Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 25 (2008) 633–658
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

A variational treatment for general elliptic equations of the flame propagation type: regularity of the free boundary ☆

Un traitement variationnel pour les équations elliptiques générales du type de celles de la propagation des flammes : régularité de la frontiére libre

Eduardo V. Teixeira

Rutgers University, Department of Mathematics, Hill Center, Busch Campus, 110 Frelinghuysen Road, Piscataway, NJ 08854-8019, USA
Departamento de Matemática, Universidade Federal do Ceará, 60455-760 Fortaleza, CE, Brasil

Received 20 April 2006; accepted 20 February 2007

Available online 24 July 2007

Abstract

We develop a variational theory to study the free boundary regularity problem for elliptic operators: $Lu = D_j(a_{ij}(x)D_iu) + b_iu_i + c(x)u = 0$ in $\{u > 0\}$, $\langle a_{ij}(x)\nabla u, \nabla u\rangle = 2$ on $\partial\{u > 0\}$. We use a singular perturbation framework to approximate this free boundary problem by regularizing ones of the form: $Lu_\varepsilon = \beta_\varepsilon(u_\varepsilon)$, where β_ε is a suitable approximation of Dirac delta function δ_0. A useful variational characterization to solutions of the above approximating problem is established and used to obtain important geometric properties that enable regularity of the free boundary. This theory has been developed in connection to a very recent line of research as an effort to study existence and regularity theory for free boundary problems with gradient dependence upon the penalization.

Résumé

Nous développons une théorie variationnelle pour l'étude du problème de la régularité de la frontière libre pour des opérateurs elliptiques : $Lu = D_j(a_{ij}(x)D_iu) + b_iu_i + c(x)u = 0$ en $\{u > 0\}$, $\langle a_{ij}(x)\nabla u, \nabla u\rangle = 2$ en $\partial\{u > 0\}$. Nous régularisons et approximons la frontière libre par une méthode de perturbation singulière de la forme : $Lu_\varepsilon = \beta_\varepsilon(u_\varepsilon)$, où β_ε est une approximation adaptée de la fonction delta de Dirac δ_0. Une caractérisation variationnelle des solutions du problème d'approximation ci-dessus est établie et employée pour obtenir les propriétés géométriques importantes qui impliquent la régularité de la frontière libre. Cette théorie a été développée en connection avec une ligne très récente de recherche comme effort pour étudier la théorie d'existence et de régularité pour des problèmes de la frontière libre avec la dépendance de gradient sur la pénalisation.

Keywords: Free boundary problems; Singular perturbation; Complete elliptic operator; Regularity theory

☆ This work was partially supported by NFS Grant DMS 0600930.
E-mail address: teixeira@math.rutgers.edu.

0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2007.02.006

1. Introduction

In this paper, we present a systematic variational approach to study existence and geometric properties of a rich class of free boundary elliptic problems. Namely, we are concerned about finding a nonnegative function u satisfying

$$\begin{aligned} &D_j\big(a_{ij}(x)D_iu\big) + b_iu_i + c(x)u = 0 \quad \text{in } \{u>0\},\\ &\big\langle a_{ij}(x)\nabla u, \nabla u\big\rangle = 2 \quad \text{on } \partial\{u>0\}. \end{aligned} \tag{1.1}$$

Our basic assumptions are: $a_{ij}(x)$ is uniform elliptic and of class C^γ, b_i, c are bounded measurable functions and $c \leqslant 0$. Our ultimate goal is to study qualitative properties of the free boundary $\partial\{u>0\}$.

We shall use a singular penalization method to generate smooth approximating solutions to our free boundary problem. More specifically, let Ω be a smooth bounded domain in $\mathbb{R}^N$ and $\varphi : \partial\Omega \to \mathbb{R}_+$ a smooth nonnegative function, $\varphi \not\equiv 0$. Consider β to be a smooth nonnegative function satisfying:

1. Support of β lies in $[0,1]$ and it is positive in $(0,1)$.
2. $\int_0^1 \beta(s)\,ds := 1$.

We then define

$$\beta_\varepsilon(s) := \frac{1}{\varepsilon}\beta\left(\frac{s}{\varepsilon}\right) \quad \text{and} \quad B_\varepsilon(s) := \int_0^s \beta_\varepsilon(\tau)\,d\tau. \tag{1.2}$$

We will be interested in appropriated limiting functions, as ε goes to zero, of solutions to

$$\begin{cases} D_j(a_{ij}(x)D_iu_\varepsilon) + b_i(u_\varepsilon)_i + c(x)u_\varepsilon = \beta_\varepsilon(u_\varepsilon) & \text{in } \Omega,\\ u_\varepsilon = \varphi & \text{on } \partial\Omega. \end{cases} \tag{1.3}$$

For each $\varepsilon > 0$ fixed, Eq. (1.3) models various problems in applied mathematics. For example, several problem in biology, such as, population dynamics, gene developments, epidemiology, among others can be modeled in terms of Eq. (1.3) (see [14]). Eq. (1.3) is also used to model, for instance, the flame propagation in a tube. For the combustion problem, however, the model becomes more accurate as $\varepsilon \to 0$, leading us to the free boundary problem (1.1). As one could expect, the mathematical analysis involved in the study of this singular limiting problem is substantially more challenging.

The problem $\Delta u_\varepsilon = \beta_\varepsilon(u_\varepsilon)$ was fully studied in the late 70's and early 80's by Lewy–Stampacchia, Caffarelli, Kinderlehrer and Nirenberg, Alt and Phillips, among others. Lederman and Wolanski in [17], gave a nice treatment for the problem $\Delta u^\varepsilon = \beta_\varepsilon(u^\varepsilon)$, with no sign restriction. Under nondegeneracy assumptions they manage to show that if free boundary has an inward unit normal in the measure-theoretic sense at a point $x_0 \in \Omega \cap \partial\{u>0\}$, then the free boundary is a $C^{1,\alpha}$ surface in a neighborhood of x_0.

Berestycki, Caffarelli and Nirenberg in [2] started the journey of analyzing uniform estimates for

$$Lu = a_{ij}(x)u_{ij} + b_i(x)u_i + c(x)u = \beta_\varepsilon(u), \tag{1.4}$$

with C^1 coefficients. Further regularity and geometric properties of the limiting free boundary could not be addressed, though. The major difficult one encounters in trying to establish finer regularity properties of the limiting free boundary problem arising as $\varepsilon \to 0$ in either (1.1) or in (1.4), is a lack of reasonable nondegeneracy condition. Notice that, for each $\varepsilon > 0$, in general, uniqueness does not hold for Eq. (1.3). Empirically speaking, it turns out that a suitable nondegeneracy of a limiting function $u_0 := \lim u_\varepsilon$ can be obtained as long as we have a "stable" way of selecting particular weak solutions to (1.3). The key strategy we suggest in this article is, that, even though, Eq. (1.3) does not have an Euler–Lagrange Functional associated to it, due to the nonzero 1st order term, one should look for weak solutions that satisfies a particular minimization property.

The parabolic version of approximating problems has been considered as well. For instance in [11], the authors study the limit $u(x,t)$ as $\varepsilon \to 0$ of the solutions $u^\varepsilon(x,t)$ of the two-phase parabolic equation $\Delta u^\varepsilon - u^\varepsilon_t = \beta_\varepsilon(u^\varepsilon)$ in $D \subset \mathbb{R}^{n+1}$. The main concern is in what sense $u(x,t)$ satisfies the limit equation (P) $\Delta u - u_t = 0$ in $D \setminus \partial\{u>0\}$, $u = 0$, $(u_\nu^+)^2 - (u_\nu^-)^2 = 2$ on $D \cap \partial\{u>0\}$.

For a didactical reason, we have chosen to present our theory for the elliptic operator $Lu = \Delta u - \vec{v}\nabla u$, with $\vec{v} \in L^\infty(\Omega, \mathbb{R}^N)$. Basically because the main difficulty of dealing with complete elliptic operators lies, as we will point out, in the fact that there is no Euler–Lagrange functional associated to the equation. The fact that we require a low regularity of the matrix a_{ij} in (1.3), namely, a_{ij} is a merely C^γ elliptic matrix, certainly brings some technical difficulties. However, in a companion work, [18], the authors present a rather complete description of the limiting problem $\operatorname{div}(A(x)\nabla u) = \beta_\varepsilon(u)$, with $A(x)$ Hölder continuous. Throughout the whole paper we shall point the corresponding result one obtains for general operators of the form $D_j(a_{ij}(x)D_i u) + b_i u_i + c(x)u$ and we shall refer to [18] for technical details.

Our paper is organized as follows. In Section 2 we present the heuristic principle that supports the paper. As we will see in Section 4, if we can, somehow, obtain a useful variational characterization of solutions to the approximating free boundary problem (1.3), it is possible to derive a linear growth away from the free boundary (Corollary 4.7). A minimization property of solutions to Eq. (1.1) also allows, via a natural perturbation argument, uniform density of the zero set $\{u > 0\}^{\mathrm{C}}$ (Theorem 5.5).

In Section 3, we show for particular cases, how one can obtain a variational characterization proposed in Section 2. Initially we discuss a fixed point argument that enables an interesting variational characterization of the form:

$$\int_\Omega \frac{1}{2}|\nabla u_\varepsilon|^2 + (\vec{v}\cdot\nabla u_\varepsilon)u_\varepsilon + B_\varepsilon(u_\varepsilon)\,dx = \min_{\xi\in H^1_\varphi} \int_\Omega \frac{1}{2}|\nabla \xi|^2 + (\vec{v}\cdot\nabla u_\varepsilon)\xi + B_\varepsilon(\xi)\,dx.$$

We specifically apply this strategy for the flame propagation equation in cylinder domains: $\Delta u - v(y)\partial_1 u = \beta_\varepsilon(u)$ in $(-a, a)\times\omega$. Here, a deep result from [3] is used to assure that, under natural conditions, there exists a solution of this equation fulfilling the above variational characterization (Theorem 3.6). Afterwards we, motivated by physical considerations, study the case when the vector field $\vec{v}$ is a potential, i.e., $\vec{v} = -\nabla\phi$. In this important physical situation, show how to obtain a variational characterization like proposed in the heuristic principle (Proposition 3.7).

Lipschitz regularity and geometric measure properties of level sets of solutions u_ε of Eq. (1.3) are derived in Section 4. Ellipticity is used together with perturbation arguments based on the variational characterization, to obtain, among other geometric properties, linear growth away from the free boundary, nondegeneracy, local behavior of the free boundary in terms of the $\mathcal{H}^{N-1}$ Hausdorff measure, uniform density. The fact that the regularity results obtained in this section are uniform in ε is very important, since those are approximating equations for our original free boundary problem.

In Section 5, we shall carry the information obtained in Section 4 over in the limit as $\varepsilon \to 0$. Those geometric properties are used, as done in [1], to assure that $\mathcal{H}^{N-1}(F(u_0)\setminus F(u_0)_{\mathrm{red}}) = 0$. The free boundary condition is derived in Section 6. In the last section, we analyze the limit of the blow-up sequence, $u_k(x) = \rho_k^{-1}u_0(x_k + \rho_k x)$, with $\rho_k \to 0$ and $x_k \to x_0$, $u_0(x_k) = 0$. We show any blow-up sequence u_k converges to the same linear function. This should be interpreted as a result concerning the asymptotic behavior of u close to the free boundary. Higher regularity of the free boundary, i.e. $C^{1,\alpha}$ regularity of $\partial_{\mathrm{red}}\{u_0 > 0\}$, for the case when $Lu = \Delta u - \vec{v}\cdot\nabla u$ follows by a small variant of the remarkable work of Luis A. Caffarelli [7,8]. For a general elliptic operator, i.e., $Lu = D_j(a_{ij}(x)D_i u) + b_i u_i + c(x)u$, under the assumption of Lipschitz continuity of a_{ij}, such a regularity result will be a consequence of a very recent and important work of Fausto Ferrari and Sandro Salsa [13].

2. A variational characterization: the heuristic principle

At the moment we are interested in studying the regularizing problem

$$\begin{cases} \Delta u - \vec{v}\cdot\nabla u = \beta_\varepsilon(u) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{2.1}$$

where $\vec{v} \in L^\infty(\Omega, \mathbb{R}^N)$. The idea is that Eq. (2.1) approximates the free boundary problem

$$\begin{aligned} &\Delta u - \vec{v}\cdot\nabla u = 0 && \text{in } \Omega^+ := \{x\in\Omega\colon\ u(x) > 0\}, \\ &|\nabla u|^2 = 2 && \text{on } \partial\Omega^+. \end{aligned} \tag{2.2}$$

It is a fruitful idea to study free boundary problems like Eq. (2.2) via regularizing problems like (2.1). Information about the original free boundary problem can often be obtained by establishing results for the approximating ones that are uniform on ε.

The key point of our strategy is to obtain a nice variational characterization of solutions of problem (2.1). In this way we shall be able to deeply investigate regularity properties of problem (2.2), via perturbation arguments.

Before presenting the core of our approach, let us recall the basic ideas of the theory of Quasi-Minima introduced by Mariano Giaquinta and Enrico Giusti in [15]. Our intention is to make a parallel between this theory and the strategy we shall introduce to properly study regularity properties of problem (2.2).

Let $F(x,u,z)$, $F:\Omega\times\mathbb{R}^m\times\mathbb{R}^{mN}\to\mathbb{R}_+$ be a nonnegative function. Let us consider the functional

$$\mathcal{F}(u,\omega):=\int_\omega F(x,u,Du)\,dx.$$

It is well known that, under natural assumptions on F, the functional $\mathcal{F}$ attains its minimum over certain Sobolev spaces. Minimizers of the above functional turn out to be much more regular than mere Sobolev functions.

