi_r(t) + \frac{1}{r} \int\limits_{\partial B_r(0)} \left\langle X_\varphi^t \wedge X^t, \frac{\partial}{\partial t} X^t \right\rangle ds. \qquad (89)$$

Proof. Without loss of generality we may only consider some arbitrary point $t^* \in (0, 2\pi)$. We obtain by (87) applied to $Y^1 := X^{t^*}$ and $Y^2 := X^t - X^{t^*}$ for an arbitrarily chosen $r \in (0, 1)$ which (87) holds for:

$$\frac{\mathcal{F}_{B_r}(X^t) - \mathcal{F}_{B_r}(X^{t^*})}{t - t^*} = 3 \int\limits_{B_r} \left\langle X_u^{t^*} \wedge X_v^{t^*}, \frac{X^t - X^{t^*}}{t - t^*} \right\rangle dw + \int\limits_{B_r} \left\langle 2X^{t^*} + X^t, \frac{(X^t - X^{t^*})_u}{t - t^*} \wedge (X^t - X^{t^*})_v \right\rangle dw$$

$$+ \frac{1}{r} \int\limits_{\partial B_r} \left\langle X^{t^*}, \frac{X^t - X^{t^*}}{t - t^*} \wedge (2X_\varphi^{t^*} - X_\varphi^t) \right\rangle ds, \qquad (90)$$

with $B_r := B_r(0)$. Firstly we consider the first integral on the right hand side. By Theorem 2 (vi) and by the analyticity of $\tilde{\tau}$ we know that

$$\frac{X^t(w) - X^{t^*}(w)}{t - t^*} \longrightarrow \frac{\partial}{\partial t} X^{t^*}(w) \quad \text{pointwise } \forall w \in B \qquad (91)$$

and $t \to t^*$. Now combining (66) for some $\sigma < \frac{1-r}{2}$ with the analyticity of $\tilde{\tau}$ and applying Cauchy's estimates to the harmonic functions $\frac{\partial}{\partial t} X^t - \frac{\partial}{\partial t} X^{t^*}$ we achieve:

$$\frac{\partial}{\partial t} X^t \longrightarrow \frac{\partial}{\partial t} X^{t^*} \quad \text{in } C^1(\overline{B_r(0)}) \qquad (92)$$

for $t \to t^*$, which implies in particular together with the mean value theorem:

$$\left\| \frac{X^t - X^{t^*}}{t - t^*} \right\|_{C^0(\overline{B_r(0)})} \leqslant \sup_{(t^*-h, t^*+h)} \left\| \frac{\partial}{\partial t} X^t \right\|_{C^0(\overline{B_r(0)})} \leqslant const.(r, h), \qquad (93)$$

for some sufficiently small chosen $h > 0$ and $|t - t^*| < h$. Hence, recalling (91) we infer by Lebesgue's convergence theorem:

$$\int_{B_r} \left\langle X_u^{t^*} \wedge X_v^{t^*}, \frac{X^t - X^{t^*}}{t - t^*} \right\rangle \mathrm{d}w \longrightarrow \int_{B_r} \left\langle X_u^{t^*} \wedge X_v^{t^*}, \frac{\partial}{\partial t} X^{t^*} \right\rangle \mathrm{d}w = \Phi_r(t^*), \tag{94}$$

for $t \to t^*$. Now we examine the second integral in (90). Using that $X^t \equiv \tilde{\psi}(\tilde{\tau}(t)) = \psi(\tilde{\tau}(t))$ due to $Z \subset K(f)$ and Corollary 3 we have by Theorem 2(i) and the analyticity of $\tilde{\tau}$ that $X^t \to X^{t^*}$ in $C^0(\overline{B})$ for $t \to t^*$. Thus together with Cauchy's estimates applied to $X^t - X^{t^*}$ we achieve:

$$X^t \longrightarrow X^{t^*} \quad \text{in } C^1\big(\overline{B_r(0)}\big) \tag{95}$$

for $t \to t^*$. Moreover we infer from (92) together with the mean value theorem:

$$\left\| \frac{X_u^t - X_u^{t*}}{t - t^*} \right\|_{C^0(\overline{B_r(0)})} \leqslant \sup_{(t^*-h,t^*+h)} \left\| \frac{\partial}{\partial t} X_u^t \right\|_{C^0(\overline{B_r(0)})} \leqslant const.(r, h), \tag{96}$$

for some sufficiently small chosen $h > 0$ and $|t - t^*| < h$. Hence, together with (95) we achieve by Lebesgue's convergence theorem:

