An existence result for a free boundary shallow water model using a Lagrangian scheme

Un résultat d'existence pour un problème de shallow water à frontière libre en utilisant un schéma lagrangien

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Abstract

We propose a free boundary shallow water model for which we give an existence theorem. The proof uses an original Lagrangian discrete scheme in order to build a sequence of approximate solutions. The properties of this scheme allow to treat the difficulties linked to the boundary motion. These approximate solutions verify some compactness results which allow us to pass to the limit in the discrete problem.
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Résumé

Nous proposons un modèle de shallow water à frontière libre pour lequel nous donnons un théorème d'existence. La preuve utilise un schéma de discrétisation lagrangien original afin de construire une suite de solutions approchées. Les propriétés de ce schéma permettent de traiter les difficultés liées au mouvement de la frontière. Ces solutions approchées vérifient certaines estimations qui nous permettent de passer à la limite dans le problème discrétisé.
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1. Introduction

This paper deals with the behavior of a fluid defined in a domain depending on time. The model we propose can be used in various applications such as fluid-structure interaction problems [12] or the simulation of propagation problems, for instance the simulation of a spilled oil slick [15] or a fire spread [1]. To characterize the fluid motion we

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consider a shallow water problem with free boundary, the motion of the boundary being characterized by a boundary operator A (some boundary operators are used in V.A. Solonnikov [19], J.T. Beale [2]). This operator allows to conserve a smooth enough domain and consequently to use classical properties of Sobolev spaces. To solve the bi-dimensional fluid equations $(\mathcal{P})$, we propose a Lagrangian scheme. Euler scheme is not appropriate for the discretization of this kind of problem since we work on a noncylindrical domain. Moreover the Lagrangian description allows to follow each particle in its motion and thus to take naturally into account the boundary variations. Numerous papers propose to solve Navier–Stokes equations in a moving domain by using the Arbitrary Lagrangian Eulerian method. We can cite for instance J. Donéa et al. [7] who give a survey of this method. Let us mention on the subject our recent work [15] in which we deal with a shallow water problem with free boundary by using the ALE method and considering that the operator A is zero (the case $A = 0$ is considered in V.A. Solonnikov [18]). In particular we use this method to describe the behavior of a pollutant slick at the sea surface.

Our survey follows a series of papers [9–12], dealing with models defined on a domain depending on time. To solve the problem, the above papers use a method based on a fixed point theorem. The originality of our new approach is to circumvent the use of such a fixed point. Numerically it allows to decrease drastically the computational time. Our purpose is to solve the shallow water problem by using a very simple linear scheme where the total derivative is approached with a finite difference approximation to which we add a regularizing operator B depending on the discretization step and vanishing as this step goes to 0^+ [10]. The Lagrangian description is well adapted to describe the boundary motion. The operator B gives the necessary compactness to justify all the calculations and to pass to the limit inside the equations. Moreover, this operator gives a meaning to the discretization since it allows to show that a particle does not leave the domain from a time step to another. The Lagrangian discretization allows us to circumvent the difficulties linked to the nonlinear terms (advection) and leads us to solve a "nice" linear stationary problem.

At time t, the fluid occupies a bounded domain Ω_t of $\mathbb{R}^2$ with boundary γ_t. We denote by γ_0 the boundary of the fluid at initial time. Assuming that γ_0 is smooth enough, we define the deformed boundary as follows: $\gamma_t := \{x = X + d(X, t), X \in \gamma_0\}$, where d corresponds to the displacement $d(X, t) = \Gamma(t, 0, X) - X$, where $\Gamma(t, s, x)$ denotes the Lagrangian flow, i.e. the position at time t of the particle located at x at time s. This deformation has a meaning if the corresponding Lagrangian flow $X \mapsto \Gamma(t, 0, X) = X + d(X, t)$ is a diffeomorphism from γ_0 onto $\gamma_t := \Gamma(t, 0, \gamma_0)$, so that all what follows will hold as long as $\det \mathcal{J}(X, t) \neq 0$ on γ_0, (where $\mathcal{J}(X, t)$ is the Jacobian matrix associated to the transformation $X \mapsto \Gamma(t, 0, X)$), and Γ is one-to-one on γ_0. Thus we define $\Gamma(0, t, x)$ by $\Gamma(0, t, .) = \Gamma(t, 0, .)^{-1}$ and $\Gamma(t, s, x) = \Gamma(t, 0, \Gamma(0, s, x))$. Thanks to operator A, we shall see afterwards that d is bounded in $W^{1,\infty}(0, T; W^{1,\infty}(\gamma_0))$ by a bound depending proportionally on the initial data. Thus, if we consider small data, Γ verifies the previous conditions and the deformation has a meaning (see P.G. Ciarlet [4], B. Desjardins et al. [6]).

