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Strong solutions for a compressible fluid model of Korteweg type

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Abstract

We prove existence and uniqueness of local strong solutions for an isothermal model of capillary compressible fluids derived by J.E. Dunn and J. Serrin (1985). This nonlinear problem is approached by proving maximal regularity for a related linear problem in order to formulate a fixed point equation, which is solved by the contraction mapping principle. Localising the linear problem leads to model problems in full and half space, which are treated by Dore–Venni Theory, real interpolation and $\mathcal{H}^\infty$-calculus. For these steps, it is decisive to find conditions on the inhomogeneities that are necessary and sufficient.

Résumé

Nous prouvons l'existence et l'unicité de fortes solutions locales pour un modèle de fluides compressibles isothermes avec capillarité, dérivé par J.E. Dunn et J. Serrin (1985). L'idée de la démonstration consiste à montrer la régularité maximale d'un problème linéaire apparenté, afin de formuler un problème de point fixe, résolu par la suite par le principe de la contraction. Le problème linéaire est transféré par le principe de la localisation aux problèmes de modèle correspondants sur l'espace entier et sur le demi-espace. Ceux-là peuvent être traités à l'aide de la théorie de Dore–Venni, de l'interpolation réelle et du calcul des $\mathcal{H}^\infty$. Pour ces étapes, il est essentiel de trouver les conditions nécessaires et suffisantes pour les inhomogénéités.

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1. Introduction

In this paper we study a nonlinear system of partial differential equations that describes the dynamics of a non-thermal, compressible fluid exhibiting viscosity and capillarity. The density $\rho > 0$ of the fluid and its velocity field $u \in \mathbb{R}^n$ are governed by the equations

$$\begin{aligned} &\partial_t(\rho u) + \nabla \cdot \big(\rho u \otimes u + P(\rho)\mathbf{I}\big) = \nabla \cdot (\mathbf{S} + \mathbf{K}) + \rho f, \\ &\partial_t \rho + \nabla \cdot (\rho u) = 0, \end{aligned} \tag{1.1}$$

where the viscous stress tensor $\mathbf{S}$ and the Korteweg stress tensor $\mathbf{K}$ are

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$$\mathbf{S} = \lambda\nabla\cdot u\mathbf{I} + 2\mu\mathbf{D}(u),$$
$$\mathbf{K} = \frac{\kappa}{2}\big(\Delta\rho^2 - |\nabla\rho|^2\big)\mathbf{I} - \kappa\nabla\rho\otimes\nabla\rho \tag{1.2}$$

with $\mathbf{D}(u) = (\nabla u + (\nabla u)^T)/2$ the strain and $\mathbf{I}$ the unit tensor, λ and μ are viscosity coefficients, κ is a capillarity coefficient, and $P(\rho)$ and f denote pressure and external forces. The purpose of the paper is to prove existence and uniqueness of strong solutions to (1.1) in both bounded and unbounded domains locally in time, with initial and boundary conditions

$$\begin{aligned}
&u = 0, \quad (t,x) \in [0,T_0]\times\partial\Omega,\\
&\partial_\nu\rho = 0, \quad (t,x) \in [0,T_0]\times\partial\Omega,\\
&u = u_0(x), \quad (t,x) \in \{0\}\times\Omega,\\
&\rho = \rho_0(x), \quad (t,x) \in \{0\}\times\Omega.
\end{aligned} \tag{1.3}$$

To prepare for stating our main result, we compute $\nabla\cdot\mathbf{K}$ and $\nabla\cdot\mathbf{S}$ as

$$\nabla\cdot\mathbf{K} = \kappa\rho\nabla\Delta\rho + \big(\rho\Delta\rho + |\nabla\rho|^2/2\big)\nabla\kappa + \nabla\kappa\cdot\nabla\rho\otimes\nabla\rho,$$
$$\nabla\cdot\mathbf{S} = \mu\Delta u + (\lambda+\mu)\nabla\nabla\cdot u + \nabla\cdot u\nabla\lambda + 2\mathbf{D}(u)\cdot\nabla\mu,$$

and rewrite (1.1) in the form

$$\begin{aligned}
&\rho\partial_t u - \mu\Delta u - (\lambda+\mu)\nabla\nabla\cdot u - \kappa\rho\nabla\Delta\rho = H(u,\rho), \quad (t,x)\in[0,T_0]\times\Omega,\\
&\partial_t\rho + \rho\nabla\cdot u + u\cdot\nabla\rho = 0, \quad (t,x)\in[0,T_0]\times\Omega,
\end{aligned} \tag{1.4}$$

with

$$H(u,\rho) = \rho f - \rho\nabla u\cdot u - \nabla P(\rho) + \big(\rho\Delta\rho + |\nabla\rho|^2/2\big)\nabla\kappa + \nabla\kappa\cdot\nabla\rho\otimes\nabla\rho + \nabla\cdot u\nabla\lambda + 2\mathbf{D}(u)\cdot\nabla\mu.$$

In the case of constant coefficients, $H(u,\rho) = \rho f - \rho\nabla u\cdot u - \nabla P(\rho)$.

The main result of this paper ensures existence and uniqueness on a maximal interval $[0, t_{\max})$, with $0 < t_{\max} \leqslant T_0$ depending on the initial value (u_0, ρ_0):

Theorem 1.1. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, with C^3-boundary, $\Gamma := \partial\Omega$, and J_0 denote the compact time interval $[0, T_0]$, $T_0 > 0$. Let $n + 2 < p < \infty$ and suppose that*

(1) $\mu, \lambda, \kappa \in \mathrm{C}(J_0; \mathrm{C}^1(\overline{\Omega}))$ *and* $\mu(t,x) > 0$, $\kappa(t,x) > 0$, $2\mu(t,x) + \lambda(t,x) > 0$ *for* $(t,x) \in J_0\times\overline{\Omega}$;
(2) $P \in \mathrm{C}^{2-}(\mathbb{R}_+;\mathbb{R})$;
(3) $f \in X = \mathrm{L}_p(J_0; \mathrm{L}_p(\Omega;\mathbb{R}^n))$;
(4) $u_0 \in \mathrm{B}^{2-2/p}_{pp}(\Omega;\mathbb{R}^n)$, $\rho_0 \in \mathrm{B}^{3-2/p}_{pp}(\Omega)$, $\rho_0(x) > 0$ *for all* $x \in \overline{\Omega}$;
(5) *compatibility conditions*: $u_0 = 0$ *in* $\mathrm{B}^{2-3/p}_{pp}(\Gamma;\mathbb{R}^n)$, $\partial_\nu\rho_0 = 0$ *in* $\mathrm{B}^{2-3/p}_{pp}(\Gamma)$.

Then there exists $t_{\max} \in (0, T_0]$ such that for any $T < t_{\max}$ the nonlinear problem (1.4), (1.3) *admits a unique solution (u,ρ) on $J = [0,T]$ in the maximal regularity class $Z_1(J)\times Z_2(J)$ with*

$$Z_1(J) := \mathrm{H}^1_p\big(J; \mathrm{L}_p(\Omega;\mathbb{R}^n)\big) \cap \mathrm{L}_p\big(J; \mathrm{H}^2_p(\Omega;\mathbb{R}^n)\big),$$
$$Z_2(J) := \mathrm{H}^{3/2}_p\big(J; \mathrm{L}_p(\Omega)\big) \cap \mathrm{L}_p\big(J; \mathrm{H}^3_p(\Omega)\big).$$

In particular, if $f \in \mathrm{C}(J_0\times\Omega;\mathbb{R}^n)$ we have

$$u \in \mathrm{C}^1\big((0,t_{\max}); \mathrm{C}(\Omega;\mathbb{R}^n)\big) \cap \mathrm{C}\big((0,t_{\max}); \mathrm{C}^2(\Omega;\mathbb{R}^n)\big),$$
$$\rho \in \mathrm{C}^{3/2}\big((0,t_{\max}); \mathrm{C}(\Omega)\big) \cap \mathrm{C}\big((0,t_{\max}); \mathrm{C}^3(\Omega)\big).$$

Moreover, the map $(u_0,\rho_0) \to (u,\rho)(t)$ defines a local semiflow on the natural phase space $\mathrm{B}^{2-2/p}_{pp}(\Omega;\mathbb{R}^n)\times \mathrm{B}^{3-2/p}_{pp}(\Omega)$ in the autonomous case.

Remark 1.1. The condition on p, which is used to guarantee that nonlinear terms, such as

$$\rho\partial_t u, \quad \rho\nabla u\cdot u, \quad \rho\nabla\Delta\rho, \quad \text{etc.},$$

are defined in certain L_p-spaces, is chosen rather generous and could still be weakened by means of more refined considerations all nonlinear terms, for instance, in order to further minimise the smoothness assumptions on the initial data. But this is not our goal here. We thus require $p > n+2$ several times. The importance of this restriction relies on the following considerations. Assuming that v belongs to $Z_1(J) = \mathrm{H}^1_p(J;\mathrm{L}_p(p)\Omega;\mathbb{R}^n) \cap \mathrm{L}_p(J;\mathrm{H}^2_p(\Omega;\mathbb{R}^n))$, with $J = [0,T]$ (or $\mathbb{R}_+$) and Ω a bounded or unbounded domain with sufficiently smooth boundary. Then, by the *mixed derivative theorem* which is due to Sobolevskij [19], see also [17], we obtain for all $p \in (1,\infty)$ and $\theta \in (0,1)$ the embedding $Z_1(J) \hookrightarrow \mathrm{H}^\theta_p(J;\mathrm{H}^{2(1-\theta)}_p(\Omega;\mathbb{R}^n))$ for $0<\theta<1$. For instance, choosing $\theta = 1/2$ we conclude $v \in \mathrm{H}^{1/2}_p(J;\mathrm{H}^1_p(\Omega;\mathbb{R}^n))$ and this means that ∇v still possesses time regularity. Starting from this general embedding one can address the validity of

$$\mathrm{H}^\theta_p\big(J;\mathrm{H}^{2(1-\theta)}_p(\Omega;\mathbb{R}^n)\big) \hookrightarrow \mathrm{C}^\alpha\big(J;\mathrm{C}^\beta(\Omega;\mathbb{R}^n)\big), \quad \text{with pair } (\alpha,\beta)\in\mathbb{R}^2_+.$$

If $p > n+2$, Sobolev embeddings show that $v \in \mathrm{C}^{1/2}(J;\mathrm{C}(\Omega;\mathbb{R}^n)) \cap \mathrm{C}(J;\mathrm{C}^1(\Omega;\mathbb{R}^n))$ and thus

$$\|v\|_{\mathrm{C}^{1/2}(J;\mathrm{C}(\Omega;\mathbb{R}^n))\cap\mathrm{C}(J;\mathrm{C}^1(\Omega;\mathbb{R}^n))} \leqslant C\|v\|_{Z_1(J)}.$$

For the case of a compact time interval $J = [0,T]$, $T>0$, the constant C can be chosen uniformly in T, for all v that vanish at $t=0$; this can be easily seen by an extension argument. The above embeddings also show that for $p > n+2$ the regularity class $Z_1(J)$ forms a Banach algebra. Similar investigations for the regularity class $Z_2(J)$ lead to

$$Z_2(J) \hookrightarrow \mathrm{H}^{\theta 3/2}_p\big(J;\mathrm{H}^{3(1-\theta)}_p(\Omega)\big) \hookrightarrow \mathrm{C}^1\big(J;\mathrm{C}(\Omega)\big)\cap\mathrm{C}^{1/2}\big(J;\mathrm{C}^1(\Omega)\big)\cap\mathrm{C}\big(J;\mathrm{C}^2(\Omega)\big), \quad \theta\in(0,1),$$

where $p > n+2$ is again needed to prove embeddings in continuous spaces.

In the following Section 2, we derive a linear problem corresponding to (1.4) and prove maximal L_p-regularity. The proof primarily consists of solving related full and half space problems and is performed in detail. In Section 3 we solve the nonlinear problem (1.4) via the contraction mapping principle, after rearranging the problem into a fixed point equation with the aid of maximal regularity of the linear problem. Only estimates showing self-mapping are carried out, since the estimates for contraction are very similar to the case of self-mapping. Section 4 gathers various remarks including arguments which show that and how Theorem 1 carries over to unbounded domains.

Model (1.1) can be found in the paper [8] by Dunn and Serrin as well as in [2,4] and [10]. For the whole-space problem, $\Omega = \mathbb{R}^n$, Danchin and Desjardins in [5] proved existence and uniqueness in critical Besov spaces of type $\mathrm{B}^{n/2}_{21}(\mathbb{R}^n)$. These spaces are close to $\mathrm{L}_2(\mathbb{R}^n)$ and have the advantage that the embedding $\mathrm{B}^{n/2}_{21}(\mathbb{R}^n) \hookrightarrow \mathrm{L}_\infty(\mathbb{R}^n)$ holds. The results of [5] comprise (i) global solutions for sufficiently small data and initial data close enough to stable equilibria, and (ii) local well-posedness for initial densities bounded away from zero and without stability assumption on the pressure law. Comparable earlier existence results for strong solutions in the whole space $\mathbb{R}^n$ due to Hattori and Li [11,12] were restricted to the case of constant coefficients and very regular initial data $(\rho_0,u_0)\in\mathrm{H}^s_2(\mathbb{R}^n)\times\mathrm{H}^{s-1}_2(\mathbb{R}^n;\mathbb{R}^n)$ with $s \geqslant n/2+4$. In [3], Bresch, Desjardins, and Lin obtained existence of global weak solutions in a periodic or strip domain and studied various dependencies of the viscosity and capillarity coefficients on the density as well as related shallow water and lubrication models.

