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Well-posedness theory for geometry-compatible hyperbolic conservation laws on manifolds

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Abstract

Motivated by many applications (geophysical flows, general relativity), we attempt to set the foundations for a study of entropy solutions to non-linear hyperbolic conservation laws posed on a (Riemannian or Lorentzian) manifold. The flux of the conservation laws is viewed as a vector-field on the manifold and depends on the unknown function as a parameter. We introduce notions of entropy solutions in the class of bounded measurable functions and in the class of measure-valued mappings. We establish the well-posedness theory for conservation laws on a manifold, by generalizing both Kruzkov's and DiPerna's theories originally developed in the Euclidian setting. The class of *geometry-compatible* (as we call it) conservation laws is singled out as an important case of interest, which leads to robust L^p estimates independent of the geometry of the manifold. On the other hand, general conservation laws solely enjoy the L^1 contraction property and leads to a unique contractive semi-group of entropy solutions. Our framework allows us to construct entropy solutions on a manifold via the vanishing diffusion method or the finite volume method.

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1. Introduction

The theoretical work on discontinuous solutions to non-linear hyperbolic conservation laws has been restricted so far to problems set in the Euclidian space $\mathbb{R}^n$. Motivated by numerous applications, to geophysical fluid flows (shallow-water equations on the surface on the Earth) and general relativity (Euler–Einstein equations describing neutron stars and black holes) in particular, we attempt in the present paper to set the foundations for a study of weak solutions (including shock waves) to hyperbolic conservation laws posed on a Riemannian or Lorentzian manifold. Recall that Kruzkov's theory [6] deals with equations posed in the Euclidian space $\mathbb{R}^n$ and provides existence, uniqueness, and stability of $L^1 \cap L^\infty$ entropy solutions to non-linear hyperbolic balance laws in several space dimensions. In addition, in the case of conservation laws with "constant flux", depending solely on the conservative variable but not

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on the time and space variables, solutions are known to satisfy the maximum principle as well as the L^1 contraction and total variation diminishing (TVD) properties.

The above stability features play a central role in the theory and numerical analysis of hyperbolic conservation laws. The purpose of this paper is to introduce a suitable generalization of Kruzkov's theory when solutions are defined on a manifold rather than on $\mathbb{R}^n$. An investigation of the interplay between the geometry of manifolds with limited regularity and the analysis of hyperbolic partial differential equations (for non-smooth functions) appears to be particularly challenging. We hope that the framework initiated here will lead to further works on this important subject.

Let $(\mathbf{M}^n, g)$ be an oriented, compact, n-dimensional Riemannian manifold. One of our tasks will be to define a suitable class of conservation laws posed on $\mathbf{M}^n$ to which the stability properties of the Euclidian space could possibly ex tend. It should be observed that, on a manifold, there is no concept of "constant flux" – in the sense that the flux would be "independent" of the space (and time) variable, since as we will see below the flux is a section of the tangent bundle $T\mathbf{M}^n$. Indeed, the flux at a point $x \in \mathbf{M}^n$ belongs to the tangent space $T_x\mathbf{M}^n$ to the manifold at that point. The Euclidian space is special in that it is possible to choose the flux $f(u)$ at an arbitrary point $x \in \mathbb{R}^n$ and to parallel-transport it to the whole space $\mathbb{R}^n$; the vector field generated in this fashion remains constant in x and, therefore, smooth. By contrast, if a tangent vector is selected at one point of the manifold one cannot, in general, associate with it by parallel-transport a unique smooth vector field on the manifold; this is a consequence of the curvature of $\mathbf{M}^n$. As an example, we recall that any vector field on S^2 (the unit sphere in $\mathbb{R}^3$, an important example arising in geophysical flows and motivating our study) must have two critical points, at least, or else must be discontinuous.

An outline of this p aper is as follows. In Section 3, we introduce the class of *geometry-compatible conservation laws* on $\mathbf{M}^n$ based on divergence-free flux, which we single out as a class of particular interest. We then define the notions of entropy pair and entropy solution in the function space $L^\infty(\mathbb{R}_+ \times \mathbf{M}^n)$. In Section 4, we show the existence of entropy solutions via the so-called vanishing diffusion method, and establish that solutions to geometry-compatible conservation laws enjoy L^p stability properties (for all p) that are *independent of the geometry* of the manifold. Next, Section 5 is devoted to the general well-posedness theory on a Riemannian manifold, and covers both L^∞ solutions (following Kruzkov [6]) and measure-valued solutions (following DiPerna [3]).

In turn, we obtain a versatile framework allowing us to establish the convergence toward the unique entropy solution of any sequence of approximate solutions satisfying all of the entropy inequalities (possibly up to some small error). Our geometry-independent bounds should be useful for the numerical analysis of stable and robust, shock-capturing schemes adapted to hyperbolic equations posed on manifolds. In this direction, the follow-up paper [1] will cover the convergence of finite volume methods based on non-Cartesian meshes, and the derivation of total variation diminishing (TVD) estimates.

The remaining of the paper contains several generalizations of interest. In Section 6 we show that the well-posedness theory can be developed in the function space L^1 for *general* conservation laws that need not be geometry-compatible; we prove that the L^1 contraction property hold at this level of generality, but that for $p > 1$ the L^p stability properties may be violated. Finally, in Section 7 we discuss the well-posedness theory for conservation laws posed on a Lorentzian manifold, including for instance the Schwarzschild spacetime.

2. Preliminaries

In this preliminary section, we briefly review some key concepts and results from Kruzkov's theory in the Euclidian space. Consider the *hyperbolic conservation law*

$$\partial_t u + \sum_{j=1}^{n} \partial_j \big(f^j(u, \cdot) \big) = 0, \quad u = u(t,x) \in \mathbb{R},\ t > 0,\ x \in \mathbb{R}^n, \tag{2.1}$$

where $\partial_j := \partial/\partial x^j$ and $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is a given smooth mapping, referred to as the *flux* of (2.1). Observe that in (2.1) (as well as in (2.3) below) the divergence operator acts on the x-coordinate and takes into account the dependence in $x = (x^j)$ of both the solution $u = u(t,x)$ and the flux $f = (f^j(\bar{u}, x))$, that is

$$\sum_{j=1}^{n} \partial_j \big(f^j\big(u(t,x),x\big)\big) = \sum_{j=1}^{n} \Big(\big(\partial_u f^j\big)\big(u(t,x),x\big)\partial_j u(t,x) + \big(\partial_j f^j\big)\big(u(t,x),x\big)\Big).$$

As is well known [2,5–7,9,12], solutions of (2.1) are generally discontinuous and must be understood in the weak sense of distributions. In addition, for the sake of uniqueness one should restrict attention to *entropy solutions* characterized by an infinite family of inequalities. For simplicity, assume first that the flux is *divergence-free,* that is

$$\sum_{j=1}^{n} \big(\partial_j f^j\big)(\bar{u}, x) = 0, \quad \bar{u} \in \mathbb{R},\ x \in \mathbb{R}^n. \tag{2.2}$$

(This assumption is motivated by Lemma 2.1 below.) When the condition (2.2) is imposed, the *entropy inequalities* associated with Eq. (2.1) take the form

$$\begin{aligned} &\partial_t U(u) + \sum_{j=1}^{n} \partial_j \big(F^j(u,\cdot)\big) \leqslant 0,\\ &U'' \geqslant 0,\\ &F^j(\bar{u}, x) := \int^{\bar{u}} \partial_u U(u') \partial_u f^j(u', x)\, \mathrm{d}u', \quad x \in \mathbb{R}^n,\ \bar{u} \in \mathbb{R},\ j = 1, \dots, n. \end{aligned} \tag{2.3}$$

Relying on Kruzkov's theory [6] and the arguments given later in this paper, one can check that, given any divergence-free flux and initial data $u_0 \in L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$, the initial value problem (2.1),

$$u(0, x) = u_0(x), \quad x \in \mathbb{R}^n, \tag{2.4}$$

admits a unique entropy solution $u \in L^\infty\big(\mathbb{R}_+, L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)\big)$ which, moreover, satisfies the L^p *stability properties*

$$\big\|u(t)\big\|_{L^p(\mathbb{R}^n)} \leqslant \|u_0\|_{L^p(\mathbb{R}^n)}, \quad t \geqslant 0,\ p \in [1, \infty], \tag{2.5}$$

together with the L^1 *contraction property:* for any two entropy solutions u, v

$$\big\|v(t) - u(t)\big\|_{L^1(\mathbb{R}^n)} \leqslant \big\|v(t') - u(t')\big\|_{L^1(\mathbb{R}^n)}, \quad 0 \leqslant t' \leqslant t. \tag{2.6}$$

Furthermore, one has the following regularity property: If the initial data has bounded total variation, that is $u_0 \in BV(\mathbb{R}^n)$, then the solution has bounded variation for all times, that is $u \in L^\infty(\mathbb{R}_+, BV(\mathbb{R}^n))$.

In the Euclidian setting, the condition (2.2) arises in the following way.

Lemma 2.1. *A necessary and sufficient condition for every smooth solution of* (2.1) *to satisfy the infinite list of additional* conservation laws (2.3) *is that the divergence-free condition* (2.2) *holds.*

Proof. This follows readily by multiplying the conservation law (2.1) by the derivative $\partial_u U(u)$ of an arbitrary function $U(u)$, using the chain rule, and taking advantage of the fact that the function U is arbitrary. Namely, we have

$$\begin{aligned} \sum_{j=1}^{n} \partial_j \big(F^j(u,\cdot)\big) &= \sum_{j=1}^{n} \big(\partial_u F^j\big)(u,\cdot)\partial_j u + \big(\partial_j F^j\big)(u,\cdot)\\ &= \sum_{j=1}^{n} \big(\partial_u U\, \partial_u f^j\big)(u,\cdot)\partial_j u + \big(\partial_j F^j\big)(u,\cdot), \end{aligned}$$

so that

$$\partial_t U(u) + \partial_j \big(F^j(u,\cdot)\big) = \sum_{j=1}^{n} \big(\partial_j F^j\big)(u,\cdot) - \partial_u U(u)\big(\partial_j f^j\big)(u,\cdot).$$

Now, by imposing that the right-hand side of the above identity vanishes for all entropy U and by differentiating in u the corresponding relation while using the definition of F^j in (2.3), we obtain

$$\partial_{uu} U(u)\big(\partial_j f^j\big)(u,\cdot) = 0.$$

Since U is arbitrary, the desired claim follows. □

Observe that Kruzkov's theory in $\mathbb{R}^n$ also covers flux that need not satisfy our condition (2.2), and actually applies to *balance laws* of the general form

$$\partial_t u + \sum_{j=1}^{n} \partial_j \big(f^j(u,\cdot) \big) = h(u,\cdot), \quad u = u(t,x) \in \mathbb{R}, \tag{2.7}$$

where $f = (f^j(\bar{u},t,x))$ and $h = h(\bar{u},t,x)$ are given smooth mappings. Provided h is Lipschitz continuous in the u-variable (uniformly in t,x), the initial value problem associated with (2.7) admits a globally defined, unique, entropy solution $u \in L^\infty_{\mathrm{loc}}(\mathbb{R}_+ \times \mathbb{R}^n)$. Note, however, that the general equation (2.7) does not enjoy the L^p stability, L^1 contraction and TVD properties. Most of the literature on scalar conservation laws is restricted to the case of a constant flux $f = f(\bar{u})$ which, clearly, arises as a special case of (2.2). As pointed out earlier, there is no such concept as a "constant flux" in the context of manifolds, and a suitable generalization of (2.2) will arise in our analysis.