We set $Q = \bigcup_{t \in (0,T)} \Omega_t \times \{t\}$, $\Sigma = \bigcup_{t \in (0,T)} \gamma_t \times \{t\}$ and n the exterior unit normal to Ω_t on γ_t. We suppose that the fluid is governed by the following shallow water problem

$$(\mathcal{P}) \quad \begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla)u - \mu \Delta u + \nabla h = 0 & \text{in } Q, \\ \frac{\partial h}{\partial t} + \operatorname{div}(hu) = 0 & \text{in } Q, \end{cases}$$

where u is the velocity, h is the fluid thickness and μ is the diffusion coefficient. In order to set the boundary conditions, we introduce the Lagrangian description of the velocity, $U : \gamma_0 \times (0, T) \to \mathbb{R}^2$, $(X, T) \mapsto u(\Gamma(t, 0, X), t)$. On boundary γ_0 we have

$$U(X, t) = u\big(X + d(X, t), t\big) = \frac{\partial d(X, t)}{\partial t}, \tag{1}$$

and we characterize the boundary motion γ_t by a condition on the normal component of the fluid stress tensor σ

$$\sigma\big(X + d(X, t), t\big) n\big(X + d(X, t), t\big) |\det \mathcal{J}|(X, t) = A\big(\partial U(X, t)/\partial t\big) \quad \text{on } \gamma_0 \times (0, T), \tag{2}$$

where A is an operator defined on γ_0 and vector valued, which takes into account the stress applied to the fluid on the boundary. We assume that A is the square of the Laplace–Beltrami operator which ensures that

$$\int_{\gamma_0} A(v) \cdot v = \int_{\gamma_0} A^{1/2}(v) \cdot A^{1/2}(v) = \|v\|^2_{H^2(\gamma_0)}.$$

Notice that this assumption on the mathematical operator A is necessary to keep a smooth free boundary (for more details about this kind of boundary operator see V.A. Solonnikov [19] and R. Dautray, J.L. Lions [5]).

The equations are completed by the initial conditions

$$h_0 \log h_0 \in L^1(\Omega_0), \quad h_0 \geqslant 0, \tag{3}$$

$$u_0 \in H^{5/2}(\Omega_0). \tag{4}$$

2. Preliminary results

2.1. Energy estimates

In this section we are going to state and prove some *a priori* estimates for the problem $(\mathcal{P})$.

Lemma 1. *Let* (u, h) *be a classical solution of problem* $(\mathcal{P})$*. As P.L. Lions in* [14] *or P. Orenga in* [16]*, we assume that*

$$M_0 = \frac{1}{2}\|u_0\|_{L^2(\Omega_0)}^2 + \int_{\Omega_0} h_0 \log h_0 + \frac{1}{e}\,\mathrm{meas}\left(\bigcup_{t\in(0,T)}\Omega_t\right) + \frac{1}{2}\|u_0\|_{H^2(\gamma_0)}^2 < \beta \min\left(\left(\frac{2\mu}{C_{GN}}\right)^2 ; \left(\frac{2\alpha}{TC_{GN}}\right)^2\right), \tag{5}$$

$$\|u_0\|_{L^2(\Omega_0)} < \min\left(2\frac{\mu}{C_{GN}} ; 2\frac{\alpha}{C_{GN}T}\right), \tag{6}$$

where α and β are two positive numbers such that $\alpha + \beta = 1/2$, e is the classical Neper constant and C_{GN} is the best constant satisfying Gagliardo–Niremberg inequality

$$\|u\|_{L^4(\Omega_t)}^2 \leqslant C_{GN}\|u\|_{L^2(\Omega_t)}\|u\|_{H^1(\Omega_t)}. \tag{7}$$