We now briefly comment on the regularity classes we use. Since we are interested in strong solutions, i.e., all derivatives are supposed to be in $\mathrm{L}_p(J;\mathrm{L}_p(\Omega))$, our choices of Z_1 and Z_2 are probably obvious to the reader, with the possible exception of the first part of the class Z_2. To understand this constraint, i.e. $\partial_t\rho \in \mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\Omega))$ as well as $\partial_t\rho \in \mathrm{L}_p(J;\mathrm{H}^1_p(\Omega))$ being a consequence of the embedding $Z_2 \hookrightarrow \mathrm{H}^1_p(J;\mathrm{H}^1_p(\Omega))$, let us assume that ρ lies in $\mathrm{H}^1_p(J;\mathrm{L}_p(\Omega)) \cap \mathrm{L}_p(J;\mathrm{H}^3_p(\Omega))$ which firstly seems natural by the equations. But in this case, and for $p > n+2$, there holds

$$u\cdot\nabla\rho + \rho\nabla\cdot u \in \mathrm{H}^{1/2}_p\big(J;\mathrm{L}_p(\Omega)\big)\cap\mathrm{L}_p\big(J;\mathrm{H}^1_p(\Omega)\big)$$

by virtue of the embeddings

$$\mathrm{H}^1_p\big(J;\mathrm{L}_p(\Omega)\big)\cap\mathrm{L}_p\big(J;\mathrm{H}^2_p(\Omega)\big) \hookrightarrow \mathrm{H}^{1/2}_p\big(J;\mathrm{H}^1_p(\Omega)\big)\cap\mathrm{L}_p\big(J;\mathrm{H}^2_p(\Omega)\big),$$
$$\mathrm{H}^1_p\big(J;\mathrm{L}_p(\Omega)\big)\cap\mathrm{L}_p\big(J;\mathrm{H}^3_p(\Omega)\big) \hookrightarrow \mathrm{H}^{1/2}_p\big(J;\mathrm{H}^{3/2}_p(\Omega)\big)\cap\mathrm{L}_p\big(J;\mathrm{H}^3_p(\Omega)\big),$$

and the fact that all these spaces are Banach algebras. These embeddings arise from the *mixed derivative theorem*. By maximal regularity, we might expect that $\partial_t\rho$ also belongs to $Z_{1/2} := \mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\Omega)) \cap \mathrm{L}_p(J;\mathrm{H}_p^1(\Omega))$ due to the conservation equation, which explains the choice of the regularity class Z_2. Hence, for seeking (u,ρ) in $Z_1 \times Z_2$ we have to consider the momentum equation in $X := \mathrm{L}_p(J;\mathrm{L}_p(\Omega;\mathbb{R}^n))$ and the conservation equation in $Z_{1/2}$. Then, by well-known trace theorems (cf. [1,6,15]) the data u_0 and ρ_0 necessarily satisfy

$$u_0 \in \mathrm{B}_{pp}^{2-2/p}(\Omega;\mathbb{R}^n), \qquad \rho_0 \in \mathrm{B}_{pp}^{3-2/p}(\Omega).$$

As usual, here and in the sequel H_p^s denote the Bessel potential spaces and B_{pq}^s the Besov spaces which coincide with the Slobodeckij spaces W_p^s for $q = p$, $s \notin \mathbb{N}$ (fractional Sobolev spaces), see [20,21]. Furthermore, if $\mathcal{F}$ is any function space then we set ${}_0\mathcal{F} := \{v \in \mathcal{F}\colon\ v_{t=0} = 0\}$, whenever traces exist.

We finally remark that the positivity of ρ persists from that of its data ρ_0 at least for a sufficiently short time.

2. The linear problem

2.1. Linearisation

We are looking for a "good" linearisation approximating the nonlinear problem in some respects. For this purpose we introduce the auxiliary function $\tilde\rho(t,x) > 0$, for all $(t,x) \in J \times \overline{\Omega}$, satisfying $\tilde\rho(0) = \rho_0 \in \mathrm{B}_{pp}^{3-2/p}(\Omega)$ and being regular as needed, e.g. $\tilde\rho \in Z_2$ which implies $\tilde\rho \in \mathrm{C}(J;\mathrm{B}_{pp}^{3-2/p}(\Omega))$ and thus $\tilde\rho \in \mathrm{C}(J \times \overline{\Omega})$ for $p > (n+2)/3$ due to Sobolev embedding. To assure positivity of $\tilde\rho$ we have to demand $\rho_0(x) > 0$ in $\overline{\Omega}$. We rewrite the nonlinear problem (1.4) as follows

$$\begin{aligned}
&\partial_t u - \tilde\mu\Delta u - (\tilde\lambda + \tilde\mu)\nabla\nabla\cdot u - \kappa\nabla\Delta\rho = F(u,\rho) + b(u,\rho), \quad (t,x) \in J \times \Omega,\\
&\partial_t\rho + \tilde\rho\nabla\cdot u = G(u,\rho), \quad (t,x) \in J \times \Omega,\\
&u = 0, \quad (t,x) \in J \times \Gamma,\\
&\partial_\nu\rho = 0, \quad (t,x) \in J \times \Gamma,\\
&u = u_0, \quad (t,x) \in \{0\} \times \Omega,\\
&\rho = \rho_0, \quad (t,x) \in \{0\} \times \Omega,
\end{aligned} \tag{2.1}$$

where all nonlinear terms (both lower and higher order) were summarised into $F(u,\rho)$ and $G(u,\rho)$ reading as

$$\begin{aligned}
F(u,\rho) &= \tilde\rho^{-1}\big[-\rho\nabla u\cdot u - \nabla P(\rho) - P(\rho)\nabla\lambda + \big(\rho\Delta\rho + |\nabla\rho|^2/2\big)\nabla\kappa\\
&\qquad + \nabla\kappa\cdot\nabla\rho\otimes\nabla\rho + (\tilde\rho - \rho)\partial_t u - (\tilde\rho - \rho)\kappa\nabla\Delta\rho\big],\\
G(u,\rho) &= -u\cdot\nabla\rho + (\tilde\rho - \rho)\nabla\cdot u,
\end{aligned} \tag{2.2}$$

and the linear terms of lower order are summed up to

$$b(u,\rho) = \tilde\rho^{-1}\big[\rho f + 2\mathbf{D}(u)\cdot\nabla\mu + \nabla\cdot u\nabla\lambda\big]. \tag{2.3}$$

The functions $\tilde\mu := \mu\tilde\rho^{-1}$ and $\tilde\lambda := \lambda\tilde\rho^{-1}$ denote the new viscosity coefficients, which inherit all properties of μ and λ, i.e. $\tilde\mu, \tilde\lambda \in \mathrm{C}(J;\mathrm{C}^1(\overline{\Omega}))$ and $\tilde\mu$, $2\tilde\mu + \tilde\lambda$ are again bounded below.

2.2. Maximal regularity for the linearised system

In this section we show maximal regularity for the linear problem, that means, we provide necessary and sufficient conditions for the inhomogeneities which entail existence and uniqueness of solutions in $Z_1 \times Z_2$. Having in mind the linearisation (2.1) the linear problem for (v,ϱ) reads

$$\begin{aligned}
&\partial_t v - \mu\Delta v - (\lambda + \mu)\nabla\nabla\cdot v - \kappa\nabla\Delta\varrho = f(t,x), \quad (t,x) \in J \times \Omega,\\
&\partial_t\varrho + \beta\nabla\cdot v = g(t,x), \quad (t,x) \in J \times \Omega,\\
&v = h(t,x), \quad (t,x) \in J \times \Gamma,
\end{aligned}$$

$$\partial_\nu \varrho = \sigma(t,x), \quad (t,x) \in J \times \Gamma,$$
$$v = u_0(x), \quad (t,x) \in \{0\} \times \Omega,$$
$$\varrho = \rho_0(x), \quad (t,x) \in \{0\} \times \Omega, \tag{2.4}$$

where the coefficients are again denoted by their original notations and $\tilde{\rho}$ has been replaced by a more general function β. The main theorem in this section is

Theorem 2.1. *Let Ω be an open bounded domain in $\mathbb{R}^n$ with C^3-boundary Γ. Let $J = [0,T]$, $0 < T < \infty$, and $1 < p < \infty$, $p \neq 3/2$. Suppose that μ, λ, $\kappa \in \mathrm{C}(J \times \overline{\Omega})$, $\beta \in \mathrm{C}^{1/2}(J;\mathrm{C}(\overline{\Omega})) \cap \mathrm{C}(J;\mathrm{C}^1(\overline{\Omega}))$ and $\mu > 0$, $2\mu + \lambda > 0$, $\kappa > 0$, $\beta > 0$ for all $(t,x) \in J \times \overline{\Omega}$. Then problem* (2.4) *has exactly one solution (v,ϱ) in*

$$Z_1 \times Z_2 = \mathrm{H}^1_p\big(J;\mathrm{L}_p(\Omega;\mathbb{R}^n)\big) \cap \mathrm{L}_p\big(J;\mathrm{H}^2_p(\Omega;\mathbb{R}^n)\big) \times \mathrm{H}^{3/2}_p\big(J;\mathrm{L}_p(\Omega)\big) \cap \mathrm{L}_p\big(J;\mathrm{H}^3_p(\Omega)\big),$$

if and only if the data f, g, h, σ, u_0, ρ_0 satisfy the following conditions

(1) $f \in \mathrm{L}_p(J;\mathrm{L}_p(\Omega;\mathbb{R}^n))$;
(2) $g \in Z_{1/2} := \mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\Omega)) \cap \mathrm{L}_p(J;\mathrm{H}^1_p(\Omega))$;
(3) $h \in Y(\mathbb{R}^n) := \mathrm{B}^{1-1/2p}_{pp}(J;\mathrm{L}_p(\Gamma;\mathbb{R}^n)) \cap \mathrm{L}_p(J;\mathrm{B}^{2-1/p}_{pp}(\Gamma;\mathbb{R}^n))$;
(4) $\sigma \in Y := \mathrm{B}^{1-1/2p}_{pp}(J;\mathrm{L}_p(\Gamma)) \cap \mathrm{L}_p(J;\mathrm{B}^{2-1/p}_{pp}(\Gamma))$;
(5) $u_0 \in \mathrm{B}^{2-2/p}_{pp}(\Omega;\mathbb{R}^n)$, $\rho_0 \in \mathrm{B}^{3-2/p}_{pp}(\Omega)$;
(6) $h_{|t=0} = u_{0|\Gamma}$ *in* $\mathrm{B}^{2-3/p}_{pp}(\Gamma;\mathbb{R}^n)$ *and* $\sigma_{|t=0} = \partial_\nu \rho_{0|\Gamma}$ *in* $\mathrm{B}^{2-3/p}_{pp}(\Gamma)$*, in case* $p > 3/2$.

Moreover, the linear operator $\mathcal{L}$ defined by the left-hand side of (2.4) *is an isomorphism between the regularity class $Z_1 \times Z_2$ and the space of data including the compatibility conditions, i.e. $\mathcal{L} \in \mathcal{L}is(Z_1 \times Z_2, D)$ with*

$$D := \big\{\xi \in X \times Z_{1/2} \times Y(\mathbb{R}^n) \times Y \times \mathrm{B}^{2-2/p}_{pp}(\Omega;\mathbb{R}^n) \times \mathrm{B}^{3-2/p}_{pp}(\Omega)\colon\ \xi \text{ satisfies condition } (6)\big\}$$

Proof. *First step – the necessity part*. Suppose (v,ϱ) solves (2.4) and belongs to $Z_1 \times Z_2$. Then it follows $f = \partial_t v - \mu\Delta v - (\lambda+\mu)\nabla\nabla\cdot v - \kappa\nabla\Delta\varrho \in \mathrm{L}_p(J;\mathrm{L}_p(\Omega;\mathbb{R}^n))$ due to the regularities of v, ϱ and the continuity of coefficients. By virtue of the embeddings

$$Z_1 \hookrightarrow \mathrm{H}^{1/2}_p\big(J;\mathrm{H}^1_p(\Omega;\mathbb{R}^n)\big) \cap \mathrm{L}_p\big(J;\mathrm{H}^2_p(\Omega;\mathbb{R}^n)\big), \qquad Z_2 \hookrightarrow \mathrm{H}^{3/2}_p\big(J;\mathrm{L}_p(\Omega)\big) \cap \mathrm{H}^1_p\big(J;\mathrm{H}^1_p(\Omega;\mathbb{R}^n)\big)$$

we see $\partial_t\varrho, \nabla\cdot v \in Z_{1/2}$ and thus $\partial_t\varrho + \beta\nabla\cdot v \in Z_{1/2}$ as well, since β lies in a multiplicator space of $Z_{1/2}$. Conditions (3)–(5) are obtained by well-known trace theorems (cf. [1,6,15]), which we shall encounter later on. Finally, for $p > 3/2$ the compatibility conditions $h_{|t=0} = u_0$ and $\sigma_{|t=0} = \partial_\nu\rho_0$ on Γ must hold in $\mathrm{B}^{2-3/p}_{pp}(\Gamma;\mathbb{R}^n)$ and $\mathrm{B}^{2-3/p}_{pp}(\Gamma)$, respectively. The value $p = 3/2$ has been excluded since the trace theorems leading to 6 are not true for this value.