Definition 2.1. A function $u\in W^{1,p}(\Omega,\mathbb{R}^m)$ is a quasi-minimum of the functional $\mathcal{F}$, with constant $Q\geqslant 1$ (briefly: a Q-minimum), if for every $v\in W^{1,p}(\Omega,\mathbb{R}^m)$, with $K:=\operatorname{supp}(u-v)\Subset\Omega$, we have

$$\mathcal{F}(u,K)\leqslant Q\mathcal{F}(v,K).$$

One of the main accomplishment of this theory is that, not only minimizers, i.e. 1-minima, are special functions, but also Q-minima are regular functions as well. We will focus our attention in another property of the Q-minimum theory, which probably is its main motivation. Consider the system of partial differential equations in divergence form

$$\frac{\partial}{\partial x_i}A^i_\alpha\big(x,u(x),Du(x)\big)-B_\alpha\big(x,u(x),Du(x)\big)=0. \tag{2.3}$$

We recall that a function $u\in W^{1,p}_{\mathrm{loc}}(\Omega,\mathbb{R}^m)$ is weak solution of Eq. (2.3) if for every $\varphi\in W^{1,p}_0(\Omega,\mathbb{R}^m)$ we have

$$\int_\Omega A^i_\alpha\big(x,u(x),Du(x)\big)D_i\varphi^\alpha+B_\alpha\big(x,u(x),Du(x)\big)\varphi^\alpha\,dx=0.$$

The most common way of finding a weak solution to Eq. (2.3) is to minimize, or more generally to obtain a critical point of an associated functional. However, sometimes it is not possible to find such a functional. For instance, in general, problem (2.1) does not (immediately) allow a minimization characterization. However, it can be proven that a weak solution of Eq. (2.3) is a Q-minimum of

$$\mathcal{F}(u,\omega):=\int_\omega |Du|^p+b(x)|u|^\gamma+a(x)\,dx,$$

where a and b are intrinsically related to the behavior of the nonlinearities A_α and B_α. For more details see [15].

To summarize (and justify this apparent digression) let us highlight that, even though, in general, Eq. (2.3) does not admit a variational characterization, weak solutions satisfy a sort of minimization property for a specific functional. This information can be explored to prove regularity results for weak solutions of Eq. (2.3).

Let us return to our original purpose. Heuristically, our purpose is to find a solution to the regularizing free boundary problem

$$\begin{cases}\Delta u-\vec{v}\cdot\nabla u=\beta_\varepsilon(u) & \text{in } \Omega,\\ u=\varphi & \text{on } \partial\Omega,\end{cases}$$

that has a useful variational characterization to be described now.

Definition 2.2. Let μ be a Radon measure. We denote by $H^1(\Omega,d\mu)$ the set of functions $\psi\in H^1(\Omega)$ such that ψ and its weak derivatives belong to $L^2(\Omega,d\mu)$. We also denote by $H^1_\varphi(\Omega,d\mu):=\{\psi\in H^1(\Omega,d\mu)\mid \psi\equiv\varphi \text{ on } \partial\Omega\}$.

For reason that will become clear later, we are driven to consider solutions to the above problem that minimize

$$\mathcal{F}_\varepsilon(\zeta):=\int_\Omega\left(\frac{1}{2}|\nabla\zeta|^2+\sigma_\varepsilon(x)\zeta+B_\varepsilon(\zeta)\right)d\mu_\varepsilon, \tag{2.4}$$

among $H^1_\varphi(\Omega, d\mu_\varepsilon)$. Here $\sigma_\varepsilon : \Omega \to \mathbb{R}$ is a bounded function and μ_ε is a positive Radon measure which is absolutely continuous w.r.t. the Lebesgue measure. From the Radon–Nikodym Theorem, there exists an integrable function F_ε, such that $\mu_\varepsilon = F_\varepsilon\, dx$. In general, the function σ_ε and the measure μ_ε depend on ε and on the vector field $\vec{v}$. The reasonable properties on σ_ε and μ_ε are:

1. The functions σ_ε is uniformly locally bounded, i.e., for any subset $\tilde{\Omega} \Subset \Omega$, there exists a constant $C(\tilde{\Omega})$, independently of ε, such that,
$$\|\sigma_\varepsilon\|_{L^\infty(\tilde{\Omega})} \leqslant C(\tilde{\Omega}), \quad \forall \varepsilon > 0.$$
2. There exist universal constants $0 < c < C < \infty$ such that $c \leqslant F_\varepsilon \leqslant C$ for almost everywhere $x \in \Omega$ and $\{F_\varepsilon\}$ is relatively compact in $L^2(dx)$.

The above conditions can be relaxed from the mathematical point of view; however, these properties are satisfied by the physical problems we are concerned with, as we shall see in the next section.

3. Motivation and special cases

The intention of this section is to justify the variational characterization assumed in (2.4). Here we shall explore two situations for which one can obtain a variational characterization as suggested in the preceding section. We point out that the settings we shall explore in this section arise from very natural physical considerations.

In Subsection 3.1, we explore a fixed point idea, which can be widely applied, as long as the problem has some special geometry that allows, in some sense, uniqueness results. In Subsection 3.2, we show how to obtain a useful variational characterization like in (2.4) when the field $\vec{v}$ is a potential. This is a quite natural assumption for many physical problems equations (2.1) and (2.2) model.

3.1. Fixed point argument

The mathematical fundaments of this approach is the following. For each $f \in H^1_\varphi(\Omega)$ let us define the functional $E_f : H^1_\varphi(\Omega) \to \mathbb{R}$ by
$$E_f(\xi) := \int_\Omega \frac{1}{2}|\nabla \xi|^2 + (\vec{v} \cdot \nabla f)\xi + B_\varepsilon(\xi)\, dx.$$

Proposition 3.1. *For each $f \in H^1_\varphi(\Omega)$ there exists a $u_f \in H^1_\varphi(\Omega)$ such that*
$$E_f(u_f) = \min_{H^1_\varphi(\Omega)} E_f.$$
Furthermore $u_f \in H^2_\varphi(\Omega)$ and
$$\|u_f\|_{H^2(\Omega)} \leqslant C\big\{\|\beta_\varepsilon(u_f) + \vec{v} \cdot \nabla f\|_2 + \|u_f\|_2 + C_\varphi\big\}. \tag{3.1}$$

Proof. Initially, notice that E_f is coercive. Indeed,
$$E_f(\xi) \geqslant \frac{1}{2}\|\nabla \xi\|_2^2 - C_\Omega \big\||\vec{v} \cdot \nabla f|\big\|_2 \cdot \|\nabla \xi\|_2 - C(\Omega, \vec{v}, f, \varphi), \tag{3.2}$$
where we have used Poincaré and Hölder inequalities. Furthermore, it is classical to show E_f is weakly lower semicontinuous. It guarantees the existence of a minimizer u_f. The fact that $u_f \in H^2_\varphi(\Omega)$ as well as estimate (3.1) come from standard elliptic regularity theory. □

Let E be a set. Recall 2^E denotes the set of all subsets of E. We shall define the map $\mathfrak{F} : H^1_\varphi(\Omega) \to 2^{H^1_\varphi(\Omega)}$ by
$$\mathfrak{F}(f) := \Big\{u_f \in H^1_\varphi(\Omega) \;\Big|\; E_f(u_f) = \min_{H^1_\varphi(\Omega)} E_f\Big\}.$$

Because, in principal, there is no uniqueness for minimizers of the functional E_f, the operator $\mathfrak{F}$ as defined above is indeed multi-valued, justifying therefore why its target space is $2^{H^1_\varphi(\Omega)}$. Notice that if we can find a fixed point u_ε for the multi-valued map $\mathfrak{F}$, i.e., if we can obtain a u_ε such that $u_\varepsilon \in \mathfrak{F}$, simply by differentiating the functional E_{u_ε}, we would see that u_ε is a solution to problem (2.1). More important than that is the fact that u_ε would satisfy the variational characterization

$$E_{u_\varepsilon}(u_\varepsilon) = \min_{H^1_\varphi} E_{u_\varepsilon}.$$

Comparing the above variational characterization to (2.4), we obtain:

1. $\sigma_\varepsilon = \vec{v} \cdot \nabla u_\varepsilon$,
2. The estimate $\|\sigma_\varepsilon\|_{L^\infty(\tilde{\Omega})} \leqslant C(\tilde{\Omega})$, for any $\tilde{\Omega} \Subset \Omega$, comes from uniform Lipschitz estimate of solutions of Eq. (2.1), to be established at the beginning of the next section.
3. $d\mu_\varepsilon = dx$.

The next proposition is a hope of finding a fixed point for $\mathfrak{F}$.

Proposition 3.2. *The operator $\mathfrak{F}$ is compact.*

Proof. Let f_n be a sequence in $H^1_\varphi(\Omega)$ converging to f in the weak-$H^1_\varphi(\Omega)$ sense. For any selection $u_n \in \mathfrak{F}(f_n)$, we need to show, up to a subsequence, $u_n \to u$ strongly in $H^1(\Omega)$ and $u \in \mathfrak{F}(f)$. Notice, first of all, that by (3.2), $\|\nabla u_n\|_2 \leqslant C$. This information, together with estimate (3.1) implies $\|u_n\|_{H^2(\Omega)}$ is bounded. Thus, up to a subsequence, we might suppose $u_n \to u$ strongly in $H^1_\varphi(\Omega)$. Finally, we have, for any $\xi \in H^1_\varphi(\Omega)$ and any $n \geqslant 1$.

$$\begin{aligned}
E_f(\xi) &= \int_\Omega \frac{1}{2}|\nabla \xi|^2 + (\vec{v} \cdot \nabla f)\xi + B_\varepsilon(\xi)\,dx \\
&= \int_\Omega \frac{1}{2}|\nabla \xi|^2 + (\vec{v} \cdot \nabla f_n)\xi + B_\varepsilon(\xi)\,dx + \mathrm{o}(1) \\
&\geqslant \int_\Omega \frac{1}{2}|\nabla u_n|^2 + (\vec{v} \cdot \nabla f_n)u_n + B_\varepsilon(u_n)\,dx + \mathrm{o}(1) \\
&= E_f(u) + \mathrm{o}(1),
\end{aligned}$$

which implies $u \in \mathfrak{F}(f)$ as desired. □

Unfortunately, well known fixed point results for maps does not work, in general, for multi-valued maps. At this moment, some other information, that in general will come from the geometry of our problem, need to be used to assure existence of a fixed point to the operator $\mathfrak{F}$. It could be the case that symmetries of the problem and special properties of vector field $\vec{v}$ enable to find a continuous section of the multi-valued map $\mathfrak{F}$. This is the case, for instance when, for each $f \in H^1_\varphi(\Omega)$, the set $\mathfrak{F}(f)$ is convex. In this case, Michael's Theorem can be applied to assure the existence of a continuous, actually due to Proposition 3.2, a compact selection. Let us mention that, by scaling, we might suppose, without lost of generality, that $2C_\Omega\|\vec{v}\|_\infty = L < 1$, where C_Ω stands for the optimal constant in the Poincaré inequality.

Theorem 3.3. *Suppose, there exists a continuous selection $\hat{\mathfrak{F}}$ of $\mathfrak{F}$. Then, for each $\varepsilon > 0$, there exists a fixed point u_ε for $\mathfrak{F}$. Furthermore, such a fixed point is a solution to*

$$\begin{cases} \Delta u_\varepsilon - \vec{v} \cdot \nabla u_\varepsilon = \beta_\varepsilon(u_\varepsilon) & \text{in } \Omega, \\ u_\varepsilon = \varphi & \text{on } \partial\Omega, \end{cases} \tag{3.3}$$

and it can be variationally characterized by

$$E_{u_\varepsilon}(u_\varepsilon) = \min_{\xi \in \mathcal{H}} E_{u_\varepsilon}(\xi).$$

Proof. For each $f \in H^1_\varphi(\Omega)$, it follows from the minimizing property of $\hat{\mathfrak{F}}(f)$ that

$$E_f\big(\hat{\mathfrak{F}}(f)\big) \leqslant E_f(\psi), \quad \forall \psi \in H^1_\varphi(\Omega).$$

From now on in this proof, let us fix a $\psi \in H^1_\varphi(\Omega)$. The above inequality implies

$$\begin{aligned}\int_\Omega \big|\nabla\hat{\mathfrak{F}}(f)\big|^2 &\leqslant \int_\Omega |\nabla\psi|^2 + 2|\vec{v}||\nabla f|\big|\hat{\mathfrak{F}}(f) - \psi\big|\,dx + |\Omega| \\ &\leqslant L\|\nabla f\|_2\big(\big\|\nabla\hat{\mathfrak{F}}(f)\big\|_2 + \|\nabla\psi\|_2\big) + \big(|\Omega| + \|\nabla\psi\|_2^2\big).\end{aligned}$$

That is

$$\big\|\nabla\hat{\mathfrak{F}}(f)\big\|_2^2 \leqslant L\|\nabla f\|_2\big(\big\|\nabla\hat{\mathfrak{F}}(f)\big\|_2 + \|\nabla\psi\|_2\big) + \big(|\Omega| + \|\nabla\psi\|_2^2\big). \tag{3.4}$$

Let us denote by ρ the positive root of the quadratic polynomial

$$\mathcal{P}(t) := (L-1)t^2 + \big(\|\nabla\psi\|_2 + Lt\big)\|\nabla\psi\|_2 + |\Omega| = 0.$$