Then, under assumptions (3), (4) *on the data, and for a finite time T, h, u, and d verify the following a priori estimates*

$$u \in L^2\big(0, T; H^1(\Omega_t)\big) \cap L^\infty\big(0, T; L^2(\Omega_t)\big), \tag{8}$$

$$h \log h \in L^\infty\big(0, T; L^1(\Omega_t)\big), \quad h \geqslant 0, \tag{9}$$

$$U \in L^\infty\big(0, T; H^2(\gamma_0)\big), \tag{10}$$

$$d \in W^{1,\infty}\big(0, T; H^2(\gamma_0)\big), \quad \det \mathcal{J} \neq 0, \tag{11}$$

$$h \in L^2(Q). \tag{12}$$

Proof. We multiply equation $(\mathcal{P})_1$ by u and we use Leibniz formula. We obtain

$$\frac{1}{2}\frac{d}{dt}\|u\|_{L^2(\Omega_t)}^2 - \frac{1}{2}\int_{\gamma_t}|u|^2 u \cdot n - \frac{1}{2}\int_{\Omega_t}|u|^2\,\mathrm{div}\,u + \frac{1}{2}\int_{\gamma_t}|u|^2 u \cdot n$$

$$+ \mu\|Du\|_{L^2(\Omega_t)}^2 + \int_{\Omega_t}\nabla h \cdot u - \mu\int_{\gamma_t}\frac{\partial u}{\partial n} \cdot u = 0.$$

The term $\int_{\Omega_t}\nabla h \cdot u$ is treated as follows

$$\int_{\Omega_t}\nabla h \cdot u = \int_{\Omega_t}\nabla \log h \cdot hu$$

$$= -\int_{\Omega_t}\log h\,\mathrm{div}(hu) + \int_{\gamma_t}h\log h u \cdot n$$

$$= \int_{\Omega_t}\log h\frac{\partial h}{\partial t} + \int_{\gamma_t}h\log h u \cdot n$$

$$= \int_{\Omega_t} \frac{\partial}{\partial t}(h \log h - h) + \int_{\gamma_t} h \log h u \cdot n$$

$$= \frac{d}{dt} \int_{\Omega_t} h \log h - \frac{d}{dt} \int_{\Omega_t} h - \int_{\gamma_t} h \log h u \cdot n + \int_{\gamma_t} h u \cdot n + \int_{\gamma_t} h \log h u \cdot n.$$

Thus, noticing that the continuity equation gives $\frac{d}{dt} \int_{\Omega_t} h = 0$, we obtain

$$\int_{\Omega_t} \nabla h \cdot u = \frac{d}{dt} \int_{\Omega_t} h \log h + \int_{\gamma_t} h u \cdot n.$$

We estimate $\int_{\Omega_t} |u|^2 \operatorname{div} u$ using the Gagliardo–Niremberg inequality

$$\int_{\Omega_t} |u|^2 \operatorname{div} u \leqslant C_{GN} \|u\|_{L^2(\Omega_t)} \|u\|_{H^1(\Omega_t)}^2.$$

So, writing the boundary terms $\int_{\gamma_t} h u \cdot n - \mu \int_{\gamma_t} \frac{\partial u}{\partial n} \cdot u = \int_{\gamma_t} u \cdot \sigma n$ on γ_0 and using the boundary condition (2), we obtain

$$\frac{1}{2} \frac{d}{dt} \|u\|_{L^2(\Omega_t)}^2 + \mu \|Du\|_{L^2(\Omega_t)}^2 + \frac{d}{dt} \int_{\Omega_t} h \log h + \frac{1}{2} \frac{d}{dt} \int_{\gamma_0} |A^{1/2} U|^2 \leqslant C_{GN} \|u\|_{L^2(\Omega_t)} \|u\|_{H^1(\Omega_t)}^2.$$

Then, we integrate over $(0, t)$, $t \in (0, T)$. We write

$$C_{GN} \int_0^t \|u\|_{L^2(\Omega_t)} \big(\|u\|_{L^2(\Omega_t)}^2 + \|Du\|_{L^2(\Omega_t)}^2 \big)$$

$$\leqslant C_{GN} \|u\|_{L^\infty(0,t;L^2(\Omega_t))} \|Du\|_{L^2(0,t;L^2(\Omega_t))}^2 + C_{GN} T \|u\|_{L^\infty(0,t;L^2(\Omega_t))}^3.$$