Second step – the sufficiency part. We follow the strategy for general parabolic problems. The starting point is localisation w.r.t. space: we choose a partition of unity $\varphi_j \in \mathrm{C}^\infty_0(\mathbb{R}^n)$, $j = 1,\dots,N$, with $0 \leqslant \varphi_j \leqslant 1$ and supp $varphi_j =: U_j$, such that the domain is covered

$$\overline{\Omega} \subset \bigcup_{j=1}^N U_j.$$

After multiplying all equations of (2.4) by each φ_j and commuting φ_j with differential operators we obtain local problems for $(u_j,\rho_j) := (\varphi_j u, \varphi_j \rho)$, $j = 1,\dots,N$. Considering local coordinates in $\overline{\Omega} \cap U_j$ and coordinate transformations θ_j which are C^3-diffeomorphisms to smoothness assumptions on the boundary the original problem is reduced to a finite number of so-called full-space problems related to $U_j \subset \mathring{\Omega}$ $(U_j \cap \partial\Omega = \varnothing)$ and half-space problems for $U_j \cap \partial\Omega \neq \varnothing$. Further, the transformed differential operators enjoy the same ellipticity properties etc. as before, i.e. the principal part remains unchanged. Note that the transformation induces isomorphisms between Sobolev spaces, i.e.

$$\theta_j : \mathrm{H}^k_p(\Omega \cap U_j; E) \longrightarrow \mathrm{H}^k_p\big(\mathbb{R}^n_+ \cap \theta_j(U_j); E\big), \quad E \text{ any Banach space,}$$

for each $p \in [1, \infty]$ and $k = 0, 1, 2, 3$. For these (full- and half-space) problems unique solutions will be available, and after summing up all local solutions we obtain a fixed point equation which can be solved first on a small time interval (!). Proceeding in this way the problem can be solved on the entire interval $[0, T]$ after finitely many steps. As to literature of localisation techniques for bounded domains, we refer to [15,6]; a very detailed description of these techniques, with application to an example, can be found in [23] and [14]. We start with

(a) The full-space problem. In this case we are concerned with

$$\begin{aligned}
&\partial_t u - \mu\Delta u - (\lambda+\mu)\nabla\nabla\cdot u - \kappa\Delta\nabla\rho = f(t,x), \quad (t,x) \in J \times \mathbb{R}^n,\\
&\partial_t \rho + \beta\nabla\cdot u = g(t,x), \quad (t,x) \in J \times \mathbb{R}^n,\\
&u = u_0(x), \quad (t,x) \in \{0\} \times \mathbb{R}^n,\\
&\rho = \rho_0(x), \quad (t,x) \in \{0\} \times \mathbb{R}^n,
\end{aligned} \tag{2.5}$$

in $\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^n;\mathbb{R}^n)) \times \mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\mathbb{R}^n)) \cap \mathrm{L}_p(J;\mathrm{H}_p^1(\mathbb{R}^n))$ with $J = [0, T]$, and look for unique solutions (u, ρ) in the maximal regularity class $Z_1 \times Z_2$ defined by

$$\begin{aligned}
&Z_1 := \mathrm{H}_p^1\big(J;\mathrm{L}_p(\mathbb{R}^n;\mathbb{R}^n)\big) \cap \mathrm{L}_p\big(J;\mathrm{H}_p^2(\mathbb{R}^n;\mathbb{R}^n)\big),\\
&Z_2 := \mathrm{H}_p^{3/2}\big(J;\mathrm{L}_p(\mathbb{R}^n)\big) \cap \mathrm{L}_p\big(J;\mathrm{H}_p^3(\mathbb{R}^n)\big).
\end{aligned}$$

Theorem 2.2. *Let $J = [0, T]$ and $1 < p < \infty$. Assume that μ, $2\mu + \lambda$, κ, β are positive. Then problem* (2.5) *has exactly one solution (u, ρ) in the space $Z_1 \times Z_2$ if and only if the data f, g, u_0, ρ_0 satisfy the following conditions*

(1) $f \in \mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^n;\mathbb{R}^n))$;
(2) $g \in Z_{1/2} := \mathrm{H}_p^{1/2}(J;\mathrm{L}_p(\mathbb{R}^n)) \cap \mathrm{L}_p(J;\mathrm{H}_p^1(\mathbb{R}^n))$;
(3) $u_0 \in \mathrm{B}_{pp}^{2-2/p}(\mathbb{R}^n;\mathbb{R}^n)$ *and* $\rho_0 \in \mathrm{B}_{pp}^{3-2/p}(\mathbb{R}^n)$.

Proof. (i) *The necessity part.* It is easy to verify the regularities of data the f and g. As regards the initial data, we exemplarily show how to deal with ρ_0. For this, we draw on a general trace theorem, which can be found in [17] and [22], reading as follows. Assume that $I = [0, T]$ or $\mathbb{R}_+$, $s > 1/p$ and B is an $\mathcal{R}$-sectorial operator[1] in a Banach space X of class $\mathcal{HT}$,[2] i.e. the Hilbert transform is bounded on $\mathrm{L}_q(\mathbb{R}; X)$ for some (and then all) $q \in (1, \infty)$ or equivalently X belongs to the class *UMD* (unconditional martingale differences), with $\mathcal{R}$-angle $\phi_B^{\mathcal{R}} < \pi$. Then there holds

$$\mathrm{H}_p^s(I;X) \cap \mathrm{L}_p\big(I;D(B^s)\big) \hookrightarrow \mathrm{C}\big(I;D_B(s-1/p,p)\big),$$

with $D_B(s - 1/p, p) := (X, D(B))_{s-1/p,p}$, for $0 < s - 1/p < 1$, denoting real interpolation between Banach spaces X and $D(B)$, and in case of $1 + \alpha > s - 1/p > 1$, $\alpha > 0$, we set $D_B(s - 1/p, p) := \{x \in X : B^\alpha x \in (X, D(B))_{s-\alpha-1/p,p}\}$. Applying this result with $B = 1 - \Delta$ and $s = 3/2$ we then obtain $\rho_0 \in D_B(3/2 - 1/p, p)$ which is equivalent to $B^{1/2}\rho_0 \in D_B(1 - 1/p, p) = (\mathrm{L}_p(\mathbb{R}^n), \mathrm{H}_p^2(\mathbb{R}^n))_{1-1/p,p} = \mathrm{B}_{pp}^{2-2/p}(\mathbb{R}^n)$. However this means $\rho_0 \in B^{-1/2}\mathrm{B}_{pp}^{2-2/p}(\mathbb{R}^n) = \mathrm{B}_{pp}^{3-2/p}(\mathbb{R}^n)$, which finishes the necessity part of the proof.

(ii) *The sufficiency part.* Suppose that the data belong to the prescribed spaces. Further on, we may assume w.l.o.g. that $\beta = 1$ as in the other case we can set $\tilde{u} := \beta u$ and $\tilde{\kappa} = \beta\kappa$. Multiplying the momentum equation with β we obtain a problem for $(\tilde{u}, \rho)$ having the precisely identical form of (2.5), but with $\beta = 1$ and $\tilde{\kappa}$ in place of κ. Next, we define $\gamma := 2\mu + \lambda$ and set $w := \nabla \cdot u$. We then deduce an equation for the auxiliary function w by applying the divergence operator to the first equation and $z\Delta$, with $z \in \mathbb{C}\backslash\{0\}$, to the second one. This leads to a problem for (w, ρ) reading as

$$\begin{aligned}
&\partial_t w - \gamma\Delta w - \kappa\Delta\Delta\rho = \nabla\cdot f(t,x), \quad (t,x) \in J \times \mathbb{R}^n,\\
&\partial_t z\Delta\rho + z\Delta w = z\Delta g(t,x), \quad (t,x) \in J \times \mathbb{R}^n,\\
&w = \nabla\cdot u_0(x), \quad (t,x) \in \{0\} \times \mathbb{R}^n,\\
&\rho = \rho_0(x), \quad (t,x) \in \{0\} \times \mathbb{R}^n.
\end{aligned}$$

[1] In the notion of sectorial operators one replaces the condition of boundedness by means of $\mathcal{R}$-boundedness.

[2] Let $1 < q < \infty$ and $(\Omega, \Sigma, d\mu)$ a measure space, then $\mathrm{L}_q(\Omega, d\mu; Y)$, Y any Hilbert space, is a Banach space of class $\mathcal{HT}$.

Summing up both differential equations entails

$$\partial_t w - (\gamma - z)\Delta w + \partial_t(z\Delta\rho) - \frac{\kappa}{z}\Delta(z\Delta\rho) = \nabla\cdot f + z\Delta g, \quad (t,x)\in J\times\mathbb{R}^n.$$

Next, we are looking for a complex number z, such that $\gamma - z = \kappa/z$ holds. This condition leads to an equation for z solved by

$$z_{1,2} = \frac{2\mu+\lambda}{2} \pm \sqrt{\frac{(2\mu+\lambda)^2}{4} - \kappa}.$$

It is easy to see that by the assumptions $2\mu+\lambda>0$ and $\kappa>0$ we have $z_{1,2}\in\Sigma_\phi$ with $\phi<\pi/2$. In the following we take $z=z_1$ and thus $z_2=\gamma-z_1=\kappa/z_1$. Now, we are able to solve the problem for $q := w + z_1\Delta\rho$ with initial value $q_0 = \nabla\cdot u_0 + z_1\Delta\rho_0$. Note that q_0 lies in $\mathrm{B}^{1-1/p}_{pp}(\mathbb{R}^n)$ because of the regularities of u_0 and ρ_0. Then, the problem for q takes the form

$$\begin{aligned}&\partial_t q - z_2\Delta q = \nabla\cdot f + z_1\Delta g, \quad (t,x)\in J\times\mathbb{R}^n,\\ &q = q_0, \quad (t,x)\in\{0\}\times\mathbb{R}^n.\end{aligned}$$

Set $G=\partial_t$ with domain $D(G) = {}_0\mathrm{H}^1_p(J;\mathrm{L}_p(\mathbb{R}^n))$ and let A denote the natural extension of $-\Delta$ to $\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^n))$ with domain $D(A)=\mathrm{L}_p(J;\mathrm{H}^2_p(\mathbb{R}^n))$; then G is invertible and belongs to $\mathcal{BIP}(\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^n)))$ [3] with power angle $\theta_G\leqslant\pi/2$, and $z_2A\in\mathcal{BIP}(\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^n)))$ with power angle $\theta_{z_2A}=\phi_{z_2}<\pi/2$. Since both operators commute, the Dore–Venni Theorem, see [7,17,18], yields that $G+z_2A$ with domain $D(G)\cap D(A)$ is invertible and belongs to $\mathcal{BIP}(\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^n)))$ with power angle $\theta_{G+z_2A}\leqslant\pi/2$. Thus the unique solution is given by

$$q = w + z_1\Delta\rho = \mathrm{e}^{-z_2A\cdot t}(\nabla\cdot u_0 + z_1\Delta\rho_0) + \nabla\cdot(G+z_2A)^{-1}(f+z_1\nabla g)$$

and belongs to the regularity class $Z_{1/2}$. To see this for the last term, you have to take into account the regularity assumptions of f and g as well as the embedding

$$Z_1 \hookrightarrow \mathrm{H}^{1/2}_p\big(J;\mathrm{H}^1_p(\mathbb{R}^n;\mathbb{R}^n)\big)\cap\mathrm{L}_p\big(J;\mathrm{H}^2_p(\mathbb{R}^n;\mathbb{R}^n)\big).$$

Since q_0 lies in $\mathrm{B}^{1-1/p}_{pp}(\mathbb{R}^n)$, we have $(1+A)^{-1/2}q_0\in\mathrm{B}^{2-1/p}_{pp}(\mathbb{R}^n)=D_{zA}(1-1/p,p)$ and this gives rise to $\mathrm{e}^{-zA\cdot t}(1+A)^{-1/2}q_0\in Z_1$. Using the above embedding once more we conclude $(1+A)^{1/2}\mathrm{e}^{-zA}(1+A)^{-1/2}q_0=\mathrm{e}^{-zA}q_0\in Z_{1/2}$. At length, we are in the position to solve the problem of ρ reading as

$$\begin{aligned}&\partial_t\rho - z_1\Delta\rho = g - q, \quad (t,x)\in J\times\mathbb{R}^n,\\ &\rho=\rho_0, \quad (t,x)\in\{0\}\times\mathbb{R}^n,\end{aligned}$$

where the inhomogeneity $g-q\in Z_{1/2}$ is given. This problem is solved by $\rho=\mathrm{e}^{-z_1At}\rho_0+(G+z_1A)^{-1}(g-q)$ and it lefts to verify $\rho\in Z_2$. We have already seen that $q\in Z_{1/2}$ and this leads to $(G+z_1A)^{-1}(g-q)\in Z_2$. To prove $\mathrm{e}^{-z_1At}\rho_0\in Z_2$, we first set $\rho_1 := \mathrm{e}^{-z_1At}(1+A)^{1/2}\rho_0$ which lies in Z_1 as $(1+A)^{1/2}\rho_0\in\mathrm{B}^{2-2/p}_{pp}\mathbb{R}^n$. Hence, we have $\mathrm{e}^{-z_1At}\rho_0\in\mathrm{H}^1_p(J;\mathrm{H}^1_p(\mathbb{R}^n))\cap\mathrm{L}_p(J;\mathrm{H}^3_p(\mathbb{R}^n))\hookrightarrow\mathrm{H}^{1/2}_p(J;\mathrm{H}^2_p(\mathbb{R}^n))$ and $A\mathrm{e}^{-z_1At}\rho_0\in\mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\mathbb{R}^n))$. In virtue of the identity $\partial_t\mathrm{e}^{-z_1At}\rho_0=-z_1A\mathrm{e}^{-z_1At}\rho_0$ and remarks before, we get $\partial_t\mathrm{e}^{-z_1At}\rho_0\in\mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\mathbb{R}^n))$.