We claim that $\hat{\mathfrak{F}}$ maps B_ρ into itself, where $B_\rho := \{\xi \in H^1_\varphi(\Omega) \mid \|\nabla\xi\|_2 \leqslant \rho\}$. Indeed, suppose $\|\nabla f\|_2 \leqslant \rho$. It follows from inequality (3.4) that

$$\begin{aligned}\big\|\nabla\hat{\mathfrak{F}}(f)\big\|_2^2 &\leqslant L\rho\big(\big\|\nabla\hat{\mathfrak{F}}(f)\big\|_2 + \|\nabla\psi\|_2\big) + \big(|\Omega| + \|\nabla\psi\|_2^2\big) \\ &= L\rho\big\|\nabla\hat{\mathfrak{F}}(f)\big\|_2 + \big(\|\nabla\psi\|_2 + L\rho\big)\|\nabla\psi\|_2 + |\Omega|.\end{aligned} \tag{3.5}$$

Now, if we assume, by contraction, $\|\nabla\hat{\mathfrak{F}}(f)\|_2 > \rho$, we would obtain, from (3.5)

$$\big\|\nabla\hat{\mathfrak{F}}(f)\big\|_2^2 < L\big\|\nabla\hat{\mathfrak{F}}(f)\big\|_2^2 + \big(\|\nabla\psi\|_2 + L\big\|\nabla\hat{\mathfrak{F}}(f)\big\|_2\big)\|\nabla\psi\|_2 + |\Omega|.$$

However, by the suitable choose of ρ,

$$(L-1)t^2 + \big(\|\nabla\psi\|_2 + Lt\big)\|\nabla\psi\|_2 + |\Omega| < 0,$$

for any $t > \rho$ and the claim is proven. Finally, since $\hat{\mathfrak{F}}$ is a compact operator, Schauder fixed point theorem applies. This finishes the proof. □

Let us discuss another way of finding a fixed point to the operator $\mathfrak{F}$. Let $\mathbb{S}$ be the set of all solutions of Eq. (2.1). It is simple to show, $\mathbb{S}$ is nonempty and compact in $H^1_\varphi(\Omega)$. Furthermore, as long as β_ε is regular, functions in $\mathbb{S}$ are sufficiently regular. For each $u \in \mathbb{S}$, consider the functional E_u as defined above. Our first observation is that u is a critical point of E_u. Indeed, this follows from differentiating E_u and using the fact that $u \in \mathbb{S}$. Furthermore, any minimizer f of E_u is a solution of

$$\begin{cases}\Delta f - \vec{v}\cdot\nabla u = \beta_\varepsilon(f) & \text{in } \Omega, \\ f = \varphi & \text{on } \partial\Omega.\end{cases} \tag{3.6}$$

After this comment, we notice that, if the geometry of our problem enables a solution $u \in \mathbb{S}$ such that either, E_u has a unique critical point or Eq. (3.6) has a unique solution, then $u \in \mathfrak{F}(u)$ and hence, u admits the variational characterization as in Theorem 3.3. The former can be explored by critical point theory such as Morse theory in infinite dimensional spaces, topological deformation arguments among others. We shall focus our attention to the latter case, i.e., the geometry of our problem enables uniqueness to Eq. (3.6).

The "right" geometry for this approach is present in cylinders, where, Berestycki and Nirenberg have proven several deep results about monotonicity, symmetry and antisymmetry of solutions of semi-linear elliptic equation in cylindrical domains. See for instance [3–6]. The type of symmetry results provided in the above literature, may, at least in our case, yield uniqueness for Eq. (3.6). Let us discuss a special case.

We shall use the following powerful result:

Theorem 3.4. *(See Berestycki and Nirenberg [3].) Let $\Omega = (-a,a)\times\omega$ and u be a $C^2(\bar{\Omega})$ solution of*

$$\begin{cases}\Delta u - f(x,u,\nabla u) = 0 & \textit{in } (-a,a)\times\omega, \\ u = \varphi & \textit{on } \partial[(-a,a)\times\omega].\end{cases} \tag{3.7}$$

Assume φ is continuous and

$$\varphi(x_1, y) \leqslant \varphi(x_1', y) \quad \text{for } x_1 \leqslant x_1'.$$

Assume also, $f(x, u, p)$ is continuous, Lipschitz in the variables (u, p) and satisfies

$$f(x, u, p) \text{ is nondecreasing in } x_1 \text{ for } p_1 \geqslant 0.$$

Finally, suppose u satisfies

$$\begin{cases} \varphi(-a, y) \leqslant u(x_1, y) \leqslant \varphi(a, y) \quad \text{for } -a < x_1 < a,\ y \in \omega \text{ and} \\ \forall x_1 \text{ in } (-a, a), \exists y \in \omega \text{ such that } \varphi(-a, y) < u(x_1, y). \end{cases}$$

Then u is strictly increasing in x_1 in Ω. Furthermore it is unique, i.e., if $\bar{u}$ is another solution of (3.7) *satisfying the above conditions, then $\bar{u} = u$.*

Here is the specific case we are interested in. Let $\Omega = (-a, a) \times \omega$ and $\varphi(-a, y) \equiv 0$ and $\varphi(a, y) \equiv A$, where A is a positive constant. Consider the following special case of Eq. (2.1).

$$\begin{cases} \Delta u - v(y)\partial_1 u = \beta_\varepsilon(u) & \text{in } (-a, a) \times \omega, \\ u = \varphi & \text{on } \partial[(-a, a) \times \omega], \end{cases} \tag{3.8}$$

where v is a positive function. This equation models the flame propagation in the cylinder $(-a, a) \times \omega$.

Proposition 3.5. *Eq.* (3.8) *has a unique solution for any ε small enough.*

Proof. Let ε be small enough such that $\beta_\varepsilon(A) = 0$. The idea is to show that $0 < u < A$. Let $M = \max u$. Suppose initially $M > A$. Thus, in some ball $B \subset \Omega$,

$$\Delta u - v(y)\partial_1 u = 0 \quad \text{in } B.$$

By the maximum principle, $u \equiv M$ in B. By connectedness of Ω, we find $u \equiv M$ in Ω, which is a contradiction. Let us now define $\zeta := u - A$. Notice that $\zeta \leqslant 0$. Furthermore

$$\begin{aligned} \Delta\zeta - v(y)\partial_1\zeta &= \beta_\varepsilon(u) \\ &= \beta_\varepsilon'(\theta)\zeta, \end{aligned}$$

once $\beta_\varepsilon(A) = 0$. By Serrin's maximum principle (see, for instance, [16] Theorem 2.10), either $\zeta < 0$ or $\zeta \equiv 0$. Since the latter cannot happen, we conclude $u < A$. Analogously we can show $u > 0$. The proposition is proven with the aid of Theorem 3.4. □

Notice that the solution of problem (3.8) is a subharmonic function. We now can state the following result

Theorem 3.6. *Let u be the solution of problem* (3.8). *Suppose u is convex in the x_1 direction. Then $u \in \mathfrak{F}(u)$ and thus it satisfies the desired variational characterization.*

Proof. The proof follows the same steps of Proposition 3.5. Indeed, let f be a minimizer of the functional E_u. As we have commented before, f satisfies

$$\Delta f - v(y)\partial_1 u = \beta_\varepsilon(f).$$

Since u is convex in the x_1 direction, we still can apply Theorem 3.4 to conclude, as we did on Proposition 3.5, uniqueness result for the above equation. Thus $f \equiv u$. □

3.2. The potential case

Motivated by well known physical assumptions, we are guided to study the case when the vector field $\vec{v}$ is a potential, i.e.,

$$\vec{v} = -\nabla\phi, \tag{3.9}$$

for some $\phi \in W^{1,\infty}(\Omega)$. In the variational characterization proposed in (2.4), let

1. $\sigma_\varepsilon \equiv 0$,
2. $d\mu_\varepsilon = d\mu := \mathrm{e}^\phi\, dx$.

Thus, the functional we consider for the potential case is

$$\mathcal{F}_\varepsilon(\zeta) := \int_\Omega \big(|\nabla\zeta|^2 + B_\varepsilon(\zeta)\big)\mathrm{e}^\phi\, dx. \tag{3.10}$$

Next proposition provides the existence of solutions of (2.1) that satisfies a nice variational characterization in the spirit of (2.4).

Proposition 3.7. *The above functional has a minimizer in $H^1_\varphi(\Omega, d\mu)$. Furthermore such a minimizer is a solution to problem* (2.1).

Proof. The fact that $\mathcal{F}_\varepsilon$ has a minimizer follows from the same step as Proposition 3.1. Let u be a minimizer, $h \in C_0^\infty(\Omega)$ and consider

$$j(\gamma) := \mathcal{F}_\varepsilon(u + \gamma h).$$

From the fact that $j'(0) = 0$ we obtain

$$\int_\Omega \big(\nabla u \nabla h + \beta_\varepsilon(u)h\big)\, d\mu = 0. \tag{3.11}$$

We now compute

$$\operatorname{div}\big(h\mathrm{e}^\phi \nabla u\big) = \mathrm{e}^\phi \nabla u \nabla h + \mathrm{e}^\phi[\nabla\phi\nabla u]h + [\mathrm{e}^\phi \Delta u]h.$$

Thus (3.11) can be rewritten, using integration by parts and taking into account (3.9),

$$\int_\Omega \big[\mathrm{e}^\phi \Delta u - \mathrm{e}^\phi \vec{v}\cdot\nabla u - \mathrm{e}^\phi \beta_\varepsilon(u)\big]h\, dx = 0. \tag{3.12}$$

Since (3.12) is true for every $h \in C_0^\infty(\Omega)$ and $\mathrm{e}^\phi > 0$, we conclude u satisfies the partial differential Eq. (2.1). □

4. Uniform Lipschitz regularity and some geometric measure properties of level sets

In this section we shall explore some geometric properties of solutions of

$$\begin{cases} \Delta u_\varepsilon - \vec{v}\cdot\nabla u_\varepsilon = \beta_\varepsilon(u_\varepsilon) & \text{in } \Omega \quad \text{(PDE)}, \\ u_\varepsilon = \varphi & \text{on } \partial\Omega, \end{cases}$$

that admit a variational characterization as in (2.4) i.e.,

$$\mathcal{F}_\varepsilon(u_\varepsilon) = \min_{\xi \in H^1_\varphi(\Omega, d\mu_\varepsilon)} \mathcal{F}_\varepsilon(\xi),$$

where

$$\mathcal{F}_\varepsilon(\zeta) := \int_\Omega \left(\frac{1}{2}|\nabla\zeta|^2 + \sigma_\varepsilon(x)\zeta + B_\varepsilon(\zeta)\right) d\mu_\varepsilon.$$

From now on, we shall denote $L\xi = \Delta\xi - \vec{v}\cdot\nabla\xi$. Notice that $Lu_\varepsilon = 0$ in $\{u_\varepsilon \leqslant 0\}$, and since, $u_\varepsilon > 0$ on $\partial\Omega$, by the maximum principle, $u_\varepsilon \geqslant 0$ in Ω.

Remark 4.1 *(Lipschitz Renormalization).* Suppose u is a solution the PDE $Lu = \beta_\varepsilon(u)$ in some ball $B_r(x_0)$. Then the function $w : B_{r/\varepsilon}(0) \to \mathbb{R}$, defined as

$$w(y) = \frac{1}{\varepsilon} u(x_0 + \varepsilon y),$$

satisfies

$$\Delta w - \varepsilon(\vec{v}^{\varepsilon} \cdot \nabla w) = \beta_1(w) \quad \text{in } B_{r/\varepsilon}(0),$$

with $\vec{v}^{\varepsilon} := \vec{v}(x_0 + \varepsilon x)$. Furthermore, $\nabla w(0) = \nabla u(x_0)$.

Let us point out that Remark 4.1 suggests that we should expect to get a uniform Lipschitz estimate, since proving u is Lipschitz in $B_r(x_0)$ is equivalent to proving w in Lipschitz in $B_{r/\varepsilon}$.

The next lemma takes care of Lipschitz estimate in a region where u_ε is small.

Lemma 4.2. *Let $x \in \tilde{\Omega} \Subset \Omega$ satisfy $0 \leqslant u_\varepsilon(x) \leqslant \varepsilon$. Then*

$$\left|\nabla u_\varepsilon(x)\right| \leqslant C,$$

where C does not depend on ε.

Proof. Let $2\delta = \operatorname{dist}(x, \partial\Omega) \geqslant \operatorname{dist}(\partial\tilde{\Omega}, \partial\Omega) = c > 0$. Consider the Lipschitz renormalization suggested in Remark 4.1, $w : B_{\delta/\varepsilon}(0) \to \mathbb{R}$,

$$w(y) = \frac{1}{\varepsilon} u(x + \varepsilon y).$$

Applying Schauder's estimate for the elliptic operator $Lw = \Delta w - \varepsilon(\vec{v} \cdot \nabla w)$, we find, in particular, that

$$\left|\nabla w(0)\right| \leqslant C\left(\|w\|_{L^\infty(B_{\delta/2\varepsilon}(0))} + 1\right).$$

However, by Harnack inequality

$$\|w\|_{L^\infty(B_{\delta/2\varepsilon}(0))} \leqslant C w(0) = C.$$

Hence,

$$\left|\nabla w(0)\right| = \left|\nabla u_\varepsilon(x)\right| \leqslant C,$$

where C depends only on dimension, $\tilde{\Omega}$ and $\|\vec{v}\|_\infty$. □

Hereafter, for any $0 < \alpha < \sup_{\partial\Omega} \varphi$, Ω_α, will stand for the set $\{u_\varepsilon > \alpha\}$, i.e.,

$$\Omega_\alpha := \left\{x \in \Omega \mid u_\varepsilon(x) > \alpha\right\}.$$

Next lemma provides uniform Lipschitz estimate over Ω_ε.

