

The unstable spectrum of the Navier–Stokes operator in the limit of vanishing viscosity

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Abstract

The Navier–Stokes equations for the motion of an incompressible fluid in three dimensions are considered. A partition of the evolution operator into high frequency and low frequency parts is derived. This decomposition is used to prove that the eigenvalues of the Navier–Stokes operator in the inviscid limit converge precisely to the eigenvalues of the Euler operator beyond the essential spectrum.

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1. Introduction

The equations of motion governing an incompressible fluid with viscosity ε are the Navier–Stokes equations

$$\frac{\partial q_\varepsilon}{\partial t} = -(q_\varepsilon \cdot \nabla)q_\varepsilon - \nabla p_\varepsilon + \varepsilon \Delta q_\varepsilon + F_\varepsilon, \quad (1.1a)$$

$$\nabla \cdot q_\varepsilon = 0, \quad (1.1b)$$

where q_ε denotes the n -dimensional velocity vector, p_ε denotes the pressure and F_ε is an external force vector. Here n can be any integer with $n \geq 2$, but the case $n = 3$ is of the most interest.

The same equations with zero viscosity are the Euler equations

$$\frac{\partial q}{\partial t} = -(q \cdot \nabla)q - \nabla p, \quad (1.2a)$$

$$\nabla \cdot q = 0. \quad (1.2b)$$

An important connection between the Euler and the Navier–Stokes systems is the behavior of (1.1) in the limit of vanishing viscosity (i.e. $\varepsilon \rightarrow 0$). This limit is likely to be crucial in the understanding of many physical problems of fluid flow, such as the transition to turbulence. It is clear, since the types of the two systems are very different ((1.1) is parabolic and (1.2) is degenerate hyperbolic), that the limit of vanishing viscosity is a subtle and singular limit. There

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are a number of partial results for the nonlinear system as $\varepsilon \rightarrow 0$. The history of such results is briefly surveyed in the appendix of the book of Temam [11].

In this present paper we address the connections between the spectra of the linearized Navier–Stokes operators in the inviscid limit and the spectrum of the linearized Euler operator. The results are closely tied to issues of linear, and even nonlinear, instabilities for fluid flows (c.f. Yudovich [16]).

Let $u(x, \varepsilon)$ be an arbitrary steady solution of (1.1):

$$0 = -(u \cdot \nabla)u - \nabla P + \varepsilon \Delta u + F_\varepsilon, \quad (1.3a)$$

$$\nabla \cdot u = 0. \quad (1.3b)$$

We assume that $u(x, \varepsilon)$ and F_ε are infinitely smooth vector valued functions on the torus \mathbb{T}^n with regular dependence on $\varepsilon \in [0, \varepsilon_0)$ and that $\lim_{\varepsilon \rightarrow 0} F_\varepsilon = 0$. For the sake of simplicity we will present the proof of the theorems only for the case where $u(x)$ has no dependence on ε . The more general results follow from similar arguments.

The linearized Navier–Stokes equations for the evolution of a small perturbation velocity $v(x, t)$ are

$$\frac{\partial v}{\partial t} = -(u \cdot \nabla)v - (v \cdot \nabla)u - \nabla p + \varepsilon \Delta v, \quad (1.4a)$$

$$\nabla \cdot v = 0. \quad (1.4b)$$

The corresponding linearized Euler equations are

$$\frac{\partial v}{\partial t} = -(u \cdot \nabla)v - (v \cdot \nabla)u - \nabla p, \quad (1.5a)$$

$$\nabla \cdot v = 0. \quad (1.5b)$$

We will study general classes of differential operators on \mathbb{T}^n which include the operators of the fluid equations defined by (1.4) and (1.5). We will investigate the relationship between the unstable point spectrum of the inviscid operator and the eigenvalues of the viscous operator in the limit of vanishing viscosity.

For a general equilibrium $u(x)$ the Euler operator defined in (1.5) is nonself-adjoint, nonelliptic and degenerate. Hence, contrary to the case of the elliptic Navier–Stokes operator given by (1.4), standard spectral results for elliptic operators do not apply to the Euler operator. However in the past decade considerable progress has been made in understanding the structure of the spectrum of the Euler operator using techniques of geometric optics. A survey of these results is given in Friedlander and Lipton-Lifschitz [4]. In particular, Vishik [12] obtained an explicit, and often computable, expression for the essential spectral radius of the Euler evolution operator in terms of a geometric quantity that can be considered as a “fluid” Lyapunov exponent. Recently Shvydkoy [10] has extended these results to a general class of advective PDEs with pseudodifferential bounded perturbation. In [10] the evolution operator for the Euler equation is partitioned into high frequency and low frequency parts.

In Section 3 of this present paper we show that an analogous construction can be achieved for the evolution operator of the Navier–Stokes equation. This result requires certain explicit estimates on the symbols of PDOs on the torus that we present in Appendix A. We note that the decomposition for the Navier–Stokes operator converges to the decomposition of the Euler operator in the limit of vanishing viscosity.

In Section 4 we prove the result relating the unstable spectrum of the Navier–Stokes equations as the viscosity goes to zero with the unstable eigenvalues of the Euler equation. We remark (see Theorem 4.3) on the consequence of this result for nonlinear instability. We first prove a result for spectral convergence in the inviscid limit for the semigroups. We then prove that beyond the limit of the essential spectrum of the inviscid operator the eigenvalues of the viscous operator converge precisely to those of the inviscid operator. A key step is to use the decomposition established in Section 3 to split off a finite dimensional subspace corresponding to growing modes. An analogous argument was used by Lyashenko and Friedlander [7] to obtain a sufficient condition for instability in the limit of vanishing viscosity for a class of operators satisfying certain boundedness and accretive properties. The properties required in [7] do not hold in general for the Euler operator (1.5) (although they do hold for the coupled rotating fluid/body system as noted in [7]). In this present paper we adapt the arguments of [7] to a wider class of operators that include the generic Euler and Navier–Stokes operators themselves.

2. Formulation of the spectral result

Let us consider the operator on the right hand side of Eq. (1.4a):

$$\mathbf{L}^\varepsilon f = -(u \cdot \nabla) f - (f \cdot \nabla) u - \nabla p + \varepsilon \Delta f. \tag{2.1}$$

Here $u \in C^\infty(\mathbb{T}^n)$ is a divergence-free time independent vector field. It follows from the perturbation theory that \mathbf{L}^ε generates a C_0 -semigroup $\{\mathbf{G}_t^\varepsilon\}_{t \geq 0}$ over the space $L^2(\mathbb{T}^n)$ of divergence-free vector fields, here denoted by L^2_{div} .