Since ρ is completely determined, we are able to solve the problem for u, which can be written as follows

$$\begin{aligned}&\partial_t u - \mu\Delta u = (\lambda+\mu)\nabla(g-\partial_t\rho) + \kappa\Delta\nabla\rho + f, \quad (t,x)\in J\times\mathbb{R}^n,\\ &u=u_0 \quad (t,x)\in\{0\}\times\mathbb{R}^n.\end{aligned}$$

Here, the right-hand side, which we now call $\tilde{f}$, is known and belongs to $\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^n;\mathbb{R}^n))$. Thus, the solution u, given by $u=\mathrm{e}^{-\mu A\cdot t}u_0+(G+\mu A)^{-1}\tilde{f}$, lies in Z_1. □

(b) The half-space problem. The next problem concerns with

[3] Definition of class $\mathcal{BIP}$: Suppose $A\in\mathcal{S}(X)$. Then A is said to admit bounded imaginary powers if $A^{\mathrm{i}s}\in\mathcal{B}(X,)$ for each $s\in\mathbb{R}$, and there is a constant $C>0$ such that $|A^{\mathrm{i}s}|\leqslant C$ for $|s|\leqslant 1$.The class of such operators will be denoted by $\mathcal{BIP}(X)$.

$$\begin{aligned}
&\partial_t u-\mu\Delta u-(\lambda+\mu)\nabla\nabla\cdot u-\kappa\Delta\nabla\rho=f(t,x,y),\quad (t,x,y)\in J\times\mathbb{R}^n_+,\\
&\partial_t\rho+\beta\nabla\cdot u=g(t,x,y),\quad (t,x,y)\in J\times\mathbb{R}^n_+,\\
&u=h(t,x),\quad (t,x,y)\in J\times\mathbb{R}^{n-1}\times\{0\},\\
&\partial_y\rho=\sigma(t,x),\quad (t,x,y)\in J\times\mathbb{R}^{n-1}\times\{0\},\\
&u=u_0(x,y),\quad (t,x,y)\in\{0\}\times\mathbb{R}^n_+,\\
&\rho=\rho_0(x,y),\quad (t,x,y)\in\{0\}\times\mathbb{R}^n_+,
\end{aligned}\tag{2.6}$$

in $\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^n_+;\mathbb{R}^n))\times\mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\mathbb{R}^n_+))\cap\mathrm{L}_p(J;\mathrm{H}^1_p(\mathbb{R}^n_+))$ with $\mathbb{R}^n_+:=\mathbb{R}^{n-1}\times\mathbb{R}_+$, and we look for unique solutions (u,ρ) in the maximal regularity space $Z_1\times Z_2$ defined by

$$\begin{aligned}
&Z_1:=\mathrm{H}^1_p\big(J;\mathrm{L}_p(\mathbb{R}^n_+;\mathbb{R}^n)\big)\cap\mathrm{L}_p\big(J;\mathrm{H}^2_p(\mathbb{R}^n_+;\mathbb{R}^n)\big),\\
&Z_2:=\mathrm{H}^{3/2}_p\big(J;\mathrm{L}_p(\mathbb{R}^n_+)\big)\cap\mathrm{L}_p\big(J;\mathrm{H}^3_p(\mathbb{R}^n_+)\big).
\end{aligned}$$

Theorem 2.3. *Let $J=[0,T]$ and $1<p<\infty$, $p\neq 3/2$. Assuming that μ, $2\mu+\lambda$, κ, β are positive. Then problem* (2.6) *has exactly one solution (u,ρ) in the regularity class $Z_1\times Z_2$ if and only if the data f, g, h, σ, u_0, ρ_0 satisfy the following conditions*

(1) $f\in\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^n_+;\mathbb{R}^n))$;
(2) $g\in Z_{1/2}:=\mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\mathbb{R}^n_+))\cap\mathrm{L}_p(J;\mathrm{H}^1_p(\mathbb{R}^n_+))$;
(3) $h\in Y(\mathbb{R}^n):=\mathrm{B}^{1-1/2p}_{pp}(J;\mathrm{L}_p(\mathbb{R}^{n-1};\mathbb{R}^n))\cap\mathrm{L}_p(J;\mathrm{B}^{2-1/p}_{pp}(\mathbb{R}^{n-1};\mathbb{R}^n))$;
(4) $\sigma\in Y:=\mathrm{B}^{1-1/2p}_{pp}(J;\mathrm{L}_p(\mathbb{R}^{n-1}))\cap\mathrm{L}_p(J;\mathrm{B}^{2-1/p}_{pp}(\mathbb{R}^{n-1}))$;
(5) $u_0\in\mathrm{B}^{2-2/p}_{pp}(\mathbb{R}^n_+;\mathbb{R}^n)$ *and* $\rho_0\in\mathrm{B}^{3-2/p}_{pp}(\mathbb{R}^n_+)$;
(6) $h_{|t=0}=u_{0|y=0}\in\mathrm{B}^{2-3/p}_{pp}(\mathbb{R}^{n-1};\mathbb{R}^n)$ *and* $\sigma_{|t=0}=\partial_y\rho_{0|y=0}\in\mathrm{B}^{2-3/p}_{pp}(\mathbb{R}^{n-1})$ *in case* $p>3/2$.

Proof. (i) *The necessity part.* We only discuss the regularities of h and σ. Assume that (u,ρ) belongs to $Z_1\times Z_2$ satisfying problem (2.6). Extend u and ρ in t to $\mathbb{R}$ by any smooth extension to $\mathbb{R}_+$ and then by symmetry, i.e. $u(t)=u(-t)$ and $\rho(t)=\rho(-t)$ for $t\leqslant 0$. Then $\nabla\rho$ also belongs to Z_1 due to the embedding $Z_2\hookrightarrow\mathrm{H}^1_p(J;\mathrm{H}^1_p(\mathbb{R}^n_+))$, but now with $J=\mathbb{R}$. We employ the trace theorem, which was stated in the previous proof, with $I=\mathbb{R}_+$, $s=2$ and $B=(G+A)^{1/2}$, $G=\partial_t$ with domain $D(G)=\mathrm{H}^1_p(\mathbb{R})$ and $A=1-\Delta$ with domain $D(A)=\mathrm{H}^2_p(\mathbb{R}^{n-1})$, and consider B in $X=\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^{n-1}))$ with natural domain $D(B)=D(G^{1/2})\cap D(A^{1/2})=\mathrm{H}^{1/2}_p(J;\mathrm{L}_p(\mathbb{R}^{n-1}))\cap\mathrm{L}_p(J;\mathrm{H}^1_p(\mathbb{R}^{n-1}))$. We then obtain the embedding $\mathrm{H}^2_p(\mathbb{R}_+;X)\cap\mathrm{L}_p(\mathbb{R}_+;D(B^2))\hookrightarrow\mathrm{C}(\mathbb{R}_+;D_B(2-1/p,p))$. Further, interpolation rules provide

$$\begin{aligned}
D_B(2-1/p,p)&=B^{-1}D_B(1-1/p,p)=B^{-1}\mathrm{B}^{1/2-1/2p}_{pp}\big(J;\mathrm{L}_p(\mathbb{R}^{n-1})\big)\cap\mathrm{L}_p\big(J;\mathrm{B}^{1-1/p}_{pp}(\mathbb{R}^{n-1})\big)\\
&=\mathrm{B}^{1-1/2p}_{pp}\big(J;\mathrm{L}_p(\mathbb{R}^{n-1})\big)\cap\mathrm{L}_p\big(J;\mathrm{B}^{2-1/p}_{pp}(\mathbb{R}^{n-1})\big),
\end{aligned}$$

and after restricting to $t\in J$ this shows 4.

(ii) *The sufficiency part.* As in the proof of Theorem 2.3, we may assume w.l.o.g. $\beta=1$. Next, we study the problem

$$\begin{aligned}
&\partial_t u-\mu\Delta u-(\lambda+\mu)\nabla\nabla\cdot u-\kappa\Delta\nabla\rho=f(t,x,y),\quad (t,x,y)\in J\times\mathbb{R}^{n-1}\times\mathbb{R}_+,\\
&\partial_t\rho+\nabla\cdot u=g(t,x,y),\quad (t,x,y)\in J\times\mathbb{R}^{n-1}\times\mathbb{R}_+,\\
&u=h_0(t,x),\quad (t,x,y)\in J\times\mathbb{R}^{n-1}\times\{0\},\\
&\partial_y\rho=\sigma_0(t,x),\quad (t,x,y)\in J\times\mathbb{R}^{n-1}\times\{0\},\\
&u=0,\quad (t,x,y)\in\{0\}\times\mathbb{R}^{n-1}\times\mathbb{R}_+,\\
&\rho=0,\quad (t,x,y)\in\{0\}\times\mathbb{R}^{n-1}\times\mathbb{R}_+,
\end{aligned}\tag{2.7}$$

with the modified inhomogeneities $h_0(t,x):=h(t,x)-\bar h(t,x)$, $\sigma_0(t,x):=\sigma(t,x)-\bar\sigma(t,x)$, which are supposed to have the same regularity as h, σ as well as $h_{0|t=0}=0$, $\sigma_{0|t=0}$ to ensure compatibility. Later on, we shall see which functions $\bar{h}$ and $\bar{\sigma}$ can be chosen. Exactly as in the previous section, we set $w := \nabla \cdot u$, $\gamma := 2\mu + \lambda$, $z_{1,2} = \gamma/2 \pm \sqrt{(\gamma/2)^2 - \kappa}$ and proceed to obtain an equation for $q = w + z_1 \Delta\rho$. In fact, applying $\nabla\cdot$ to the first equation, $z_1\Delta$ to the second equation, and summing up these new equations we arrive at

$$\begin{aligned}
&-\partial_y^2 q + F_2^2 q = z_2^{-1} \nabla \cdot (f + z_1 \nabla g), \quad y > 0,\\
&q = q_0, \quad y = 0.
\end{aligned}$$

Here, $F_2^2 = z_2^{-1} G + A$ was set with domain $D(F_2^2) = D(G) \cap D(A) = {}_0\mathrm{H}_p^1(J; \mathrm{L}_p(\mathbb{R}^{n-1})) \cap \mathrm{L}_p(J; \mathrm{H}_p^2(\mathbb{R}^{n-1}))$. Due to the Dore–Venni Theorem, we know that F_2^2 belongs to the class $\mathcal{BIP}(\mathrm{L}_p(J; \mathrm{L}_p(\mathbb{R}^{n-1})))$ with power angle $\theta_{F^2} = \theta_{z_2^{-1}G} \leqslant \pi/2 + |\arg(z_2)| < \pi$ and F^2 is invertible. Further, let $B = \partial_y^2$ with domain $D(B) = \mathrm{H}_p^2(\mathbb{R})$, $R : \mathrm{L}_p(\mathbb{R}_+) \to \mathrm{L}_p(\mathbb{R})$ denote the operator of odd extension, i.e. $(Rf)(y) = -f(-y)$ for $y < 0$, and $P_+ : \mathrm{L}_p(\mathbb{R}) \to \mathrm{L}_p(\mathbb{R}_+)$ the restriction operator to $\mathbb{R}_+$. Splitting f into tangential and normal components, i.e. setting

$$f = (f_1, f_2), \quad f_1 \in \mathrm{L}_p\big(J; \mathrm{L}_p(\mathbb{R}_+^n; \mathbb{R}^{n-1})\big) \quad \text{and} \quad f_2 \in \mathrm{L}_p\big(J; \mathrm{L}_p(\mathbb{R}_+^n; \mathbb{R})\big),$$

we can write the unique solution q as

$$q = q_1 + \mathrm{e}^{-F_2 \cdot y} q_0, \tag{2.8}$$

where q_0 is not known yet and will be determined later, and

$$\begin{aligned}
q_1 &= P_+ \frac{1}{z_2}(F_2^2 + B)^{-1} R\nabla \cdot (f + z_1 \nabla g) = P_+ \frac{1}{z_2} F_2^{-1} \int_{-\infty}^{\infty} \mathrm{e}^{-F_2|y-s|} R\nabla \cdot (f + z_1\nabla g)\, ds\\
&= \nabla_x \cdot \frac{1}{2z_2} F_2^{-1} \int_0^\infty \big[\mathrm{e}^{-F_2|y-s|} - \mathrm{e}^{-F_2(y+s)}\big](f_1 + z_1 \nabla_x g)\, ds\\
&\quad + \partial_y \frac{1}{2z_2} F_2^{-1} \int_0^\infty \big[\mathrm{e}^{-F_2|y-s|} + \mathrm{e}^{-F_2(y+s)}\big](f_2 + z_1 \partial_s g)\, ds.
\end{aligned} \tag{2.9}$$

By virtue of the necessary assumptions on f and g we deduce $q_1 \in \nabla \cdot [D(F_2^2) \cap D(B)] = {}_0Z_{1/2}$ which includes $q_{1|t=0} = 0$. To attain $q \in_0 Z_{1/2}$ as well, we have to assure that $q_0 \in D_{F_2}(1 - 1/p, p) = {}_0\mathrm{B}_{pp}^{1/2-1/2p}(J; \mathrm{L}_p(\mathbb{R}^{n-1})) \cap \mathrm{L}_p(J; \mathrm{B}_{pp}^{1-1/p}(\mathbb{R}^{n-1}))$. We now derive an equation for ρ by utilising the solution formula (2.8) to replace $w = \nabla \cdot u$ in the second equation of (2.7). In doing so, we obtain

$$\begin{aligned}
&-\partial_y^2 \rho + F_1^2 \rho = z_1^{-1}(g - q_1) - z_1^{-1} \mathrm{e}^{-F_2 \cdot y} q_0, \quad y > 0,\\
&\partial_y \rho = \sigma_0, \quad y = 0,
\end{aligned} \tag{2.10}$$

with $F_1^2 = z_1^{-1} G + A$ and $D(F_1^2) = D(F_2^2)$. This problem is uniquely solved by

$$\begin{aligned}
\rho &= -\mathrm{e}^{-F_1 \cdot y} F_1^{-1} \sigma_0 + \frac{1}{2z_1} F_1^{-1} \int_0^\infty \big[\mathrm{e}^{-F_1 \cdot |y-s|} + \mathrm{e}^{-F_1 \cdot (y+s)}\big]\big(g - q_1 - \mathrm{e}^{-F_2 \cdot s} q_0\big)\, ds\\
&:= \rho_1 - \frac{1}{2z_1} F_1^{-1} \int_0^\infty \big[\mathrm{e}^{-F_1 \cdot |y-s|} + \mathrm{e}^{-F_1 \cdot (y+s)}\big] \mathrm{e}^{-F_2 \cdot s} q_0\, ds.
\end{aligned} \tag{2.11}$$