Furthermore, noticing that $\int_{\Omega_t} h(t) \log h(t) \geqslant - \operatorname{meas}(\bigcup_{t \in (0,T)} \Omega_t)/e$, we obtain

$$\left(\alpha - \frac{C_{GN}}{2} T \|u\|_{L^\infty(0;t;L^2(\Omega_t))} \right) \|u\|_{L^\infty(0,t;L^2(\Omega_t))}^2 + \beta \|u\|_{L^\infty(0;t;L^2(\Omega_t))}^2 + \frac{1}{2} \|U\|_{L^\infty(0,t;H^2(\gamma_0))}$$

$$+ \left(\mu - \frac{C_{GN}}{2} \|u\|_{L^\infty(0,t;L^2(\Omega_t))} \right) \|u\|_{L^2(0,t;L^2(\Omega_t))}^2 + \|h \log h\|_{L^\infty(0,t;L^1(\Omega_t))}$$

$$\leqslant \frac{1}{2} \|u_0\|_{L^2(\Omega_0)}^2 + \int_{\Omega_0} h_0 \log h_0 + \frac{1}{e} \operatorname{meas} \left(\bigcup_{t \in (0,T)} \Omega_t \right) + \frac{1}{2} \|u_0\|_{H^2(\gamma_0)}^2 = M_0, \tag{13}$$

with $\alpha + \beta = 1/2$. Now, we have to verify that $\alpha - C_{GN} T/2 \|u\|_{L^\infty(0;t;L^2(\Omega_t))} > 0$ and $\mu - C_{GN}/2 \|u\|_{L^\infty(0,t;L^2(\Omega_t))} > 0$. To show this last point, we recall that u_0 verifies $\|u_0\|_{L^2(\Omega_0)} < \min(2\mu/C_{GN}; 2\alpha/(C_{GN} T))$. In finite dimension (at least), there exists $t_1 > 0$ such that for all $t \in {]}0, t_1{[}$, $\|u(t)\|_{L^2(\Omega_t)} < \min(2\mu/C_{GN}; 2\alpha/(C_{GN} T))$. Supposing that there exists t_1 such that $\|u(t_1)\|_{L^2(\Omega_t)} = \min(2\mu/C_{GN}; 2\alpha/(C_{GN} T))$, for instance $\|u(t_1)\|_{L^2(\Omega_t)} = 2\mu/C_{GN}$, then estimate (13) at time t_1 leads to

$$\beta \|u\|_{L^\infty(0,t;L^2(\Omega_t))}^2 = \beta \left(\frac{2\mu}{C_{GN}} \right)^2 \leqslant M_0,$$

which contradicts (5). We obtain a similar contradiction if $\|u(t_1)\|_{L^2(\Omega_t)} = 2\alpha/(C_{GN} T)$), thus estimates (8)–(10) are proved.

Remark 2. From relation (1) and estimate on U, we deduce that $d \in W^{1,\infty}(0, T; H^2(\gamma_0))$ and consequently that the boundary is of class C^1. Thus, for all t, we can give a meaning to the trace of a function of $H^1(\Omega_t)$. Notice also that the bound on d allows to ensure that $\det \mathcal{J} \neq 0$ and to give a meaning to the deformation.

To obtain the bound L^2 on h, we introduce the gradient operator $\overline{\nabla}$ in Q and we set $W = \sum_{i=1}^{4} w_i$ with

$$w_1 = \left(\frac{\partial u_1}{\partial t}, \frac{\partial u_2}{\partial t}, 0 \right), \qquad w_2 = (u \nabla u_1, u \nabla u_2, 0),$$

$$w_3 = \mu \left(-\frac{\partial \operatorname{div} u}{\partial x_1} - \frac{\partial \operatorname{curl} u}{\partial x_2}, -\frac{\partial \operatorname{div} u}{\partial x_2} + \frac{\partial \operatorname{curl} u}{\partial x_1}, 0 \right), \qquad w_4 = \big(0, 0, \operatorname{div}(hu) \big).$$