The first order advective operator \mathbf{L}^0 was treated in [10,12]. It was shown that the action of \mathbf{G}_t^0 on shortwave localized envelopes of the form

$$f_\delta(x) = b_0(x) e^{i\xi_0 \cdot x / \delta} + O(\delta), \quad \delta \ll 1$$

is described by the asymptotic formula

$$\mathbf{G}_t^0 f_\delta(x) = \mathbf{B}_t(\varphi_{-t}(x), \xi_0) f_\delta(\varphi_{-t}(x)) + O(\delta), \tag{2.2}$$

as $\delta \rightarrow 0$. In this formula φ_t is the integral flow of the steady field u , and \mathbf{B}_t is the fundamental matrix solution of the amplitude equation

$$b_t = \left(2 \frac{\xi \otimes \xi}{|\xi|^2} - \text{id} \right) \partial u(x) b, \tag{2.3}$$

where (x, ξ) evolve according to bicharacteristic system of equations

$$\begin{cases} x_t = u(x), \\ \xi_t = -\partial u^\top(x) \xi. \end{cases} \tag{2.4}$$

In order to satisfy the incompressibility condition we assume that

$$b_0(x) \cdot \xi_0 = 0$$

throughout the support of b_0 . One can show that the nonautonomous dynamical system (2.3) preserves this constraint too.

It was proved by Vishik [12] that the exponential instabilities of the amplitude equation (2.3) not only cause exponential instability of the semigroup \mathbf{G}^0 via (2.2), but also create the essential spectrum of the semigroup operator \mathbf{G}_t^0 in the unstable region. More precisely, the following formula for the essential spectral radius holds:

$$r_{\text{ess}}(\mathbf{G}_t^0) = e^{t\mu}, \tag{2.5}$$

where μ is the maximal Lyapunov exponent of the dynamical system (2.3). The main result of this present article states that beyond this limit of the essential spectrum the eigenvalues of \mathbf{L}^ε converge precisely to the eigenvalues of \mathbf{L}^0 (and, of course, by spectral mapping the same is true for the semigroups). Even stronger, we show convergence of the corresponding spectral subspaces.

For a closed operator \mathbf{L} we use the following notation:

$$\sigma_a^+(\mathbf{L}) = \{ \lambda \in \sigma(\mathbf{L}) : \text{Re } \lambda > a \},$$

and we denote by $m_a(\lambda, \mathbf{L})$ the algebraic multiplicity of λ .

Theorem 2.1. *Suppose that $\sigma_\mu^+(\mathbf{L}^0) \neq \emptyset$. Then*

- (i) *there exists $\varepsilon_0 > 0$ such that $\sigma_\mu^+(\mathbf{L}^\varepsilon) \neq \emptyset$ for all $0 \leq \varepsilon < \varepsilon_0$,*
- (ii) *for any $\lambda \in \sigma_\mu^+(\mathbf{L}^0)$ and any sufficiently small $r > 0$ there is $\varepsilon_r > 0$ such that for all $\varepsilon < \varepsilon_r$ one has*

$$m_a(\lambda, \mathbf{L}^0) = \sum_{\substack{\lambda' \in \sigma_\mu^+(\mathbf{L}^\varepsilon) \\ |\lambda - \lambda'| < r}} m_a(\lambda', \mathbf{L}^\varepsilon), \tag{2.6}$$

(iii) we have the limit

$$\lim_{\varepsilon \rightarrow 0} \sum_{\substack{\lambda' \in \sigma_{\mu}^+(\mathbf{L}^\varepsilon) \\ |\lambda - \lambda'| < r}} \mathbf{P}_{\lambda'}^\varepsilon = \mathbf{P}_\lambda^0, \tag{2.7}$$

where $\mathbf{P}_\lambda^\varepsilon$ denotes the Riesz projection onto the spectral subspace corresponding to λ .

We note that (iii) \Rightarrow (ii) \Rightarrow (i). So, it suffices to prove only part (iii). The proof heavily relies on the results of the next section, and will be finished in Section 4. In Appendix A we state some of the general facts on PDO’s in the way that is convenient to use in the subsequent arguments.

3. High frequency decomposition

In this section we prove a high frequency decomposition of the semigroup operator \mathbf{G}_t^ε . We show that \mathbf{G}_t^ε is given by the sum of a global pseudo-differential operator (PDO) of order 0 on the torus shifted by the flow φ_t , an operator of norm $O(\sqrt{\varepsilon})$ and a compact operator that behaves like a PDO of order -1 uniformly in ε . We introduce the following notation. As before, let φ_t denote the flow generated by u on \mathbb{T}^n , and χ_t the phase flow generated by the bicharacteristic system of Eqs. (2.4) on $\mathbb{T}^n \times \mathbb{R}^n \setminus \{0\}$. One can check that χ_t is given by

$$\chi_t : (x_0, \xi_0) \rightarrow (\varphi_t(x_0), \partial\varphi_t^{-\top}(x_0)\xi_0). \tag{3.1}$$

Here $\partial\varphi_t^{-\top}$ denotes the Jacobian matrix of the flow inversely transposed. We thus can write the amplitude equation (2.3) as follows

$$b_t = \mathbf{a}_0(\chi_t(x_0, \xi_0))b, \tag{3.2}$$

where

$$\mathbf{a}_0(x, \xi) = \left(2 \frac{\xi \otimes \xi}{|\xi|^2} - \mathbf{id} \right) \partial u(x). \tag{3.3}$$

The fundamental matrix solution of the amplitude equation (2.3), which we denoted $\mathbf{B}_t(x, \xi)$, is a smooth linear cocycle over the flow χ_t (see [1]). We call it the b -cocycle. Clearly, the b -cocycle is a smooth 0-homogeneous in ξ symbol of class \mathcal{S}^0 . We consider the operator of composition with the inverse flow φ_{-t} :

$$\Phi_t f = f \circ \varphi_{-t}, \tag{3.4}$$

and the orthogonal (Leray) projector:

$$\Pi : L^2 \rightarrow L^2_{\text{div}}, \tag{3.5}$$

which is a Fourier multiplier with symbol

$$\mathbf{p}(\xi) = \mathbf{id} - \frac{\xi \otimes \xi}{|\xi|^2}. \tag{3.6}$$

We use $\text{Op}[\mathbf{a}]$ to denote a global pseudo-differential operator on the torus with symbol \mathbf{a} :

$$\text{Op}[\mathbf{a}]f(x) = \sum_{k \in \mathbb{Z}^n \setminus \{0\}} e^{ik \cdot x} \mathbf{a}(x, k) \hat{f}(k). \tag{3.7}$$

The following decomposition was proved in [10] in the case of $\varepsilon = 0$:

$$\mathbf{G}_t^0 = \mathbf{H}_t^0 + \mathbf{U}_t^0, \tag{3.8}$$

where

$$\mathbf{H}_t^0 = \Pi \Phi_t \text{Op}[\mathbf{B}_t] \tag{3.9}$$

and \mathbf{U}_t^0 is a compact operator, which behaves like a PDO of order -1 (hence the asymptotic formula (2.2)). For an arbitrary positive ε formula (3.8) can be generalized as follows.