Observe that the solution belongs to the regularity class ${}_0\mathrm{H}_p^{3/2}(J; \mathrm{L}_p(\mathbb{R}_+^n)) \cap \mathrm{L}_p(J; \mathrm{H}_p^3(\mathbb{R}_+^n))$ as long as $F_1^{-1}\sigma_0 \in D_{F_1}(3 - 1/p, p) = {}_0\mathrm{B}_{pp}^{3/2-1/2p}(J; \mathrm{L}_p(\mathbb{R}^{n-1})) \cap \mathrm{L}_p(J; \mathrm{B}_{pp}^{3-1/p}(\mathbb{R}^{n-1}))$ and $g - q_1 - \mathrm{e}^{-F_2 \cdot y} q_0 \in Z_{1/2}$. Note that g and q_1 satisfy this regularity by assumption and formula (2.9), respectively. To determine the unknown function q_0, we consider the trace at $y = 0$ of $\partial_t \rho + \nabla \cdot u = g$, i.e. we study the equation

$$\partial_t \rho_{|y=0} + \partial_y u_{2|y=0} = g_{|y=0} - \nabla_x \cdot h_{01}, \quad (t, x) \in J \times \mathbb{R}^{n-1} \tag{2.12}$$

in

$$Y_{1/2} := \mathrm{B}_{pp}^{1/2-1/2p}\big(J;\mathrm{L}_p(\mathbb{R}^{n-1})\big)\cap \mathrm{L}_p\big(J;\mathrm{B}_{pp}^{1-1/p}(\mathbb{R}^{n-1})\big), \tag{2.13}$$

which follows due to the regularity of g, h. Here we have set $u=(u_1,u_2)$, u_1 comprising the first $n-1$ components of u and u_2 the last one, and in the same manner $h_0=(h_{01},h_{02})$ was decomposed. In order to solve this equation in question for q_0, we still need a solution formula for u_2. Therefore we consider the equation for u

$$\partial_t u - \mu\Delta u = f + \kappa\Delta\nabla\rho + (\lambda+\mu)\nabla w, \tag{2.14}$$

and simplify the right-hand side as follows

$$\begin{aligned}
\kappa\Delta\nabla\rho + (\lambda+\mu)\nabla w &= \nabla\big(\kappa\Delta\rho + (\gamma-\mu)\big[-z_1\Delta\rho + q_1 + \mathrm{e}^{-F_2\cdot y}q_0\big]\big)\\
&= \nabla\left(\left(\frac{\kappa}{z_1}-\gamma+\mu\right)z_1\Delta\rho + (\gamma-\mu)\big[q_1+\mathrm{e}^{-F_2\cdot y}q_0\big]\right)\\
&= \nabla\big((\mu-z_1)z_1\Delta\rho + (\gamma-\mu)\big[q_1+\mathrm{e}^{-F_2\cdot y}q_0\big]\big)\\
&= \nabla\big((\mu-z_1)\big(\partial_t\rho - g + q_1 + \mathrm{e}^{-F_2\cdot y}q_0\big) + (\gamma-\mu)\big[q_1+\mathrm{e}^{-F_2\cdot y}q_0\big]\big)\\
&= (z_1-\mu)\nabla g + (\mu-z_1)\nabla\partial_t\rho + z_2\nabla q_1 + z_2\nabla\mathrm{e}^{-F_2\cdot y}q_0.
\end{aligned}$$

There, we exploited the identities $w + z_1\Delta\rho = q_1 + \mathrm{e}^{-F_2\cdot y}q_0$ and $z_1\Delta\rho = \partial_t\rho - g + q_1 + \mathrm{e}^{-F_2\cdot y}q_0$, see (2.10), and used several times the relations between z_1, z_2, γ, κ. Eq. (2.14) is eventually equivalent to

$$\partial_t u - \mu\Delta u = \bar f - (z_1-\mu)\nabla\partial_t\rho + z_2\nabla\mathrm{e}^{-F_2\cdot y}q_0,$$

with $\bar f := (\bar f_1,\bar f_2) := f + (z_1-\mu)\nabla g + z_2\nabla q_1 \in \mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^n_+;\mathbb{R}^n))$. Then the problem for the last component of u takes the form

$$\begin{aligned}
&-\partial_y^2 u_2 + F_\mu^2 u_2 = \mu^{-1}\bar f_2 - \mu^{-1}(z_1-\mu)\partial_y\partial_t\rho - z_2F_2\mathrm{e}^{-F_2\cdot y}q_0, \quad y>0,\\
&u_2 = h_{02}, \quad y=0,
\end{aligned}$$

with $F_\mu^2 := \mu^{-1}G + A$ and $D(F_\mu^2) = D(F_2^2)$. Putting

$$\bar u_2 := \mathrm{e}^{-F_\mu\cdot y}h_{02} + \frac{1}{2\mu}F_\mu^{-1}\int_0^\infty\big[\mathrm{e}^{-F_\mu\cdot|y-s|} - \mathrm{e}^{-F_\mu\cdot(y+s)}\big]\bar f_2\,ds,$$

which is completely known, the solution can be represented in the following way

$$u_2 = \bar u_2 - \frac{1}{2\mu}F_\mu^{-1}\int_0^\infty\big[\mathrm{e}^{-F_\mu\cdot|y-s|} - \mathrm{e}^{-F_\mu\cdot(y+s)}\big]\big((z_1-\mu)\partial_s\partial_t\rho + z_2F_2\mathrm{e}^{-F_2\cdot s}q_0\big)\,ds.$$

Note that $\bar u_2$ lies in ${}_0Z_1$ by the regularity assumptions of h_0, f. Now, we are able to compute $\partial_y u_{2|y=0}$ to find a determining equation for q_0. It holds

$$\partial_y u_{2|y=0} = \partial_y\bar u_{2|y=0} - \frac{z_2}{\mu}F_2(F_2+F_\mu)^{-1}q_0 - \frac{z_1-\mu}{\mu}\int_0^\infty \mathrm{e}^{-F_\mu\cdot s}\partial_s\partial_t\rho(t,s)\,ds,$$

and by formula (2.11) the expression $\partial_s\partial_t\rho(t,s)$ can be computed, resulting in

$$\partial_s\partial_t\rho(t,s) = \partial_s\partial_t\rho_1(t,s) + \frac{z_1^{-1}\kappa}{z_1-z_2}F_2\big[\mathrm{e}^{-F_1\cdot s} - \mathrm{e}^{-F_2\cdot s}\big]q_0.$$

Note that by the Dore–Venni Theorem the operator $F_\mu + F_2$ is invertible and belongs to $\mathcal{BIP}(\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^{n-1})))$ with power angle $\theta_{F_\mu+F_2} = \theta_{F_2} \leqslant \pi/4 + |\arg(z_2^{-1})|/2 < \pi/2$. Thus we have

$$\partial_y u_{2|y=0} = \partial_y \bar{u}_{2|y=0} - \frac{z_1-\mu}{\mu}\int_0^\infty \mathrm{e}^{-F_\mu \cdot s}\partial_s \partial_t \rho_1(t,s)\,ds - \frac{z_2}{\mu}F_2(F_2+F_\mu)^{-1}q_0$$
$$- \frac{z_1^{-1}\kappa}{\mu}\frac{z_1-\mu}{z_1-z_2}F_2\int_0^\infty \mathrm{e}^{-F_\mu\cdot s}\big[\mathrm{e}^{-F_1\cdot s}-\mathrm{e}^{-F_2\cdot s}\big]q_0\,ds$$
$$= u_{22} - \frac{z_2}{\mu}F_2(F_2+F_\mu)^{-1}q_0 - \frac{z_1^{-1}\kappa}{\mu}\frac{z_1-\mu}{z_1-z_2}F_2\big((F_1+F_\mu)^{-1}-(F_2+F_\mu)^{-1}\big)q_0,$$

$u_{22} \in Y_{1/2}$ comprising the first two terms of $\partial_y u_{2|y=0}$. Owing to the above remark, the operators $(F_1+F_2)^{-1}$ and $(F_\mu+F_1)^{-1}$ are bounded as well. Using

$$\partial_t \rho_{|y=0} = \partial_t \rho_{1|y=0} - z_1^{-1}F_1^{-1}(F_1+F_2)^{-1}Gq_0$$

and the above computations we obtain, due to constraint (2.12), an equation for q_0 reading

$$-z_1^{-1}F_1^{-1}(F_1+F_2)^{-1}Gq_0 - \frac{z_2}{\mu}F_2(F_2+F_\mu)^{-1}q_0 - \frac{z_1^{-1}\kappa}{\mu}\frac{z_1-\mu}{z_1-z_2}F_2\big((F_\mu+F_1)^{-1}-(F_\mu+F_2)^{-1}\big)q_0$$
$$= g_{|y=0} - \nabla_x\cdot h_1 - \partial_t\rho_{1|y=0} - u_{22}.$$

By virtue of the identities

$$(F_\mu+F_1)^{-1}-(F_\mu+F_2)^{-1} = \frac{z_1-z_2}{\kappa}G(F_1+F_2)^{-1}(F_\mu+F_1)^{-1}(F_\mu+F_2)^{-1}$$

and

$$F_1^{-1}-(F_\mu+F_1)^{-1}(F_\mu+F_2)^{-1}F_2 = F_\mu(F_\mu+F_1+F_2)F_1^{-1}(F_\mu+F_1)^{-1}(F_\mu+F_2)^{-1}$$

as well as $z_1 z_2 = \kappa$, the equation for q_0 is equivalent, with setting

$$\psi := -\mu z_1[g_{|y=0} - \nabla_x\cdot h_1 - \partial_t\rho_{1|y=0} - u_{22}], \tag{2.15}$$

to

$$\mathcal{S}(F_1+F_2)^{-1}q_0 = z_1^{-1/2}(F_\mu+F_2)(F_\mu+F_1+F_2)^{-1}\psi, \tag{2.16}$$

where

$$\mathcal{S} := \mu z_1^{-1/2}F_\mu F_1^{-1}(F_\mu+F_1)^{-1}G + \kappa z_1^{-1/2}F_2(F_1+F_2)(F_\mu+F_1+F_2)^{-1}$$
$$+ z_1^{1/2}F_2(F_\mu+F_1+F_2)^{-1}(F_1+F_2)^{-1}G.$$

Let us first remark that ψ belongs to $Y_{1/2} = \mathrm{B}_{pp}^{1/2-1/2p}(J;\mathrm{L}_p(\mathbb{R}^{n-1})) \cap \mathrm{L}_p(J;\mathrm{B}_{pp}^{1-1/p}(\mathbb{R}^{n-1}))$. Further, the operator $(F_\mu+F_2)(F_\mu+F_1+F_2)^{-1}$ is an isomorphism in $\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^n))$, which can be seen by the Dore–Venni Theorem, and after using a shift argument also in $\mathrm{H}_p^s(J;\mathrm{H}_p^r(\mathbb{R}^n))$, $r,s\in\mathbb{R}$. Finally, real interpolation between these spaces gives rise to isomorphy in $\mathrm{B}_{pp}^s(J;\mathrm{H}_p^r(\Omega))$ as well as in $\mathrm{H}_p^s(J;\mathrm{B}_{pp}^r(\Omega))$. Thus, it can be shown that $\mathcal{R} := z_1^{-1/2}(F_\mu + F_2)(F_\mu+F_1+F_2)^{-1}$ is an isomorphism in $Y_{1/2}$.

The next purpose is to prove boundedness of the operator $(F_1+F_2)\mathcal{S}^{-1}$ with the aid of the joint functional calculus, the natural analogue of McIntosh's $\mathcal{H}^\infty$-calculus. For proofs and details we refer to [16] and [13]. For $X = \mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^{n-1}))$ the pair (G,A) admits a bounded joint $\mathcal{H}^\infty$-calculus, more precisely, for each $\delta\in(0,\pi/2)$, there exists $C_\delta>0$ such that for all $f\in\mathcal{H}^\infty(\Sigma_{\pi/2+\delta}\times\Sigma_\delta)$, $f(G,A)\in\mathcal{B}(X)$ and $|f(G,A)|_{\mathcal{B}(X)} \leqslant C_\delta |f|_\infty^{\pi/2+\delta,\delta}$. Looking at the symbol

$$\big((z_1^{-1}\zeta+\eta)^{1/2}+(z_2^{-1}\zeta+\eta)^{1/2}\big)s(\zeta,\eta)^{-1},$$

i.e. $G=\partial_t$ and $A=-\Delta$ are replaced by the complex numbers ζ and η, respectively, we have to show

$$\big|s(\zeta,\eta)\big| \geqslant C\big(|\zeta|+|\eta|\big)^{1/2}, \quad (\zeta,\eta)\in\Sigma_{\pi/2+\delta}\times\Sigma_\delta \tag{2.17}$$

entailing

$$\left|\left((z_1^{-1}\zeta+\eta)^{1/2}+(z_2^{-1}\zeta+\eta)^{1/2}\right)s(\zeta,\eta)^{-1}\right|\leqslant C, \tag{2.18}$$

which in turn gives rise to boundedness of $(F_1+F_2)\mathcal{S}^{-1}$ in $\mathrm{L}_p(J;\mathrm{L}_p(\mathbb{R}^{n-1}))$. Observe that the symbol of $\mathcal{S}$ can be written as $s(\zeta,\eta)=\zeta^{1/2}s_0(\lambda)$, with $\lambda:=\eta/\zeta$ in $\Sigma_{\pi/2+2\delta}$ and s_0 given by

$$\begin{aligned}
s_0(\lambda)&=s_{01}(\lambda)+s_{02}(\lambda)+s_{03}(\lambda)\\
&=z_1^{-1/2}\frac{\mu(\mu^{-1}+\lambda)^{1/2}}{(z_1^{-1}+\lambda)^{1/2}([\mu^{-1}+\lambda)^{1/2}+(z_1^{-1}+\lambda)^{1/2}]}+z_1^{-1/2}\frac{\kappa(z_2^{-1}+\lambda)^{1/2}[(z_1^{-1}+\lambda)^{1/2}+(z_2^{-1}+\lambda)^{1/2}]}{(\mu^{-1}+\lambda)^{1/2}+(z_1^{-1}+\lambda)^{1/2}+(z_2^{-1}+\lambda)^{1/2}}\\
&\quad+\frac{z_1^{1/2}(z_2^{-1}+\lambda)^{1/2}}{[(z_1^{-1}+\lambda)^{1/2}+(z_2^{-1}+\lambda)^{1/2}][(\mu^{-1}+\lambda)^{1/2}+(z_1^{-1}+\lambda)^{1/2}+(z_2^{-1}+\lambda)^{1/2}]}.
\end{aligned}$$