3. Geometry-compatible conservation laws

3.1. Background and notation

Throughout this paper, $(\mathbf{M}^n, g)$ is a compact, oriented, n-dimensional Riemannian manifold. As usual, the tangent space at a point $x \in \mathbf{M}^n$ is denoted by $T_x\mathbf{M}^n$ and the tangent bundle by $T\mathbf{M}^n := \bigcup_{x\in\mathbf{M}^n} T_x\mathbf{M}^n$, while the cotangent bundle is denoted by $T^\star\mathbf{M}^n = T_x^\star\mathbf{M}^n$. The metric structure is determined by a positive-definite, 2-covariant tensor field g, that is, at each $x \in \mathbf{M}^n$, g_x is a inner product on $T_x\mathbf{M}^n$.

In local coordinates $x = (x^j)$, the derivations $\partial_j := \frac{\partial}{\partial x^j}$ form a basis of the tangent space $T_x\mathbf{M}^n$, while the differential forms $\mathrm{d}x^j$ determine a basis of the cotangent space $T_x^\star\mathbf{M}^n$. Here and below, we use Einstein's summation convention on repeated indices so, for instance, in local coordinates

$$g = g_{ij}\,\mathrm{d}x^i\,\mathrm{d}x^j, \quad g_{ij} = g(\partial_i,\partial_j).$$

The notation $\langle X_x, Y_x\rangle_{g_x} := g_x(X_x,Y_x)$ may also be used for the inner product of $X_x, Y_x \in T_x\mathbf{M}^n$.

We denote by (g^{ij}) the inverse of the positive definite matrix (g_{ij}). The metric tensor will be used to raise and lower indices, so that to each vector $X = (X^j)$ one associates the covector (X_j) via

$$X_j := g_{ij}X^i.$$

Recall that the differential $\mathrm{d}f$ of a function $f : \mathbf{M}^n \to \mathbb{R}$ is the section of the cotangent bundle $T_x^\star\mathbf{M}^n$, that is $\mathrm{d}f_x \in T_x^\star\mathbf{M}^n$, defined by

$$\mathrm{d}f_x(X_x) = X_x(f), \quad X_x \in T_x\mathbf{M}^n.$$

We denote by $\mathrm{d}V_g$ the volume form on $\mathbf{M}^n$, which in local coordinates reads

$$\mathrm{d}V_g = \sqrt{|g|}\,\mathrm{d}x^1\cdots\mathrm{d}x^n,$$

with $|g| := \det(g_{ij}) > 0$. The gradient of a function h is obtained from dh by using the isomorphism between $T\mathbf{M}^n$ and $T^*\mathbf{M}^n$ via the Riemannian scalar product. In local coordinates, the gradient of a function h is the vector field

$$\mathrm{grad}_g\, h = \big(\nabla^j h\big) = \frac{\partial}{\partial x^j} := g^{ij}\partial_i h \frac{\partial}{\partial x^j}.$$

To the metric tensor g one associates a covariant derivative operator ∇, characterized by the condition $\nabla g = 0$. In particular, the covariant derivative of a vector field $X = X^j\frac{\partial}{\partial x^j}$ is the $(1,1)$-tensor field whose coefficients in local coordinates are

$$\nabla_j X^i := \partial_j X^i + \Gamma^i_{kj}X^k, \quad i,j = 1,\dots,n,$$

where the Christoffel's symbols are determined from the metric tensor by

$$\Gamma^i_{kj} := \frac{1}{2}g^{il}(\partial_k g_{lj} + \partial_j g_{kj} - \partial_l g_{kj}).$$

The divergence of the vector field X is the function

$$\operatorname{div}_g X := \nabla_j X^j = \partial_j X^j + \Gamma^j_{kj} X^k. \tag{3.1}$$

Recall the duality relation

$$\int_{\mathbf{M}}^{n} g(\operatorname{grad}_g h, X)\, \mathrm{d}V_g = -\int_{\mathbf{M}^n} h \operatorname{div}_g X\, \mathrm{d}V_g, \tag{3.2}$$

which is valid for any smooth function h and vector field X, at least. Interestingly enough, it follows from (3.2) that, in local coordinates,

$$\operatorname{div}_g X = \Big(\sqrt{|g|}\Big)^{-1} \partial_j \Big(\sqrt{|g|} X^j\Big), \tag{3.3}$$

which shows that the divergence operator *on vector fields* depends upon $|g|$, only.

3.2. A class of conservation laws

The observation made earlier in Lemma 2.1 in the Euclidian setting provides us the motivation for the definition that we now introduce. As was already pointed out in the introduction, although Kruzkov's theory applies to general balance laws (2.7), the subclass defined by (2.1) and the condition (2.2) leads to many properties (maximum principle, L^1 contraction, L^p stability) which are very desirable features that we attempt to guarantee on the manifold $\mathbf{M}^n$. At this stage of the discussion, we restrict attention to smooth functions satisfying in a classical sense the partial differential equations under consideration.

Definition 3.1.

(1) A *flux* on the manifold $\mathbf{M}^n$ is a vector field $f = f_x(\bar{u})$ depending upon the parameter $\bar{u}$ (the dependence in both variables being smooth).
(2) The *conservation law associated with the flux* f on $\mathbf{M}^n$ is

$$\partial_t u + \operatorname{div}_g\big(f(u)\big) = 0, \tag{3.4}$$

where the unknown is the scalar-valued function $u = u(t, x)$ defined for $t \geqslant 0$ and $x \in \mathbf{M}^n$ and the divergence operator is applied, for each fixed time t, to the vector field $x \mapsto f_x(u(t, x)) \in T_x\mathbf{M}^n$.
(3) A flux is called *geometry-compatible* if it satisfies the condition

$$\operatorname{div}_g f_x(\bar{u}) = 0, \quad \bar{u} \in \mathbb{R},\ x \in \mathbf{M}^n. \tag{3.5}$$

We will also refer to (3.4)–(3.5) as a geometry-compatible conservation law. Let us emphasize that (3.4) is a geometric partial differential equation which depends on the geometry of the manifold, only, and is independent of any particular system of local coordinates on M. In particular, all estimates derived for solutions of (3.4) must take a coordinate-independent form, while, in the proofs, it will often be convenient to introduce a particular chart.

Observe that, by the divergence theorem on Riemannian manifolds ([11], for instance), all sufficiently smooth solutions of (3.4) satisfy the balance law

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_S u(t, x)\, \mathrm{d}V_g(x) = \int_{\partial S} g_x\big(f_x\big(u(t, x)\big), \nu_x\big)\, \mathrm{d}V_{\partial S}(x),$$

for every smooth, n-dimensional sub-manifold $S \subseteq \mathbf{M}^n$ with boundary ∂S. Here, ν denotes the outward unit normal to S, and $V_{\partial S}$ is the volume form induced on the boundary by restricting the metric g to the tangent space $T(\partial S)$.

In local coordinates, the flux components $f^j_x(\bar{u})$ depend upon $\bar{u}, x$ and we will denote by $\partial_u f^j_x(\bar{u})$ and $\partial_k f^j_x(\bar{u})$ its $\bar{u}$- and x^k-derivatives, respectively. Before we proceed, observe that:

Lemma 3.2. *Let $f = f_x(\bar{u})$ be a geometry-compatible flux on $\mathbf{M}^n$. In local coordinates the conservation law* (3.4) *takes the* (*nonconservative*) *form*

$$\partial_t u(t,x) + \big(\partial_u f_x^j\big)\big(u(t,x)\big)\partial_j u(t,x) = 0. \tag{3.6}$$

Proof. It follows from the definition of the divergence operator (3.1) that the conservation law (3.4) can be written as

$$\partial_t u(t,x) + \partial_j\big(f_x^j\big(u(t,x)\big)\big) + \Gamma_{kj}^j(x) f_x^k\big(u(t,x)\big) = 0.$$

The function $\partial_j(f_x^j(u(t,x))$ is the sum of partial derivatives with respect to the local coordinate x of the composite map $x \mapsto f_x(u(t,x))$, that is

$$\partial_j\big(f_x^j\big(u(t,x)\big)\big) = \big(\partial_u f_x^j\big)\big(u(t,x)\big)\partial_j u(t,x) + \big(\partial_j f_x^j\big)\big(u(t,x)\big).$$

On the other hand, the condition in Definition 3.1 yields

$$\big(\partial_j f_x^j\big)(\bar{u},x) + \Gamma_{kj}^j(x) f_x^k(\bar{u}) = 0, \quad \bar{u} \in \mathbb{R},\ x \in \mathbf{M}^n.$$

Writing this identity with $\bar{u} = u(t,x)$ and combining it with the above observations lead us to (3.6). □

3.3. Weak solutions

We will be interested in measurable and bounded functions $u \in L^\infty(\mathbb{R}_+ \times \mathbf{M}^n)$ satisfying (3.4) in the sense of distributions and assuming a prescribed initial condition $u_0 \in L^\infty(\mathbf{M}^n)$ at the time $t = 0$:

$$u(0,x) = u_0(x), \quad x \in \mathbf{M}^n. \tag{3.7}$$

For the sake of uniqueness, we need an analogue of the entropy inequalities (2.3) proposed by Kruzkov in the Euclidian space $\mathbb{R}^n$. These inequalities are derived as follows.