Theorem 3.1. For any $0 \leq t < T$ and $0 \leq \varepsilon < \varepsilon_0$ the following decomposition holds:

$$\mathbf{G}_t^\varepsilon = \mathbf{H}_t^\varepsilon + \sqrt{\varepsilon} \mathbf{T}_t^\varepsilon + \mathbf{U}_t^\varepsilon, \tag{3.10}$$

where

$$\mathbf{H}_t^\varepsilon = \mathbf{\Pi} \mathbf{\Phi}_t \text{Op}[\boldsymbol{\tau}_t^\varepsilon], \tag{3.11}$$

$$\boldsymbol{\tau}_t^\varepsilon(x, \xi) = \mathbf{B}_t(x, \xi) \exp \left\{ -\varepsilon \int_0^t |\partial \varphi_s^{-\top}(x) \xi|^2 ds \right\}, \tag{3.12}$$

the family $\{\mathbf{T}_t^\varepsilon\}_{0 \leq \varepsilon < \varepsilon_0, 0 \leq t < T}$ is uniformly bounded, and $\{\mathbf{U}_t^\varepsilon\}_{0 \leq \varepsilon < \varepsilon_0, 0 \leq t < T}$ is uniformly compact.

By a uniform compact family we mean the following.

Definition 3.2. Let $\psi_N(\xi)$ be the characteristic function of the ball $\{|\xi| < N\}$. Define the projection multiplier $\mathbf{P}_N f = (\psi_N \hat{f})^\vee$. We say that a family of operators $\{\mathbf{U}_t\}_{t \in I}$ on L^2 , or its subspace invariant with respect to \mathbf{P}_N , is uniformly compact if

$$\lim_{N \rightarrow \infty} \sup_{t \in I} \|\mathbf{U}_t - \mathbf{U}_t \mathbf{P}_N\| = 0. \tag{3.13}$$

The rest of the section is devoted to the proof of Theorem 3.1.

Step 1. First we show that the linearized Navier–Stokes operator \mathbf{L}^ε can be written as follows

$$\mathbf{L}^\varepsilon f = -(u \cdot \nabla) f + \mathbf{A} f + \varepsilon \Delta f, \tag{3.14}$$

where \mathbf{A} is a PDO with principal symbol given by (3.3). Indeed, since the Lie bracket $-(u \cdot \nabla) f + (f \cdot \nabla) u$ is divergence-free, we have

$$\begin{aligned} -(u \cdot \nabla) f - (f \cdot \nabla) u - \nabla p &= \mathbf{\Pi} (-(u \cdot \nabla) f + (f \cdot \nabla) u) - 2\mathbf{\Pi} ((f \cdot \nabla) u) \\ &= -(u \cdot \nabla) f + (f \cdot \nabla) u - 2\mathbf{\Pi} (\partial u f). \end{aligned}$$

The principal symbol of the last composition is the product of the principal symbols. Thus, we obtain (4.19).

Step 2. We notice that the theorem easily reduces to the case when $\mathbf{\Pi} = \mathbf{I}$. Indeed, consider the semigroup \mathbf{G}_t^ε acting on the whole space L^2 . Suppose (3.10) holds on all L^2 . Since \mathbf{G}_t^ε leaves L^2_{div} invariant, by applying $\mathbf{\Pi}$ and restricting to L^2_{div} , we see that (3.10) holds on L^2_{div} too.

Step 3. Using the fact that the b -cocycle solves the amplitude equation (3.2) we find the evolution equation for \mathbf{H}_t^ε by straightforward differentiation:

$$\frac{d}{dt} \mathbf{H}_t^\varepsilon = -(u \cdot \nabla) \mathbf{H}_t^\varepsilon + \mathbf{\Phi}_t \text{Op}[\mathbf{a}_0 \circ \chi_t \boldsymbol{\tau}_t^\varepsilon] - \varepsilon \mathbf{\Phi}_t \text{Op}[|\partial \varphi_t^{-\top}(x) \xi|^2 \boldsymbol{\tau}_t^\varepsilon]. \tag{3.15}$$

We compare the second term on the right hand side with $\mathbf{A} \mathbf{H}_t^\varepsilon$. First, the change of variables rule implies

$$\mathbf{A} \mathbf{\Phi}_t = \mathbf{\Phi}_t \mathbf{A}'$$

where $\mathbf{A}' = \text{Op}[\mathbf{a}_0 \circ \chi_t] + \text{Op}[\mathbf{a}'_t]$ with $\mathbf{a}'_t \in \mathcal{S}^{-1}$ uniformly in $t < T$. The latter follows from the fact that $\varphi_t \in C^\infty(\mathbb{T}^n)$ uniformly in $-T < t < T$. Hence we obtain

$$\mathbf{A} \mathbf{H}_t^\varepsilon = \mathbf{\Phi}_t \text{Op}[\mathbf{a}_0 \circ \chi_t] \text{Op}[\boldsymbol{\tau}_t^\varepsilon] + \mathbf{\Phi}_t \text{Op}[\mathbf{a}'_t] \text{Op}[\boldsymbol{\tau}_t^\varepsilon]. \tag{3.16}$$

Step 4. Let us show that the symbols $\boldsymbol{\tau}_t^\varepsilon$ satisfy a uniformity condition in the x -variable.

Lemma 3.3. For any multi-index α there exists a constant B_α independent of $0 \leq \varepsilon < \varepsilon_0$ and $t < T$ such that

$$\sup_{x \in \mathbb{T}^n, \xi \neq 0} |\partial_x^\alpha \boldsymbol{\tau}_t^\varepsilon(x, \xi)| \leq B_\alpha. \tag{3.17}$$

Proof. By the Leibnitz rule,

$$\partial_x^\alpha \tau_t^\varepsilon = \sum_{\alpha' \leq \alpha} \partial_x^{\alpha-\alpha'} \mathbf{B}_t(x, \xi) \partial_x^{\alpha'} \exp \left\{ -\varepsilon \int_0^t |\partial \varphi_s^{-\top}(x) \xi|^2 ds \right\}.$$

Hence, by the uniform boundedness of the b -cocycle,

$$|\partial_x^\alpha \tau_t^\varepsilon| \leq C_{t,\alpha} \sup_{\alpha' \leq \alpha, x, \xi} \left| \partial_x^{\alpha'} \exp \left\{ -\varepsilon \int_0^t |\partial \varphi_s^{-\top}(x) \xi|^2 ds \right\} \right|.$$