Lemma 4.3. *Let $x_0 \in \Omega_\varepsilon$. If we denote by $r = \operatorname{dist}(x_0, \partial\Omega_\varepsilon)$, then*

$$\left|\nabla u_\varepsilon(x_0)\right| \leqslant \frac{C}{r},$$

where C depends only on dimension, $\|\vec{v}\|_\infty$ and $\|\varphi\|_\infty$.

Proof. Since β_ε is supported in $[0, \varepsilon]$, we have that

$$Lu_\varepsilon = \Delta u_\varepsilon - \vec{v} \cdot \nabla u_\varepsilon = 0 \quad \text{in } \Omega_\varepsilon.$$

We then apply interior Schauder estimate to the elliptic operator L and obtain

$$\left|\nabla u_\varepsilon(x_0)\right| \leqslant C\left(\frac{\sup|u_\varepsilon|}{r}\right).$$

Finally, by the maximum principle, we have $\sup|u_\varepsilon| \leqslant \|\varphi\|_\infty$. □

We now have to care about a universal bound of the gradient of u_ε for points $x \in \Omega_\varepsilon$ that are close to $\Gamma_\varepsilon := \{x \in \Omega\colon\ u_\varepsilon(x) = \varepsilon\}$, since the estimate given by Lemma 4.3 could blow up when x approaches Γ_ε. If is worthwhile to comment that, for each $\varepsilon > 0$, we can find a $1 \leqslant c_\varepsilon \leqslant 2$, so that $\Gamma_{c_\varepsilon \varepsilon}$ is smooth (Sard's Theorem) and thus we can

control the Lipschitz norm of u_ε up to the boundary of $\Omega_{c_\varepsilon \varepsilon}$. However, since we do not know yet any smoothness of the limiting free boundary, such a control could deteriorate as $\varepsilon \to 0$.

The idea will be to obtain an estimate of $u_\varepsilon(x)$ in terms of the distance of $\operatorname{dist}(x, \Gamma_\varepsilon)$ and afterwards employ Schauder estimate and Harnack inequality. Here are the details:

Let $\chi \in \Gamma_\varepsilon$, r be small enough so that $B(\chi, r) \subset \Omega$. Fix $x_0 \in B(\chi, r)$ and call $h = \operatorname{dist}(x_0, \Gamma_\varepsilon)$. Set λ so that $u_\varepsilon(x_0) = \lambda \cdot h$. We now consider the Lipschitz renormalization

$$w(y) := \frac{1}{h} u_\varepsilon(x_0 + hy).$$

Notice that

1. $w(0) = \lambda$;
2. $w \geqslant \varepsilon / h$ in B_1;
3. There exists a $Y_1 \in \partial B_1$ such that $w(Y_1) = \varepsilon / h$;
4. $\Delta w - h(\vec{v}^h \cdot \nabla w) = 0$ in B_1, for $\vec{v}^h(x) := \vec{v}(x_0 + hx)$;
5. $\nabla w(y) = \nabla u_\varepsilon(x_0 + hy)$;
6. From Lemma 4.2, $|\nabla w(Y_1)| \leqslant C$ and C is universal.

By Harnack inequality, there exists a universal constant $c > 0$ such that

$$w(y) \geqslant c\lambda \quad \forall y \in B_{1/2}.$$

Consider z to be the following function

$$\begin{cases} \Delta z - h(\vec{v} \cdot \nabla z) = 0 & \text{in } B_1 \setminus \overline{B_{1/2}}, \\ z|_{\partial B_1} = 0, \quad z|_{\partial B_{1/2}} = 1. \end{cases} \tag{4.1}$$

By Hopf's Lemma and $C^{1,1}$ regularity up to the boundary, there exists a $\delta > 0$ such that

$$z_\nu \geqslant \delta \quad \text{on } \partial B_1,$$

where ν denotes the inward normal vector $-x$. From maximum principle, $w \geqslant c\lambda z$ in $B_1 \setminus B_{1/2}$; therefore

$$C \geqslant \left|\nabla w(Y_1)\right| \geqslant w_\nu(Y_1) \geqslant c\lambda z_\nu(Y_1) \geqslant \delta c \lambda.$$

We have concluded that $u_\varepsilon(x_0) \leqslant C \cdot \operatorname{dist}(x_0, \Gamma_\varepsilon)$, for a universal constant $C > 0$. Finally by Schauder estimate and Harnack inequality,

$$\left|\nabla u(x_0)\right| \leqslant C \frac{u(x_0)}{\operatorname{dist}(x_0, \Gamma_\varepsilon)} \leqslant C.$$

Combining the above with Lemmas 4.2 and 4.3, we obtain

Theorem 4.4 *(Uniform Lipschitz Estimate). Let $\tilde{\Omega} \Subset \Omega$. Then there exists a constant C depending only on dimension, $\|\vec{v}\|_\infty$, $\|\varphi\|_\infty$ and $\tilde{\Omega}$, such that*

$$\sup_{\tilde{\Omega}} |\nabla u_\varepsilon| \leqslant C.$$

Remark 4.5. For a general elliptic operator like in (1.3), we exchange in (4.1) $\Delta z - h(\vec{v} \cdot \nabla z)$ by $L_h u = D_j(a_{ij}(x_0 + hx) D_i u) + b_i(x) u_i + c(x) u$. In this case, the best we can assure is $C^{1,\alpha}$ regularity up to the boundary. However, this is enough to carry out the same computation above.

Let us turn our attention to nondegeneracy. The next theorem implies, as we will see in its corollary, linear growth way from the free boundary. We remark that this is an important geometric property that allows a deeper understanding of regularity properties of the free boundary. Here the variational characterization plays a crucial role.

Theorem 4.6 *(Linear Growth). There exist universal constants $c_1 > 1$ and c_2 such that, if $x_0 \in \tilde{\Omega} \Subset \Omega$ and $u_\varepsilon(x_0) = \lambda \geqslant c_1\varepsilon$, then*

$$\operatorname{dist}(x_0, \partial\Omega_\varepsilon) \leqslant c_2\lambda.$$

Proof. Let us call $d = \operatorname{dist}(x_0, \partial\Omega_\varepsilon)$. Suppose $\lambda = \alpha d$. We want to show $\alpha \geqslant \underline{c} > 0$. To this end, let us make a Lipschitz renormalization

$$w(y) := \frac{1}{d} u_\varepsilon(x_0 + dy).$$

If we define $\vec{\nu} : B_1(0) \to \mathbb{R}^N$ by $\vec{\nu}(y) := d\vec{v}(x_0 + dy)$, w satisfies

$$w \geqslant \varepsilon/d \quad \text{and} \quad \Delta w - (\vec{\nu} \cdot \nabla w) = \beta_{\varepsilon/d}(w) = 0 \quad \text{in } B_1(0),$$

because $\beta_{\varepsilon/d}$ is supported in $[0, \varepsilon/d]$.

Furthermore $w(0) = \alpha$. By Harnack inequality we have

$$\underline{c}\alpha \leqslant w \leqslant \bar{c}\alpha \quad \text{in } B_{1/2}.$$

Now, let ψ be a cut-off function fulfilling $\psi \equiv 0$ in $B_{1/4}$, $\psi \equiv 1$ in $B_1 \setminus B_{1/2}$ and define

$$\zeta = \begin{cases} \min(w, \bar{c}\alpha\psi) & \text{in } B_{1/2}, \\ w & \text{in } B_1 \setminus B_{1/2}. \end{cases}$$

Notice that, by the Change of Variables Theorem, w minimizes the functional

$$\mathcal{E}(\xi) := \int_{B_1} \left\{ \frac{1}{2}\left|\nabla\xi(y)\right|^2 + d\sigma(x_0 + d \cdot y)\xi(y) + B_\varepsilon\big(d \cdot w(y)\big) \right\} F_\varepsilon(x_0 + d \cdot y)\, dy,$$

among all $\xi \in w + H_0^1(B_1, d\mu_\varepsilon)$. Hence, since ζ competes against w in the above problem, we have

$$\mathcal{E}(\zeta) \geqslant \mathcal{E}(w).$$

Writing this down, we find

$$\int_{B_1} \frac{1}{2}\left(|\nabla\zeta|^2 - \frac{1}{2}|\nabla w|^2 \right) d\mu_\varepsilon + \int_{B_1} \big(d \cdot \sigma(\zeta - w)\big)\, d\mu_\varepsilon \geqslant \int_{B_1} \big(B_\varepsilon(d \cdot w) - B_\varepsilon(d \cdot \zeta)\big)\, d\mu_\varepsilon. \tag{4.2}$$

However,

$$\int_{B_1} \frac{1}{2}\big(|\nabla\zeta|^2 - \frac{1}{2}|\nabla w|^2\big)\, d\mu_\varepsilon = \frac{1}{2} \int_{B_{1/2} \cap \{\bar{c}\alpha\psi \leqslant w\}} \left(\bar{c}^2\alpha^2|\nabla\psi|^2 - \frac{1}{2}|\nabla w|^2 \right) d\mu_\varepsilon \leqslant C\alpha^2,$$

$$\int_{B_1} \big(d \cdot \sigma(\zeta - w)\big)\, d\mu_\varepsilon = \frac{1}{2} \int_{B_{1/2} \cap \{\bar{c}\alpha\psi \leqslant w\}} \big(d \cdot \sigma(\bar{c}\alpha\psi - w)\big)\, d\mu_\varepsilon \leqslant C\alpha,$$

and, since $w \geqslant \zeta$ and B_ε is an increasing function,

$$\begin{aligned} \int_{B_1} \big(B_\varepsilon(d \cdot w) - B_\varepsilon(d \cdot \zeta)\big)\, d\mu_\varepsilon &\geqslant \int_{B_{1/4}} \big(B_\varepsilon(d \cdot w) - B_\varepsilon(d \cdot \zeta)\big)\, d\mu_\varepsilon \\ &= \int_{B_{1/4}} B_\varepsilon(d \cdot w)\, d\mu_\varepsilon \\ &\geqslant \mu_\varepsilon(B_{1/4}) B_\varepsilon(d\underline{c}\alpha) \\ &= c B_\varepsilon(\underline{c}\lambda) \\ &\geqslant c B_\varepsilon(\underline{c}c_1\varepsilon) \\ &\geqslant c > 0. \end{aligned}$$

Putting those inequalities together, we finish the proof. □

Corollary 4.7. *Let $x \in \Omega_{c_1\varepsilon} := \{y \in \Omega \mid u_\varepsilon(x) > c_1\varepsilon\}$. Then there exists a universal constant C, such that*

$$C^{-1}\operatorname{dist}(x, \partial\Omega_\varepsilon) \leqslant u_\varepsilon(x) \leqslant C\operatorname{dist}(x, \partial\Omega_\varepsilon).$$

Proof. The first inequality follows immediately Theorem 4.6. The estimate by above is consequence of uniform Lipschitz continuity. Indeed, from the definition of $\Omega_{c_1\varepsilon}$ and Theorem 4.4, we have

$$c_1\varepsilon \leqslant u_\varepsilon(x) \leqslant C\operatorname{dist}(x, \partial\Omega_\varepsilon) + \varepsilon.$$

Thus,

$$u_\varepsilon(x) \leqslant \frac{c_1}{c_1 - 1} C\operatorname{dist}(x, \partial\Omega_\varepsilon). \qquad \square$$

Regarding nondegeneracy, making use only of the ellipticity of the operator $Lu = \Delta u - \vec{v}\nabla u$, the uniform Lipschitz regularity (Theorem 4.4) and the linear growth away from the free boundary (Corollary 4.7) we obtain (see, for instance, [9], page 593, Lemma 7) the following strong nondegeneracy result.

Theorem 4.8 *(Strong Nondegeneracy). Let $\tilde{\Omega} \Subset \Omega$. There exist constants c_1 and c_2 which depend only on dimension, $\|\vec{v}\|_\infty$ and $\tilde{\Omega}$ such that, if $x_0 \in \tilde{\Omega}$ and*

$$u_\varepsilon(x_0) \geqslant c_1\varepsilon,$$

then

$$\sup_{B_\rho(x_0)} u_\varepsilon \geqslant c_2\rho.$$

Corollary 4.9. *Let $\tilde{\Omega} \Subset \Omega$. There exist constants c_1, c_2 and c_3, depending only on dimension, $\|\vec{v}\|_\infty$ and $\tilde{\Omega}$, such that if $x_0 \in \tilde{\Omega}$ with*

$$u_\varepsilon(x_0) = \lambda \geqslant c_1\varepsilon \quad \textit{and} \quad \rho \geqslant c_2\lambda,$$

then

$$\left|B_\rho(x_0) \cap \{u_\varepsilon > \lambda\}\right| \geqslant c_3\rho^N.$$

Proof. Let $y \in B_{\rho/2}(x_0)$ be such that $u_\varepsilon(y) \geqslant c\rho$. Such a point exists by Theorem 4.8. Then, by continuity, we may assume, for ε small enough,

$$\operatorname{dist}(y, \partial\Omega_\varepsilon) \geqslant c_1\rho.$$

By Harnack inequality, for $\bar{c}$ small enough so that $B_{\rho\bar{c}(y)} \subset B_{c_1\rho}(y) \cap B_\rho(x_0)$, there holds

$$u_\varepsilon(x) \geqslant \frac{c\rho}{2} \geqslant \frac{cc_2\lambda}{2} > \lambda, \quad \text{in } B_{\bar{c}\rho}(y),$$

if c_2 is taken large enough. Thus,

$$B_{\bar{c}\rho}(y) \subset B_\rho(x_0) \cap \{u_\varepsilon > \lambda\}.$$

This finishes the proof. $\square$

Definition 4.10. Let $\delta > 0$ and E be a set in $\mathbb{R}^N$. We define the δ-strip of E by

$$\mathcal{N}_\delta(E) := \left\{x \in \mathbb{R}^N\colon \operatorname{dist}(x, E) < \delta\right\}.$$

As we shall see, some geometric information of the limiting free boundary will be concluded in terms the of δ-strip of $\Omega_{c(\varepsilon)}$, for some convenient modulus of continuity $c(\varepsilon)$.