First of all, consider smooth solutions $u = u(t,x)$ of (3.4) and multiply the conservation law by the derivative $\partial_u U(u)$ of an arbitrary function U. We obtain

$$\partial_t U\big(u(t,x)\big) + \partial_u U\big(u(t,x)\big)\operatorname{div}_g\big(f_x\big(u(t,x)\big)\big) = 0,$$

which suggests to search for a vector field $F = F_x(\bar{u})$ such that

$$\partial_u U\big(u(t,x)\big)\operatorname{div}_g\big(f_x\big(u(t,x)\big)\big) = \operatorname{div}_g\big(F_x\big(u(t,x)\big)\big).$$

This relation should hold for all functions $u = u(t,x)$, and is equivalent to impose that the flux components $f_x^j(\bar{u})$ satisfy the two partial differential equations

$$\begin{aligned}
&\partial_u U(\bar{u})\partial_u f_x^j(\bar{u}) = \partial_u F_x^i(\bar{u}), \quad 1 \leqslant j \leqslant n,\\
&\partial_u U(\bar{u})\big(\partial_j f_x^j(\bar{u}) + \Gamma_{kj}^j(x) f_x^k(\bar{u})\big) = \partial_j F_x^j(\bar{u}) + \Gamma_{kj}^j(x) F_x^k(\bar{u}),
\end{aligned} \tag{3.8}$$

for all $\bar{u} \in \mathbb{R}$ and $x \in \mathbf{M}^n$. The first equation implies that F_x is given by

$$F_x(\bar{u}) := \int^{\bar{u}} \partial_w U(w)\partial_w f_x(w)\,\mathrm{d}w \in T_x\mathbf{M}^n. \tag{3.9}$$

Differentiating the second equation with respect to $\bar{u}$ and using (3.9) we find

$$\partial_{uu} U(\bar{u})\big(\partial_j f_x^j(\bar{u}) + \Gamma_{kj}^j(x) f_x^k(\bar{u})\big) = 0.$$

Since U is arbitrary, a necessary and sufficient condition for (3.8) to hold is that $f_x(\bar{u})$ satisfies (3.5). Note in passing that necessarily $F_x(\bar{u})$ satisfies also this condition.

This derivation provides us with an analogue of the definition (2.3) of the Euclidian case. In turn, smooth solutions of any geometry-compatible conservation law (3.4) satisfy the additional conservation laws

$$\partial_t U\big(u(t,x)\big) + \operatorname{div}_g\big(F_x\big(u(t,x)\big)\big) = 0, \tag{3.10}$$

where U is arbitrary and F is given by (3.9).

We are now in position to introduce the notion of entropy solution which generalizes Kruzkov's notion to the case of manifolds. We require the equalities (3.10) to hold as inequalities only, as this arises naturally by the vanishing diffusion method. (See Section 4.)

Definition 3.3. Let $f = f_x(\bar{u})$ be a geometry-compatible flux on the Riemannian manifold (M, g).

(1) A convex entropy/entropy-flux pair is a pair (U, F) where $U : \mathbb{R} \to \mathbb{R}$ is a convex function, and $F = F_x(\bar{u})$ is the vector field depending on the parameter $\bar{u}$ defined by

$$F_x(\bar{u}) := \int^{\bar{u}} \partial_{u'} U(u') \partial_{u'} f_x(u') \, \mathrm{d}u', \quad \bar{u} \in \mathbb{R}, \ x \in \mathbf{M}^n. \tag{3.11}$$

(2) A function $u_0 \in L^\infty(\mathbf{M}^n)$ being given, a function $u \in^\infty \big(\mathbb{R}_+, L^\infty(\mathbf{M}^n)\big)$ is called an entropy solution to the initial value problem (3.4)–(3.7) if the following entropy inequalities hold

$$\iint_{\mathbb{R}_+ \times \mathbf{M}^n} \big(U\big(u(t,x)\big)\partial_t \theta(t,x) + g_x\big(F_x\big(u(t,x)\big), \operatorname{grad}_g \theta(t,x)\big)\big) \, \mathrm{d}V_g(x) \, \mathrm{d}t$$
$$+ \int_{\mathbf{M}}^n U\big(u_0(x)\big)\theta(0,x) \, \mathrm{d}V_g(x) \geqslant 0, \tag{3.12}$$

for every convex entropy/entropy flux pair (U, F) and all smooth functions $\theta = \theta(t, x) \geqslant 0$ compactly supported in $[0, \infty) \times \mathbf{M}^n$.

4. Vanishing diffusion method on manifolds

In the present section, we construct solutions of the conservation law (3.4) via the vanishing diffusion method. We establish the existence of smooth solutions to a regularized version of (3.4) taking into account a small diffusion term, and then establish that these solutions converge to an entropy solution in the sense of Definition 3.3 as the diffusion tends to zero. For convenience, we provide a proof that is based on a uniform total variation bound, and refer the reader to the forthcoming section for an alternative approach based on the concept of measure-valued solution.

Recall that $\mathbf{M}^n$ endowed with the metric tensor g is a compact, oriented, n-dimensional Riemannian manifold. We denote by $L^p(\mathbf{M}^n; \mathrm{d}V_g)$ the usual Lebesgue spaces on the manifold $(\mathbf{M}^n, g)$. The Sobolev space $H^1(\mathbf{M}^n; \mathrm{d}V_g)$ is the space of all functions $h \in L^2(\mathbf{M}^n; \mathrm{d}V_g)$ such that $\operatorname{grad}_g h \in L^2(\mathbf{M}^n; \mathrm{d}V_g)$ – where the gradient is defined in the distributional sense, via the formula (3.2). The spaces $H^k(\mathbf{M}^n; \mathrm{d}V_g)$ for $k \geqslant 2$ are defined similarly.

The *total variation* of a function $h \in L^1(\mathbf{M}^n; \mathrm{d}V_g)$ is defined by

$$TV(h) := \sup_{\|\psi\|_{L^\infty(\mathbf{M}^n)} \leqslant 1} \int_{\mathbf{M}}^n h \operatorname{div}_g \psi \, \mathrm{d}V_g,$$

where ψ describes all C^1 vector fields on the manifold. We denote by $BV(\mathbf{M}^n; \mathrm{d}V_g) \subset L^1(\mathbf{M}^n; \mathrm{d}V_g)$ the space of all functions h with finite total variation, endowed with the norm $\|h\|_{L^1(\mathbf{M}^n; \mathrm{d}V_g)} + TV(h)$. For material on BV functions we refer to [4,12]. In particular, it is well known that the inclusion map of $BV(\mathbf{M}^n; \mathrm{d}V_g)$ in $L^1(\mathbf{M}^n; \mathrm{d}V_g)$ is compact.

It is clear that, for all smooth functions $h : \mathbf{M}^n \to \mathbb{R}$,

$$TV(h) = \int_{\mathbf{M}}^n |\operatorname{grad}_g h|_g \, \mathrm{d}V_g,$$

where $|\cdot|_g$ denotes the Riemannian norm associated with g. Observe also that, by taking a partition of unity determining a finite covering of $\mathbf{M}^n$ by coordinate patches we can easily "localize" the concept of total variation on the manifold, as follows.

Lemma 4.1. *There exist finitely many* (*smooth*) *vector fields* $X^{(1)}, \dots, X^{(L)}$ *on* $\mathbf{M}^n$, *a constant* $C_0 \geqslant 1$ (*depending only on the manifold*), *and a chart covering the manifold such that*:

(i) *each vector field* $X^{(l)}$ *is supported in a coordinate patch and*

$$\operatorname{Span}\bigl\{X_x^{(1)}, \dots, X_x^{(L)}\bigr\} := T_x\mathbf{M}^n, \quad x \in \mathbf{M}^n;$$

(ii) *and for every smooth function* $h : \mathbf{M}^n \to \mathbb{R}$

$$\frac{1}{C_0} \sum_{l=1}^{L} \int_{\mathbf{M}}^{n} \bigl|X^{(l)}(h)\bigr|_g \, \mathrm{d}V_g \leqslant TV(h) \leqslant C_0 \sum_{l=1}^{L} \int_{\mathbf{M}}^{n} \bigl|X^{(l)}(h)\bigr|_g \, \mathrm{d}V_g.$$

An initial data u_0 being given in $L^\infty(\mathbf{M}^n) \cap BV(\mathbf{M}^n; \mathrm{d}V_g)$, we want to find a solution $u^\epsilon = u^\epsilon(t, x)$ to the initial value problem

$$\partial_t u^\epsilon + \operatorname{div}_g\bigl(f_x(u^\epsilon)\bigr) = \epsilon \Delta_g u^\epsilon, \quad x \in \mathbf{M}^n, \ t \geqslant 0, \tag{4.1}$$

$$u^\epsilon(0, x) = u_0^\epsilon(x), \quad x \in \mathbf{M}^n, \tag{4.2}$$

where Δ_g denotes the Laplace operator on the manifold $\mathbf{M}^n$,

$$\begin{aligned} \Delta_g v &:= \operatorname{div}_g \operatorname{grad}_g v \\ &= g^{ij}\left(\frac{\partial^2 v}{\partial x^i \partial x^j} - \Gamma_{ij}^k \frac{\partial v}{\partial x^k}\right). \end{aligned}$$

In (4.2), $u_0^\epsilon : \mathbf{M}^n \to \mathbb{R}$ is a sequence of smooth functions satisfying

$$\begin{aligned} &\|u_0^\epsilon\|_{L^p(\mathbf{M}^n)} \leqslant \|u_0\|_{L^p(\mathbf{M}^n)}, \quad p \in [1, \infty], \\ &TV(u_0^\epsilon) \leqslant TV(u_0), \\ &\epsilon \|u_0^\epsilon\|_{H^2(\mathbf{M}^n; \mathrm{d}V_g)} \leqslant C TV(u_0), \quad \text{for some } C > 0 \text{ depending only on } \mathbf{M}^n, \\ &u_0^\epsilon \to u_0 \quad \text{a.e. on } \mathbf{M}^n. \end{aligned} \tag{4.3}$$

In the following, we use the notation $u(t)$ instead of $u(t, \cdot)$. We begin with:

Theorem 4.2 *(Regularized conservation law). Let* $f = f_x(\bar{u})$ *be a geometry-compatible flux on a Riemannian manifold* $(\mathbf{M}^n, g)$. *Given* $u_0^\epsilon \in C^\infty(\mathbf{M}^n)$ *satisfying* (4.3) *there exists a unique solution* $u^\epsilon \in C^\infty(\mathbb{R}_+ \times \mathbf{M}^n)$ *to the initial value problem* (4.1)–(4.2). *Moreover, for each* $1 \leqslant p \leqslant \infty$ *the solution satisfies*

$$\bigl\|u^\epsilon(t)\bigr\|_{L^p(\mathbf{M}^n; \mathrm{d}V_g)} \leqslant \bigl\|u^\epsilon(t')\bigr\|_{L^p(\mathbf{M}^n; \mathrm{d}V_g)}, \quad 0 \leqslant t' \leqslant t \tag{4.4}$$

and, for any two solutions u^ϵ *and* v^ϵ,

$$\bigl\|v^\epsilon(t) - u^\epsilon(t)\bigr\|_{L^1(\mathbf{M}^n; \mathrm{d}V_g)} \leqslant \bigl\|v^\epsilon(t') - u^\epsilon(t')\bigr\|_{L^1(\mathbf{M}^n; \mathrm{d}V_g)}, \quad 0 \leqslant t' \leqslant t. \tag{4.5}$$

In addition, for every convex entropy/entropy flux pair (U, F_x) *the solution* u^ϵ *satisfies the entropy inequality*

$$\partial_t U(u^\epsilon) + \operatorname{div}_g\bigl(F(u^\epsilon)\bigr) \leqslant \epsilon \Delta_g U(u^\epsilon). \tag{4.6}$$

All proofs are postponed to the end of this section.