With these notations, $(\mathcal{P})$ can be formulated under the form $\overline{\nabla} h + W = 0$. We have $u \in L^2(0, T; H^1(\Omega_t))$, then $w_1 \in H^{-1}(Q)$ and $w_3 \in H^{-1}(Q)$. Moreover, $w_2 \in L^{4/3}(Q) \subset H^{-1}(Q)$, since $u \in L^4(Q) \cap L^2(0, T; H^1(\Omega_t))$. Moreover $h \log h \in L^\infty(0, T; L^1(\Omega_t))$, then $h \in L^2(0, T; H^{-1}(\Omega_t))$ and $\operatorname{div}(hu) = -\frac{\partial h}{\partial t} \in H^{-1}(Q)$ from which we deduce that $w_4 \in H^{-1}(Q)$. Thus,

$$\overline{\nabla} h = -W \in H^{-1}(Q)$$

and so $h \in L^2(Q)$ if $h \in L^2_{\text{loc}}(Q)$. The bound on h in $L^2_{\text{loc}}(Q)$ can be obtained as in P.L. Lions [13] or in F.J. Chatelon [3]. In these references, notice that the authors establish this bound in a simple cylinder domain $(0, T) \times \Omega$. This result is still valid in a domain such as Q, since we can apply the reasoning used by F. Flori and B. Giudicelli [8]. Indeed, since the boundary is smooth enough, we can define $K \subset \bigcup_{t \in [0,T]} \Omega_t$ such that $K_t = K \cap \Omega_t$ is compact for all t. We introduce a cut-off function $\varphi \in C^\infty([0, T]; C_0^\infty(\Omega_t))$ (for instance) such that $\varphi \equiv 1$ on K and $0 \leqslant \varphi \leqslant 1$ on $\bigcup_{t \in [0,T]} \Omega_t$. Then, we can apply the arguments used by P.L. Lions or F.J. Chatelon on ϕh and finally we obtain a bound $L^2(Q)$ on ϕh. $\quad \square$

2.2. Regularization of the problem

We approach the problem $(\mathcal{P})$ by regularizing the continuity equation with the term δh^2

$$(\mathcal{P}^\delta) \quad \begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla) u - \mu \Delta u + \nabla h = 0 & \text{in } \Omega_t, \\ \frac{\partial h}{\partial t} + \operatorname{div}(hu) + \delta h^2 = 0 & \text{in } \Omega_t, \end{cases}$$

with the previous boundary condition (2). This regularization is an argument allowing us to construct the approximate solutions in the following sections and in particular to pass to the limit in the discrete equations in Section 5.

Remark 3. The bound on h in $L^2(Q)$ obtained in Lemma 1 allows to pass to the limit on δ in $(\mathcal{P}^\delta)$ and consequently to recover the solutions of $(\mathcal{P})$.

In view of the numerical scheme and to conserve the positivity of h, we renormalize the continuity equation as follows: $\partial \log h / \partial t + u \cdot \nabla \log h + \operatorname{div} u + \delta h = 0$. Thus $(\mathcal{P}^\delta)$ can be formulated as

$$(\mathcal{P}^\delta) \quad \begin{cases} \frac{\partial u}{\partial t} + (u \cdot \nabla) u - \mu \Delta u + \nabla h = 0 & \text{in } \Omega_t, \\ \frac{\partial \log h}{\partial t} + u \nabla \log h + \operatorname{div} u + \delta h = 0 & \text{in } \Omega_t. \end{cases}$$

Notice that this renormalization has a meaning since $h \in L^2(Q)$ and $u \in L^2(0, T; H^1(\Omega_t))$ (R.J. Di Perna and P.L. Lions [17], P.L. Lions, Lemma 2.3 [13]).

3. Lagrangian discretization

To prove an existence result for the problem $(\mathcal{P}^\delta)$, we build a sequence of approximate solutions by using a Lagrangian scheme. In Section 4, we shall show that these approximate solutions verify some estimations which allow to pass to the limit in the time-discretized problem in Section 5.

The Lagrangian scheme is well adapted since it allows to follow each particle in its motion and thus naturally takes into account boundary variations. First, we propose a time-discretization for the domain and we define the approximate domains Ω_k. Then we introduce the stationary problems solved on each Ω_k. As mentioned in the introduction, to pass to the limit inside the equations, we introduce in the discretization an operator $\Delta t^\alpha B u$, where $0 < \alpha < 1$, such that $D(B) \equiv H^3(\Omega_t)$ for almost all $t \in (0, T)$ and $\Delta t^\alpha B u \xrightarrow{\Delta t \to 0} 0$ in the sense of distributions.