Theorem 4.11. *If $x_0 \in \partial\Omega_{c\varepsilon}$, $\lambda > 3c\varepsilon$ and $1 \geqslant R \geqslant c_1\lambda$, then*

$$\left|\mathcal{N}_\lambda(\partial\Omega_{c\varepsilon}) \cap B_R(x_0)\right| \leqslant c_3\lambda R^{N-1},$$

where all constants are universal.

Proof. Let $G := \Omega_{c\varepsilon} \cap B_R(x_0)$ and define $w := \min\{(u_\varepsilon - c\varepsilon)^+, \lambda - c\varepsilon\}$. If we multiply (PDE) by w and integrate over G we obtain

$$-\int_G \Delta u_\varepsilon w + w\beta_\varepsilon(u_\varepsilon) = \int_G (\vec{v}\cdot\nabla u_\varepsilon)w.$$

However, if we take $c > 1$, $\beta_\varepsilon(u_\varepsilon) \equiv 0$ in G. Furthermore, by Green's formula

$$-\int_G \Delta u_\varepsilon w = \int_G \nabla u_\varepsilon \nabla w - \int_{\partial G} w(u_\varepsilon)_\mu \, d\sigma,$$

where μ is the outward unit normal vector on ∂G. Hence

$$\begin{aligned}
\int_{\{c\varepsilon<u_\varepsilon<\lambda\}\cap B_R(x_0)} |\nabla u_\varepsilon|^2 &\leqslant \int_G \nabla u_\varepsilon \nabla w \\
&= \int_{\partial G} w(u_\varepsilon)_\mu \, d\sigma + \int_G (\vec{v}\cdot\nabla u_\varepsilon)w \\
&\leqslant C_1\lambda R^{N-1} + C_2\lambda R^N \\
&\leqslant C_0\lambda R^{N-1}.
\end{aligned}$$

Our next step is to compare

$$\int_{\{c\varepsilon<u_\varepsilon<\lambda\}\cap B_R(x_0)} |\nabla u_\varepsilon|^2 \quad \text{with } \left|\{c\varepsilon < u_\varepsilon < \lambda\} \cap B_R(x_0)\right|.$$

Let $\{B_j\}$ be a finite overlapping covering of $\partial\Omega_{c\varepsilon}$ by balls of radius $c_1\lambda$ and centered at $\partial\Omega_{c\varepsilon}$. In every B_j there are sub balls B_j^1 and B_j^2 of radius r_j of order λ for which

$$u_\varepsilon \geqslant \frac{3}{4}\lambda \quad \text{in } B_j^1 \quad \text{and} \quad u_\varepsilon \leqslant \frac{2}{3}\lambda \quad \text{in } B_j^2.$$

Indeed, this follows from Lipschitz continuity and strong nondegeneracy ($r_j^1 = \frac{1}{8\,\mathrm{Lip}(u_\varepsilon)}\lambda$ and $r_j^2 = \frac{1}{3\,\mathrm{Lip}(u_\varepsilon)}\lambda$). Therefore, if $m_j := \fint_{B_j} u_\varepsilon$, then $|u_\varepsilon - m_j| \geqslant c\lambda$ at least on one of the two sub balls. Indeed, if not, there would exist sequences $x_n^k \in B_j^k$, $k = 1, 2$, such that $|u_\varepsilon(x_n^k) - m_j| < \frac{\lambda}{n}$, $\forall n$, what would imply $|u_\varepsilon(x_n^1) - u_\varepsilon(x_n^2)| \to 0$, contradicting the construction of the sub balls. Thus, by Poncaré inequality

$$c\lambda^2 \leqslant \fint_{B_j} (u_\varepsilon - m_j)^2 \leqslant \bar{c} r_j^2 \fint_{B_j} |\nabla u_\varepsilon|^2.$$

Therefore

$$\int_{B_j\cap\{c\varepsilon<u_\varepsilon<\lambda\}} |\nabla u_\varepsilon|^2 \geqslant c|B_j|.$$

By nondegeneracy,

$$B_R(x_0) \cap \{c\varepsilon < u_\varepsilon < \lambda\} \subset \bigcup B_j.$$

Finally, this implies

$$\begin{aligned}\int_{B_{4R}(x_0)\cap\{c\varepsilon<u_\varepsilon<\lambda\}} |\nabla u_\varepsilon|^2\,dx &\geqslant \int_{(\bigcup B_j)\cap\{c\varepsilon<u_\varepsilon<\lambda\}} |\nabla u_\varepsilon|^2\,dx\\ &\geqslant \frac{1}{m}\sum \int_{B_j\cap\{c\varepsilon<u_\varepsilon<\lambda\}} |\nabla u_\varepsilon|^2\,dx\\ &\geqslant \frac{c}{m}\sum |B_j|\\ &\geqslant c\big|B_R(x_0)\cap\{c\varepsilon<u_\varepsilon<\lambda\}\big|.\end{aligned}$$

We have proven, so far,

$$\big|\{c\varepsilon<u_\varepsilon<\lambda\}\cap B_R(x_0)\big| \leqslant c_3\lambda R^{N-1}.$$

Now, making use of Corollary 4.9 we obtain

$$\big|\mathcal{N}_\lambda(\partial\Omega_{c\varepsilon})\cap B_R(x_0)\big| \leqslant c\big|\mathcal{N}_\lambda(\partial\Omega_{c\varepsilon})\cap B_R(x_0)\cap\Omega_{c\varepsilon}\big| + C\delta R^{N-1}.$$

By Lipschitz regularity, for some universal large constant A,

$$\mathcal{N}_\lambda(\partial\Omega_{c\varepsilon})\cap B_R(x_0)\cap\Omega_{c\varepsilon}\subset\{c\varepsilon u_\varepsilon<A\lambda\}\cap B_R(x_0).$$

This finishes the prove of the theorem. □

5. Letting $\varepsilon\to 0$

In this section we shall study the free boundary problem (2.2) by letting $\varepsilon\to 0$. The strategy to be used is based on the following observation. If we let $\varepsilon\to 0$, then, up to a subsequence, we may assume

1. $u_\varepsilon \rightharpoonup u_0$ in $H^1_\varphi(\Omega)$.
2. $u_\varepsilon\to u_0$ uniformly over compacts.
3. $\nabla u_\varepsilon\to\nabla u_0$ locally uniformly in $\{x\in\Omega \mid u_0(x)>0\}$.
4. $\sigma_\varepsilon\rightharpoonup\sigma$ in $L^2(\Omega)$.
5. $F_\varepsilon\to F$ in $L^2(\Omega)$.

The function u_0 is our natural candidate to solve problem (2.2). We shall denote $\Omega_0:=\{x\in\Omega \mid u_0(x)>0\}$ and $F(u_0):=\partial\Omega_0\cap\Omega$.

Theorem 5.1 *(Properties of u_0). If $\Omega_0:=\{x\in\Omega \mid u_0(x)>0\}$, then*

1. *u_0 is locally Lipschitz in Ω and*
$$\Delta u_0-\vec{v}\cdot\nabla u_0=0 \quad \text{in } \Omega_0.$$
2. *u_0 is nondegenerated away from $\partial\Omega_0\cap\tilde{\Omega}$ for any $\tilde{\Omega}\Subset\Omega$. That is, there exists a constant $c=c(\tilde{\Omega},N,\|\vec{v}\|_\infty)$ such that for any $B_\rho(x)\subset\tilde{\Omega}$ centered at the free boundary $\partial\Omega_0$,*
$$\sup_{B_\rho(x)} u_0\geqslant c\rho.$$
3. *Ω_0 is the limit in the Hausdorff distance of $\Omega_{c\varepsilon}=\{u_\varepsilon>c\varepsilon\}$. That is, given $\delta>0$, for ε small enough,*
$$\tilde{\Omega}\cap\Omega_{c\varepsilon}\subset\mathcal{N}_\delta(\Omega_0)\cap\tilde{\Omega},$$
$$\tilde{\Omega}\cap\Omega_0\subset\mathcal{N}_\delta(\Omega_{c\varepsilon})\cap\tilde{\Omega}.$$
4. *$|\mathcal{N}_\delta(\partial\Omega_0)\cap B_R|\leqslant c\delta R^{N-1}$, for every $\delta>0$. In particular*
$$\mathcal{H}^{n-1}(\partial\Omega_0\cap B_R)\leqslant CR^{N-1}.$$

5. *There exists a universal constant C depending*
$$C^{-1}\operatorname{dist}\big(x_0,\{u_0=0\}\big)\leqslant u_0(x)\leqslant C\operatorname{dist}\big(x_0,\{u_0=0\}\big).$$
6. *There exist a constant $\mu=\mu(\tilde\Omega)$ such that for $x_0\in F(u_0)\cap\tilde\Omega$,*
$$\frac{|B_\rho(x_0)\cap\Omega_0|}{|B_\rho(x_0)|}\geqslant\mu.$$

Proof. The fact that u_0 is locally Lipschitz in Ω follows from uniform Lipschitz continuity of u_ε. Furthermore, in Ω_0, u_ε converges locally to u_0, say, in the $C^{1,1}$ topology. Thus, one can carry the limit over the equation (PDE) and conclude
$$\Delta u_0-\vec v\cdot\nabla u_0=0\quad\text{in }\Omega_0.$$
We have justified item (1).

Let us turn our attention to strong nondegeneracy. Let $x_0\in\tilde\Omega_0\Subset\tilde\Omega_1\Subset\Omega_0$. We know there exists a sequence of points $x_\varepsilon\to x_0$, as $\varepsilon\to 0$, with $x_\varepsilon\in\Omega_{c\varepsilon}\cap\tilde\Omega_1$. By Theorem 4.8, the fact that $u_\varepsilon(x_\varepsilon)>c\varepsilon$ implies, for any $\varrho>0$, there exists a $y_\varepsilon^\varrho\in\partial B_{\varrho/4}(x_\varepsilon)$, so that
$$u_\varepsilon(y_\varepsilon^\varrho)\geqslant c\varrho.$$
Now, for ε small, $B_{\varrho/4}(x_\varepsilon)\subset B_\varrho(x_0)$, and, up to a subsequence, y_ε^ϱ converges to some $y^\varrho\in B_\varrho(x_0)$ as $\varepsilon\to 0$. Finally, since u_ε converges uniformly to u_0, there holds
$$c\varrho\leqslant u_\varepsilon(y_\varepsilon^\varrho)\longrightarrow u_0(y^\varrho)\leqslant\sup_{B_\varrho(x_0)}u_0.$$
Let us prove item (3). Suppose the first inclusion is not true. Therefore, there would exist a sequence of points x_ε and a positive real number $\alpha>0$, satisfying

1. $\operatorname{dist}(x_\varepsilon,\Omega_0)\geqslant\alpha>0$.
2. $x_\varepsilon\in\Omega_{c\varepsilon}\cap\tilde\Omega$.
3. x_ε converges to some x_0 that is α away from Ω_0.

Item (3) means, $u_0(x_0)=0$, while $u_\varepsilon(x_\varepsilon)\geqslant c\varepsilon$. By Theorem 4.8, there exists points $y_\varepsilon\in B_{\frac{\alpha}{8}}(x_\varepsilon)$ so that
$$u_\varepsilon(y_\varepsilon)\geqslant c\frac{\alpha}{8}.$$
But, again, for $|x_\varepsilon-x_0|<\frac{\alpha}{8}$, we have $B_{\alpha/8}(x_\varepsilon)\subset B_{\alpha/2}(x_0)$. Finally, up to a subsequence, $y_\varepsilon\to y_0\in B_\alpha(x_0)$, and, since $u_\varepsilon(y_\varepsilon)\to u_0(y_0)$, we would conclude $0>c\alpha$. By a similar argument, we conclude the second inequality. Indeed, suppose the inclusion does not hold. It means, there exists a sequence $x_\varepsilon\in\Omega_0\cap\tilde\Omega$ such that $\operatorname{dist}(x_\varepsilon,\Omega_{c\varepsilon}\cap\tilde\Omega)\geqslant\alpha$, for some fixed $\alpha>0$. In particular, $u_\varepsilon(x)\leqslant c\varepsilon$ for any $x\in B_{\alpha/2}(x_\varepsilon)$. Assume $x_\varepsilon\to x_0$; then, for $|x_\varepsilon-x_0|\leqslant\frac{\alpha}{8}$, we have $B_{\alpha/8}(x_0)\subset B_{\alpha/2}(x_k)$ and thus, $B_{\alpha/8}(x_0)\subset\tilde\Omega\setminus\Omega_0$.

Item (4) follows from item (3) together with Theorem 4.11. Item (5) follows immediately from Corollary 4.7. Finally, item (6) follows as Corollary 4.9. □

We can also prove the following finer convergence result, which will be employed in the proof of the free boundary condition. This is the content of the next lemma.