One can check by induction that if $g = g(x_1, \dots, x_n)$ is a smooth function, then

$$\partial_x^\alpha (\exp(g)) = \exp(g) \sum (\partial_x^{\gamma_1} g)^{l_1} \dots (\partial_x^{\gamma_r} g)^{l_r}$$

where the sum is taken over a subset of indexes satisfying

$$|\gamma_1|l_1 + \dots + |\gamma_r|l_r = |\alpha|.$$

In our case $g = -\varepsilon \int_0^t |\partial \varphi_s^{-\top}(x) \xi|^2 ds$. Then,

$$|(\partial_x^{\gamma_1} g)^{l_1} \dots (\partial_x^{\gamma_r} g)^{l_r}| \leq C_{t,\alpha} \varepsilon^{l_1+\dots+l_r} |\xi|^{2(l_1+\dots+l_r)}.$$

Using that $|\partial \varphi_t^{-\top}(x) \xi| \geq c_t |\xi|$, we obtain

$$\left| \partial_x^{\alpha'} \exp \left\{ -\varepsilon \int_0^t |\partial \varphi_s^{-\top}(x) \xi|^2 ds \right\} \right| \leq C_{t,\alpha} e^{-c_t \varepsilon |\xi|^2} \sum (\varepsilon |\xi|^2)^{l_1+\dots+l_r} \leq C'_{t,\alpha}$$

uniformly in ε . \square

Step 5. Using Lemmas 3.3 and A.4 we immediately conclude that the family $\mathbf{U}_{t,\varepsilon}^{(1)} = \Phi_t \text{Op}[\mathbf{a}'_t] \text{Op}[\tau_t^\varepsilon]$ is uniformly compact in $0 \leq t < T$ and $0 \leq \varepsilon < \varepsilon_0$. By Theorem A.1, with $m_1 = m_2 = 0$ and $N = 4$,

$$\text{Op}[\mathbf{a}_0 \circ \chi_t] \text{Op}[\tau_t^\varepsilon] = \text{Op}[\lambda_t^\varepsilon],$$

where

$$\lambda_t^\varepsilon = (\mathbf{a}_0 \circ \chi_t) \tau_t^\varepsilon + \sum_{1 \leq |\gamma| < 4} \frac{(-1)^{|\gamma|}}{\gamma!} (\partial_\xi^\gamma \mathbf{a}_0 \circ \chi_t) (\partial_x^\gamma \tau_t^\varepsilon) + \mathbf{r}_4^{t,\varepsilon}.$$

From the estimate on the remainder (A.5) and the τ_t^ε given in (3.17), we see that the families of symbols $\{\mathbf{r}_4^{t,\varepsilon}\}$ and $\{(\partial_\xi^\gamma \mathbf{a}_0 \circ \chi_t) (\partial_x^\gamma \tau_t^\varepsilon)\}$, with $|\gamma| \geq 1$, satisfy the assumption of Lemma A.3. Hence, they contribute a uniformly compact family $\{\mathbf{U}_{t,\varepsilon}^{(2)}\}$.

Summarizing the above, we have shown the identity

$$\mathbf{A}\mathbf{H}_t^\varepsilon = \Phi_t \text{Op}[(\mathbf{a}_0 \circ \chi_t) \tau_t^\varepsilon] + \mathbf{U}_{t,\varepsilon}^{(3)}, \tag{3.18}$$

where $\{\mathbf{U}_{t,\varepsilon}^{(3)}\}$ is uniformly compact.

Step 6. Let us now consider the last term on the right hand side of (3.15):

$$-\varepsilon \Phi_t \text{Op}[|\partial \varphi_t^{-\top}(x) \xi|^2 \tau_t^\varepsilon] \tag{3.19}$$

and compare it to

$$\varepsilon \Delta \mathbf{H}_t^\varepsilon = -\varepsilon \text{Op}[|\xi|^2] \Phi_t \text{Op}[\tau_t^\varepsilon]. \tag{3.20}$$

By the change of variables rule, we obtain

$$\text{Op}[|\xi|^2] \Phi_t = \Phi_t \text{Op}[|\partial \varphi_t^{-\top}(x) \xi|^2].$$

By Theorem A.1 with $m_1 = 2, m_2 = 0, N = 6$

$$\varepsilon \text{Op}[|\partial\varphi_t^{-\top}(x)\xi|^2] \text{Op}[\tau_t^\varepsilon] = \text{Op}[\lambda_t^\varepsilon],$$

where

$$\lambda_t^\varepsilon = \varepsilon |\partial\varphi_t^{-\top}(x)\xi|^2 \tau_t^\varepsilon + \varepsilon \sum_{1 \leq |\gamma| < 6} \frac{(-1)^{|\gamma|}}{\gamma!} (\partial_\xi^\gamma |\partial\varphi_t^{-\top}(x)\xi|^2) (\partial_x^\gamma \tau_t^\varepsilon) + \varepsilon \mathbf{r}_6^{t,\varepsilon}. \tag{3.21}$$

Substitution of the first term on the right hand side of (3.21) into (3.20) gives us precisely (3.19). From (A.5), (3.17), and Theorem A.2 we see that $\varepsilon \Phi_t \text{Op}[\mathbf{r}_6^{t,\varepsilon}] = \varepsilon \mathbf{T}_{t,\varepsilon}^{(1)}$, with $\{\mathbf{T}_{t,\varepsilon}^{(1)}\}$ being uniformly bounded.

Now, for all $|\gamma| = 1$ and any α one has

$$|\partial_x^\alpha (\varepsilon \partial_\xi^\gamma |\partial\varphi_t^{-\top}(x)\xi|^2 \partial_x^\gamma \tau_t^\varepsilon)| \leq C_{t,\alpha} \varepsilon |\xi| e^{-c_t \varepsilon |\xi|^2} \leq \sqrt{\varepsilon} C'_{t,\alpha}$$

uniformly for all $0 \leq \varepsilon < \varepsilon_0, 0 \leq t < T, \xi \in \mathbb{R}^n \setminus \{0\}, x \in \mathbb{T}^n$. Hence, the terms with $|\gamma| = 1$ add up to a term of the form $\sqrt{\varepsilon} \mathbf{T}_{t,\varepsilon}^{(2)}$, where $\{\mathbf{T}_{t,\varepsilon}^{(2)}\}$ is uniformly bounded.

The terms with $|\gamma| = 2$ can be estimated as follows

$$|\partial_x^\alpha (\varepsilon \partial_\xi^\gamma |\partial\varphi_t^{-\top}(x)\xi|^2 \partial_x^\gamma \tau_t^\varepsilon)| \leq C_{t,\alpha} \varepsilon e^{-c_t \varepsilon |\xi|^2}.$$

So, they contribute a term $\varepsilon \mathbf{T}_{t,\varepsilon}^{(3)}$. And finally, all the terms with $|\gamma| > 2$ vanish.