Our next goal is to prove the strong convergence of the family $\{u^\epsilon\}_{\epsilon > 0}$ to an entropy solution of (3.4)–(3.7), as $\epsilon \to 0$. We will assume here that the initial data has bounded total variation, and we refer to the forthcoming section for a more general argument which does not require this assumption. From the estimates for the solutions obtained in Theorem 4.2 we can deduce a uniform total variation bound in space and time.

Lemma 4.3 *(BV bounds for diffusive approximations). There exists a positive constant* C_1 *depending on the geometry of* $\mathbf{M}^n$ *and* $\|u_0\|_{L^\infty(\mathbf{M}^n)}$, *only, such that the solutions* u^ϵ *given by Theorem* 4.2 *satisfy*

$$TV\big(u^\epsilon(t)\big) \leqslant \mathrm{e}^{C_1 t}\big(1 + TV(u_0)\big), \quad t \in \mathbb{R}_+, \tag{4.7}$$

$$\big\|\partial_t u^\epsilon(t)\big\|_{L^1(\mathbf{M}^n;\mathrm{d}V_g)} \leqslant TV(u_0) + \epsilon\big\|\Delta_g u_0^\epsilon\big\|_{L^2(\mathbf{M}^n;\mathrm{d}V_g)}, \quad t \in \mathbb{R}_+. \tag{4.8}$$

Observe that the right-hand side of (4.7) is independent of ϵ, while the term $\epsilon C_1 \|\Delta_g u_0^\epsilon\|_{L^2(\mathbf{M}^n)}$ arising in (4.8) is dominated by $CTV(u_0)$ thanks to our assumption (4.3). At this juncture, we refer to [1] in which a sharp version of the above estimates is derived with a specific constant C_1 depending on the geometry of the manifold.

We are now in a position to state and prove the main result of this section.

Theorem 4.4 *(Convergence of the vanishing diffusion method on a Riemannian manifold). Let $f = f_x(\bar{u})$ be a geometry-compatible flux on a Riemannian manifold $(\mathbf{M}^n, g)$. Given any initial condition $u_0 \in L^\infty(\mathbf{M}^n) \cap BV(\mathbf{M}^n; \mathrm{d}V_g)$ there exists an entropy solution $u \in L^\infty(\mathbb{R}_+ \times \mathbf{M}^n)$ to the initial value problem* (3.4)–(3.7) *in the sense of Definition* 3.3*, which is the limit of the sequence u^ϵ constructed in Theorem* 4.2 *by vanishing diffusion.*

Moreover, this solution has the following properties:

$$\big\|u(t)\big\|_{L^p(\mathbf{M}^n;\mathrm{d}V_g)} \leqslant \|u_0\|_{L^p(\mathbf{M}^n;\mathrm{d}V_g)}, \qquad t \in \mathbb{R}_+,\ 1 \leqslant p \leqslant \infty,$$

and, for some constant $C_1 > 0$ depending on $\|u_0\|_{L^\infty(\mathbf{M}^n)}$ and the geometry of $\mathbf{M}^n$ only,

$$TV\big(u(t)\big) \leqslant \mathrm{e}^{C_1 t}\big(1 + TV(u_0)\big), \quad t \in \mathbb{R}_+,$$

$$\big\|u(t) - u(t')\big\|_{L^1(\mathbf{M}^n;\mathrm{d}V_g)} \leqslant TV(u_0)|t - t'|, \quad 0 \leqslant t' \leqslant t. \tag{4.9}$$

Furthermore, if u, v are entropy solutions associated with some initial data u_0, v_0, respectively, it hold

$$\big\|v(t) - u(t)\big\|_{L^1(\mathbf{M}^n;\mathrm{d}V_g)} \leqslant \|v_0 - u_0\|_{L^1(\mathbf{M}^n;\mathrm{d}V_g)}, \quad t \in \mathbb{R}_+. \tag{4.10}$$

Proof of Theorem 4.2. By introducing a local chart of coordinates and using the condition (3.5), one can reduce Eq. (4.1) to a parabolic equation in the Euclidian space $\mathbb{R}^n$:

$$\partial_t u^\epsilon + \big(\partial_u f_x^j\big)(u^\epsilon)\partial_j u^\epsilon = \epsilon g^{ij}\big(\partial_i \partial_j u^\epsilon - \Gamma_{ij}^k \partial_k u^\epsilon\big). \tag{4.11}$$

The *local in time* existence, uniqueness, and regularity of the solution u^ϵ follows as in the Euclidian setting. By putting together several charts to cover the whole manifold, one arrives at the *local in time* well-posedness for the diffusive conservation law (4.1). Next, in order to extend the solution to arbitrary times, one needs a uniform estimate on the sup norm of the solution. This indeed follows in the form stated in (4.4) for $p = \infty$, from the standard maximum principle for parabolic equations.

Next, in every strip $[0, T] \times \mathbf{M}^n$ the dual equation associated with (4.1) reads

$$\partial_t \varphi + g_x\big(f_x(u^\epsilon), \mathrm{grad}_g\, \varphi\big) + \epsilon \Delta_g \varphi = 0,$$

for which a "terminal value" problem with data given at $t = T$ is now considered. This equation satisfies the maximum principle and, by duality, the inequality (4.4) for $p = 1$ follows. The inequality for intermediate values $p \in (1, \infty)$ is then obtained by straightforward interpolation.

We next proceed with the derivation of (4.5). The function $w^\epsilon := v^\epsilon - u^\epsilon$ satisfies the equation

$$\partial_t w^\epsilon + \mathrm{div}_g\big(b_x^\epsilon\, w^\epsilon\big) = \epsilon \Delta_g w^\epsilon, \tag{4.12}$$

where

$$b_x^\epsilon := \frac{f_x(v^\epsilon) - f_x(u^\epsilon)}{v^\epsilon - u^\epsilon}$$

is a smooth vector field on $\mathbf{M}^n$. As in the argument above, the dual equation associated with (4.12) satisfies the maximum principle and so, by duality,

$$\big\|w^\epsilon(t)\big\|_{L^1(\mathbf{M}^n;\mathrm{d}V_g)} \leqslant \big\|w^\epsilon(t')\big\|_{L^1(\mathbf{M}^n;\mathrm{d}V_g)}, \quad 0 \leqslant t' \leqslant t,$$

which is precisely (4.5).

Finally, to show (4.6) we multiply (4.1) by $\partial_u U(u^\epsilon)$, where (U, F) is a convex entropy pair in the sense of Definition 3.3. We get

$$\partial_t U(u^\epsilon) + \operatorname{div}_g F_x(u^\epsilon) = \epsilon \partial_u U(u^\epsilon) \Delta_g u^\epsilon.$$

In view of the definition of Δ_g we can write in local coordinates

$$\partial_u U(u^\epsilon) \Delta_g u^\epsilon = \Delta_g U(u^\epsilon) - \partial_u^2 U(u^\epsilon) g^{jk} \partial_j u^\epsilon \partial_k u^\epsilon. \tag{4.13}$$

Since $\partial_u^2 U$ is non-negative (U being convex) and the matrix (g^{jk}) is positive definite (since g is a Riemannian metric tensor), the second term in the right-hand side of (4.13) is non-positive. This establishes the desired inequality (4.6). □

Proof of Lemma 4.3. To establish (4.7) we rely on Lemma 4.1 which shows that, without loss of generality, it is sufficient to estimate $\|\partial_1(\psi u^\epsilon(t))\|_{L^1(\mathbf{M}^n; \mathrm{d}V_g)}$ for every smooth ψ supported in a coordinate patch $\mathcal{O}$ with $\|\psi\|_{L^\infty(\mathbf{M}^n)} \leqslant 1$. Multiplying (4.11) by ψ and differentiating with respect to x^1, the function $w^\epsilon := \partial_1(\psi u^\epsilon)$ is found to satisfy

$$\partial_t w^\epsilon(t,x) + \partial_j\big(\big(\partial_u f_x^j\big)\big(u^\epsilon(t,x)\big) w^\epsilon(t,x)\big) = A(x, u^\epsilon, \operatorname{grad}_g u^\epsilon)(t,x) + \epsilon \Delta_g w^\epsilon(t,x),$$

where A depends linearly on $\operatorname{grad}_g u^\epsilon$ and smoothly upon x, u^ϵ (and is supported in $\mathcal{O}$). We now multiply the identity above by $\operatorname{sgn}(w^\epsilon)$ and integrate over $\mathbf{M}^n$. At this juncture we note that, since $w^\epsilon \in H^2(\mathbf{M}^n; \mathrm{d}V_g)$ at least,

$$\operatorname{sgn}(w^\epsilon)\partial_t w^\epsilon = \partial_t |w^\epsilon|, \quad \operatorname{sgn}(w^\epsilon)\partial_j w^\epsilon = \partial_j |w^\epsilon|, \quad \operatorname{sgn}(w^\epsilon)\Delta_g w^\epsilon \leqslant \Delta_g |w^\epsilon|,$$

where the latter inequality must be understood in the sense of distributions.

Relying on the bound $\|u^\epsilon(t)\|_{L^\infty(\mathbf{M}^n)} \leqslant \|u_0\|_{L^\infty(\mathbf{M}^n)}$ which follows from (4.4), we then deduce that there exists a constant $C_1 > 0$ (depending on both $\|u_0\|_{L^\infty(\mathbf{M}^n)}$ and the geometry of $\mathbf{M}^n$) such that

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbf{M}}^n \big|w^\epsilon(t)\big| \, \mathrm{d}V_g \leqslant C_1 + C_1 \int_{\mathbf{M}}^n \big|\operatorname{grad}_g u^\epsilon(t)\big|_g \, \mathrm{d}V_g.$$

This estimate can now be repeated for each of the (finitely many) vector fields $X^{(1)}, \ldots, X^{(L)}$, and the desired conclusion (4.7) finally follows from Gronwall's lemma.

To establish (4.8) we differentiate (4.11) with respect to t and note that

$$\operatorname{div}_g\big(\partial_t f_x(u^\epsilon)\big) = \partial_t \operatorname{div}_g\big(f_x(u^\epsilon)\big),$$
$$\Delta_g\big(\partial_t f_x(u^\epsilon)\big) = \partial_t \Delta_g\big(f_x(u^\epsilon)\big).$$

Also, the vector field $\partial_t f_x(u^\epsilon(t,x))$ satisfies the condition (3.5) (with respect to the explicit dependence on x).