Lemma 5.2. *Let $\tilde\Omega\Subset\Omega$. Then $u_\varepsilon\to u_0$ in $H^1(\tilde\Omega)$.*

Proof. Let us fix $\tilde\Omega\Subset\Omega$. We may assume $\tilde\Omega$ is smooth. We already know $u_\varepsilon\rightharpoonup u_0$ in $H^1(\tilde\Omega)$. Thus is it sufficient to show, by uniform convexity of H^1, that
$$\limsup_{\varepsilon\to 0}\int_{\tilde\Omega}|\nabla u_\varepsilon|^2\,dx\leqslant\int_{\tilde\Omega}|\nabla u_0|^2\,dx.\tag{5.1}$$

To this end, let us multiply (PDE) by u_ε and integrate over $\tilde{\Omega}$. By doing that, we find

$$0 \geqslant -\int_{\tilde{\Omega}} \beta_\varepsilon(u_\varepsilon)u_\varepsilon = \int_{\tilde{\Omega}} |\nabla u_\varepsilon|^2 + (\vec{v}\cdot\nabla u_\varepsilon)u_\varepsilon - \int_{\partial\tilde{\Omega}} u_\varepsilon(u_\varepsilon)_\mu \, d\sigma.$$

Hence

$$\int_{\tilde{\Omega}} |\nabla u_\varepsilon|^2 \leqslant \int_{\partial\tilde{\Omega}} u_\varepsilon(u_\varepsilon)_\mu \, d\sigma - \int_{\tilde{\Omega}} (\vec{v}\cdot\nabla u_\varepsilon)u_\varepsilon.$$

Notice, however,

$$\int_{\partial\tilde{\Omega}} u_\varepsilon(u_\varepsilon)_\mu \, d\sigma - \int_{\tilde{\Omega}} (\vec{v}\cdot\nabla u_\varepsilon)u_\varepsilon \to \int_{\partial\tilde{\Omega}\cap\{u_0>0\}} u_0(u_0)_\mu \, d\sigma - \int_{\tilde{\Omega}\cap\{u_0>0\}} (\vec{v}\cdot\nabla u_0)u_0,$$

since $\nabla u_\varepsilon \to \nabla u_0$ locally uniformly in $\{u_0 > 0\}$. Now, for any $\delta > 0$, we have, since $\Delta u_0 - \vec{v}\cdot\nabla u_0 = 0$ in $\{u_0 > 0\}$,

$$\int_{\tilde{\Omega}\cap\{u_0>\delta\}} |\nabla u_0|^2 = \int_{\partial[\tilde{\Omega}]\cap\{u_0>\delta\}} u_0(u_0)_\mu \, d\sigma - \int_{\tilde{\Omega}\cap\{u_0>\delta\}} (\vec{v}\cdot\nabla u_0)u_0.$$

Letting $\delta \to 0$ we conclude (5.1). □

Our next step is to obtain a variational characterization for the limiting function u_0. As we have anticipated, this information shall yield uniform density of Ω_0 and Ω_0^c and thus the Hausdorff measure totality of the reduced free boundary.

Definition 5.3. Let $\mu := F\,dx$, where F is the L^2 limit of F_ε. For each ball $B \subseteq \Omega$, we consider the functional

$$E_0(B,\xi) := \int_B \left\{ \frac{1}{2}|\nabla\xi|^2 + \sigma(x)\xi + \chi_{\{\xi>0\}} \right\} d\mu, \tag{5.2}$$

where σ is the weak limit of σ_ε in L^2.

Theorem 5.4 *(Variational Characterization of u_0). The function u_0 is a local minimizer of E_0 over H^1, i.e., for any $\xi \in H^1_{\mathrm{loc}}$ with $\xi = u_0$ on ∂B, for some ball $B \subseteq \Omega$, there holds $E_0(B,\xi) \geqslant E_0(B,u_0)$.*

Proof. Suppose by contraction there is a ball $B_r \Subset \Omega$ and a function $\xi \in H^1$ with $\xi \equiv u_0$ on ∂B_r and

$$E_0(B_r,\xi) \leqslant E_0(B_r,u_0) - \delta,$$

for some $\delta > 0$. Fix $h > 0$ and define

$$\xi_\varepsilon^h := \begin{cases} u_0 + \dfrac{|x|-r}{h}(u_\varepsilon - u_0) & \text{in } B_{r+h}\setminus B_r, \\ \xi & \text{in } B_r. \end{cases}$$

In $B_{r+h}\setminus B_r$, we have $\nabla\xi_\varepsilon^h = \nabla u_0 + \frac{|x|-r}{h}(\nabla u_\varepsilon - \nabla u_0) + \frac{(u_\varepsilon-u_0)x}{|x|h}$. Therefore,

$$|\nabla\xi_\varepsilon^h|^2 \leqslant C + 2\frac{|u_\varepsilon - u_0|^2}{h^2} \quad \text{in } B_{r+h}\setminus B_r.$$

Thus,

$$\begin{aligned} \Xi_h^\varepsilon &:= \int_{B_{r+h}\setminus B_r} \left\{ \frac{|\nabla\xi_\varepsilon^h|^2}{2} + \sigma_\varepsilon\cdot\xi_\varepsilon^h + B_\varepsilon(\xi_\varepsilon^h) \right\} F_\varepsilon\,dx \\ &\leqslant C\mu(B_{r+h}\setminus B_r) + \frac{1}{h^2}\int_{B_{r+h}\setminus B_r} |u_\varepsilon - u_0|^2\,d\mu_\varepsilon + \mathrm{o}(1) \\ &\leqslant Cr^{N-1}h + \mathrm{o}(1). \end{aligned}$$

Now we can estimate

$$\begin{aligned} E^\varepsilon(B_{r+h},\xi_\varepsilon^h) &:= \int_{B_{r+h}} \left\{ \frac{|\nabla \xi_\varepsilon^h|^2}{2} + \sigma_\varepsilon \cdot \xi_\varepsilon^h + B_\varepsilon(\xi_\varepsilon^h) \right\} F_\varepsilon \, dx \\ &= \Xi_h^\varepsilon + \int_{B_r} \left\{ \frac{|\nabla \xi|^2}{2} + \sigma \cdot \xi + B_\varepsilon(\xi) \right\} F \, dx + \mathrm{o}(1) \\ &\leqslant C r^{N-1} h + \mathrm{o}(1) + E_0(B_r,\xi), \end{aligned}$$

since $B_\varepsilon(\xi) \leqslant \chi_{\{\xi>0\}}$. Thus

$$\limsup_{\varepsilon\to 0} E^\varepsilon(B_{r+h},\xi_\varepsilon^h) \leqslant E_0(B_r,\xi) + C r^{N-1} h. \tag{5.3}$$

On the other hand,

$$E^\varepsilon(B_{r+h},\xi_\varepsilon^h) = E^\varepsilon(B_r,\xi) + \omega_1(\varepsilon) + \omega_2(h) \geqslant E^\varepsilon(B_r,u_\varepsilon) + \omega_1(\varepsilon) + \omega_2(h), \tag{5.4}$$

where ω_1 and ω_2 are modulus of continuity. Furthermore, since, $u_\varepsilon \to u_0$ almost everywhere, we have

$$\mu\big(\{u_0>0\}\cap B_r\big) \leqslant \liminf_{\varepsilon\to 0} \mu\big(\{u_\varepsilon>0\}\cap B_r\big).$$

Moreover,

$$\int_{B_r} B_\varepsilon(u_\varepsilon) F_\varepsilon \, dx \leqslant \mu_\varepsilon\big(\{u_\varepsilon>0\}\cap B_r\big) = \mu\big(\{u_\varepsilon>0\}\cap B_r\big) + \mathrm{o}(1).$$

We then conclude,

$$E_0(B_r,u_0) \leqslant \liminf_{\varepsilon\to 0} E^\varepsilon(B_r,u_\varepsilon). \tag{5.5}$$

Combining inequalities (5.3), (5.4) and (5.5), we obtain

$$E_0(B_r,u_0) \leqslant E_0(B_r,\xi) + \omega(h),$$

for some modulus of continuity ω. This finally implies the theorem. □

The next theorem provides the last geometric measure property we shall need. It gives a uniform density of the zero phase of the solution u_0 along the free boundary points $x_0 \in \partial\{u_0>0\}$. It is worthwhile to notice that by non-degeneracy and Lipschitz continuity, we already know $|\Omega_0 \cap B_r(x_0)| \sim r^N$, for any ball $B_r(x_0)$ centered at a generic free boundary point.

Theorem 5.5. *Let $\tilde\Omega \Subset \Omega$ and $x_0 \in \partial\Omega_0 \cap \tilde\Omega$. Then there exists a constant $c = c(\tilde\Omega, N, \|\vec v\|_\infty)$ such that*

$$\big|\Omega_0^C \cap B_r(x_0)\big| \geqslant c r^N$$

for $r \leqslant \operatorname{dist}(\Omega_0, \partial\Omega)$ small enough.

Proof. Consider the auxiliary function h given by

$$\begin{cases} \Delta h = 0 & \text{in } B_r(x_0), \\ h = u_0 & \text{on } \partial B_r(x_0). \end{cases}$$

By the Variational Characterization of u_0 established in Theorem 5.4, we have

$$\int_{B_r(x_0)} \left\{ \frac{|\nabla u_0|^2}{2} + \sigma u_0 + \chi_{\{u_0>0\}} \right\} d\mu \leqslant \int_{B_r(x_0)} \left\{ \frac{|\nabla h|^2}{2} + \sigma h \right\} d\mu + \mu\big(B_r(x_0)\big).$$

From the above inequality we derive

$$\frac{1}{2}\int_{B_r(x_0)} \left\{|\nabla u_0|^2 - |\nabla h|^2\right\} d\mu \leqslant \mu\big(\{u=0\}\cap B_r(x_0)\big) - \int_{B_r(x_0)} \sigma(u_0-h)\,d\mu. \tag{5.6}$$

We also have, by Poincaré inequality for balls,

$$\int_{B_r(x_0)} \left\{|\nabla u_0|^2 - |\nabla h|^2\right\} d\mu \geqslant c\int_{B_r(x_0)} \left|\nabla(u_0-h)\right|^2 dx \geqslant \frac{c}{r^2}\int_{B_r(x_0)} (u_0-h)^2\,dx. \tag{5.7}$$

Furthermore, since $u_0(x),\ h(x)\leqslant Cr$ in $B_r(x_0)$, we have

$$\left|\int_{B_r(x_0)} \sigma(u_0-h)\,d\mu\right| \leqslant C\int_{B_r(x_0)} |u_0-h|\,dx \leqslant Cr^{N+1}. \tag{5.8}$$

We also have, by nondegeneracy and Lipschitz continuity, there exists a universal $0<\kappa<1$ and a point $y\in B_r(x_0)$ so that $B_{\kappa r}(y)\subset B_r(x_0)$ and there holds

$$u > cr \quad \text{in } B_{\kappa r}(y).$$

In this way, we obtain

$$h(x_0) = \fint_{\partial B_r(x_0)} u_0 \geqslant cr,$$

and thus, by Harnack inequality,

$$h(x)\geqslant cr \quad \text{in } B_{r/2}(x_0). \tag{5.9}$$

Finally, by Lipschitz regularity,

$$u_0(x)\leqslant c\delta r \quad \text{in } B_{\delta r}(x_0). \tag{5.10}$$

From (5.9) and (5.10) we conclude

$$h-u_0\geqslant cr \quad \text{in } B_{\delta r}(x_0), \tag{5.11}$$

for a uniform δ small enough. Combining (5.6)–(5.8) and (5.11) we obtain

$$\begin{aligned}\left|\{u=0\}\cap B_r(x_0)\right| &\geqslant c\mu\big(\{u=0\}\cap B_r(x_0)\big)\\ &\geqslant \frac{c}{r^2}\int_{B_r(x_0)} (u_0-h)^2\,dx - Cr^{N+1}\\ &\geqslant cr^N - Cr^{N+1}\\ &\geqslant cr^N,\end{aligned}$$

if r is small enough. □

This is the final ingredient we needed to state

Theorem 5.6. *Let $F(u_0)$ denote the free boundary $\partial\{u_0>0\}$. Then there holds*:

1. *The $N-1$ Hausdorff measure of $F(u_0)$ is finite. Moreover, there exists positive constants c and C, such that, for any ball $B(x,r)$ centered at $F(u_0)$, there holds*

$$cr^{N-1}\leqslant \mathcal{H}^{N-1}\big(F(u_0)\cap B(x,r)\big)\leqslant Cr^{N-1}.$$

2. $\mathcal{H}^{N-1}(F(u_0)\setminus F(u_0)_{\mathrm{red}})=0.$

3. *There is a Borel function q_{u_0} such that*

$$\Delta u_0 - \vec{v} \cdot \nabla u_0 = q_{u_0} \mathcal{H}^{N-1} \lfloor F(u_0),$$

in the sense that, for any $\zeta \in C_0^\infty(\Omega)$,

$$-\int_\Omega \langle \nabla u_0, \nabla \zeta + \zeta \vec{v} \rangle \, dx = \int_{\Omega \cap F(u_0)} \zeta q_{u_0} \, d\mathcal{H}^{N-1}.$$

Furthermore, there exists positive constants c and C, such that,

$$c \leqslant q_{u_0} \leqslant C.$$

Proof. Item (1) follows from Theorem 5.1 and isoperimetric inequality. Item (2) follows from a standard argument in Geometric Measure Theory. Item (3) is a representation of $\Delta u_0 - \vec{v} \cdot \nabla u_0$ as a measure supported in $F(u_0)$. This is derived as in [1], Theorem 4.5. □

6. Free boundary condition

At this point we have found a local minimizer of the functional

$$E_0(\xi) := \int_\Omega \left\{ \frac{1}{2} |\nabla \xi|^2 + \sigma(x)\xi + \chi_{\{\xi > 0\}} \right\} d\mu.$$

Such a minimizer is locally Lipschitz in Ω and nondegenerated in its set of positively Ω_0. We also know u_0 satisfies

$$\Delta u_0 - \vec{v} \cdot \nabla u_0 = 0 \quad \text{in } \Omega_0 := \{u_0 > 0\}.$$

Furthermore, Theorem 5.6 assures that the reduced free boundary $F(u_0)_{\text{red}}$ has total measure. The advantage of dealing with $F(u_0)_{\text{red}}$ rather than the whole free boundary $F(u_0)$ is that, the former encloses all the important geometric measure properties needed to give sense most of the classical computations. Our next step, therefore, is to study the behavior of ∇u_0 on $F(u_0)_{\text{red}}$. This is the contents of the next theorem.