Step 7. Thus, the evolution equation (3.15) takes the form

$$\frac{d}{dt} \mathbf{H}_t^\varepsilon = -(u \cdot \nabla) \mathbf{H}_t^\varepsilon + \mathbf{A} \mathbf{H}_t^\varepsilon + \varepsilon \Delta \mathbf{H}_t^\varepsilon + \sqrt{\varepsilon} \mathbf{T}_{t,\varepsilon}^{(4)} + \mathbf{U}_{t,\varepsilon}^{(4)}, \tag{3.22}$$

where $\{\mathbf{T}_{t,\varepsilon}^{(4)}\}$ is uniformly bounded, and $\{\mathbf{U}_{t,\varepsilon}^{(4)}\}$ is uniformly compact. By the Duhamel principle one gets

$$\mathbf{H}_t^\varepsilon = \mathbf{G}_t^\varepsilon + \sqrt{\varepsilon} \int_0^t \mathbf{G}_{t-s}^\varepsilon \mathbf{T}_{s,\varepsilon}^{(4)} ds + \int_0^t \mathbf{G}_{t-s}^\varepsilon \mathbf{U}_{s,\varepsilon}^{(4)} ds. \tag{3.23}$$

It remains to observe that the family $\{\mathbf{G}_t^\varepsilon\}_{0 \leq \varepsilon \leq \varepsilon_0, 0 \leq t \leq T}$ itself is uniformly bounded, and hence, the integrals define operators \mathbf{T}_t^ε and \mathbf{U}_t^ε with the desired properties.

This finishes the proof of Theorem 3.1.

4. Proof of Theorem 2.1

Let us recall that μ is the maximal Lyapunov exponent of the b -cocycle, which determines the essential spectral radius for the semigroup operator \mathbf{G}_t^0 through formula (2.5). Let us fix a $\delta > 0$. We can find a large t such that

$$\sup_{x,\xi} |\mathbf{B}_t(x, \xi)| < \frac{1}{4} e^{t(\mu+\delta)}.$$

Then by the sharp Gårding inequality, for N large enough, we get

$$\|\mathbf{H}_t^\varepsilon - \mathbf{H}_t^\varepsilon \mathbf{P}_N\| \leq 2 \sup_{x,\xi} |\tau_t^\varepsilon(x, \xi)| < \frac{1}{2} e^{t(\mu+\delta)}, \tag{4.1}$$

for all $0 \leq \varepsilon < \varepsilon_0$. By the uniform compactness we also have

$$\|\mathbf{U}_t^\varepsilon - \mathbf{U}_t^\varepsilon \mathbf{P}_N\| < \frac{1}{3} e^{t(\mu+\delta)}. \tag{4.2}$$

Let us fix N for which both (4.1) and (4.2) hold, and split the semigroup \mathbf{G}_t^ε into the sum

$$\mathbf{G}_t^\varepsilon = \mathbf{G}_{t,\varepsilon}^- + \mathbf{G}_{t,\varepsilon}^+, \tag{4.3}$$

where we denote

$$\mathbf{G}_{t,\varepsilon}^- = \mathbf{H}_t^\varepsilon (\mathbf{I} - \mathbf{P}_N) + \sqrt{\varepsilon} \mathbf{T}_t^\varepsilon + \mathbf{U}_t^\varepsilon (\mathbf{I} - \mathbf{P}_N), \quad (4.4)$$

$$\mathbf{G}_{t,\varepsilon}^+ = \mathbf{H}_t^\varepsilon \mathbf{P}_N + \mathbf{U}_t^\varepsilon \mathbf{P}_N. \quad (4.5)$$

So, $\mathbf{G}_{t,\varepsilon}^+$ is nonzero only on the finite-dimensional subspace $\text{Im } \mathbf{P}_N$, and in view of (4.1) and (4.2), we have the estimate $\|\mathbf{G}_{t,\varepsilon}^-\| < \frac{5}{6} e^{t(\mu+\delta)}$ for all sufficiently small ε . This implies that the resolvent $(\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1}$ exists and has the power series expansion whenever $|z| > e^{t(\mu+\delta)}$.

Lemma 4.1. *The convergence*

$$\lim_{\varepsilon \rightarrow 0} (\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1} = (\mathbf{G}_{t,0}^- - z\mathbf{I})^{-1} \quad (4.6)$$

holds in the strong operator topology uniformly on compact subsets of $\{|z| > e^{t(\mu+\delta)}\}$.

Proof. Observe that $\mathbf{L}^\varepsilon f \rightarrow \mathbf{L}^0 f$ for all $f \in C^\infty(\mathbb{T}^n)$, and $C^\infty(\mathbb{T}^n)$ is a core of the generator \mathbf{L}^0 . Hence, by [5, Theorem 7.2], $\mathbf{G}_t^\varepsilon \rightarrow \mathbf{G}_t^0$ strongly. It is straightforward to prove that $\mathbf{H}_t^\varepsilon \rightarrow \mathbf{H}_t^0$ strongly, which by virtue of the decomposition (3.10) also implies that $\mathbf{U}_t^\varepsilon \rightarrow \mathbf{U}_t^0$. We therefore obtain convergence $\mathbf{G}_{t,\varepsilon}^- \rightarrow \mathbf{G}_{t,0}^-$ in the strong operator topology.

Thus, the conclusion of the lemma follows from the preceding remarks. \square

Observe that for any $|z| > e^{t(\mu+\delta)}$ and $0 \leq \varepsilon < \varepsilon_0$ the identity

$$\mathbf{G}_t^\varepsilon f = z f \quad (4.7)$$

can be written as

$$f + (\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1} \mathbf{G}_{t,\varepsilon}^+ f = 0. \quad (4.8)$$

This is equivalent to the system of equations

$$[\mathbf{P}_N + \mathbf{P}_N (\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1} \mathbf{G}_{t,\varepsilon}^+ \mathbf{P}_N] f'_N = 0, \quad (4.9)$$

$$f''_N = -(\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1} \mathbf{G}_{t,\varepsilon}^+ \mathbf{P}_N f'_N, \quad (4.10)$$

where $f'_N = \mathbf{P}_N f$ and $f''_N = (\mathbf{I} - \mathbf{P}_N) f$. We see that f''_N can be found from (4.10) if f'_N is known. So, the original eigenvalue problem (4.7) is equivalent to the finite-dimensional Eq. (4.9), which in turn has a solution at $z = z_0$ if and only if z_0 is a root of the analytic function

$$g(z, \varepsilon) = \det \left\| (e_j, e_k) + (\mathbf{P}_N (\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1} \mathbf{G}_{t,\varepsilon}^+ \mathbf{P}_N e_j, e_k) \right\|_{j,k=1}^K, \quad (4.11)$$

where $\{e_1, \dots, e_K\}$ is an orthonormal basis of $\text{Im } \mathbf{P}_N$. By Lemma 4.1, we have

$$\lim_{\varepsilon \rightarrow 0} g(z, \varepsilon) = g(z, 0) \quad (4.12)$$

uniformly on compact sets in $\{|z| > e^{t(\mu+\delta)}\}$.