It therefore follows that the function $z^\epsilon = \partial_t(u^\epsilon)$ satisfies

$$\partial_t z^\epsilon(t,x) + \frac{\partial}{\partial y_j}\big(\big(\partial_u f_x^j\big)\big(u^\epsilon(t,y)\big) z^\epsilon(t,y)\big)_{y=x} = \epsilon \Delta_g z^\epsilon(t,x). \tag{4.14}$$

(In other words, the explicit dependence of $(\partial_u f_x^j)(u^\epsilon(t,x))$ on x is not differentiated.) Note that

$$\operatorname{sgn}(z^\epsilon)\frac{\partial}{\partial y_j}\big(\big(\partial_u f_x^j\big)\big(u^\epsilon(t,y)\big) z^\epsilon(t,y)\big)_{y=x} = \frac{\partial}{\partial y_j}\big(\big(\partial_u f_x^j\big)\big(u^\epsilon(t,x)\big)\big|z^\epsilon(t,x)\big|\big)_{y=x}.$$

Defining the vector field $r_x := (\partial_u f_x)(u^\epsilon(t,x))|z^\epsilon(t,x)|$ we have, by the condition (3.5), that the second term in the left-hand side of (4.14) satisfies

$$\partial_j\big(\big(\partial_u f_x^j\big)\big(u^\epsilon(t,x)\big)\big|z^\epsilon(t,x)\big|\big) = \operatorname{div}_g r_x.$$

Multiplying (4.14) by $\operatorname{sgn}(z^\epsilon)$ and integrating over $\mathbf{M}^n$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbf{M}}^n \big|z^\epsilon(t)\big| \, \mathrm{d}V_g \leqslant 0.$$

In turn, we can write

$$\int_{\mathbf{M}}^{n} \left|z^{\epsilon}(t)\right| \mathrm{d}V_g \leqslant \int_{\mathbf{M}}^{n} \left|z^{\epsilon}(0)\right| \mathrm{d}V_g.$$

Alternatively, this inequality can be derived from (4.5) by choosing $v(t,x) = v(t+\alpha, x)$ with $\alpha \to 0$. Using (4.1) we can estimate the above term at $t = 0$:

$$\begin{aligned}\int_{\mathbf{M}}^{n} \left|z^{\epsilon}(0)\right| \mathrm{d}V_g &\leqslant \| \operatorname{grad}_g u_0^{\epsilon} \|_{L^1(\mathbf{M}^n;\mathrm{d}V_g)} + \epsilon \| \Delta_g u_0^{\epsilon} \|_{L^2(\mathbf{M}^n;\mathrm{d}V_g)} \\ &\leqslant TV(u_0^{\epsilon}) + \epsilon \| \Delta_g u_0^{\epsilon} \|_{L^2(\mathbf{M}^n;\mathrm{d}V_g)}.\end{aligned}$$

Finally, applying this estimate to each of the vector fields $X^{(1)}, \ldots, X^{(L)}$, we arrive at the estimate (4.8). This completes the proof of Lemma 4.3. □

Proof of Theorem 4.4. In view of the sup norm estimate (4.4) and the uniform, space L^1 estimates (4.7), we see that the sequence $u^{\epsilon}(t)$ is uniformly bounded in $BV(\mathbf{M}^n; \mathrm{d}V_g)$ for every time t and is therefore compact in $L^1(\mathbf{M}^n; \mathrm{d}V_g)$. Applying this argument at all rational times t and then picking up a diagonal sequence we can ensure that a subsequence $u^{\epsilon_j}(t)$ converges to some limit $u(t)$ in the L^1 norm, as $\epsilon_j \to 0$, for all rational t. Next, in view of the uniform time estimate (4.8), the limit $u(t)$ extends to all values of the time variable, with $u^{\epsilon_j}(t) \to u(t)$ in $L^1(\mathbf{M}^n; \mathrm{d}V_g)$. Letting $\epsilon_j \to 0$ in the inequalities (4.6) one then deduces that u satisfies all of the entropy inequalities

$$\partial_t U(u) + \operatorname{div}_g\big(F(u)\big) \leqslant 0$$

in the weak sense (3.12).

In fact, by working in a localized coordinate patch and using the entropy formulation (4.6) with $U(u) = |u-k|$, $F_x(u) = \operatorname{sgn}(u-k)(f_x(u) - f_x(k))$, one can repeat the Kruzkov classical theorem in order to obtain a "localized" L^1 contraction property. By patching together (finitely many) coordinate patches we obtain a global estimate of the form

$$\big\| v(t) - u(t) \big\|_{L^1(\mathbf{M}^n;\mathrm{d}V_g)} \leqslant C \| v_0 - u_0 \|_{L^1(\mathbf{M}^n;\mathrm{d}V_g)}, \quad t \in [0,T], \tag{4.15}$$

where $C > 0$ depends on T (and $\mathbf{M}^n$). This is sufficient to imply uniqueness of the entropy solution $u(t)$, hence the convergence of the whole family u^{ϵ}. At this point we can invoke the parabolic estimate (4.5) and conclude that $C = 1$, thus establishing the L^1 contraction property (4.10). All the other estimates now follow from Theorem 4.2 and Lemma 4.3. □

5. The well-posedness theory

Theorem 4.4 assumes that the initial data has bounded variation and provides the existence of (locally BV) entropy solutions constructed by vanishing diffusion. In the present section, we obtain a generalization to L^∞ initial data and also establish the uniqueness of the entropy solution in this larger class. We emphasize that the class of L^∞ solutions is completely natural for geometry-compatible conservation laws, since the L^∞ estimate is independent of the geometry while the BV estimate (4.9), in general, depends upon it.

Our generalization of Kruzkov's theory to manifolds is as follows.

Theorem 5.1 *(Well-posedness theory in L^∞ for geometry-compatible conservation laws). Let $f = f_x(\bar{u})$ be a geometry-compatible flux on a compact, oriented, Riemannian manifold $(\mathbf{M}^n, g)$. Given any initial data $u_0 \in L^\infty(\mathbf{M}^n)$ there exists a unique entropy solution $u \in L^\infty(\mathbb{R}_+ \times \mathbf{M}^n)$ to the initial value problem* (3.4)–(3.7) *in the sense of Definition* 3.3. *Moreover, for each $1 \leqslant p \leqslant \infty$ the solution satisfies*

$$\big\| u(t) \big\|_{L^p(\mathbf{M}^n;\mathrm{d}V_g)} \leqslant \| u_0 \|_{L^p(\mathbf{M}^n;\mathrm{d}V_g)}, \quad t \in \mathbb{R}_+, \tag{5.1}$$

and, given any two entropy solutions u, v associated with some initial data u_0, v_0, respectively,

$$\big\| v(t) - u(t) \big\|_{L^1(\mathbf{M}^n;\mathrm{d}V_g)} \leqslant \| v_0 - u_0 \|_{L^1(\mathbf{M}^n;\mathrm{d}V_g)}, \quad t \in \mathbb{R}_+. \tag{5.2}$$

Furthermore, we will see in the proof that the following inequality holds in the sense of distributions

$$\partial_t|v-u| + \operatorname{div}_g\big(\operatorname{sgn}(u-v)\big(f_x(v)-f_x(u)\big)\big) \leqslant 0. \tag{5.3}$$

Following DiPerna [3] we introduce the (larger) class of entropy measure-valued solutions and, in fact, establish a much stronger version of Theorem 5.1. As observed in [3], Kruzkov's arguments take a simpler form in the measure-valued setting. We consider *measure-valued maps* $\nu = \nu_{t,x}$, that is, weakly measurable mappings $(t,x) \in \mathbb{R}_+ \times \mathbf{M}^n \mapsto \nu_{t,x}$ taking their values in the space of probability measures on $\mathbb{R}$ with support included in a fixed compact interval of $\mathbb{R}$. The action of the measure ν on a function U will be denoted by

$$\langle \nu_{t,x}, U\rangle := \int_{\mathbb{R}} U(\bar{u})\,\mathrm{d}\nu_{t,x}(\bar{u}).$$

The weak measurability property means that the map $\langle \nu_{t,x}, U\rangle$ is measurable in (t,x) for each U.

Definition 5.2. Let $f = f_x(\bar{u})$ be a geometry-compatible flux on a Riemannian manifold $(\mathbf{M}^n, g)$. Given any initial condition $u_0 \in L^\infty(\mathbf{M}^n)$, a measure-valued map $(t,x) \in \mathbf{M}^n \times \mathbb{R}_+ \mapsto \nu_{t,x}$ is called an entropy measure-valued solution to the initial value problem (3.4)–(3.7) if, for every convex entropy/entropy flux pair (U, F_x) (see (3.11)),

$$\iint\limits_{\mathbb{R}_+\times\mathbf{M}^n} \big(\langle \nu_{t,x}, U\rangle \partial_t\theta(t,x) + g_x\big(\langle \nu_{t,x}, F_x\rangle, \operatorname{grad}_g\theta(t,x)\big)\big)\,\mathrm{d}V_g(x)\,\mathrm{d}t + \int\limits_{\mathbf{M}}^{n} U\big(u_0(x)\big)\theta(0,x)\,\mathrm{d}V_g(x) \geqslant 0, \tag{5.4}$$

for every smooth function $\theta = \theta(t,x) \geqslant 0$ compactly supported in $[0,+\infty)\times\mathbf{M}^n$.

We will now prove that:

Theorem 5.3 *(The well-posedness theory in the measure-valued class for geometry-compatible conservation laws). Let $f = f_x(\bar{u})$ be a geometry-compatible flux on a compact, oriented, Riemannian manifold $(\mathbf{M}^n, g)$. Let u_0 be in $L^\infty(\mathbf{M}^n)$ and ν be an entropy measure-valued solution (in the sense of Definition* 5.2*) to the initial value problem* (3.4)–(3.7)*. Then, for almost every (t,x), the measure $\nu_{t,x}$ is a Dirac mass, i.e. of the form*

$$\nu_{t,x} = \delta_{u(t,x)},$$

where the function $u \in L^\infty(\mathbb{R}_+ \times \mathbf{M}^n)$ is the unique entropy solution to the problem (3.4)–(3.7) *in the sense of Definition* 3.4*. Moreover, the solution satisfies the properties* (5.1)–(5.3)*, and the initial data is assumed in the strong sense*

$$\limsup_{t\to 0+} \int\limits_{\mathbf{M}}^{n} \big|u(t,x)-u_0(x)\big|\,\mathrm{d}V_g(x) = 0. \tag{5.5}$$

We have already shown in Section 4 that, provided the initial data have bounded variation, entropy solutions can be constructed via vanishing diffusion. The proof relies on the compactness the inclusion of BV into L^1. Thanks to Theorem 5.3, we can now provide an alternative, more general proof which is valid for L^∞ initial data. In view of the uniform L^∞ estimate (i.e., the maximum principle which only requires the initial data to be in L^∞), vanishing diffusion approximations u^ϵ generate a measure-valued solution ν. Moreover, ν can be easily checked to satisfy all of the entropy inequalities and, in turn, Theorem 5.1 follows from Theorem 5.3.

Finally, we also have the following generalization of Theorem 4.4.

Corollary 5.4. *The convergence result of the vanishing diffusion approximations in Theorem* 4.4 *remains valid if the initial data u_0 is solely in $L^\infty(\mathbf{M}^n)$. All estimates therein, except for the total variation estimate, still hold.*

In the course of proving Theorem 5.3, we will need some a priori regularity of measure-valued solutions, especially the fact that the initial data u_0 is automatically assumed in a strong sense.