$$\int_{B_r} \left\langle 2X^{t^*} + X^t, \frac{(X^t - X^{t^*})_u}{t - t^*} \wedge \big(X^t - X^{t^*}\big)_v \right\rangle \mathrm{d}w \longrightarrow 0 \tag{97}$$

for $t \to t^*$. Finally we examine the third integral in (90). We deduce from (95) especially: $X_\varphi^t \to X_\varphi^{t^*}$ in $C^0(\partial B_r(0))$ for $t \to t^*$. Thus together with (91) and (93) we infer again by Lebesgue's convergence theorem:

$$\int_{\partial B_r} \left\langle X^{t^*}, \frac{X^t - X^{t^*}}{t - t^*} \wedge \big(2X_\varphi^{t^*} - X_\varphi^t\big) \right\rangle \mathrm{d}s \longrightarrow \int_{\partial B_r} \left\langle X^{t^*}, \frac{\partial}{\partial t} X^{t^*} \wedge X_\varphi^{t^*} \right\rangle \mathrm{d}s$$

$$= \int_{\partial B_r} \left\langle X_\varphi^{t^*} \wedge X^{t^*}, \frac{\partial}{\partial t} X^{t^*} \right\rangle \mathrm{d}s,$$

for $t \to t^*$. Now combining this with (90), (94) and (97) we see indeed that

$$\lim_{t \to t^*} \frac{\mathcal{F}_{B_r}(X^t) - \mathcal{F}_{B_r}(X^{t^*})}{t - t^*}$$

exists and coincides with the right hand side of (89) for any $t^* \in (0, 2\pi)$, thus $\forall t \in [0, 2\pi]/(0 \sim 2\pi)$, and for a.e. $r \in (0, 1)$. $\quad \square$

In the sequel we have to examine some notions considered in [26] in order to use Sauvigny's result, Satz 2 in [26], correctly:

Definition 2. Let $X \in \mathcal{M}(\Gamma)$ be a fixed immersed minimal surface. For any map $Y \in C^0(B, \mathbb{R}^3)$ we consider its normal component w.r.t. X:

$$Y^* := \langle Y, \xi \rangle \xi = Y - \frac{1}{E}\big(\langle Y, X_u \rangle X_u + \langle Y, X_v \rangle X_v\big) \quad \text{on } B,$$

where ξ denotes the unit normal field of X, as defined in (21).

Furthermore we have to compare our definition of the quadratic form J^X assigned to a minimal surface X with the following one, considered in [26]:

Definition 3. Let $X \in \mathcal{M}(\Gamma)$ be a fixed immersed minimal surface. For any $\phi \in C^1(B, \mathbb{R}^3) \cap H^{1,2}(B, \mathbb{R}^3)$ we define:

$$I^X(\phi) := \int_B \big|(\phi_u)^*\big|^2 + \big|(\phi_v)^*\big|^2 + \frac{2}{E}\big(\langle \phi_u, X_u \rangle \langle \phi_v, X_v \rangle - \langle \phi_u, X_v \rangle \langle \phi_v, X_u \rangle\big) \mathrm{d}w.$$

Definition 4. We term the normal space of an immersed minimal surface $X \in \mathcal{M}(\Gamma)$

$$\mathcal{N}_X := \left\{ \phi \in C^1(B, \mathbb{R}^3) \cap C^0(\overline{B}, \mathbb{R}^3) \cap \mathring{H}^{1,2}(B, \mathbb{R}^3) \mid \phi \parallel \xi \text{ on } \overline{B} \right\}.$$

As pointed out on p. 173 in [26] we are going to prove now

Lemma 6. Let $X \in \mathcal{M}(\Gamma)$ be an immersed minimal surface. For any $\phi \in \mathcal{N}_X$ there holds $\varphi := \langle \phi, \xi \rangle \in \mathring{H}^{1,2}(B) \cap C^0(\overline{B})$ and $I^X(\phi) = J^X(\varphi)$.