Lemma 4.2. *The resolvents $(\mathbf{G}_t^\varepsilon - z\mathbf{I})^{-1}$ exist and are uniformly bounded on compact subsets of $\{|z| > e^{t(\mu+\delta)}\} \setminus \sigma(\mathbf{G}_t^0)$, for $0 \leq \varepsilon < \varepsilon_0$, and the limit*

$$\lim_{\varepsilon \rightarrow 0} (\mathbf{G}_t^\varepsilon - z\mathbf{I})^{-1} = (\mathbf{G}_t^0 - z\mathbf{I})^{-1} \quad (4.13)$$

holds in the strong operator topology.

Proof. The existence of the resolvents follows readily from the convergence (4.12).

Let us fix $z \notin \sigma(\mathbf{G}_t^0)$ and observe that

$$(\mathbf{G}_t^\varepsilon - z\mathbf{I})^{-1} = [\mathbf{I} + (\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1} \mathbf{G}_{t,\varepsilon}^+]^{-1} (\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1}. \quad (4.14)$$

In view of Lemma 4.1 it suffices to show the convergence

$$\lim_{\varepsilon \rightarrow 0} [\mathbf{I} + (\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1} \mathbf{G}_{t,\varepsilon}^+]^{-1} = [\mathbf{I} + (\mathbf{G}_{t,0}^- - z\mathbf{I})^{-1} \mathbf{G}_{t,0}^+]^{-1}. \quad (4.15)$$

In the direct sum $L^2 = \text{Im } \mathbf{P}_N \oplus \text{Ker } \mathbf{P}_N$ we have the following block-representation

$$\mathbf{I} + (\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1} \mathbf{G}_{t,\varepsilon}^+ = \begin{bmatrix} \mathbf{I} + \mathbf{P}_N (\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1} \mathbf{G}_{t,\varepsilon}^+ \mathbf{P}_N & 0 \\ (\mathbf{I} - \mathbf{P}_N) (\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1} \mathbf{G}_{t,\varepsilon}^+ \mathbf{P}_N & \mathbf{I} \end{bmatrix}.$$

So,

$$[\mathbf{I} + (\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1} \mathbf{G}_{t,\varepsilon}^+]^{-1} = \begin{bmatrix} [\mathbf{I} + \mathbf{P}_N (\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1} \mathbf{G}_{t,\varepsilon}^+ \mathbf{P}_N]^{-1} & 0 \\ \mathbf{F}_{t,\varepsilon} & \mathbf{I} \end{bmatrix},$$

where

$$\mathbf{F}_{t,\varepsilon} = -(\mathbf{I} - \mathbf{P}_N) (\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1} \mathbf{G}_{t,\varepsilon}^+ \mathbf{P}_N [\mathbf{I} + \mathbf{P}_N (\mathbf{G}_{t,\varepsilon}^- - z\mathbf{I})^{-1} \mathbf{G}_{t,\varepsilon}^+ \mathbf{P}_N]^{-1}.$$

Since $g(z, \varepsilon)$ is uniformly bounded away from 0 for small ε , $\mathbf{F}_{t,\varepsilon}$ is uniformly bounded from above, and hence, so is the resolvent (4.14). The limit (4.15) now follows from the above formulas and Lemma 4.1. \square

Lemma 4.2 already proves the spectral convergence result for the semigroups. In order to prove it for the generators as stated in Theorem 2.1 we argue as follows.

Let $\lambda \in \sigma(\mathbf{L}^0)$ be arbitrary. Find a $\delta > 0$ such that $\text{Re } \lambda > \mu + \delta$, and let $t > 0$ be chosen as above to satisfy (4.1) and (4.2). Observe the following identity:

$$(\mathbf{L}^\varepsilon - \zeta \mathbf{I})^{-1} = (\mathbf{G}_t^\varepsilon - e^{\zeta t} \mathbf{I})^{-1} \int_0^t e^{\zeta(t-s)} \mathbf{G}_s^\varepsilon ds. \tag{4.16}$$

It follows from Lemma 4.2, that the resolvents $(\mathbf{L}^\varepsilon - \zeta \mathbf{I})^{-1}$ are uniformly bounded on a circle Γ of small radius r centered at λ that does not contain other points of the spectrum of \mathbf{L}^0 . Moreover,

$$\lim_{\varepsilon \rightarrow 0} (\mathbf{L}^\varepsilon - \zeta \mathbf{I})^{-1} = (\mathbf{L}^0 - \zeta \mathbf{I})^{-1}. \tag{4.17}$$

The Riesz projection on the spectral subspace corresponding to the part of the spectrum of \mathbf{L}^ε inside Γ is given by

$$\mathbf{P}^\varepsilon = \sum_{\substack{\lambda' \in \sigma(\mathbf{L}^\varepsilon) \\ |\lambda - \lambda'| < r}} \mathbf{P}_{\lambda'}^\varepsilon = \frac{1}{2\pi i} \int_{\Gamma} (\mathbf{L}^\varepsilon - \zeta \mathbf{I})^{-1} d\zeta. \tag{4.18}$$

Using (4.17) the limit $\mathbf{P}^\varepsilon \rightarrow \mathbf{P}_\lambda^0$ follows from the dominated convergence theorem. This proves statement (iii) of our theorem, and hence, (ii) and (i).

4.1. Discussion

Thus, Theorem 2.1 gives the convergence of the unstable eigenvalues of the Navier–Stokes operator to eigenvalues the Euler operator outside the essential spectrum of the latter. Our result therefore extends the theorem of Vishik and Friedlander [14] proving that a necessary condition for instability in the Navier–Stokes equations as $\varepsilon \rightarrow 0$ is an instability in the underlying Euler equations.

We note that no particular properties of the Navier–Stokes operator are used in our arguments other than the fact that it has the form

$$\mathbf{L}^\varepsilon f = -(u \cdot \nabla) f + \mathbf{A} f + \varepsilon \Delta f, \tag{4.19}$$

where \mathbf{A} is a PDO of zero order with some principal symbol \mathbf{a}_0 . We can therefore extend Theorem 2.1 to a much broader class of advective equations which include the equations of geophysical fluid dynamics describing rotating, stratified incompressible flows where the evolution operator is an operator of the type (4.19). We refer to [10] for an extended list of examples, where in particular formula (2.5) for the essential spectral radius is proved in general.