Lemma 5.5. *Let $v = v_{t,x}$ be an entropy measure-valued solution of* (3.4)–(3.7)*, where it is assumed that $u_0 \in L^\infty(\mathbf{M}^n)$ and the supports of the measures $v_{t,x}$ are all included in a fixed common interval. Then*:

(a) *For every convex function $U = U(u)$ and every smooth function $\theta = \theta(x)$, the function*

$$t \mapsto \int_{\mathbf{M}}^{n} \langle v_{t,x}, U\rangle \theta \, \mathrm{d}V_g(x) \tag{5.6}$$

has locally bounded total variation and admits a trace as $t \to 0+$.

(b) *For every function $U = U(u, x)$, which is convex in u, measurable in x, and such that $|U(u, x)| \leqslant c|u| + |\tilde{U}(x)|$ with $\tilde{U} \in L^1(\mathbf{M}^n; \mathrm{d}V_g)$ and $c \geqslant 0$,*

$$\limsup_{t\to 0+} \int_{\mathbf{M}}^{n} \big\langle v_{t,x}, U(\cdot, x)\big\rangle \mathrm{d}V_g(x) \leqslant \int_{\mathbf{M}}^{n} U\big(u_0(x), x\big) \mathrm{d}V_g(x). \tag{5.7}$$

(c) *In particular, v assumes its initial data u_0 in the following strong sense*:

$$\limsup_{t\to 0+} \int\!\!\int_{\mathbf{M}\mathbb{R}}^{n} \big|\bar{u} - u_0(x)\big| \, \mathrm{d}v_{t,x}(\bar{u}) \, \mathrm{d}V_g(x) = 0. \tag{5.8}$$

Proof of Lemma 5.5. Using in the weak formulation (5.4) a function $\theta(t, x) = \theta_1(x)\theta_2(t)$, compactly supported in $[0, \infty) \times \mathbf{M}^n$ and having $\theta_1, \theta_2 \geqslant 0$, we obtain

$$\begin{aligned}\int_0^\infty \frac{\mathrm{d}\theta_2}{\mathrm{d}t} \int_{\mathbf{M}}^{n} \langle v, U\rangle \theta_1 \, \mathrm{d}V_g(x) \, \mathrm{d}t + \theta_2(0) \int_{\mathbf{M}}^{n} U(u_0)\theta_1 \, \mathrm{d}V_g(x) &\geqslant -\int_0^\infty \theta_2 \int_{\mathbf{M}}^{n} g_x\big(\mathrm{grad}_g\, \theta_1, \langle v, F_x\rangle\big) \, \mathrm{d}V_g(x) \, \mathrm{d}t \\ &\geqslant -C_1 \int_0^\infty \theta_2 \, \mathrm{d}t,\end{aligned}$$

for some constant $C_1 > 0$ depending on θ_1 (and the common support of $v_{t,x}$). Thus the function

$$V_1(t) := -C_1 t + \int_{\mathbf{M}}^{n} \langle v_{t,x}, U\rangle \theta_1 \, \mathrm{d}V_g(x)$$

satisfies the inequality

$$-\int_0^\infty V_1(t) \frac{\mathrm{d}\theta_2}{\mathrm{d}t} \, \mathrm{d}t \leqslant \theta_2(0) \int_{\mathbf{M}}^{n} U(u_0)\theta_1 \, \mathrm{d}V_g(x). \tag{5.9}$$

Using a test-function $\theta_2 \geqslant 0$ compactly supported in $(0, \infty)$, we find

$$-\int_0^\infty V_1(t) \frac{\mathrm{d}\theta_2}{\mathrm{d}t} \, \mathrm{d}t \leqslant 0.$$

That is, the function $V_1(t)$ is decreasing and, therefore, has locally bounded total variation. Since it is uniformly bounded, $V_1(t)$ has a limit as $t \to 0+$. This proves (a).

To establish the item (b), we fix a time $t_0 > 0$ and consider the sequence of continuous functions

$$\theta_2^\epsilon(t) = \begin{cases} 1, & t \in [0, t_0], \\ (t_0 + \epsilon - t)/\epsilon, & t \in [t_0, t_0 + \epsilon], \\ 0, & t \geqslant t_0 + \epsilon. \end{cases}$$

Relying on the property (a) above, we see that

$$-\int_0^\infty V_1(t)\frac{\mathrm{d}\theta_2^\epsilon}{\mathrm{d}t}\,\mathrm{d}t \to V_1(t_0+).$$

Since $\theta_2^\epsilon(0)=1$ and t_0 is arbitrary, (5.9) yields

$$V_1(t_0) = -C_1 t_0 + \int_{\mathbf{M}}^n \langle \nu_{t_0,x}, U\rangle \theta_1\,\mathrm{d}V_g(x) \leqslant \int_{\mathbf{M}}^n U(u_0)\theta_1\,\mathrm{d}V_g(x)$$

for all $t_0>0$ and, in particular, for all $\theta_1=\theta_1(x)\geqslant 0$

$$\lim_{t\to 0+}\int_{\mathbf{M}}^n \langle \nu_{t,x}, U\rangle \theta_1\,\mathrm{d}V_g(x) \leqslant \int_{\mathbf{M}}^n U(u_0)\theta_1\,\mathrm{d}V_g(x). \tag{5.10}$$

Note that the left-hand limit exists, in view of (a).

Next, consider the set of all linear, convex and finite combinations of the form

$$\sum_j \alpha_j \theta_{1,j}(x) U_j(u),$$

where $\alpha_j\geqslant 0$, $\sum_j \alpha_j = 1$, the functions U_j are smooth and convex in u and the functions $\theta_{1,j}(x)\geqslant 0$ are smooth and compactly support, with moreover

$$\big|U_j(u)\theta_{1,j}(x)\big| \leqslant c|u| + \big|\tilde U_j(x)\big|$$

with $c\geqslant 0$ and $\tilde U_j\in L^1(\mathbf{M}^n;\mathrm{d}V_g)$. This set is dense (for the uniform topology in u and the L^1 topology in x) in the set of all functions $U=U(u,x)$ that are convex in u and measurable in x and satisfy

$$\big|U(u,x)\big| \leqslant c|u| + \big|\tilde U(x)\big|$$

for some $c>0$ and $\tilde U\in L^1(\mathbf{M}^n;\mathrm{d}V_g)$. Therefore, by density, (b) follows from (5.9).

Finally, (c) follows from (b) by choosing $U(u,x)=|u-u_0(x)|$. □

Proof of Theorem 5.3. In all of the following arguments, the inequalities should be understood in the sense of distributions. All steps can be justified rigorously by introducing test-functions in the usual way. Restricting attention in (5.4) to functions θ with compact support, we deduce that

$$\partial_t\langle \nu, U\rangle + \mathrm{div}_g\langle \nu, F\rangle \leqslant 0 \tag{5.11}$$

and so, after introducing the Kruzkov's entropies

$$\tilde U(\bar u,\bar v) = |\bar v-\bar u|, \qquad \tilde F(\bar u,\bar v) = \big(f(\bar v)-f(\bar u)\big)\,\mathrm{sgn}(\bar v-\bar u),$$

we obtain

$$\partial_t\big\langle \nu, \tilde U(\cdot,\bar v)\big\rangle + \mathrm{div}_g\big\langle \nu, \tilde F(\cdot,\bar v)\big\rangle \leqslant 0, \quad \bar v\in\mathbb{R}. \tag{5.12}$$

Let μ be another entropy measure-valued solution to (3.4). We are going to combine (5.12) together with a similar statement for μ, that is,

$$\partial_t\big\langle \mu, \tilde U(\bar u,\cdot)\big\rangle + \mathrm{div}_g\big\langle \mu, \tilde F(\bar u,\cdot)\big\rangle \leqslant 0, \quad \bar u\in\mathbb{R}. \tag{5.13}$$

Introducing the tensor product $\nu\otimes\mu = \nu_{t,x}\otimes\mu_{t,x}$ with, for instance,

$$\langle \nu_{t,x}\otimes\mu_{t,x}, \tilde U\rangle := \iint_{\mathbb{R}^2} \tilde U(\bar u,\bar v)\,\mathrm{d}\nu_{t,x}(\bar u)\,\mathrm{d}\mu_{t,x}(\bar v),$$

we can write (in the sense of distributions)

$$\partial_t\langle \nu\otimes\mu, \tilde U\rangle + \mathrm{div}_g\langle \nu\otimes\mu, \tilde F\rangle = \big\langle \nu, \partial_t\langle \mu, \tilde U\rangle + \mathrm{div}_g\langle \mu, \tilde F\rangle\big\rangle + \big\langle \mu, \partial_t\langle \nu, \tilde U\rangle + \mathrm{div}_g\langle \nu, \tilde F\rangle\big\rangle$$

and we deduce from (5.12) and (5.13) that, in the sense of distributions

$$\partial_t \langle \nu \otimes \mu, \tilde{U} \rangle + \operatorname{div}_g \langle \nu \otimes \mu, \tilde{F} \rangle \leqslant 0. \tag{5.14}$$

Next, integrating (5.14) over the manifold $\mathbf{M}^n$ we find that, for all $0 \leqslant t' \leqslant t$,

$$\int_{\mathbf{M}}^{n} \langle \nu_{t,x} \otimes \mu_{t,x}, \tilde{U} \rangle \, \mathrm{d}V_g(x) \leqslant \int_{\mathbf{M}}^{n} \langle \nu_{t',x} \otimes \mu_{t',x}, \tilde{U} \rangle \, \mathrm{d}V_g(x). \tag{5.15}$$

Letting $t' \to 0$ in (5.15) and using that the two measure-valued solutions assume the same (Dirac-mass) initial data $\delta_{u_0(x)}$ at the time $t = 0$ in the strong sense established in Lemma 5.5, we find

$$\int_{\mathbf{M}}^{n} \langle \nu_{t,x} \otimes \mu_{t,x}, \tilde{U} \rangle \, \mathrm{d}V_g(x) = 0, \quad t \in \mathbb{R}_+.$$

Therefore, $\langle \nu_{t,x} \otimes \mu_{t,x}, \tilde{U} \rangle$ vanishes for almost every (t, x), which is equivalent to saying that $\nu_{t,x}$ and $\mu_{t,x}$ coincide with the same Dirac mass, say $\delta_{u(t,x)}$ for some value $u(t, x)$. Since ν and μ are both entropy measure-valued solutions, the function $u = u(t, x)$ is an entropy solution of the problem (3.4)–(3.7) in the sense of Definition 3.3.