Theorem 6.1 *(Free boundary condition). Let $B = B_r(x_0)$ be a ball centered at the free boundary $\partial\{u > 0\}$. Then for any field $\vec{\Psi} \in H_0^1(B, \mathbb{R}^N)$ there holds*

$$\lim_{\delta \to 0} \int_{B \cap \{u_0 = \delta\}} \left[|\nabla u_0|^2 - 2 \right] \vec{\Psi} \cdot \nu \, dS = 0, \tag{6.1}$$

where ν denotes the outward normal vector on $\{u = \delta\}$ and δ is a sequence of regular values approaching 0.

Proof. Let $B_r(x_0)$ be a ball centered at the free boundary F and denote by $G = B_r(x_0) \cap \{u_0 > 0\}$. Let us denote $u = u_\varepsilon$ and $\psi \in C_0^\infty(B_r)$ be arbitrary. If we multiply equation (PDE)

$$\Delta u - \vec{v} \cdot \nabla u = \beta_\varepsilon(u)$$

by ψu_j and afterward integrate it over B_r we obtain

$$0 = \int_{B_r} \nabla u \nabla(\psi u_j) + (\vec{v} \cdot \nabla u)\psi u_j + \beta_\varepsilon(u)\psi u_j \, dx. \tag{6.2}$$

On the other hand, using integration by parts,

$$\int_{B_r} \beta_\varepsilon(u)\psi u_j \, dx = -\int_{B_r} B_\varepsilon(u)\psi_j \, dx, \tag{6.3}$$

and

$$\int_{B_r} |\nabla u|^2 \psi_j \, dx = -\int_{B_r} \partial_j \big(|\nabla u|^2\big) \psi \, dx. \tag{6.4}$$

Combining (6.2), (6.3) and (6.4), we end up with

$$\int_{B_r} B_\varepsilon(u_\varepsilon)\psi_j \, dx = \int_{B_r} -\frac{1}{2}|\nabla u_\varepsilon|^2 \psi_j + (\nabla u_\varepsilon \cdot \nabla \psi)(u_\varepsilon)_j + (\vec{v} \cdot \nabla u_\varepsilon)\psi (u_\varepsilon)_j. \tag{6.5}$$

By reflexivity, we may assume $B_\varepsilon(u_\varepsilon) \rightharpoonup \phi$ for some function $0 \leqslant \phi \leqslant 1$ in $L^2(B_r)$. But, since $B_\varepsilon(s) = 1$ for $s \geqslant \varepsilon$ we conclude $\phi \equiv 1$ in G. Now, letting $\varepsilon \to 0$ in (6.5) we find, taking into account Lemma 5.2,

$$\int_{B_r} \phi \psi_j = \int_{B_r \cap \{u_0 > \delta\}} -\frac{1}{2}|\nabla u_0|^2 \psi_j + (\nabla u_0 \cdot \nabla \psi)(u_0)_j + (\vec{v} \cdot \nabla u_0)\psi (u_0)_j \, dx + \rho(\delta), \tag{6.6}$$

where $\lim_{\delta \to 0} \rho(\delta) = 0$. Using, once more, Green's formula, taking into account that $\Delta u_0 - \vec{v} \cdot \nabla u_0 = 0$ in $B_r \cap \{u_0 > \delta\}$, we find

$$\int_{B_r \cap \{u_0 > \delta\}} \nabla u_0 \cdot \nabla\big(\psi (u_0)_j\big) + (\vec{v} \cdot \nabla u_0)\psi (u_0)_j \, dx = \int_{\{u_0 = \delta\} \cap B_r} \psi (u_0)_j \nabla u_0 \cdot \nu \, dS. \tag{6.7}$$

However, since on the right-hand side on (6.7), we are integrating over a level set of u_0, $(u_0)_i = -\nu_i |\nabla u_0|$, thus (6.7) can be rewritten as

$$\int_{B_r \cap \{u_0 > \delta\}} \nabla u_0 \cdot \nabla\big(\psi (u_0)_j\big) + (\vec{v} \cdot \nabla u_0)\psi (u_0)_j \, dx = \int_{\{u_0 = \delta\} \cap B_r} \psi |\nabla u_0|^2 \nu_j \, dS. \tag{6.8}$$

Furthermore,

$$\int_{B_r \cap \{u_0 > \delta\}} \nabla u_0 \cdot \nabla\big(\psi (u_0)_j\big) \, dx = \int_{B_r \cap \{u_0 > \delta\}} \nabla u_0 \cdot \nabla \psi (u_0)_j \, dx + \frac{1}{2} \int_{B_r \cap \{u_0 > \delta\}} \partial_j \big(|\nabla u_0|^2\big) \psi \, dx. \tag{6.9}$$

Using integration by parts again, we find,

$$\int_{B_r \cap \{u_0 > \delta\}} \partial_j \big(|\nabla u_0|^2\big) \psi \, dX = -\int_{B_r \cap \{u_0 > \delta\}} |\nabla u_0|^2 \psi_j \, dX + \int_{B_r \cap \{u_0 = \delta\}} |\nabla u_0|^2 \psi \nu_j \, dS. \tag{6.10}$$

Combining (6.6), (6.8), (6.9) and (6.10) we finally obtain

$$\int_{B_r} \phi \psi_j = \frac{1}{2} \int_{B_r \cap \{u_0 = \delta\}} |\nabla u_0|^2 \psi \nu_j \, dS + \rho(\delta). \tag{6.11}$$

It follows, in particular, that if we take ψ supported in the interior of $B_r \setminus G$, then

$$\int_{B_r} \phi \psi_j = 0.$$

Thus, ϕ is also constant in $B_r \setminus G$, say $\phi \equiv \Lambda$ in $B_r \setminus G$.

We claim $\Lambda = 0$. Indeed, let $\Delta = B(\zeta, \varrho)$ be a ball in $B_r \setminus G$ and denote by $\Delta_\sigma := B(\zeta, \varrho + \sigma)$. Here we assume $\sigma > 0$ is small enough so that Δ_σ is still contained in $B_r \setminus G$. Let η be a nonnegative $C^\infty(\Omega)$ function satisfying

- $\eta \equiv 0$ in Δ.
- $\eta \equiv 1$ in $\Omega \setminus \Delta_\sigma$.

The function ηu_ε competes with u_ε in the variational problem (2.4), and therefore, by the minimization property of u_ε, we know,

$$\mathcal{F}_\varepsilon(u_\varepsilon) \leqslant \mathcal{F}_\varepsilon(\eta u_\varepsilon).$$

Writing this down, we find,

$$\begin{aligned}\int_{\Delta_\sigma} B_\varepsilon(u_\varepsilon)\,d\mu_\varepsilon &\leqslant \int_{\Delta_\sigma}\left\{\frac{1}{2}\left[(\eta^2-1)|\nabla u_\varepsilon|^2+|\nabla\eta|^2u_\varepsilon^2\right]+\eta u_\varepsilon\nabla\eta\nabla u_\varepsilon\right\}d\mu_\varepsilon+\int_{\Delta_\sigma}\left\{(\eta-1)\sigma_\varepsilon u_\varepsilon+B_\varepsilon(\eta u_\varepsilon)\right\}d\mu_\varepsilon\\ &= \mathrm{O}(\varepsilon)+\int_{\Delta_\sigma} B_\varepsilon(\eta u_\varepsilon)\,d\mu_\varepsilon\\ &\leqslant \mathrm{O}(\varepsilon)+\mu_\varepsilon(\Delta_\sigma\setminus\Delta).\end{aligned}\tag{6.12}$$

Letting $\varepsilon\to 0$ in (6.12), we obtain

$$\mu(\Delta_\sigma)\Lambda \leqslant \mu(\Delta_\sigma\setminus\Delta),$$

and finally, letting $\sigma\to 0$ in the above inequality we conclude the claim.

As consequence of the claim, for any $\psi\in C_0^\infty(B_r)$ we have

$$\begin{aligned}\int_{B_r}\phi\psi_j\,dx &= \int_{\{u_0\geqslant\delta\}\cap B_r}\psi_j\,dx+\tilde\rho(\delta)\\ &= \int_{\{u_0=\delta\}\cap B_r}\psi\nu_j\,dS+\tilde\rho(\delta).\end{aligned}$$

Finally, this together with (6.11) finally implies the theorem. □

Notice that, at any C^1 peace of the free boundary, the free boundary condition actually holds in the strong sense

$$|\nabla u_0|^2=2.$$

Free boundary condition (6.1), even in its weak sense, gives us a hint as to who q_0 provided by Theorem 5.6 is. Indeed, if we come back to expression (6.6), knowing already that $\phi=\chi_{\{u_0>0\}}$, we obtain $q_{u_0}=\sqrt{2}$ at any C^1 part of the free boundary. In the next section we will obtain a characterization of q_{u_0} via a blow-up analysis.

7. Blow-up analysis and $C^{1,\alpha}$ regularity of the free boundary

Let $\tilde\Omega\Subset\Omega$ be fixed and $B(x_k,\rho_k)\subset\tilde\Omega$ be a sequence of balls satisfying

(i) $x_k\to x_0\in\Omega$, $u_0(x_k)=0$, $\forall k$.
(ii) $\rho_k\to 0$.

Inspired by the homogeneity observed in Remark 4.1 and by the optimal regularity, i.e., Lipschitz continuity, we are driven to consider the blow-up sequence

$$u_k(x):=\frac{1}{\rho_k}u_0(x_k+\rho_k x)\quad\text{in } B\big(x_0,d\big({\rho_k}^{-1}-\mathrm{o}(1)\big)\big),$$

where $d=\operatorname{dist}(x_0,\partial\Omega)$. Notice that, since $|\nabla u_k|$ is uniformly bounded, up to a subsequence, we might assume

- $u_k\to u_\infty$ locally uniform, as $k\to\infty$.
- $\nabla u_k\overset{*}{\rightharpoonup}\nabla u_\infty$ in $L^\infty(\mathbb{R}^N)$, when $k\to\infty$.
- $u_\infty\geqslant 0$.
- u_∞ is globally Lipschitz.

Furthermore, since

$$\Delta u_k - \rho_k(\vec{v} \cdot \nabla u_k) = 0 \quad \text{in } \{u_k > 0\}, \tag{7.1}$$

we obtain, by letting $k \to \infty$, that u_∞ is harmonic in $\{u_\infty > 0\}$. The next lemma provides further information about the blow-up sequence.

Lemma 7.1. *With the notation above, there holds*

1. $F(u_k) \to F(u_\infty)$ *locally in the Hausdorff distance, where* $F(u_k)$ *denotes* $\partial\{u_k > 0\}$ *for* $k = 1, 2, \ldots, \infty$.
2. $\chi_{\{u_k>0\}} \to \chi_{\{u_\infty>0\}}$ *in* $L^1_{\mathrm{loc}}(\mathbb{R}^N)$.
3. $\nabla u_k \to \nabla u_\infty$ *almost everywhere in* $\mathbb{R}^N$.

Moreover, if $x_k \in \partial\{u_k > 0\}$ *for all* $k \geqslant 1$, *then* $x_0 \in \partial\{u_\infty > 0\}$.

Proof. The first assertion follows from uniform nondegeneracy of u_k and fact that u_k converges uniformly to u_∞. Let us skip the technical details.

Let $\alpha \in F(u_\infty)$ be a arbitrary free boundary point. Then, there exists a sequence $\alpha_k \in F(u_k)$, such that, $\alpha_k \to \alpha$ and

$$\sup_{B_\rho \alpha_k} u_k \geqslant c\rho.$$

Therefore, carrying the limits

$$\sup_{B_\rho \alpha} u_\infty \geqslant c\rho.$$

We have verified, u_∞ is nondegenerated along $F(u_\infty)$. This implies in particular that

$$\big|\{u_\infty > 0\} \cap B_R(\alpha)\big| \geqslant cR.$$

The above, together with item (1), implies item (2).

Let us turn our attention to item (3). By (1), for any compact set E of $\{u_\infty > 0\} \cup \mathrm{Int}(\{u_\infty = 0\})$, u_k satisfies an elliptic equation in E if k is large enough, thus $\nabla u_k \to \nabla u_\infty$ uniformly in any such set as $k \to \infty$. However, $\partial\{u_\infty = 0\}$ has Lebesgue measure zero. This finishes the proof of the lemma. □

In addition, arguing as in Theorem 5.4, we also obtain a variational characterization to u_∞.

Lemma 7.2. *Let* u_k *be a blow-up sequence with respect to a sequence of balls* $B(x_k, \rho_k)$, *with* x_k *and* ρ_k *satisfying conditions* (i) *and* (ii) *described above. Then the limiting function* u_∞ *in an absolute minimizer of* $\mathfrak{J}$ *in any ball, where* $\mathfrak{J}$ *is the functional*

$$\mathfrak{J}(\xi, B) := \int_B \frac{1}{2}|\nabla \xi|^2 + \chi_{\{\xi>0\}}\, dx. \tag{7.2}$$

We are ready to characterize the blow-up limit u_∞, which, in particular, will allow a flatness improvement of the free boundary in the spirit of Alt and Caffarelli [1] and also in the viscosity sense [8].