3.1. Domain Lagrangian discretization

For the boundary motion we consider the following discretization: for all $k \in [1, \ldots, m]$, with $\Delta t = T/m$, we set

$$d_0(X) = 0 \tag{14}$$

and

$$d_k(X) = d_{k-1}(X) + u_{k-1}\big(X + d_{k-1}(X)\big)\Delta t, \tag{15}$$
$$\Gamma_k(X) = X + d_k(X), \tag{16}$$

where u_k is defined afterwards. In the same way we consider the characteristic curves defined by the equation $dx(t)/dt = u(x(t), t)$ which is discretized using the relation

$$x_{k+1} = x_k + u_k(x_k)\Delta t, \quad k \in \{0, \ldots, m-1\}, \quad \Delta t = \frac{T}{m}.$$

By recurrence, we build m approximate domains $\Omega_k = \{x_k \in \mathbb{R}^2 / x_k = x_{k-1} + u_{k-1}(x_{k-1})\Delta t, x_{k-1} \in \Omega_{k-1}\}$. We set

$$\widehat{Q}_{\Delta t} = \big\{(x, t) \in \mathbb{R}^2 \times]\Delta t, T[/ x(t) = x_k,$$
$$t \in [k\Delta t, (k+1)\Delta t[, x_i \in \Omega_i, k \in \{1, \ldots, m-1\}\big\}$$
$$\partial\widehat{Q}_{\Delta t} = \big\{(y, t) \in \mathbb{R}^2 \times]\Delta t, T[/ y(t) = \Gamma_k(X),$$
$$t \in [k\Delta t, (k+1)\Delta t[, X \in \gamma_0, k \in \{1, \ldots, m-1\}\big\}$$
$$\widetilde{Q}_{\Delta t} = \big\{(x, t) \in \mathbb{R}^2 \times]\Delta t, T[/ x(t) = x_k + (t - k\Delta t)u_k(x_k),$$
$$t \in [k\Delta t, (k+1)\Delta t[, x_i \in \Omega_i, k \in \{1, \ldots, m-1\}\big\}$$
$$\partial\widetilde{Q}_{\Delta t} = \big\{(y, t) \in \mathbb{R}^2 \times]\Delta t, T[/ y(t) = \Gamma_k(X) + (t - k\Delta t)u_k(\Gamma_k(X)),$$
$$t \in \big[k\Delta t, (k+1)\Delta t\big[, X \in \gamma_0, k \in \{1, \ldots, m-1\}\big\}.$$

3.2. Approximate problem

Let us denote by $\tilde{x}_k = x_{k-1}$ the position in Ω_{k-1} of the particle located in x_k at time $t = k\Delta t$. We approach the Lagrangian derivative in the momentum equation by $(u_k - \tilde{u}_{k-1})/\Delta t + \Delta t^\alpha Bu_k$, where $t_k = k\Delta t$, $u_k = u(x_k, t_k)$, $\tilde{u}_{k-1} = u(\tilde{x}_k, t_{k-1}) = u(x_k - \tilde{u}_{k-1}\Delta t, t_{k-1})$, $0 < \alpha < 1$, B is an operator such that $D(B) = H^3(\Omega_k)$ and $\Delta t^\alpha Bu_k \to 0$ in the distribution sense when $\Delta t \to 0^+$. We endow $\Delta t^\alpha Bu_k$ with good boundary conditions to ensure that $\Delta t^\alpha \int_{\Omega_k} Bu_k \cdot u_k = \Delta t^\alpha \|u_k\|^2_{H^3(\Omega_k)}$.

Remark 4. The condition on the normal stress tensor is "disturbed" by the Lagrangian derivative approximation, thus this condition becomes

$$\sigma\big(X + d(X, t), t\big)n\big(X + d(X, t), t\big)|\det \mathcal{J}|(X, t) + \Delta t^\alpha \operatorname{Tr}(Bu)\big(X + d(X, t), t\big)$$
$$= A\big(\partial U(X, t)/\partial t\big) \quad \text{on } \gamma_0 \times (0, T). \tag{17}$$