The next step is to establish a lower estimate for $s_0(\lambda)$ of the form

$$\left|s_0(\lambda)\right|\geqslant C\left(1+|\lambda|\right)^{1/2},\quad \lambda\in\Sigma_{\pi/2+2\delta}, \tag{2.19}$$

which implies (2.17). It is easy to check that for $\lambda\in\Sigma_{\pi/2+2\delta}$ and $\delta<\pi/2-|\arg(z_i)|$ it holds

$$\begin{aligned}
&C_1\left(1+|\lambda|\right)^{-1/2}\leqslant\left|s_{0i}(\lambda)\right|\leqslant C_2\left(1+|\lambda|\right)^{-1/2},\quad i=1,3,\\
&C_1\left(1+|\lambda|\right)^{1/2}\leqslant\left|s_{02}(\lambda)\right|\leqslant C_2\left(1+|\lambda|\right)^{1/2}.
\end{aligned}$$

Note that $\mu,\kappa>0$, $z_2=\overline{z_1}$, and z_1 lies in $\Sigma_\theta\backslash\{0\}\cap\{z\in\mathbb{C}\colon \operatorname{Im}z\geqslant 0\}$, $\theta<\pi/2$. Further, the constants C_1 and C_2 are independent of δ. Next, observe that for $\lambda\in\overline{\mathbb{C}_+}$ we have $|\arg s_{0i}(\lambda)|<\pi/2$, $i=1,2,3$. By continuity of the argument function we also attain these results for $\lambda\in\Sigma_{\pi/2+2\delta}$, provided that $|\lambda|\leqslant M$. The bound M depends on δ, more precisely, if δ tends to zero then M goes to infinity. Taking into account these considerations we infer $|\arg(s_0(\lambda))|<\pi$ for $\lambda\in\Sigma_{\pi/2+2\delta}$, $|\lambda|\leqslant M$. These observations imply the lower estimate

$$\begin{aligned}
\left|s_0(\lambda,\mu)\right|&\geqslant C\left[\left|s_{01}(\lambda)\right|+\left|s_{02}(\lambda)\right|+\left|s_{03}(\lambda)\right|\right]\geqslant C\left[\left(1+|\lambda|\right)^{-1/2}+\left(1+|\lambda|\right)^{1/2}\right]\\
&\geqslant C\left(1+|\lambda|\right)^{1/2},\quad \lambda\in\Sigma_{\pi/2+2\delta},\quad |\lambda|\leqslant M.
\end{aligned}\tag{2.20}$$

On the other hand, i.e. for $\lambda\in\Sigma_{\pi/2+2\delta}$ and $|\lambda|\geqslant M$, we accomplish

$$\begin{aligned}
\left|s_0(\lambda)\right|&\geqslant\left|s_{02}(\lambda)\right|-\left|s_{01}(\lambda)\right|-\left|s_{03}(\lambda)\right|\geqslant C_1\left(1+|\lambda|\right)^{1/2}-2C_2\left(1+|\lambda|\right)^{-1/2}\\
&\geqslant\frac{C_1}{2}\left(1+|\lambda|\right)^{1/2}+\frac{C_1}{2}-\frac{2C_2}{(1+M)^{1/2}}\geqslant C\left(1+|\lambda|\right)^{1/2},
\end{aligned}$$

if M is chosen large enough, which is possible for a sufficiently small δ. Hence, we have established a lower estimate of the same type as for $|\lambda|\leqslant M$, and together with (2.20) this yields (2.19). Then, the joint functional calculus for G and A supplies

$$(F_1+F_2)\mathcal{S}^{-1}\in\mathcal{L}is\left(\mathrm{L}_p\left(J;\mathrm{L}_p(\mathbb{R}^{n-1})\right)\right),$$

and by using shift arguments and real interpolation we get $(F_1+F_2)\mathcal{S}^{-1}\in\mathcal{L}is(Y_{1/2})$. Incorporating the mapping behaviour of $\mathcal{R}$ we eventually obtain

$$\mathcal{K}:=(F_1+F_2)\mathcal{S}^{-1}\mathcal{R}\in\mathcal{L}is(Y_{1/2})\quad\text{and}\quad q_0=\mathcal{K}\psi\in Y_{1/2}, \tag{2.21}$$

with ψ defined in (2.15). After all, we have found a unique solution $\rho\in_0 Z_2$ given by (2.11) and (2.21). This enables us to solve the problem for u, now reading as

$$\begin{aligned}
&-\partial_y^2u+F_\mu^2u=\mu^{-1}\left(\bar f-(z_1-\mu)\nabla\partial_t\rho+z_2\nabla\mathrm{e}^{-F_2\cdot y}q_0\right),\quad y>0,\\
&u=h_0,\quad y=0.
\end{aligned}$$

The unique solution u is given by

$$u=\mathrm{e}^{-F_\mu\cdot y}h_0+\frac{1}{2\mu}F_\mu^{-1}\int_0^\infty\left[\mathrm{e}^{-F_\mu|y-s|}+\mathrm{e}^{-F_\mu(y+s)}\right]\left(\bar f-(z_1-\mu)\nabla\partial_t\rho+z_2\nabla\mathrm{e}^{-F_2\cdot y}q_0\right)ds,$$

where all terms belong to ${}_0Z_1$, since $\bar{f} - (z_1 - \mu)\nabla\partial_t\rho + z_2\nabla e^{-F_2\cdot y}q_0$ lies in $L_p(J; L_p(\mathbb{R}^n_+))$ and $h_0 \in D_{F_\mu}(2 - 1/p, p) = {}_0B_{pp}^{1-1/2p}(J; L_p(\mathbb{R}^{n-1})) \cap L_p(J; B_{pp}^{2-1/p}(\mathbb{R}^{n-1}))$.

Finally, with the aid of this solution, providing a solution operator $\mathcal{L}_0$ in case of vanishing initial data, we are able to construct a solution of the starting problem (2.6) and give an answer for a choice of the inhomogeneities h_0 and σ_0. For this purpose we set $B = -\Delta_x$ with domain $D(B) = H^2_p(\mathbb{R}^{n-1})$ and $A = -\partial_y^2 + B$ with domain $D(A) = H^2_p(\mathbb{R}_+; Y) \cap {}_0H^1_p(\mathbb{R}_+; Y) \cap L_p(\mathbb{R}_+; D(B))$. Then A is sectorial, belongs to $\mathcal{BIP}(L_p(\mathbb{R}_+; L_p(\mathbb{R}^{n-1})))$ with power angle $\theta_A = 0$, and generates an analytical semigroup. Using these definitions, we put

$$
\begin{aligned}
&(u_1, \rho_1) := e^{-B\cdot t}e^{-B^{1/2}\cdot y}(u_{00}, \rho_{00}), \quad u_{00}(x) := u_0(x, 0), \quad \rho_{00}(x) := \rho_0(x, 0),\\
&(u_2, \rho_2) := \big(e^{-\mu A\cdot t}[u_0 - u_{1|_{t=0}}] + e^{-\mu A\cdot} * \big[\kappa\Delta\nabla\rho_2 - (\lambda + \mu)\nabla\partial_t\rho_2\big],\\
&\qquad e^{-z_1A\cdot t}[\rho_0 - \rho_{1|_{t=0}}] + e^{-z_1A\cdot} * e^{-z_1A\cdot t}\big[\nabla\cdot(u_0 - u_{1|_{t=0}}) + z_1\Delta(\rho_0 - \rho_{1|_{t=0}})\big]\big),
\end{aligned}
$$

where (u_2, ρ_2) solves the differential equations of (2.6) with right-hand side zero, initial value $(u_0 - u_{1|_{t=0}}, \rho_0 - \rho_{1|_{t=0}})$, and, above all, trace zero for $y = 0$. This fact results from the semigroups generated by $-z_1A$, $-z_2A$, and $-\mu A$. If we further define

$$
h_0(t, x) := h(t, x) - u_{1|_{y=0}}, \qquad \sigma_0(t, x) := \sigma(t, x) - \partial_y\rho_{1|_{y=0}} - \partial_y\rho_{2|_{y=0}},
$$

here have in mind $h_{0|t=0} = 0$ as well as $\sigma_{0|t=0} = 0$ due to compatibility, the solution of (2.6) eventually takes the form

$$
\begin{aligned}
(u, \rho) &= (u_1, \rho_1) + (u_2, \rho_2) + \mathcal{L}_0(f, g, h_0, \sigma_0, 0, 0)\\
&\quad - \mathcal{L}_0\big(\partial_t u_1 - \mu\Delta u_1 - (\lambda + \mu)\nabla\cdot\nabla u_1 - \kappa\nabla\Delta\rho_1, \partial_t\rho_1 + \nabla\cdot u_1, 0, 0, 0, 0\big),
\end{aligned}
$$

which finishes the proof of Theorem 2.3. □

3. The nonlinear problem – Proof of Theorem 1.1

3.1. Unique existence on $[0, T]$ for T sufficiently small

Firstly, we define the nonlinear operator $\mathcal{F}(u, \phi)$ being composed of the right-hand side of the nonlinear problem (2.1) by means of

$$
\mathcal{F}(u, \rho) := \big(F(u, \rho) + b(u, \rho), G(u, \rho), 0, 0\big).
$$

It is an immediate consequence of the definition of F, b, G, see (2.2) and (2.3) in Section 2.1, that the nonlinear operator $\mathcal{F}(u, \rho)$ is a mapping from $Z := Z_1 \times Z_2$ to $X \times Z_{1/2} \times Y(\mathbb{R}^n) \times Y$. In the following we want to associate (2.1) with the abstract equation

$$
\mathcal{L}(u, \rho) = \big(\mathcal{F}(u, \rho), u_0, \rho_0\big) \quad \text{in} \quad \mathcal{M}, \tag{3.1}
$$

where the space of data $\mathcal{M}$ is given by

$$
\mathcal{M} := X \times Z_{1/2} \times Y(\mathbb{R}^n) \times Y \times B_{pp}^{2-2/p}(\Omega; \mathbb{R}^n) \times B_{pp}^{3-2/p}(\Omega).
$$

By Theorem 2.1 maximal regularity of the linear problem has been proved, i.e. $\mathcal{L}$ is a continuous one-to-one mapping from the space of data to the class of maximal regularity, more precisely, we have

$$
\mathcal{L}^{-1} \in \mathcal{L}is(\mathcal{M}_c, Z_1 \times Z_2), \tag{3.2}
$$

with $\mathcal{M}_c\colon = \{\omega \in \mathcal{M}\colon\ \omega$ satisfies compatibility condition (4)$\}$. Exploitation of this fact allows a fixed point formulation of (3.1) whenever the given data lie in $\mathcal{M}_c$. Before solving this problem locally in time via the contraction mapping principle, a suitable set in $Z_1 \times Z_2$ is required in which contraction and self-mapping can be proved. For this purpose we introduce a reference function (w, ϱ) defined as the solution of the linear problem

$$
\mathcal{L}(w, \varrho) = \big(\mathcal{F}(\tilde{u}, \tilde{\rho}), u_0, \rho_0\big) \quad \text{in } \mathcal{M}_c. \tag{3.3}
$$

The functions $\tilde{u} \in Z_1$ and $\tilde{\rho} \in Z_2$, the same $\tilde{\rho}$ used in the linearisation, play the role of approximations, i.e. $\tilde{u}_{t=0} = u_0$ and $\tilde{\rho}_{t=0} = \rho_0$. Note that the right-hand side $(\mathcal{F}(\tilde{u}, \tilde{\rho}), u_0, \rho_0)$ belongs to $\mathcal{M}_c$, in particular, the compatibility conditions are satisfied. So, according to Theorem 2.1, we obtain a unique solution (w,ϱ) which belongs to the space of maximal regularity. Next we introduce a ball in $Z_1 \times Z_2$ with radius δ and centre (w,ϱ) as follows

$$\Sigma_{\delta,T} := \bigl\{(u,\rho) \in Z_1^T \times Z_2^T:\ (u,\rho)_{t=0} = (u_0,\rho_0),\ \bigl\|(u,\rho)-(w,\varrho)\bigr\|_{Z_1^T\times Z_2^T} \leqslant \delta\bigr\},$$

which is a closed subset of $Z_1 \times Z_2$. The superscript T indicates to the considered time interval $J=[0,T]$. By the contraction mapping principle we have to verify $\mathcal{L}^{-1}\mathcal{F}(\Sigma_{\delta,T}) \subset \Sigma_{\delta,T}$ and contraction of $\mathcal{L}^{-1}\mathcal{F}$ in the norm of $Z_1^T \times Z_2^T$. These two properties can be shown, provided the parameters $T \in (0,T_0]$ and $\delta > 0$ are chosen properly. For the forthcoming estimates it will be useful to introduce the auxiliary function $\psi(T)$ defined by

$$\psi(T) := \|\tilde{u} - w\|_{{}_0Z_1^T} + \|\tilde{\rho} - \rho\|_{{}_0Z_2^T}.$$

Apparently, $\psi(T)$ is bounded and tends to zero for $T \to 0$, which is caused by $(\tilde{u}-w)_{t=0} = 0$ and $(\tilde{\rho}-\varrho)_{t=0} = 0$. At length, we come to self-mapping and contraction. Let (u,ρ), $(\bar{u},\bar{\rho}) \in \Sigma_{\delta,T}$ be given. By using $\mathcal{L}^{-1} \in \mathcal{L}is(\mathcal{M}_c^T, Z_1^T \times Z_2^T)$ we may estimate as follows

$$\begin{aligned}
&\bigl\|\mathcal{L}^{-1}\bigl(\mathcal{F}(u,\rho),u_0,\rho_0\bigr) - \mathcal{L}^{-1}\bigl(\mathcal{F}(\bar{u},\bar{\rho}),u_0,\rho_0\bigr)\bigr\|_{{}_0Z_1^T\times{}_0Z_2^T}\\
&\quad\leqslant C\bigl\|(F+b)(u,\rho) - (F+b)(\bar{u},\bar{\rho})\bigr\|_{X^T} + \bigl\|G(u,\rho) - G(\bar{u},\bar{\rho})\bigr\|_{{}_0Z_{1/2}{}^T}
\end{aligned}$$

and very similar in case of self-mapping

$$\begin{aligned}
&\bigl\|\mathcal{L}^{-1}\bigl(\mathcal{F}(u,\rho),u_0,\rho_0\bigr) - (w,\varrho)\bigr\|_{{}_0Z_1^T\times{}_0Z_2^T}\\
&\quad= \bigl\|\mathcal{L}^{-1}\bigl(\mathcal{F}(u,\rho) - \mathcal{F}(\tilde{u},\tilde{\rho}),0,0\bigr)\bigr\|_{{}_0Z_1^T\times{}_0Z_2^T}\\
&\quad\leqslant C\bigl\|(F+b)(u,\rho) - (F+b)(\tilde{u},\tilde{\rho})\bigr\|_{X^T} + \bigl\|G(u,\rho) - G(\tilde{u},\tilde{\rho})\bigr\|_{{}_0Z_{1/2}{}^T}.
\end{aligned}$$

Subsequently, it is decisive that the operator norm of $\mathcal{L}^{-1}$, i.e. the constant C, is independent of the time interval. This can only be achieved in case of null initial data, which is satisfied by considering differences. This fact will also be used in the upcoming estimates in which constants occur due to embedding and interpolation inequalities.