When the function space $L^2(\mathbb{T}^n)$ is replaced by the Sobolev space $H^m(\mathbb{T}^n)$ it is possible to obtain in place of μ an analogous quantity μ_m which determines the essential spectral radius of \mathbf{G}_t^0 . The role of the b -cocycle defined by (2.3) is replaced by a new so-called $b\xi^m$ -cocycle (see [9,10]). All the arguments in this present paper remain valid for

the convergence of the spectrum of the viscous operator as $\varepsilon \rightarrow 0$ and the spectrum of the inviscid operator in $H^m(\mathbb{T}^n)$ with $\operatorname{Re} \lambda > \mu_m$. In the particular case of the two dimensional Euler equation in $H^1(\mathbb{T}^n)$ it can be shown that $\mu_1 = 0$. Hence Theorem 2.1 implies that in this example there is precise convergence of all the points of the unstable spectra of the Navier–Stokes operators to that of the Euler operator in the inviscid limit.

Justification of the linearization method (i.e proving that linear exponential stability/instability implies nonlinear stability/instability) for the Euler equation in the energy norm is an open problem (see [6,13,16]). With the help of Theorem 2.1 and the results of [3,8] it can be recast in the settings of vanishing viscosity limit as follows. We know that linear exponential instability of a smooth steady solution u to the Navier–Stokes equation implies its nonlinear instability in the Lyapunov sense in the L^2 -norm, [3]. This result holds for any viscosity $\varepsilon > 0$. Thus, using Theorem 2.1 we obtain the following statement.

Theorem 4.3. *Let $u \in C^\infty(\mathbb{T}^n)$ be a steady solution to the Euler equation, which we also consider as a solution to the Navier–Stokes equation with the force $f_\varepsilon = -\varepsilon \Delta u$. Suppose that the Euler equation linearized about u has exponentially growing solutions with a rate greater than the fluid exponent μ . Then for $\varepsilon > 0$ small enough u is nonlinearly unstable in the L^2 -norm with respect to ε -viscous perturbations.*

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Appendix A

In this section we recall a few facts about global pseudo-differential operators (PDO) on the torus defined by

$$\operatorname{Op}[\sigma]f(x) = \sum_{k \in \mathbb{Z}^n} e^{ik \cdot x} \sigma(x, k) \hat{f}(k), \quad (\text{A.1})$$

where $\mathbb{Z}^n = \mathbb{Z}^n \setminus \{0\}$, $f(x) \in \mathbb{C}^d$ and σ is a $d \times d$ -matrix valued symbol of class S^m . We write $\sigma \in S^m$ if $\sigma \in C^\infty(\mathbb{T}^n \times \mathbb{R}^n \setminus \{0\})$, where $\mathbb{R}^n \setminus \{0\} = \mathbb{R}^n \setminus \{0\}$, and

$$|\partial_x^\alpha \partial_\xi^\beta \sigma(x, \xi)| \leq A_{\alpha, \beta} |\xi|^{m-|\beta|}, \quad (\text{A.2})$$

for all $|\xi| \geq 1$, $x \in \mathbb{T}^n$, and all multi-indexes α, β . Even though in the formula (A.1) we do not need to require symbols to be defined outside the integer lattice, we do assume that they are smooth in $\xi \in \mathbb{R}^n \setminus \{0\}$. For such symbols the standard theorems of pseudo-differential calculus hold as in the case of \mathbb{R}^n (see [2]). Below we state the composition rule with an estimate on the remainder term, which can be deduced from a careful examination of the proof given in [15].

For $a, b \geq 0$, and $\{A_{\alpha, \beta}\}$ given in (A.2), let us define

$$\tilde{A}_{a, b} = \sum_{|\alpha| \leq a, |\beta| \leq b} A_{\alpha, \beta}.$$

Theorem A.1. *Suppose $\sigma \in S^{m_1}$ and $\tau \in S^{m_2}$ with the corresponding norms $\{A_{\alpha, \beta}\}$ and $\{B_{\alpha, \beta}\}$. Then*

$$\operatorname{Op}[\sigma] \circ \operatorname{Op}[\tau] = \operatorname{Op}[\lambda], \quad (\text{A.3})$$

with $\lambda \in S^{m_1+m_2}$. Moreover, for all $N \in \mathbb{N}$, λ has the following representation

$$\lambda = \sum_{|\gamma| < N} \frac{(-1)^{|\gamma|}}{\gamma!} (\partial_\xi^\gamma \sigma) (\partial_x^\gamma \tau) + \mathbf{r}_N, \quad (\text{A.4})$$

where $\mathbf{r}_N \in S^{m_1+m_2-N}$, and for $N > m_1 + 3$ satisfies the estimate

$$|\partial_x^\alpha \mathbf{r}_N(x, \xi)| \leq c \tilde{A}_{|\alpha|, N+n} \tilde{B}_{2N+|\alpha|-m_1-1, 0} |\xi|^{m_1+m_2+1-N}, \quad (\text{A.5})$$

for $|\xi| \geq 1$, where $c = c(\alpha, N, n, m_1, m_2)$ is independent of the symbols.

In (A.5) the restriction $N > m_1 + 3$ and one extra power of $|\xi|$ is needed in order to obtain the explicit bound in terms of the norms of σ and τ . We also emphasize that the estimate (A.5) uses only the x -smoothness constant of τ , and not its ξ -smoothness.

We note that in the case of the torus the L^2 -norm of a PDO is bounded by the norm of only x -derivatives.

Theorem A.2. *Suppose $\sigma \in \mathcal{S}^0$ satisfies (A.2). Then $\text{Op}[\sigma]$ is bounded on L^2 and*

$$\|\text{Op}[\sigma]\| \leq c\tilde{A}_{n+1,0} \tag{A.6}$$

where c is independent of the symbol.

The proof uses Minkowski’s inequality, and is similar to that of the next lemma.

Lemma A.3. *Let $U_\iota = \text{Op}[\sigma_\iota]$, $\iota \in I$. Suppose there exists a constant $A > 0$ independent of ι such that*

$$|\partial_x^\alpha \sigma_\iota(x, \xi)| \leq A|\xi|^{-1}, \quad |\xi| \geq 1, \quad |\alpha| \leq n + 1, \tag{A.7}$$

holds for all $\iota \in I$. Then the family $\{U_\iota\}_{\iota \in I}$ is uniformly compact.