Using these notations, we define the stationary problem

$$(\mathcal{P}_k^\delta) \quad \begin{cases} u_k - \mu\Delta t \Delta u_k + \Delta t \nabla h_k + \Delta t^{1+\alpha} Bu_k = \tilde{u}_{k-1} & \text{in } \Omega_k, \\ \log h_k + \Delta t \operatorname{div} u_k + \delta\Delta t h_k = \log \tilde{h}_{k-1} & \text{in } \Omega_k, \\ \sigma_k(\Gamma_k(X))n_k(\Gamma_k(X))|\det \mathcal{J}_k|(X) + \Delta t^\alpha \operatorname{Tr}(Bu_k)(\Gamma_k(X)) \\ \quad = A\big(\frac{u_k(\Gamma_k(X)) - u_{k-1}(\Gamma_{k-1}(X))}{\Delta t}\big) + \text{boundary conditions for the operator } \Delta t^\alpha Bu_k, & \text{on } \gamma_0, \end{cases}$$

where $\mathcal{J}_k$ is the Jacobian matrix associated to the transformation $X \mapsto \Gamma_k(X) = X + d_k(X)$, allowing to pass from γ_0 to γ_k. We shall see in the following section that $\sup_{0 \leqslant k \leqslant m} \|d_k\|_{W^{1,\infty}(\gamma_0)} \leqslant \sqrt{2K_\alpha}T$, where K_α depends proportionally on initial data. Then, if we consider small data, we deduce that $\det \mathcal{J}_k \neq 0$ and Γ_k is one to one on γ_0, and this

transformation has a meaning. In the same way, we set J_k the Jacobian matrix of the transformation $x_{k+1} = x_k + \Delta t u_k(x_k)$ allowing to pass from Ω_k to Ω_{k+1}:

$$J_k = \begin{pmatrix} 1 + \Delta t \frac{\partial u_{k1}}{\partial x_{k1}} & \Delta t \frac{\partial u_{k1}}{\partial x_{k2}} \\ \Delta t \frac{\partial u_{k2}}{\partial x_{k1}} & 1 + \Delta t \frac{\partial u_{k2}}{\partial x_{k2}} \end{pmatrix}.$$

In this case, we shall see that the term $\Delta t^\alpha B u_k$ allows us to establish that $\Delta t \, D u_k$ is bounded in $L^\infty(\Omega_k)$ by a bound which depends proportionally on initial data and $\Delta t^{(1-\alpha)/2}$. Thus if we choose Δt small enough, $\det J_k > 0$ and the transformation defined by $x_{k+1} = x_k + \Delta t u_k(x_k)$ has a meaning.

4. Compactness results

We are going now to state and prove some compactness results on the stationary solutions of the M problems $(\mathcal{P}_k^\delta)$ which allow us to pass to the limit in Section 5. To establish the estimates, we introduce the sequence M_k ($k = 1, \ldots, m$), defined by recurrence by $M_1 = D_0$ and $M_k = M_{k-1} + (2\mu \Delta t + C_2 \Delta t^{1-2\alpha} \Delta t) M_{k-1}$, where $D_0 = \int_{\Omega_0} h_0 |\det J_0| + 1/2 \int_{\Omega_0} u_0^2 |\det J_0| + 1/2 \|A^{1/2}(u_0 \circ \Gamma_0)\|_{L^2(\gamma_0)}^2$ and C_2 is defined in the proof of the following lemma. Notice that the sequence $(M_m)_{m \geqslant 1}$ (where $m = T/\Delta t$) converges to $D_0 e^{\mu T}$ when $m \to +\infty$ ($\Delta t \to 0^+$). Then for all $\Delta t < \alpha$, there exists K_α such that $M_m < K_\alpha$.

Lemma 5. *If Δt is chosen small enough ($\Delta t < \alpha$) and if we assume the condition*

$$K_\alpha < 2\left(\frac{\mu}{C_{GN}}\right)^2,\tag{18}$$

we have

$$\sup_{1 \leqslant i \leqslant k} \|u_i\|_{L^2(\Omega_i)} \leqslant 2\frac{\mu}{C_{GN}}, \qquad \sum_{i=1}^{k} \|u_i - u_{i-1}\|_{L^2(\Omega_i)}^2 \leqslant 2\left(\frac{\mu}{C_{GN}}\right)^2,$$

$$\Delta t^\alpha \sum_{i=1}^{k} \Delta t \|u_i\|_{H^3(\Omega_i)}^2 \leqslant \mathbb{C}, \qquad \sum_{i=1}^{k} \Delta t \|u_i\|_{H^1(\Omega_i)}^2 \leqslant \mathbb{C}',$$