Proof. We have $\phi = \phi^* = \langle \phi, \xi \rangle \xi$ and therefore $\phi_u = \langle \phi, \xi \rangle_u \xi + \langle \phi, \xi \rangle \xi_u$ on B. Thus recalling the fundamental equations $\xi_u \wedge \xi_v = K X_u \wedge X_v = K E \xi$ and $|\xi_u| = |\xi_v|$, $\langle \xi_u, \xi_v \rangle = 0$ on B (see Lemma 1 in [24]) we obtain $\langle \xi_u, \xi \rangle \equiv 0 \equiv \langle \xi_v, \xi \rangle$ on B in particular and conclude that $(\phi_u)^* = \langle \phi, \xi \rangle_u \xi$ on B, whence $|(\phi_u)^*|^2 = |\langle \phi, \xi \rangle_u|^2 \equiv |\varphi_u|^2$ and analogously $|(\phi_v)^*|^2 = |\varphi_v|^2$, yielding

$$\left| (\phi_u)^* \right|^2 + \left| (\phi_v)^* \right|^2 = |\nabla \varphi|^2 \quad \text{on } B, \tag{98}$$

which implies especially $\varphi \in \mathring{H}^{1,2}(B) \cap C^0(\overline{B})$ by $\phi \in \mathcal{N}_X$ and $\xi \in C^0(\overline{B}, \mathbb{R}^3) \cap C^1(B, \mathbb{R}^3)$. A brief computation yields

$$\langle \phi_u, X_u \rangle = \langle \phi, \xi \rangle \langle \xi_u, X_u \rangle \equiv -\varphi L, \qquad \langle \phi_v, X_v \rangle = \langle \phi, \xi \rangle \langle \xi_v, X_v \rangle \equiv -\varphi N,$$

and

$$\langle \phi_u, X_v \rangle = \langle \phi, \xi \rangle \langle \xi_u, X_v \rangle \equiv -\varphi M = \langle \phi_v, X_u \rangle \quad \text{on } B,$$

where we have used the notation in [2], p. 17. Therefore we arrive at:

$$\frac{2}{E} \left(\langle \phi_u, X_u \rangle \langle \phi_v, X_v \rangle - \langle \phi_u, X_v \rangle \langle \phi_v, X_u \rangle \right) = \frac{2}{E} (LN - M^2) \varphi^2 = 2KE\varphi^2,$$

(see p. 19 in [2]). Thus together with (98) we have proved $I^X(\phi) = J^X(\varphi)$. □

Theorem 9. $D^2 \tilde{f}(\tau)$ is positive semidefinite for any $\tau \in Z$.

Proof. We fix some arbitrary point $\bar{\tau}$ of Z. As in [26], pp. 174–182, we consider for an arbitrarily fixed vector $\alpha \in \mathbb{R}^N$ the following family of harmonic surfaces $Y(\cdot, \varepsilon) := X(\cdot, \bar{\tau} + \varepsilon\alpha)$ on $\overline{B}$, for $\varepsilon \in (-\varepsilon_0, \varepsilon_0)$, where $\varepsilon_0 > 0$ is chosen sufficiently small. By Satz 1 in [26] we know that

$$\phi := \left(\frac{\partial}{\partial \varepsilon} Y(\cdot, \varepsilon)|_{\varepsilon=0} \right)^* \in \mathcal{N}_{X(\cdot, \bar{\tau})}$$

and

$$\frac{d^2}{d\varepsilon^2} \tilde{f}(\bar{\tau} + \varepsilon\alpha)|_{\varepsilon=0} \equiv \frac{d^2}{d\varepsilon^2} \mathcal{D}(Y(\cdot, \varepsilon))|_{\varepsilon=0} \geqslant 2I^{X(\cdot, \bar{\tau})}(\phi). \tag{99}$$

Moreover we know that $J^{\bar{\tau}} \geqslant 0$ on $\mathring{H}^{1,2}(B)$ by definition of Z and Lemma 2. We may apply this to the function $\varphi := \langle \phi, \xi \rangle$ since Lemma 6 guarantees that $\langle \phi, \xi \rangle \in \mathring{H}^{1,2}(B) \cap C^0(\overline{B})$. Hence, we conclude by $I^X(\phi) = J^X(\varphi)$ and (99):

$$\langle \alpha, D^2 \tilde{f}(\bar{\tau})\alpha \rangle = \frac{d^2}{d\varepsilon^2} \tilde{f}(\bar{\tau} + \varepsilon\alpha)|_{\varepsilon=0} \geqslant 2J^{\bar{\tau}}(\varphi) \geqslant 0, \quad \text{for any } \alpha \in \mathbb{R}^N. \qquad □$$

Now we are able to prove

Theorem 10. There holds $\mathcal{F}(X^{(\cdot)}) \in C^1([0, 2\pi])$ with

$$\frac{d}{dt} \mathcal{F}(X^t) = 3\Phi_1(t) \quad \text{for } t \in [0, 2\pi]. \tag{100}$$

In particular, $\mathcal{F}(X^{(\cdot)})$ is strictly monotonic on $[0, 2\pi]/(0 \sim 2\pi)$.