Below, only the case of self-mapping will be carried out, since this part is more sophisticated and comprehensive compared to contraction. The forthcoming procedure can be adopted to the case of contraction, which only needs few modifications that we leave to the reader. Before proving self-mapping we collect some useful inequalities and embeddings, of which some were already used before. By using Sobolevskij's *mixed derivative theorem* and Sobolev's embeddings, see Remark 1.1, we have

$$Z_1 \hookrightarrow \mathrm{H}_p^{1/2}\bigl(J;\mathrm{H}_p^1(\Omega;\mathbb{R}^n)\bigr), \qquad Z_2 \hookrightarrow \mathrm{H}_p^1\bigl(J;\mathrm{H}_p^1(\Omega)\bigr) \cap \mathrm{H}_p^{1/2}\bigl(J;\mathrm{H}_p^2(\Omega)\bigr),$$

and for $p > n+2$

$$\begin{aligned}
&Z_1 \hookrightarrow U_1 := \mathrm{C}^{1/2}\bigl(J;\mathrm{C}(\overline{\Omega};\mathbb{R}^n)\bigr) \cap \mathrm{C}\bigl(J;\mathrm{C}^1(\overline{\Omega};\mathbb{R}^n)\bigr),\\
&Z_2 \hookrightarrow U_2 := \mathrm{C}^1\bigl(J;\mathrm{C}(\overline{\Omega})\bigr) \cap \mathrm{C}^{1/2}\bigl(J;\mathrm{C}^1(\overline{\Omega})\bigr) \cap \mathrm{C}\bigl(J;\mathrm{C}^2(\overline{\Omega})\bigr).
\end{aligned} \tag{3.4}$$

Next, let us consider the differences $\rho - \tilde{\rho}$ and $u - \tilde{u}$ which will appear several times in different norms. In case of $\mathrm{C}(J\times\Omega)$ the notation $\|\cdot\|_\infty$ is used. Further, by C, C_i, $i \in \mathbb{N}$, we denote various constants which may differ from line to line, but which are always independent of the solution (u,ρ) and, by the remarks above, independent of T. Having in mind the additional regularity we see

$$\begin{aligned}
&\|\rho - \tilde{\rho}\|_\infty = \left\|1 * \frac{d}{dt}(\rho - \tilde{\rho})\right\|_\infty \leqslant T\|\rho - \tilde{\rho}\|_{\mathrm{C}^1(J;\mathrm{C}(\Omega))} \leqslant CT\|\rho - \tilde{\rho}\|_{{}_0Z_2{}^T},\\
&\|\rho - \tilde{\rho}\|_{{}_0U_1{}^T} \leqslant CT^{1/2}\|\rho - \tilde{\rho}\|_{{}_0U_2{}^T} \leqslant CT^{1/2}\|\rho - \tilde{\rho}\|_{{}_0Z_2{}^T},\\
&\|\nabla\rho - \nabla\tilde{\rho}\|_{{}_0Z_{1/2}{}^T} \leqslant CT^{1/2}\|\nabla\rho - \nabla\tilde{\rho}\|_{{}_0Z_2{}^T},
\end{aligned}$$

and

$$\|u-\tilde u\|_{X^T}=\left\|1*\frac{d}{dt}(u-\tilde u)\right\|_{X^T}\leqslant T\|u-\tilde u\|_{\mathrm{H}^1_p(J;\mathrm{L}_p(\Omega;\mathbb{R}^n))}\leqslant T\|u-\tilde u\|_{{}_0Z_1^T},$$
$$\|u-\tilde u\|_{{}_0Z_{1/2}{}^T}\leqslant CT^{1/2}\|u-\tilde u\|_{\mathrm{H}^1_p(J;\mathrm{L}_p(\Omega;\mathbb{R}^n))\cap\mathrm{H}^{1/2}_p(J;\mathrm{H}^1_p(\Omega;\mathbb{R}^n))}\leqslant CT^{1/2}\|u-\tilde u\|_{{}_0Z_1^T},$$

and

$$\|u-\tilde u\|_{{}_0Z_1^T}+\|\rho-\tilde\rho\|_{{}_0Z_2{}^T}\leqslant\big(\delta+\psi(T)\big).$$

In the following, we shall always apply these estimates without pointing out any note of usage. Starting with the linear part $b(u,\rho)$ we get

$$\begin{aligned}\big\|b(u,\rho)-b(\tilde u,\tilde\rho)\big\|_{X^T}&\leqslant\big(4\|\tilde\rho^{-1}\nabla\mu\|_\infty+\|\tilde\rho^{-1}\nabla\lambda\|_\infty\big)\|u-\tilde u\|_{\mathrm{L}_p(J;\mathrm{H}^1_p(\Omega;\mathbb{R}^n))}\\&\quad+\|f\|_{\mathrm{L}_p(J;\mathrm{L}_p(\Omega;\mathbb{R}^n))}\|\tilde\rho^{-1}\|_\infty\|\rho-\tilde\rho\|_\infty\\&\leqslant C\big(\|u-\tilde u\|_{{}_0Z_{1/2}{}^T}+\|\rho-\tilde\rho\|_\infty\big)\leqslant C(T^{1/2}+T)\big(\delta+\psi(T)\big).\end{aligned}$$

Next, we are concerned with the nonlinearity G in $Z_{1/2}$. It is easy to verify that

$$\begin{aligned}\big\|G(u,\rho)-G(\tilde u,\tilde\rho)\big\|_{{}_0Z_{1/2}{}^T}&\leqslant\big\|(\rho-\tilde\rho)\nabla\cdot u\big\|_{{}_0Z_{1/2}{}^T}+\|\nabla\rho\cdot u-\nabla\tilde\rho\cdot\tilde u\|_{{}_0Z_{1/2}{}^T}\\&\leqslant\big\|(\rho-\tilde\rho)\big\|_{{}_0U_1{}^T}\|\nabla\cdot u\|_{Z^T_{1/2}}+\|\nabla\rho-\nabla\tilde\rho\|_{{}_0Z_{1/2}{}^T}\|u\|_{U_1^T}+\|\nabla\tilde\rho\|_{U_1^T}\|u-\tilde u\|_{{}_0Z_{1/2}{}^T}\\&\leqslant C_1T^{1/2}\big(\delta+\psi(T)\big)\big(\delta+\|\nabla\cdot w\|_{Z^T_{1/2}}\big)+C_2T^{1/2}\big(\delta+\psi(T)\big)\big(\delta+\|w\|_{U_1^T}\big)\\&\quad+C_3T^{1/2}\big(\delta+\psi(T)\big)\\&\leqslant CT^{1/2}\big(\delta+\psi(T)\big).\end{aligned}$$

In the end, it is left to observe F in X^T. A first estimate gives

$$\begin{aligned}\big\|F(u,\rho)-F(\tilde u,\tilde\rho)\big\|_{X^T}&\leqslant C_1\|\rho-\tilde\rho\|_\infty\big\|(u,\rho)\big\|_{Z_1^T\times Z_2^T}+C_2\|\rho\nabla u\cdot u-\tilde\rho\nabla\tilde u\cdot\tilde u\|_{X^T}\\&\quad+C_3\big\|\nabla P(\rho)-\nabla P(\tilde\rho)\big\|_{X^T}+C_4\big\||\nabla\rho|^2-|\nabla\tilde\rho|^2\big\|_{X^T}\\&\quad+C_5\|\rho\Delta\rho-\tilde\rho\Delta\tilde\rho\|_{X^T}+C_6\big\||\nabla\rho|\nabla\rho-|\nabla\tilde\rho|\nabla\tilde\rho\big\|_{X^T},\end{aligned}$$

where the constants C_i, $i=1,\dots,7$, contain various norms of the given functions $\tilde\rho^{-1}$, κ, $\nabla\kappa$ and $\nabla\lambda$. To hold down the effort we consider the first three terms and the fifth one only, but the remaining terms can be dealt in the same manner. Proceeding as before we achieve for the first term

$$\|\rho-\tilde\rho\|_\infty\big\|(u,\rho)\big\|_{Z_1^T\times Z_2^T}\leqslant CT\big(\delta+\psi(T)\big)\big(\delta+\big\|(w,q)\big\|_{Z_1^T\times Z_2^T}\big)\leqslant CT\big(\delta+\psi(T)\big),$$

and for the second one

$$\begin{aligned}\|\nabla u\cdot\rho u-\nabla\tilde\rho\tilde u\cdot\tilde u\|_{X^T}&\leqslant\|u\|_{\mathrm{C}(J;\mathrm{C}^1(\Omega;\mathbb{R}^n))}\|\rho u-\tilde\rho\tilde u\|_{X^T}+\|\tilde\rho\tilde u\|_\infty\|u-\tilde u\|_{\mathrm{L}_p(J;\mathrm{H}^1_p(\Omega;\mathbb{R}^n))}\\&\leqslant\|u\|_{U_1^T}\big(\|\rho-\tilde\rho\|_\infty\|u\|_{X^T}+\|\tilde\rho\|_\infty\|u-\tilde u\|_{X^T}t\big)+C\|u-\tilde u\|_{{}_0Z_{1/2}{}^T}\\&\leqslant C_1\|u\|_{U_1^T}\big[T\big(\delta+\psi(T)\big)\big(T\delta+\|w\|_{X^T}\big)+T\big(\delta+\psi(T)\big)\big]+C_2T^{1/2}\big(\delta+\psi(T)\big)\\&\leqslant C\big(\big(\delta+\|w\|_{U_1^T}\big)T+T^{1/2}\big)\big(\delta+\psi(T)\big)\leqslant C(T+T^{1/2})\big(\delta+\psi(T)\big).\end{aligned}$$

In order to treat $\nabla P(\rho)-\nabla P(\tilde\rho)$, we have to take into account the local Lipschitz condition $(P2)$ for P' leading to

$$\begin{aligned}\big\|P'(\rho)\nabla\rho-P'(\tilde\rho)\nabla\tilde\rho\big\|_{X^T}&\leqslant\|\nabla\rho\|_\infty\big\|P'(\rho)-P'(\tilde\rho)\big\|_{X^T}+\big\|P'(\tilde\rho)\big\|_\infty\|\nabla\rho-\nabla\tilde\rho\|_{X^T}\\&\leqslant C_1\big(\delta+\|\nabla q\|_\infty\big)T^2\|\rho-\tilde\rho\|_{{}_0Z_2^T}+C_2T^{3/2}\|\rho-\tilde\rho\|_{{}_0Z_2^T}\\&\leqslant C\big(\delta+\psi(T)\big)(T^2+T^{3/2}).\end{aligned}$$

At last, we tackle the lower order term $\rho\Delta\rho-\tilde\rho\Delta\tilde\rho$ in X^T. Using the techniques as before, we achieve

$$\begin{aligned}\|\rho\Delta\rho-\tilde\rho\Delta\tilde\rho\|_{X^T}&\leqslant\|\rho-\tilde\rho\|_{X^T}\|\rho\|_{\mathrm{C}(J;\mathrm{C}^2(\Omega))}+\|\tilde\rho\|_\infty\|\rho-\tilde\rho\|_{\mathrm{L}_p(J;\mathrm{H}^2_p(\Omega))}\\&\leqslant C_1T^2\|\rho-\tilde\rho\|_{{}_0Z_2^T}\big(\delta+\|q\|_{\mathrm{C}(J;\mathrm{C}^2(\Omega))}\big)+C_2T^{1/2}\|\rho-\tilde\rho\|_{{}_0\mathrm{H}^{1/2}_p(J;\mathrm{H}^2_p(\Omega))}\\&\leqslant C(T^2+T^{1/2})\big(\delta+\psi(T)\big).\end{aligned}$$

All in all, we have shown

$$\big\|\mathcal{L}^{-1}\big(\mathcal{F}(u,\rho),u_0,\rho_0\big)-(w,\varrho)\big\|_{{}_0Z_1^T\times{}_0Z_2^T}=C(T^{1/2}+T+T^{3/2}+T^2)\big(\delta+\psi(T)\big).$$

If we choose $T\in(0,T_0]$ sufficiently small, we succeed in estimating the above inequality by δ, i.e. $\mathcal{L}^{-1}\mathcal{F}$ is a self-mapping. Hence, the contraction mapping principle yields a unique fixed point of the nonlinear equation (3.1) in $\Sigma_{\delta,T}$, which is the unique strong solution on $J=[0,T]$ in the regularity class $Z_1^T\times Z_2^T$.