We establish some additional properties of the entropy solution u, as follows. Taking $U(u) := |u|^p$ in (5.11) and integrating over the manifold $\mathbf{M}^n$ yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbf{M}}^{n} U\big(u(t, x)\big) \, \mathrm{d}V_g(x) = \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbf{M}}^{n} \langle \nu_{t,x}, U \rangle \, \mathrm{d}V_g(x) \leqslant 0,$$

which leads to (5.1). Using (5.15) with two distinct solutions $\nu_{t,x} = \delta_{u(t,x)}$ and $\mu_{t,x} = \delta_{v(t,x)}$ gives the L^1 contraction property (5.3). This completes the proof of Theorem 5.3. □

6. General conservation laws and balance laws

6.1. L^1 semi-group of entropy solutions on manifolds

Consider first the case of a conservation law posed on the one-dimensional torus $T^1 = [0, 1]$

$$\partial_t u + \frac{1}{k} \partial_x \big(k f(u)\big) = 0, \quad u = u(t, x) \in \mathbb{R}, \ t \in \mathbb{R}_+, \ x \in [0, 1], \tag{6.1}$$

where $k = k(x)$ is a given, positive function and $f : \mathbb{R} \to \mathbb{R}$ is a convex function. In [10], a suitable generalization of Lax's explicit formula [7] was introduced for (6.1). When k is not a constant, the characteristics of (6.1) are not straight lines but curves $s \mapsto X(s) = X(s; y)$ ($s \geqslant 0$, $y \in \mathbb{R}$), given by

$$\partial_s X(s) = \partial_u f\big(u\big(s, X(s)\big)\big), \qquad X(0) = y.$$

It was observed that along a characteristic the function $v = v(s; y) := u(s, X(s; y))$ is such that the "weighted flux" $k(X) f(v)$ is constant. By introducing suitable left- and right-inverses of the function f, say $f_{\pm}^{-1}$ and then solving the equation $k(X) f(v) = c$, it follows that the whole family of all characteristic curves is described by

$$\partial_s X(s) = \big(\partial_u f \circ f_{\pm}^{-1}\big)\left(\frac{c}{k(X)}\right), \qquad X(0) = y. \tag{6.2}$$

The following result was derived from a detailed analysis of this family of curves:

Theorem 6.1 *(See* [8,10]*). The periodic, entropy solutions with bounded variation to the conservation law* (6.1) *are given by a generalization of Lax's explicit formula via a minimization problem along the curves* (6.2)*. Moreover, any two solutions u, v satisfy the L^1 contraction property*

$$\big\| v(t) - u(t) \big\|_{L^1(0,1)} \leqslant \big\| v(t') - u(t') \big\|_{L^1(0,1)}, \quad 0 \leqslant t' \leqslant t.$$

In view of the discussion in the previous sections, this result may seem surprising since the geometry-compatibility condition is not satisfied here (except in the trivial case where k is a constant). Theorem 6.1 motivates us to extend now our theory on Riemannian manifolds to general conservation laws that need not be geometry-compatible.

From now on we consider a general conservation law (3.4) associated with an arbitrary flux f_x. First of all, we stress that the notion of entropy pair (U, F_x) should still be defined by the same conditions (3.11) as in the geometry-compatible case, but now we no longer have (3.12). Instead, an entropy solution should be characterized by the entropy inequalities

$$\partial_t U(u) + \operatorname{div}_g\big(F(u)\big) - (\operatorname{div}_g F)(u) \leqslant 0 \tag{6.3}$$

in the sense of distributions, for every entropy pair (U, F_x). The term $(\operatorname{div}_g F)(\bar u)$ is defined by applying the divergence operator to the vector field $x \to F_x(\bar u)$, for every fixed $\bar u$.

Theorem 6.2. *Let $f = f_x(\bar u)$ be an arbitrary (not necessarily geometry-compatible) flux on a Riemannian manifold $(\mathbf{M}^n, g)$, satisfying the linear growth condition (for some constant $C_0 > 0$)*

$$\big|f_x(\bar u)\big|_g \leqslant C_0\big(1 + |\bar u|\big), \quad \bar u \in \mathbb{R},\ x \in \mathbf{M}^n. \tag{6.4}$$

Then there exists a unique contractive, semi-group of entropy solutions

$$u_0 \in L^1\big(\mathbf{M}^n\big) \mapsto u(t) := S_t u_0 \in L^1\big(\mathbf{M}^n\big)$$

to the initial value problem (3.4)–(3.7).

The condition (6.4) is required for the flux term (3.4) to be an integrable function on $\mathbf{M}^n$. Note that no uniform L^p estimate is now available, not even in the L^1 norm, since the trivial function $u \equiv 0$ need not be a solution of the conservation law. The L^1 norm of a solution is finite for each time t, but generally *grows as t increases*. This result shows that the contraction property is more fundamental than all of the other stability properties derived earlier for geometry-compatible conservation laws.

For general conservation laws, the stationary solutions $\tilde u$, determined by

$$\operatorname{div}_g\big(f_x\big(\tilde u(x)\big)\big) = 0, \quad x \in \mathbf{M}^n,$$

represent possible asymptotic states of time-dependent solutions.

Proof. The semi-group is constructed first over functions with bounded variation and then extended by density to the whole of L^1. To this end, we re-visit the proof of Theorem 5.3, and check that the key inequality (5.14), needed in the derivation of the L^1 contraction property, remains true regardless of the geometry-compatibility property. Note in passing that the stronger statement (5.11) is *no longer valid* under the present assumptions, and was precisely needed to ensure the stability in all L^p spaces, which is no longer true here.

It thus remains to establish (5.14) for solutions u, v of (3.4) with bounded variation. We rely on standard regularity results (Federer [4], Volpert [12]) for such functions: pointwise values $u_\pm, v_\pm$ can be defined almost everywhere with respect to the Hausdorff measure $\mathcal{H}_n$ on $\mathbb{R}_+ \times \mathbf{M}^n$. These pointwise values coincide at points of approximate continuity. We compute the entropy dissipation measure

$$\mu := \partial_t \tilde U(u, v) + \operatorname{div}_g\big(\tilde F(u, v)\big),$$

and distinguish between the set $\mathcal{C}_{u,v}$ of points of approximate continuity for both u and v, and the set $\mathcal{S}_{u,v}$ of points where u has an approximate jump and v is approximately continuous, or vice-versa. The $\mathcal{H}_n$-Hausdorff measure of the set $\mathcal{N}_{u,v} := \mathbb{R}_+ \times \mathbf{M}^n \setminus \mathcal{C}_{u,v} \cup \mathcal{S}_{u,v}$ is zero. Using standard calculus for BV functions we can write

$$\mu_{|\mathcal{C}_{u,v}} := \operatorname{sgn}(u - v)(\partial_t u - \partial_t v) + \operatorname{sgn}(u - v)\big(\operatorname{div}_g\big(f(u)\big) - \operatorname{div}_g\big(f(v)\big)\big) = 0.$$

On the other hand, if $B \subset \mathcal{S}_{u,v}$ is a Borel subset consisting of points where (for instance) v is approximately continuous then

$$\begin{aligned}\mu(B) := \int_B \big(&n_t|u_+ - v| + g_x\big(n_x, \operatorname{sgn}(u_+ - v)\big(f_x(u_+) - f_x(v)\big)\big)\\ &- n_t|u_- - v| - g_x\big(n_x, \operatorname{sgn}(u_- - v)\big(f_x(u_-) - f_x(v)\big)\big)\big)\,\mathrm{d}\mathcal{H}_n(t, x),\end{aligned}$$

where $n = (n_t(t,x), n_x(t,x) \in \mathbb{R} \times T_x\mathbf{M}^n$ is the propagation speed vector associated with the discontinuity in u. Now, by relying on the entropy inequalities, $\mu\{(t,x)\}$ can easily be checked to be non-positive. In turn, the measure μ is also non-positive on $\mathcal{S}_{u,v}$ and, in turn, on the whole of $\mathbb{R}_+ \times \mathbf{M}^n$. The argument is complete. □

6.2. General balance laws

It is not difficult to generalize the well-posedness theory to the balance law

$$\partial_t u + \operatorname{div}_g\big(f(u,\cdot)\big) = h(u,\cdot),$$

where $f = f_x(\bar{u}, t)$ is a family of vector fields on $\mathbf{M}^n$ depending (smoothly) on the time variable t and on the parameter $\bar{u}$, and $h = h(\bar{u}, t, x)$ is a smooth function. Our previous results should be modified to take into account the dependence of f in t and the effect of the source-term h. For instance, for L^∞ initial data, the L^p estimate (5.1) should be replaced with

$$\big\|u(t)\big\|_{L^p(\mathbf{M}^n;\mathrm{d}V_g)} \leqslant C_p(T)\big\|u(t')\big\|_{L^p(\mathbf{M}^n;\mathrm{d}V_g)} + C'(T), \quad 0 \leqslant t' \leqslant t \leqslant T,$$

where the constant $C_p(T) \geqslant 1$ in general depends on T and $C'(T)$ also depends upon g. Similarly, the contraction property should be replaced by

$$\big\|v(t) - u(t)\big\|_{L^1(\mathbf{M}^n;\mathrm{d}V_g)} \leqslant C_0(T)\|v_0 - u_0\|_{L^1(\mathbf{M}^n;\mathrm{d}V_g)}, \quad t \in \mathbb{R}_+.$$

for some constant $C_0(T)$. We omit the details.

7. Conservation laws on Lorentzian manifolds

7.1. Globally hyperbolic Lorentzian manifold

Motivated by the mathematical developments in general relativity, we now extend our theory to Lorentzian manifolds.

Let $(\mathbf{M}^{n+1}, g)$ be a time-oriented, $(n+1)$-dimensional Lorentzian manifold, g being a pseudo-Riemannian metric tensor on $\mathbf{M}^{n+1}$ with signature $(-,+,\ldots,+)$. Recall that tangent vectors X on a Lorentzian manifold can be separated into time-like vectors ($g(X,X) < 0$), null vectors ($g(X,X) = 0$), and space-like vectors ($g(X,X) > 0$). A vector field is said to be time-like, null, or space-like if the corresponding property hold at every point. The null cone separates time-like vectors into future-oriented and past-oriented ones; it is assumed here that the manifold is time-oriented, i.e. a consistent orientation can be chosen throughout the manifold.

Let ∇ be the Levi-Cevita connection associated with the Lorentzian metric g so that, in particular, $\nabla g = 0$. The divergence div_g operator is defined in a standard way which is formally similar to the Riemannian case.

Definition 7.1. A *flux* on the manifold $\mathbf{M}^{n+1}$ is a vector field $x \mapsto f_x(\bar{u}) \in T_x\mathbf{M}^{n+1}$, depending on a parameter $\bar{u} \in \mathbb{R}$. The *conservation law* on $(\mathbf{M}^{n+1}, g)$ associated with f is

$$\operatorname{div}_g\big(f(u)\big) = 0, \quad u : \mathbf{M}^{n+1} \to \mathbb{R}. \tag{7.1}$$

It is said to be geometry compatible if f satisfies the condition

$$\operatorname{div}_g f_x(\bar{u}) = 0, \quad \bar{u} \in \mathbb{R},\ x \in \mathbf{M}^{n+1}. \tag{7.2}$$

Furthermore, f is said to be a time-like flux if

$$g_x\big(\partial_u f_x(\bar{u}), \partial_u f_x(\bar{u})\big) < 0, \quad x \in \mathbf{M}^{n+1},\ \bar{u} \in \mathbb{R}. \tag{7.3}$$

Note that our terminology here differs from the one in the Riemannian case, where the conservative variable was singled out.