Proof. Let $f \in L^2$, $\|f\| = 1$, and $\text{supp } \hat{f} \subset \{|k| \geq N\}$. We obtain

$$\|U_\iota f\|^2 = \sum_{q \in \mathbb{Z}^n} \left| \sum_{|k| \geq N} \hat{\sigma}_\iota(q - k, k) \hat{f}(k) \right|^2 \tag{A.8}$$

$$\lesssim \sum_{q \in \mathbb{Z}^n} \left| \sum_{k \neq q, |k| \geq N} \hat{\sigma}_\iota(q - k, k) \hat{f}(k) \right|^2 + \sum_{|k| \geq N} |\hat{\sigma}_\iota(0, k) \hat{f}(k)|^2 \tag{A.9}$$

$$\leq A^2 \sum_{q \in \mathbb{Z}^n} \left(\sum_{k \neq q, |k| \geq N} \frac{|\hat{f}(k)|}{|k||q - k|^{n+1}} \right)^2 + A^2 \sum_{|k| \geq N} \frac{|\hat{f}(k)|^2}{|k|^2} \tag{A.10}$$

$$\leq N^{-2} A^2 \left(\sum_{q \in \mathbb{Z}^n} |q|^{-n-1} \right)^2 + N^{-2} A^2 \lesssim N^{-2} A^2. \tag{A.11}$$

This proves the lemma. \square

Lemma A.4. *Let $\{U_\iota\}_{\iota \in I}$ be as in Lemma A.3. Let $V_\kappa = \text{Op}[\tau_\kappa]$, $\kappa \in K$, be another family such that there is a constant $B > 0$ independent of κ such that*

$$|\partial_x^\alpha \tau_\kappa(x, \xi)| \leq B, \quad |\xi| \geq 1, \quad |\alpha| \leq n + 1 \tag{A.12}$$

holds for all $\kappa \in K$. Then the family $\{U_\iota V_\kappa\}_{\iota \in I, \kappa \in K}$ is uniformly compact.

Proof. Let $N > 0$, $f \in L^2$ with $\text{supp } \hat{f} \subset \{|k| \geq N\}$ be fixed. Let $|q| < N/2$. Using the Cauchy–Schwartz inequality we estimate

$$\begin{aligned} |(\widehat{V_\kappa f})(q)| &= \left| \sum_{|k| \geq N} \hat{\tau}_\kappa(q - k, k) \hat{f}(k) \right| \leq B \sum_{|k| \geq N} \frac{|\hat{f}(k)|}{|q - k|^{n+1}} \\ &\leq B \sum_{|p| > N/2} \frac{|\hat{f}(q - p)|}{|p|^{n+1}} \leq B \|f\| \left(\sum_{|p| > N/2} |p|^{-2(n+1)} \right)^{1/2} \\ &\lesssim N^{-1} \|f\|. \end{aligned}$$

So, for any fixed $M > 0$ we have

$$\lim_{N \rightarrow \infty} \|\mathbf{P}_M V_\kappa (\mathbf{I} - \mathbf{P}_N)\| = 0$$

uniformly in $\kappa \in K$. Observe that

$$\|U_\iota V_\kappa - U_\iota V_\kappa P_N\| \leq \|U_\iota(\mathbf{I} - P_M)V_\kappa(\mathbf{I} - P_N)\| + \|U_\iota P_M V_\kappa(\mathbf{I} - P_N)\|.$$

Thus, using uniform compactness of U_ι 's and uniform boundedness of V_κ 's, which follows from Theorem A.2, we can choose M large enough to make the first summand small uniformly in N, ι, κ . Letting $N \rightarrow \infty$ we make the second summand small too. This finishes the proof. \square

References

- [1] C. Chicone, Y. Latushkin, *Evolution Semigroups in Dynamical Systems and Differential Equations*, American Mathematical Society, Providence, RI, 1999, MR 2001e:47068.
- [2] D. Dickinson, T. Gramchev, M. Yoshino, First order pseudodifferential operators on the torus: Normal forms, diophantine phenomena and global hypoellipticity, *Ann. Univ. Ferrara, Nuova Ser., Sez. VII* 41 (1996) 51–64.
- [3] S. Friedlander, N. Pavlović, R. Shvydkoy, Nonlinear instability for the Navier–Stokes equations, *Commun. Math. Phys.* 264 (2006) 335–347.
- [4] S. Friedlander, A. Lipton-Lifschitz, Localized instabilities in fluids, in: *Handbook of Mathematical Fluid Dynamics*, vol. II, North-Holland, Amsterdam, 2003, pp. 289–354, MR MR1984155 (2004g:76072).
- [5] J.A. Goldstein, *Semigroups of Linear Operators and Applications*, The Clarendon Press Oxford University Press, New York, 1985, MR 87c:47056.
- [6] Z. Lin, Nonlinear instability of ideal plane flows, *Int. Math. Res. Not.* 41 (2004) 2147–2178, MR MR2078852.
- [7] A.A. Lyashenko, S.J. Friedlander, A sufficient condition for instability in the limit of vanishing dissipation, *J. Math. Anal. Appl.* 221 (2) (1998) 544–558, MR MR1621750 (99g:76065).
- [8] D.H. Sattinger, The mathematical problem of hydrodynamic stability, *J. Math. Mech.* 19 (1969/1970) 817–979, MR MR0261182 (41 #5798).
- [9] R. Shvydkoy, Cocycles and Mañé sequences with an application to ideal fluids, *J. Diff. Eq.* 229 (2006) 49–62.
- [10] R. Shvydkoy, The essential spectrum of advective equations, *Commun. Math. Phys.* 265 (2006) 507–545.
- [11] R. Temam, *Navier–Stokes Equations*, AMS Chelsea Publishing, Providence, RI, 2001. Theory and Numerical Analysis, Reprint of the 1984 edition. MR MR1846644 (2002j:76001).
- [12] M. Vishik, Spectrum of small oscillations of an ideal fluid and Lyapunov exponents, *J. Math. Pures Appl.* (9) 75 (6) (1996) 531–557, MR 97k:35203.
- [13] M. Vishik, S. Friedlander, Nonlinear instability in two dimensional ideal fluids: the case of a dominant eigenvalue, *Comm. Math. Phys.* 243 (2) (2003) 261–273, MR MR2021907 (2004k:76055).
- [14] M. Vishik, S. Friedlander, Dynamo theory methods for hydrodynamic stability, *J. Math. Pures Appl.* (9) 72 (2) (1993) 145–180, MR 95a:76036.
- [15] M.W. Wong, *An introduction to Pseudo-Differential Operators*, World Scientific Publishing Co. Inc., Teaneck, NJ, 1991, MR MR1100930 (92b:47083).
- [16] V.I. Yudovich, *The Linearization Method in Hydrodynamical Stability Theory*, Translations of Mathematical Monographs, vol. 74, American Mathematical Society, Providence, RI, 1989. Translated from the Russian by J.R. Schulenberger. MR MR1003607 (90h:76001).