Proof. As $\operatorname{trace}(\tilde{\tau}) = K(\tilde{f})^1_{\tau^*}$ we have $\nabla \tilde{f}(\tilde{\tau}(t)) \equiv 0$ implying $\tilde{f}(\tilde{\tau}(t)) \equiv const.$, where we used that $K(\tilde{f})^1_{\tau^*} = Z$ is an analytic curve, and therefore:

$$0 \equiv \frac{d^2}{dt^2} \tilde{f}(\tilde{\tau}(t)) = \left\langle \frac{d\tilde{\tau}}{dt}, D^2(\tilde{f})(\tilde{\tau}(t)) \frac{d\tilde{\tau}}{dt} \right\rangle, \tag{101}$$

on $[0, 2\pi]/(0 \sim 2\pi)$. Moreover Theorem 9 yields the positive semidefiniteness of $D^2(\tilde{f})(\tilde{\tau}(t)) \ \forall t \in [0, 2\pi]$. Together with the symmetry of $D^2(\tilde{f})(\tilde{\tau}(t))$ we achieve the existence of a symmetric root of $D^2(\tilde{f})(\tilde{\tau}(t))$, i.e. there exists some symmetric matrix $R(t)$ with $D^2(\tilde{f})(\tilde{\tau}(t)) = R(t) \cdot R(t)$, which yields together with (101):

$$0 \equiv \left\langle \frac{d\tilde{\tau}}{dt}, D^2(\tilde{f})(\tilde{\tau}(t)) \frac{d\tilde{\tau}}{dt} \right\rangle = \left\langle \frac{d\tilde{\tau}}{dt}, R(t)^\top \cdot R(t) \frac{d\tilde{\tau}}{dt} \right\rangle = \left| R(t) \frac{d\tilde{\tau}}{dt} \right|^2,$$

$\forall t \in [0, 2\pi]$. Hence, we arrive at

$$D^2(\tilde{f})(\tilde{\tau}(t)) \frac{d\tilde{\tau}}{dt} = R(t) \cdot R(t) \frac{d\tilde{\tau}}{dt} \equiv R(t) 0 = 0 \quad \forall t \in [0, 2\pi].$$

Thus on account of Satz 1 in [14] we can conclude that

$$\left\langle \xi^t, \sum_{k=1}^{N} \frac{d\tilde{\tau}_k}{dt} X^t_{\tau_k} \right\rangle \in \operatorname{Ker}(A^{\tilde{\tau}(t)}), \tag{102}$$

with $\xi^t := X^t_u \wedge X^t_v / |X^t_u \wedge X^t_v|$. Now due to $\kappa(\tilde{\tau}(\cdot)) \equiv 0$ by definition of Z we infer from Corollary 2 that the functions $\{X^t_{\tau_l}\}_{l \in \{1,\dots,N\}}$ are linearly independent on B for any $t \in [0, 2\pi]$. Hence, by the regularity of the parametrization $\tilde{\tau}$ of Z, i.e. by $d\tilde{\tau}/dt \neq 0$ on $[0, 2\pi]$, we gain that

$$\sum_{k=1}^{N} \frac{d\tilde{\tau}_k}{dt} X^t_{\tau_k} \not\equiv 0 \quad \text{on } B, \tag{103}$$

$\forall t \in [0, 2\pi]$. Furthermore in [14] Heinz assigned to every $\tau \in K(\tilde{f})$ the linear map

$$C^\tau : V^\tau := \left\{ \sum_{k=1}^{N} \alpha_k X_{\tau_k}(\cdot, \tau) \ \middle| \ (\alpha_1, \dots, \alpha_N) \in \operatorname{Ker}(D^2 \tilde{f}(\tau)) \right\} \longrightarrow \operatorname{Ker}(A^\tau),$$

defined by $Y \mapsto \langle \xi(\cdot, \tau), Y \rangle$. By (5.7), (5.7') and (5.17) in [14] we know the formula

$$\dim \operatorname{Ker}(C^\tau) = 2\kappa(\tau) - \sharp(\{\text{branch points of } X(\cdot, \tau)\} \cap \{e^{i\tau_l}\}_{l=1,\dots,N}),$$

for any $\tau \in K(\tilde{f})$. Now by $\kappa(\tilde{\tau}(t)) = 0$ we infer $\dim \operatorname{Ker}(C^{\tilde{\tau}(t)}) = 0$, i.e. that $C^{\tilde{\tau}(t)}$ is injective $\forall t \in [0, 2\pi]$, which implies by (102), (103) and $\lambda_{\min}(A^{\tilde{\tau}(t)}) = 0$:

$$\left\langle \xi^t, \frac{\partial}{\partial t} X^t \right\rangle = \left\langle \xi^t, \sum_{k=1}^{N} \frac{d\tilde{\tau}_k}{dt} X^t_{\tau_k} \right\rangle \in ES_{(\lambda_{\min}=0)}(A^{\tilde{\tau}(t)}) \setminus \{0\}$$