Theorem 7.3 *(Characterization of* u_∞*). Suppose* $x_0 \in \partial\{u_0 > 0\}_{\mathrm{red}}$ *and let* $\nu(x_0)$ *denote the inward unit normal vector on* $\partial\{u_0 > 0\}_{\mathrm{red}}$ *at* x_0. *Then, for any blow-up sequence* u_k *with* $x_k \to x_0$, *we have*

$$u_\infty(x) = \sqrt{2}\big\langle x, \nu(x_0)\big\rangle^+.$$

Proof. Let us suppose, without loss of generality, $\nu(x_0) = e_N$. We know the blow-up limits of Ω_0 and Ω_0^C are the half planes

$$\{x_N > 0\} \quad \text{and} \quad \{x_N < 0\},$$

respectively. Thus, $u_\infty(x) = 0$ if $x_N \leqslant 0$. However, since u_∞ is harmonic in $x_N > 0$ and globally Lipschitz, Liouville's Theorem assures u_∞ is a linear function. Hence

$$u_\infty(x) = \alpha x_N^+, \quad \text{for some } \alpha > 0.$$

Now, it is not hard to show, using the minimization property provided by Lemma 7.2, that $\alpha = \sqrt{2}$. Indeed, we will show that any local minimizer of functional (7.2) satisfies the same free boundary condition established in Theorem 6.1. For that, consider the strip

$$S(M,b) := \{x = (x', x_n) \mid |x'| < M,\ b < x_n < 1\},$$

with $M > 0$, $|b| < 1$, We perform a simple perturbation argument as follows: let $\psi(x')$ be a smooth function, $0 \leqslant \psi \leqslant 1$, satisfying, $\psi(x') \equiv 1$ in $|x'| < M - 1$ and $\psi(x') \equiv 0$ in $|x'| > M$. Define

$$W(x) := \left[\frac{\alpha}{1-b}(x_n - b)\psi(x') + \alpha x_n^+\big(1 - \psi(x')\big)\right]^+.$$

Notice that W agrees with V on the boundary of $S(M,b)$. From Lemma 7.2,

$$\frac{1}{2}\int\limits_{S(M,b)} \big|\nabla W(x)\big|^2 - \big|\nabla V(x)\big|^2\,dx \geqslant -M^{n-1}b.$$

Now,

$$\int\limits_{S(M-1,b)} \big|\nabla W(x)\big|^2 - \big|\nabla V(x)\big|^2\,dx = M^{n-1}\left(\frac{\alpha^2}{1+b} - \alpha^2\right).$$

Certainly,

$$\int\limits_{S(M,b)\setminus S(M-1,b)} \big|\nabla W(x)\big|^2 - \big|\nabla V(x)\big|^2\,dx \leqslant |b|C,$$

for a universal constant C. Putting these information together, we obtain

$$|b|C - M^{n-1}\alpha^2\frac{b}{1+b} + 2M^{n-1}b \geqslant 0.$$

Dividing the above expression by M^{n-1} and letting $M \to \infty$, we find

$$-\alpha^2\frac{b}{1+b} + 2b \geqslant 0.$$

Finally diving the this inequality by $|b|$ and letting $b \to 0$ from both sides, we conclude $-\alpha^2 + 2 = 0$ and the theorem is proven. □

Let us summarize the geometric properties we have proven so far about the free boundary $\partial\{u_0 > 0\}$:

1. *Elliptic equation satisfied on Ω_0*: In $\Omega_0 := \{u_0 > 0\}$, there holds
$$\Delta u_0 - \vec{v}\cdot\nabla u_0 = 0.$$
2. *Linear growth away from the free boundary and Lipschitz continuity*:
$$c\,\mathrm{dist}\big(x, F(u_0)\big) \leqslant u_0(x) \leqslant C\cdot\mathrm{dist}\big(x, F(u_0)\big).$$
3. *Nondegeneracy*:
$$\sup_{B_\rho} u_0 \geqslant c\rho.$$
4. $N-1$ *Hausdorff measure property of the free boundary*: For any ball $B(x,R)$ centered at $F(u_0)$
$$cR^{N-1} \leqslant \mathcal{H}^{N-1}\big(F(u_0)\cap B(x,R)\big) \leqslant CR^{N-1}.$$

5. *Uniform density of* Ω_0 *and* Ω_0^C: If $x_0 \in F(u_0)$,

$$\left|\Omega_0^C \cap B(x_0, R)\right| \sim R^N \quad \text{and} \quad \left|\Omega_0 \cap B(x_0, R)\right| \sim R^N.$$

6. *Classification of global profiles*: If $\nu(x_0)$ denote the inward unit normal vector on $F(u_0)_{\text{red}}$ at x_0, then the blow-up sequence converges to

$$u_\infty(x) = \sqrt{2}\left\langle x, \nu(0)\right\rangle^+.$$

In particular, the Borel function q_{u_0} provided by Theorem 5.6 is constant. Indeed, $q_{u_0} \equiv \sqrt{2}$.

Higher regularity of the free boundary, i.e., $C^{1,\alpha}$ regularity of $\partial_{\text{red}}\{u_0 > 0\}$, follows now by a small variant of the last section in [1], where it is done for the Laplacian. At this stage, the term $\vec{v} \cdot \nabla u_0$ does not add any substantial difficult anymore. It is simple to carry it out through the arguments in [1] and therefore we skip the details.

Here it is important to point out that if we are working with the elliptic operator $Lu = \Delta u - \vec{v} \cdot \nabla u$, then a similar argument as the one used in [8] permits us to interpret our free boundary condition in the sense of the celebrated works of Luis Caffarelli [7–9] (free boundary condition in the viscosity sense). In [7,8], it is proven, for the Laplacian operator, that, under the hypotheses above listed, the free boundary is $C^{1,\alpha}$ (flatness implies Lipschitz regularity [8] and Lipschitz free boundaries are $C^{1,\alpha}$, [7]). Again, the term $\vec{v} \cdot \nabla u_0$ is simple carried out through the arguments in [7,8] and thus we skip the details.

For a general elliptic operator as in (1.3), following the lines of Theorem 6.1, we obtain the following free boundary condition in the integral sense:

$$\lim_{\delta \to 0} \int_{B \cap \{u_0 = \delta\}} \left[\left\langle a_{ij}(x) Du, Du\right\rangle - 2\right] \vec{\Psi} \cdot \nu \, dS = 0, \tag{7.3}$$

for any field $\vec{\Psi} \in H_0^1(B, \mathbb{R}^N)$. It is also possible to show, such a free boundary condition also holds in the viscosity sense.

Let us re-emphasize, our approach can be successfully applied for any general elliptic operators of the form

$$Lu = D_j\left(a_{ij}(x) D_i u\right) + b_i u_i + cu,$$

with a_{ij} Hölder continuous, $b,\ c \in L^\infty$ and $c \leqslant 0$, as long as we can find solutions of the approximating free boundary problem

$$\begin{cases} D_j\left(a_{ij}(x) D_i u\right) + b_i u_i + cu = \beta_\varepsilon(u) & \text{in } \Omega, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{7.4}$$

that satisfy a useful variational characterization of the form

$$\mathfrak{F}_\varepsilon(u_\varepsilon) = \min_{\zeta \in \mathfrak{G}} \mathfrak{F}_\varepsilon(\zeta) \tag{7.5}$$

for some appropriated functional space $\mathfrak{G}$, where

$$\mathfrak{F}_\varepsilon(\zeta) := \int_\Omega \left(\frac{1}{2}\left\langle a_{ij}(x) D\zeta, D\zeta\right\rangle + \sigma_\varepsilon(x)\zeta + B_\varepsilon(\zeta)\right) d\mu_\varepsilon \tag{7.6}$$

and σ_ε and μ_ε are like in (2.4). The generalization of the regularity theory developed in [7,8] for free boundary problems with general elliptic operators as (1.1) is much more involved. However, C. Cerutti, F. Fausto and S. Salsa, in [12] showed $C^{1,\alpha}$ regularity of Lipschitz free boundaries for operators of the form

$$\mathcal{L}u = a_{ij}(x) D_{ij} u + b_i(x) u_i,$$

with Hölder continuous coefficients. In a work under preparation, F. Fausto and S. Salsa complete the bridge (at least for a large class of problems we are particularly interested in) between Flatness and $C^{1,\alpha}$ regularity for elliptic operators in divergence form. In this paper, they study the implication Flat free boundaries are Lipschitz for operators of the form

$$\mathcal{L}u = D_j\left(a_{ij}(x) D_i u\right) + b_i u_i,$$

with Lipschitz coefficients. Thus, if we assume, $a_{ij}, b_i \in \mathrm{Lip}(\Omega)$, $c = 0$, the free boundary obtained as the limiting process described in this present paper is $C^{1,\alpha}$ smooth around any point of the reduced free boundary.

In connection to the regularity theory for our free boundary problem, it would be interesting to try to derive a similar monotonicity formula as in [20]. We intend to come back to this issue in a future project.

To finish, let us register that it seems our approach would naturally generalize to study nonisotropic singular equations, that is, when the singular term also depends upon direction, as long as some variational characterization can be established. The simplest generalization would be to consider free boundary problems that are limit of $\partial_j(a_{ij}(x)u_i) + b_i u_i + cu = Q(x)\beta_\varepsilon(u)H(Du)$, where Q is a bounded and positive function and, say, $H(t) = \mathrm{o}(t^2)$ as $t \to \infty$. Uniform Lipschitz regularity of a family of solutions to

$$\Delta u = \beta_\varepsilon(u)H(Du),$$

has been established by Caffarelli, Jerison and Kenig in [10], with aid of a rather powerful monotonicity formula developed there. D. Moreira has recently studied this problem by a least supersolution method, obtaining a nice geometric description of the limiting free boundary. The complete study of nonisotropic singular perturbation problems for more general (non-linear) elliptic equations is currently in progress. The first advances on fully nonlinear singular elliptic equations have been recently obtained in [19].

Acknowledgements

The author would like to thank professor Luis A. Caffarelli for bringing this problem to his attention. He is also grateful to professor Sandro Salsa for some useful discussions and to the anonymous referee for such a careful review.

References

[1] H. Alt, L. Caffarelli, Existence and regularity for a minimum problem with free boundary, J. Reine Angew. Math. 325 (1981) 105–144.
[2] H. Berestycki, L.A. Caffarelli, L. Nirenberg, Uniform estimates for regularization of free boundary problems, in: Analysis and Partial Differential Equations, in: Lecture Notes in Pure and Appl. Math., vol. 122, Dekker, New York, 1990, pp. 567–619.
[3] H. Berestycki, L. Nirenberg, Monotonicity, symmetry and antisymmetry of solutions of semilinear elliptic equations, J. Geom. Phys. 5 (2) (1988) 237–275.
[4] H. Berestycki, L. Nirenberg, Some qualitative properties of solutions of semilinear elliptic equations in cylindrical domains, in: Analysis, et cetera, Academic Press, Boston, MA, 1990, pp. 115–164.
[5] H. Berestycki, L. Nirenberg, Travelling front solutions of semilinear equations in n dimensions, in: Frontiers in Pure and Applied Mathematics, North-Holland, Amsterdam, 1991, pp. 31–41.
[6] H. Berestycki, L. Nirenberg, On the method of moving planes and the sliding method, Bol. Soc. Brasil. Mat. (N.S.) 22 (1) (1991) 1–37.
[7] L.A. Caffarelli, A Harnack inequality approach to the regularity of free boundaries. I. Lipschitz free boundaries are $C^{1,\alpha}$, Rev. Mat. Iberoamericana 3 (2) (1987) 139–162.
[8] L.A. Caffarelli, A Harnack inequality approach to the regularity of free boundaries. II. Flat free boundaries are Lipschitz, Comm. Pure Appl. Math. 42 (1) (1989) 55–78.
[9] L.A. Caffarelli, A Harnack inequality approach to the regularity of free boundaries. III. Existence theory, compactness, and dependence on X, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 15 (4) (1988) 583–602 (1989).
[10] L.A. Caffarelli, D. Jerison, C.E. Kenig, Some new monotonicity theorems with applications to free boundary problems, Ann. of Math. (2) 155 (2) (2002) 369–404.
[11] L.A. Caffarelli, C. Lederman, N. Wolanski, Pointwise and viscosity solutions for the limit of a two phase parabolic singular perturbation problem, Indiana Univ. Math. J. 46 (3) (1997) 719–740.
[12] M.C. Cerutti, F. Ferrari, S. Salsa, Two-phase problems for linear elliptic operators with variable coefficients: Lipschitz free boundaries are $C^{1,\gamma}$, Arch. Ration. Mech. Anal. 171 (2004) 329–348.
[13] F. Ferrari, S. Salsa, Regularity of the free boundary in two-phase problems for linear elliptic operators, Adv. Math. 214 (1) (2007) 288–322.
[14] P.C. Fife, Aspects of Reacting and Diffusing Systems, Lecture Notes in Biomath., vol. 28, Springer-Verlag, New York, 1979.
[15] M. Giaquinta, E. Giusti, Quasi-Minima, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984) 79–107.
[16] Q. Han, F. Lin, Elliptic Partial Differential Equations, Courant Lecture Notes in Mathematics, vol. 1, New York University, Courant Institute of Mathematical Sciences, New York; American Mathematical Society, Providence, RI, 1997.
[17] C. Lederman, N. Wolanski, Viscosity solutions and regularity of the free boundary for the limit of an elliptic two phase singular perturbation problem, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 27 (2) (1998) 253–288 (1999).
[18] D. Moreira, E.V. Teixeira, A singular free boundary problem for elliptic equations in divergence form, Calc. Var. Partial Differential Equations 29 (2) (2007) 161–190.
[19] E.V. Teixeira, Optimal regularity of viscosity solutions of fully nonlinear singular equations and their limiting free boundary problems, Mat. Contemp., submitted for publication.
[20] G.S. Weiss, Partial regularity for a minimum problem with free boundary, J. Geom. Anal. 9 (2) (1999) 317–326.