We are interested in the initial-value problem associated with (7.1). We fix a space-like hypersurface $\mathcal{H}_0 \subset \mathbf{M}^{n+1}$ and a measurable and bounded function u_0 defined on $\mathcal{H}_0$. Then, we search for a function $u = u(x) \in L^\infty(\mathbf{M}^{n+1})$ satisfying (7.1) in the distributional sense and such that the (weak) trace of u on $\mathcal{H}_0$ coincides with u_0:

$$u_{|\mathcal{H}_0} = u_0. \tag{7.4}$$

It is natural to require that the vectors $\partial_u f_x(\bar{u})$, which determine the propagation of waves in solutions of (7.1), are time-like and future-oriented.

We assume that the manifold $\mathbf{M}^{n+1}$ is *globally hyperbolic*, in the sense that there exists a foliation of $\mathbf{M}^{n+1}$ by space-like, compact, oriented hypersurfaces $\mathcal{H}_t$ ($t \in \mathbb{R}$):

$$\mathbf{M}^{n+1} = \bigcup_{t\in\mathbb{R}} \mathcal{H}_t.$$

Any hypersurface $\mathcal{H}_{t_0}$ is referred to as a *Cauchy surface* in $\mathbf{M}^{n+1}$, while the family $\mathcal{H}_t$ ($t \in \mathbb{R}$) is called an *admissible foliation associated with* $\mathcal{H}_{t_0}$. The future of the given hypersurface will be denoted by

$$\mathbf{M}^{n+1}_+ := \bigcup_{t\geqslant 0} \mathcal{H}_t.$$

Finally we denote by n^t the future-oriented, normal vector field to each $\mathcal{H}_t$, and by g^t the induced metric. Finally, along $\mathcal{H}_t$, we denote by X^t the normal component of a vector field X, thus $X^t := g(X, n^t)$.

Definition 7.2. A flux $F = F_x(\bar{u})$ is called a convex entropy flux associated with the conservation law (7.1) if there exists a convex function $U : \mathbb{R} \to \mathbb{R}$ such that

$$F_x(\bar{u}) = \int_0^{\bar{u}} \partial_u U(u')\partial_u f_x(u')\,\mathrm{d}u', \quad x \in \mathbf{M}^{n+1},\ \bar{u} \in \mathbb{R}.$$

A measurable and bounded function $u = u(x)$ is called an entropy solution of the geometry-compatible conservation law (7.1)–(7.2) if the following entropy inequality

$$\int_{\mathbf{M}^{n+1}_+} g\big(F(u), \operatorname{grad}_g \theta\big)\,\mathrm{d}V_g + \int_{\mathcal{H}_0} g_0\big(F(u_0), n_0\big)\theta_{\mathcal{H}_0}\,\mathrm{d}V_{g_0} \geqslant 0. \tag{7.5}$$

for all convex entropy flux $F = F_x(\bar{u})$ and all smooth functions $\theta \geqslant 0$ compactly supported in $\mathbf{M}^{n+1}_+$.

In particular, in (7.5) the inequality

$$\operatorname{div}_g\big(F(u)\big) \leqslant 0$$

holds in the distributional sense.

Theorem 7.3. *Consider a geometry-compatible conservation law* (7.1)–(7.2) *posed on a globally hyperbolic Lorentzian manifold* $\mathbf{M}^{n+1}$. *Let* $\mathcal{H}_0$ *be a Cauchy surface in* $\mathbf{M}^{n+1}$, *and* $u_0 : \mathcal{H}_0 \to \mathbb{R}$ *be a measurable and bounded function. Then, the initial-value problem* (7.1)–(7.4) *admits a unique entropy solution* $u = u(x) \in L^\infty(\mathbf{M}^{n+1})$. *For every admissible foliation* $\mathcal{H}_t$, *the trace* $u_{\mathcal{H}_t}$ *exists and belong to* $L^1(\mathcal{H}_t)$, *and the functions*

$$\big\| F^t(u_{\mathcal{H}_t}) \big\|_{L^1(\mathcal{H}_t)},$$

are non-increasing in time, for any convex entropy flux F. *Moreover, given any two entropy solutions* u, v, *the function*

$$\big\| f^t(u_{\mathcal{H}_t}) - f^t(v_{\mathcal{H}_t}) \big\|_{L^1(\mathcal{H}_t)} \tag{7.6}$$

is non-increasing in time.

We emphasize that, in the Lorentzian case, no time-translation property is available in general, contrary to the Riemannian case. Hence, no time-regularity is implied by the L^1 contraction property.

As the proof is very similar to the one in the Riemannian case, we will content with sketching the proof. Introduce a local chart

$$x = \big(x^\alpha\big) = \big(t, x^j\big), \qquad g := g_{\alpha\beta}\,\mathrm{d}x^\alpha\,\mathrm{d}x^\beta,$$

where by convention Greek indices describe $0, 1, \dots, n$ and Latin indices describe $1, \dots, n$. By setting $f =: (f_x^\alpha(\bar{u}))$ and using local coordinates, the conservation law (7.1) reads

$$\partial_\alpha\big(|g_x|^{1/2} f_x^\alpha\big(u(x)\big)\big) = 0, \tag{7.7}$$

where $|g| := \det(g_{\alpha\beta})$. Thanks to the assumption on f, for all *smooth* solutions (7.7) takes the equivalent form

$$g_x\big((\partial_u f_x)\big(u(x)\big), \operatorname{grad}_g u(x)\big) := \big(\partial_u f_x^\alpha\big)\big(u(x)\big)(\partial_\alpha u)(x) = 0. \tag{7.8}$$

In other words, setting $x = (t, \bar{x})$ and $f = (f^t, f^j)$,

$$\big(\partial_u f_x^t\big)\big(u(t,\bar{x})\big)\partial_t u(t,\bar{x}) + \big(\partial_u f_x^j\big)\big(u(t,\bar{x})\big)(\partial_j u)(t,\bar{x}) = 0,$$

in which, since $\partial_u f$ is future-oriented and time-like, the coefficient in front of the time-derivative is positive

$$\partial_u f_x^0(\bar{u}) > 0, \quad \bar{u} \in \mathbb{R}, x \in \mathbf{M}^{n+1}.$$

To proceed with the construction of the entropy solutions, we add a vanishing diffusion term, as follows:

$$\operatorname{div}_g\big(f(u^\epsilon)\big) = \epsilon \bar{\Delta}_{\bar{g}} u^\epsilon, \tag{7.9}$$

where $\bar{\Delta}_{\bar{g}}$ is the Laplace operator on the leaves $\mathcal{H}_t$ of the foliation, that is in coordinates

$$\big(\partial_u f_x^0\big)\big(u^\epsilon(t,\bar{x})\big)\partial_t u^\epsilon(t,\bar{x}) + \big(\partial_u f_x^j\big)\big(u^\epsilon(t,\bar{x})\big)(\partial_j u^\epsilon)(t,\bar{x}) = \epsilon g^{ij}(t,\bar{x})\big(\partial_i\partial_j u^\epsilon - \Gamma_{ij}^k \partial_k u^\epsilon\big)(t,\bar{x}). \tag{7.10}$$

In view of (7.10), we see that all of the estimates follow similarly as in the Riemannian case. For instance, multiplying (7.10) by a convex function U, the entropy inequality now takes the form

$$\begin{aligned}
&\partial_t\big(F_x^0\big(u^\epsilon(t,\bar{x})\big)\big) + \partial_j\big(F_x^j\big(u^\epsilon(t,\bar{x})\big)\big)\\
&\quad = \partial_u U\big(u(t,\bar{x})\big)\big(\partial_u f_x^0\big)\big(u^\epsilon(t,\bar{x})\big)\partial_t u^\epsilon(t,\bar{x}) + \partial_u U\big(u(t,\bar{x})\big)\big(\partial_u f_x^j\big)\big(u^\epsilon(t,\bar{x})\big)\big(\partial_j u^\epsilon\big)(t,\bar{x})\\
&\quad = \epsilon\partial_u U\big(u(t,\bar{x})\big) g^{ij}(t,\bar{x})\big(\partial_i\partial_j u^\epsilon - \Gamma_{ij}^k\partial_k u^\epsilon\big)(t,\bar{x})\\
&\quad = \epsilon\bar{\Delta}_{\bar{g}} U(u^\epsilon) - \epsilon\partial_u^2 U(u^\epsilon) g^{jk}\partial_j u^\epsilon \partial_k u^\epsilon.
\end{aligned} \tag{7.11}$$

The metric $\bar{g}$ induced on the space-like leaves $\mathcal{H}_t$ is positive-definite and, therefore, the latter term above if non-positive and we conclude that, given a geometry compatible flux f and for every convex entropy flux F

$$\operatorname{div}_g\big(F(u^\epsilon)\big) = \partial_t\big(F_x^0\big(u^\epsilon(t,\bar{x})\big)\big) + \partial_j\big(F_x^j\big(u^\epsilon(t,\bar{x})\big)\big) \leqslant \epsilon\bar{\Delta}_{\bar{g}} U(u^\epsilon).$$

Note that the regularization (7.9) does depend on the specific foliation under consideration. However, by the contraction property (7.6) the limiting solution is unique and independent of the chosen regularization mechanism.

7.2. Schwarzschild spacetime

In the context of general relativity, the Schwarzschild metric represents a spherically symmetric empty spacetime surrounding a black hole with mass m and is one of the most important example of Lorentzian metrics. The outer communication region of the Schwarzschild spacetime is a $1+3$-Lorentzian manifold with boundary, described in the so-called Schwarzschild coordinates (t, r, ω) by

$$g = -\left(1 - \frac{2m}{r}\right)\mathrm{d}t^2 + \left(1 - \frac{2m}{r}\right)^{-1}\mathrm{d}r^2 + r^2\,\mathrm{d}\omega^2,$$

with $t > 0$ and $r > 2m$, while ω describes the 2-sphere. There is an apparent (but not a physical) singularity in the metric coefficients at $r = 2m$, which corresponds to the horizon of the spacetime. This spacetime is spherically symmetric, that is invariant under the group of rotations operating on the space-like 2-spheres given by keeping t and r constant. The part $r^2\,\mathrm{d}\omega^2$ of the metric is the canonical metric on the 2-spheres of symmetry. The spacetime under consideration is static, since the vector field ∂_t is a time-like Killing vector. Moreover, this metric is asymptotic to the flat metric when $r \to \infty$. Theorem 7.3 extends to the exterior of the Schwarzschild spacetime, by observing that along the boundary $r = 2m$ the characteristics of the hyperbolic equation are outgoing.

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