3.2. Continuation and regularity

In order to carry out the process of continuation, we have to ensure that $(u(T),\rho(T))$ belongs to $\mathrm{B}^{2-2/p}_{pp}(\Omega;\mathbb{R}^n)\times\mathrm{B}^{3-2/p}_{pp}(\Omega)$ and $\rho(T,x)>0$, $\forall x\in\overline{\Omega}$. Note that the regularity follows directly from the trace theorem included in Theorem 2.1. The positivity of $\rho(T)$ is required to guarantee the positivity of the coefficients of partial differential operators in the linearisation (2.1). Hence, the maximal interval of existence $[0,t_{\max})$ is characterised by the conditions $\lim_{t\to t_{\max}}(u(t),\rho(t))$ does not exist in $\mathrm{B}^{2-2/p}_{pp}(\Omega;\mathbb{R}^n)\times\mathrm{B}^{3-2/p}_{pp}(\Omega;\mathbb{R}_+)$ or $\rho(t_{\max},x)>0$ is not fulfilled for all $x\in\overline{\Omega}$, since otherwise we may apply Theorem 1.1 once again with initial value

$$\big(u(t_{\max}),\rho(t_{\max})\big)=\lim_{t\to t_{\max}}\big(u(t),\rho(t)\big)$$

to obtain a contradiction to maximality. If λ,μ,κ,f are constant in t, problem (1.1) is autonomous, hence translation is invariant, which by uniqueness of solution shows that the map $(u_0,\rho_0)\to(u,\rho)(t)$ is a local semiflow in the phase space $\mathrm{B}^{2-2/p}_{pp}(\Omega;\mathbb{R}^n)\times\mathrm{B}^{3-2/p}_{pp}(\Omega)$.

Employing the results and methods of Escher, Prüss and Simonett [9] provides classical solutions in $(0,t_{\max})\times\Omega$ if $f\in\mathrm{C}(J_0\times\Omega)$, since in that case the right-hand side ρf lies in $\mathrm{C}(0,t_{\max})\times\Omega$ due to $\rho\in Z_2(J)\hookrightarrow U_2$, see (3.4). Thus, the proof of Theorem 1.1 is complete. □

4. Extensions and remarks

We first remark that our method to prove strong well-posedness extends to problem (1.1) with coefficients depending sufficiently smooth on the unknown functions ρ and $\nabla\rho$. In fact, due to the embeddings (3.4) these functions are continuous and the gradients of μ, λ, and κ, which appear in the nonlinearity $F(u,\rho)$, remain of lower order; e.g. assuming that κ depends on ρ, $\nabla\rho$ then we have

$$\begin{aligned}&\kappa\big(t,x,\rho(t,x),\nabla\rho(t,x)\big)\in\mathrm{C}(J\times\overline{\Omega})\quad\text{for }\rho\in Z_2\hookrightarrow\mathrm{C}\big(J;\mathrm{C}^2(\overline{\Omega})\big),\quad p>n+2,\\&\nabla\kappa\big(t,x,\rho(t,x),\nabla\rho(t,x)\big)=\nabla_x\kappa+\partial_\rho\kappa\nabla\rho+\partial_{\nabla\rho}\kappa\cdot\nabla^2\rho\in\mathrm{C}(J\times\overline{\Omega};\mathbb{R}^n).\end{aligned}$$

Notice that a dependency on u of the coefficients, which could be possible from a mathematical viewpoint, infringes upon the Galilean invariance. In view of these remarks, the linearisation in Section 2.1 can be applied, but now, with nonlinearity

$$\begin{aligned}F(u,\rho)=\tilde\rho^{-1}\big\{&-u\nabla u-\nabla P(\rho)-P(\rho)\nabla\lambda(\rho,\nabla\rho)+2\mathbf{D}(u)\cdot\mu(\rho,\nabla\rho)+\nabla\cdot u\nabla\lambda(\rho,\nabla\rho)\\&+\big(\rho\Delta\rho+|\nabla\rho|^2/2\big)\nabla\kappa(\rho,\nabla\rho)+\nabla\kappa(\rho,\nabla\rho)\cdot\nabla\rho\otimes\nabla\rho-\big[\mu(\tilde\rho,\nabla\tilde\rho)-\mu(\rho,\nabla\rho)\big]\Delta u\\&-\big[\lambda(\tilde\rho,\nabla\tilde\rho)-\lambda(\rho,\nabla\rho)\big]\nabla\nabla\cdot u+[\tilde\rho-\rho]\partial_tu-\big[\kappa(\tilde\rho,\nabla\tilde\rho)\tilde\rho-\kappa(\rho,\nabla\rho)\rho\big]\nabla\Delta\rho\big\}.\end{aligned}$$

The coefficients of the left-hand side of (2.1) take the form $\tilde\mu(t,x):=\mu(t,x,\tilde\rho,\nabla\tilde\rho)/\tilde\rho$, $\tilde\lambda(t,x):=\lambda(t,x,\tilde\rho,\nabla\tilde\rho)/\tilde\rho$, and $\tilde\kappa(t,x):=\kappa(t,x,\tilde\rho,\nabla\tilde\rho)$. As to proving contraction and self-mapping in Section 3, some additional estimates have to be carried out, where the Lipschitz continuity has to be taken into account. Therefore, we obtain the following result

Theorem 4.1. *Let Ω be a bounded domain in $\mathbb{R}^n$, $n \geqslant 2$, with C^3-boundary, $\Gamma := \partial\Omega$, and J_0 denote the compact time interval* $[0, T_0]$. *Let $n+2 < p < \infty$ and suppose that*

(1) $\mu, \lambda, \kappa \in \mathrm{C}(J_0; \mathrm{C}^1(\overline{\Omega}; \mathrm{C}^{2-}(\mathbb{R}_+ \times \mathbb{R}^n)))$,
(2) $\mu(z) \geqslant \bar{\mu} > 0$, $\kappa(z) \geqslant \bar{\kappa} > 0$, $2\mu(z) + \lambda(z) \geqslant \bar{\lambda} > 0$ *for all* $z \in J_0 \times \overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^n$,

and the assumptions (2)–(5) *of Theorem* 1.1 *are satisfied. Then, the same assertions of Theorem* 1.1 *hold true for problem* (2.1).

Further, in problem (1.1) we prescribe zero boundary conditions for u and ρ. In the proof of existence and uniqueness for the linearisation (2.4) these conditions do not play any role, for we have studied the linear problem with general inhomogeneities on the boundary. This makes considering inhomogeneous or nonlinear boundary conditions possible, e.g. $b_D(t,x,u) = 0$ and $b_N(t,x,\nabla\rho) = 0$ on Γ, whose linearisations lead to boundary conditions of type $u = B_D(t,x,u)$ and $\partial_\nu\rho = B_N(t,x,\nabla\rho)$, but now with new "good" nonlinearities B_D and B_N. As to solving this nonlinear problem, similar estimates as for the nonlinear operator $F(u,\rho)$, see in Section 3.1, have to be carried out for B_D and B_N, but now, in the trace spaces $Y(\mathbb{R}^n)$ and Y, respectively. For more detail of treating nonlinear boundary conditions we refer to [22] and [14].

Now, we will briefly perform how one can tackle unbounded domains Ω with compact boundary (or $\mathbb{R}^n$). In this case, the assumption $\rho_0 \in \mathrm{B}^{3-2/p}_{pp}(\Omega)$ with $\rho_0(x) > 0$ for all $x \in \Omega$ does not imply the existence of a constant $c_0 > 0$ such that $\rho_0(x) \geqslant c_0 > 0$ for all $x \in \overline{\Omega}$. On the other hand, such a lower estimate does not go with the regularity class $\mathrm{B}^{3-2/p}_{pp}(\Omega)$ as we require for Theorem 1.1. Thus, we assume that there exist constants $\bar{\rho} > 0$ and $c_0 > -1$ such that $\rho_0 - \bar{\rho} \in \mathrm{B}^{3-2/p}_{pp}(\Omega)$ and $(\rho_0(x) - \bar{\rho})/\bar{\rho} \geqslant c_0$ for all $x \in \overline{\Omega}$. This implies, for p large enough, $\rho_0 \in \mathrm{C}(\Omega)$ and $\rho_0(x) \geqslant 1 + c_0 > 0$ for all $x \in \overline{\Omega}$. Introducing the density fluctuation $\varrho(t,x) = (\rho(t,x) - \bar{\rho})/\bar{\rho}$ we study the problem (1.4) for (u, ϱ), which takes the form

$$
\begin{aligned}
&(\varrho+1)\partial_t u - \bar{\mu}\Delta u - (\bar{\lambda} + \bar{\mu})\nabla\cdot\nabla u - \bar{\kappa}(\varrho+1)\nabla\Delta\varrho = H(u,\varrho), \quad (t,x) \in J \times \Omega,\\
&\partial_t\varrho + (\varrho+1)\nabla\cdot u = -u\cdot\nabla\varrho, \quad (t,x) \in J \times \Omega,\\
&u = 0, \quad \partial_\nu\varrho = 0, \quad (t,x) \in J \times \Gamma,\\
&u = u_0, \quad (t,x) \in \{0\} \times \Omega,\\
&\varrho = (\rho_0 - \bar{\rho})/\bar{\rho}, \quad (t,x) \in \{0\} \times \Omega.
\end{aligned} \tag{4.1}
$$

Here we used the notations $\bar{\mu} = \mu/\bar{\rho}$, $\bar{\lambda} = \lambda/\bar{\rho}$, $\bar{\kappa} = \kappa\bar{\rho}$, and

$$
\begin{aligned}
H(u,\varrho) = {}&(\varrho+1)f - \nabla P\big(\bar{\rho}(\varrho+1)\big) - (\varrho+1)u\cdot\nabla u + \big[(\varrho+1)\Delta\varrho + |\nabla\varrho|^2/2\big]\nabla\bar{\kappa}\\
&+ \nabla\bar{\kappa}\cdot\nabla\varrho\otimes\nabla\varrho + \big[\nabla\cdot u - P\big(\bar{\rho}(\varrho+1)\big)\big]\nabla\bar{\lambda} + 2\mathbf{D}(u)\cdot\nabla\bar{\mu}.
\end{aligned}
$$

Of course, boundary conditions are omitted in case $\Omega = \mathbb{R}^n$. If we now take a look at the highest order terms containing the factor $(\varrho+1)$, it becomes clear why the second condition for ρ_0 is needed. In fact, $\varrho_{|t=0} + 1 = (\rho_0 - \bar{\rho})/\bar{\rho} + 1 \geqslant 1 + c_0 > 0$ and so the linearisation in Section 2.1 is applicable. Further, coefficients depending on t and x as well as on ρ and $\nabla\rho$ can be admitted by means of supplementing the conditions

$$
\begin{aligned}
&\lim_{|x|\to\infty} a\big(t,x,v(t,x),\nabla v(t,x)\big) =: \lim_{|x|\to\infty} b(t,x) = b_\infty(t), \quad \forall t \in J,\ v \in \mathrm{C}\big(J; \mathrm{C}^1(\overline{\Omega})\big),\\
&\quad a \in \{\mu,\lambda,\kappa\} \subset \mathrm{C}\big(J; \mathrm{C}^1\big(\overline{\Omega}; \mathrm{C}^{2-}(\mathbb{R}_+ \times \mathbb{R}^n)\big)\big), \quad b_\infty \in \{\mu_\infty, \lambda_\infty, \kappa_\infty\} \subset \mathrm{C}(J),
\end{aligned} \tag{4.2}
$$

which are required for the process of localisation. These investigations lead to

Theorem 4.2. *Let Ω be $\mathbb{R}^n$ or an unbounded domain in $\mathbb{R}^n$, $n \geqslant 2$, with compact C^3-boundary, $\Gamma := \partial\Omega$. Let J_0 denote the compact time interval* $[0, T_0]$ *and $n+2 < p < \infty$. Suppose that*

(1) $\mu, \lambda, \kappa \in \mathrm{C}(J_0; \mathrm{C}^1(\overline{\Omega}; \mathrm{C}^{2-}(\mathbb{R}_+ \times \mathbb{R}^n)))$ *and satisfy* (4.2);
(2) $\mu(z) \geqslant \bar{\mu} > 0$, $\kappa(z) \geqslant \bar{\kappa} > 0$, $2\mu(z) + \lambda(z) \geqslant \bar{\lambda} > 0$ *for all* $z \in J_0 \times \overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^n$;

(3) $f \in X = \mathrm{L}_p(J_0; \mathrm{L}_p(\Omega; \mathbb{R}^n))$;
(4) $u_0 \in \mathrm{B}_{pp}^{2-2/p}(\Omega; \mathbb{R}^n)$, $\rho_0 - \bar{\rho} \in \mathrm{B}_{pp}^{3-2/p}(\Omega)$, *with a positive constant* $\bar{\rho}$, *and there exists* $c > 0$, *so that* $\rho_0(x) \geqslant c$ *for all* $x \in \overline{\Omega}$;

and in case $\Omega \neq \mathbb{R}^n$:

(5) *compatibility conditions*: $u_0 = 0$ *in* $\mathrm{B}_{pp}^{2-3/p}(\Gamma; \mathbb{R}^n)$, $\partial_\nu \rho_0 = 0$ *in* $\mathrm{B}_{pp}^{2-3/p}(\Gamma)$.

Then the same assertions of Theorem 1.1 *hold true for problem* (4.1).

A forthcoming paper treats a model with temperature where the balance equations for mass and momentum are supplemented by one for energy.

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