$\forall t \in [0, 2\pi]$. Hence, we infer from Theorem 7(ii) and $|X^t_u \wedge X^t_v| = \frac{1}{2}|DX^t|^2 > 0$ on B:

$$\left| \left\langle X^t_u \wedge X^t_v, \frac{\partial}{\partial t} X^t \right\rangle \right| > 0 \quad \text{on } B, \ \forall t \in [0, 2\pi]. \tag{104}$$

Now we choose some arbitrary sequence of radii $r_n \nearrow 1$ such that formula (89) holds for each r_n and conclude together with (88) that

$$\Phi_{r_n}(t) \leqslant \Phi_{r_{n+1}}(t) \quad \text{or} \quad \Phi_{r_n}(t) \geqslant \Phi_{r_{n+1}}(t) \quad \forall t \in [0, 2\pi], \tag{105}$$

$\forall n \in \mathbb{N}$. Lebesgue's convergence theorem guarantees that $\Phi_{r_n}(t) \to \Phi_1(t)$ pointwise for any $t \in [0, 2\pi]$. Since we know by (74) and (88) that the functions Φ_r are continuous on $[0, 2\pi]$ for any $r \in (0, 1)$ we can apply Dini's theorem

to the monotonic sequence in (105) yielding $\Phi_{r_n} \to \Phi_1$ in $C^0([0, 2\pi])$ for $n \to \infty$. If we insert also the convergence (75) into formula (89) we thus obtain

$$\frac{d}{dt} \mathcal{F}_{B_{r_n}}\left(X^{(\cdot)}\right) \longrightarrow 3\Phi_1 \quad \text{in } C^0([0, 2\pi]), \tag{106}$$

for $n \to \infty$, which especially implies the equicontinuity of $\{\mathcal{F}_{B_{r_n}}(X^{(\cdot)})\}$ on $[0, 2\pi]$. Furthermore due to estimate (61) we can infer from Lebesgue's convergence theorem that $\mathcal{F}_{B_{r_n}}(X^t) \to \mathcal{F}_B(X^t)$ pointwise $\forall t \in [0, 2\pi]$ and for $n \to \infty$. Hence, combining this with the proof of Arzela–Ascoli's theorem and (106) we obtain

$$\mathcal{F}_{B_{r_n}}\left(X^{(\cdot)}\right) \longrightarrow \mathcal{F}_B\left(X^{(\cdot)}\right) \quad \text{in } C^1([0, 2\pi]),$$

for $n \to \infty$, which shows by (106):

$$\frac{d}{dt} \mathcal{F}_B(X^t) = \lim_{n \to \infty} \frac{d}{dt} \mathcal{F}_{B_{r_n}}(X^t) = 3\Phi_1(t) \quad \forall t \in [0, 2\pi].$$

Now together with (104) and (74) we can conclude that $\frac{d}{dt}\mathcal{F}_B(X^{(\cdot)}) > 0$ or < 0 on $[0, 2\pi]$, i.e. that $\mathcal{F}_B(X^{(\cdot)})$ is strictly monotonic on $[0, 2\pi]$. $\quad\square$

Since we trivially have $\mathcal{F}_B(X^0) = \mathcal{F}_B(X^{2\pi})$ by $\tilde{\tau}(0) = \tilde{\tau}(2\pi)$ in contradiction to the strict monotonicity of $\mathcal{F}_B(X^{(\cdot)})$ on $[0, 2\pi]$ we finally proved our main result under the additional condition that Γ is not contained in a plane.

If the polygon Γ is contained in a plane, then it is well known that $\mathcal{M}(\Gamma)$ consists of a single element on account of the imposed three-point condition. Hence, the first statement of Theorem 1 is proved, and together with the compactness of the set $(\mathcal{M}_s(\Gamma), \|\cdot\|_{C^0(\bar{B})})$, on account of Theorems 3 and 4, we immediately infer its finiteness.

Acknowledgements

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