$$\sup_{1 \leqslant i \leqslant k} \|h_i\|_{L^1(\Omega_i)} \leqslant 2\left(\frac{\mu}{C_{GN}}\right)^2, \qquad \sum_{i=1}^{k} \Delta t \|h_i\|_{L^2(\Omega_i)}^2 \leqslant \mathbb{C}''(\delta),$$

$$\sup_{1 \leqslant i \leqslant k} \|u_i \circ \Gamma_i\|_{H^2(\gamma_0)} \leqslant \sqrt{2K_\alpha} \leqslant 2\frac{\mu}{C_{GN}},$$

$$\sum_{i=1}^{k} \|u_i \circ \Gamma_i - u_{i-1} \circ \Gamma_{i-1}\|_{H^2(\gamma_0)}^2 \leqslant 2\left(\frac{\mu}{C_{GN}}\right)^2,$$

where $\mathbb{C}$, $\mathbb{C}'$ and $\mathbb{C}''(\delta)$ are independent of Δt.

Proof. We give the estimates for $k = 1$, $k = 2$ and we generalize for all k.

Estimates for $k = 1$

We multiply the momentum equation $(\mathcal{P}_1^\delta)_1$ by u_1 and we integrate over Ω_1. Taking into account the boundary conditions described in the previous section (Eq. (17)) we obtain

$$\frac{1}{2}\|u_1\|_{L^2(\Omega_1)}^2 + \frac{1}{2}\|u_1 - \tilde{u}_0\|_{L^2(\Omega_1)}^2 + \mu \Delta t \|D u_1\|_{L^2(\Omega_1)}^2 - \Delta t \int_{\Omega_1} h_1 \operatorname{div} u_1 + \Delta t^{1+\alpha} \|u_1\|_{H^3(\Omega_1)}^2$$

$$+ \int_{\gamma_0} \left[A(u_1 \circ \Gamma_1) - A(u_0 \circ \Gamma_0) \right] \cdot u_1 \circ \Gamma_1 = \frac{1}{2}\int_{\Omega_1} |\tilde{u}_0|^2 = \frac{1}{2}\int_{\Omega_0} |u_0|^2 |\det J_0|.$$

We have

$$\int_{\gamma_0} \left[A(u_1 \circ \Gamma_1) - A(u_0 \circ \Gamma_0) \right] \cdot u_1 \circ \Gamma_1 = \int_{\gamma_0} \left| A^{\frac{1}{2}}(u_1 \circ \Gamma_1) \right|^2 - \int_{\gamma_0} A^{\frac{1}{2}}(u_0 \circ \Gamma_0) \cdot A^{\frac{1}{2}}(u_1 \circ \Gamma_1)$$

$$= \frac{1}{2} \left\| A^{\frac{1}{2}}(u_1 \circ \Gamma_1) \right\|_{L^2(\gamma_0)}^2 + \frac{1}{2} \left\| A^{\frac{1}{2}}(u_1 \circ \Gamma_1 - u_0 \circ \Gamma_0) \right\|_{L^2(\gamma_0)}^2 - \frac{1}{2} \left\| A^{\frac{1}{2}}(u_0 \circ \Gamma_0) \right\|_{L^2(\gamma_0)}^2.$$

The term $-\Delta t \int_{\Omega_1} h_1 \operatorname{div} u_1$ is estimated by using the continuity equation

$$-\Delta t \int_{\Omega_1} h_1 \operatorname{div} u_1 = \int_{\Omega_1} h_1 \log \frac{h_1}{\tilde{h}_0} - \delta \Delta t \int_{\Omega_1} h_1^2 = \int_{\Omega_1} h_1 \log \frac{h_1}{\tilde{h}_0} + \delta \Delta t \|h_1\|_{L^2(\Omega_1)}^2,$$

and we write

$$\int_{\Omega_1} h_1 \log \frac{h_1}{\tilde{h}_0} + \delta \Delta t \|\nabla h_1\|_{L^2(\Omega_1)}^2 \geqslant \int_{\Omega_1} (h_1 - \tilde{h}_0) + \delta \Delta t \|h_1\|_{L^2(\Omega_1)}^2.$$

Moreover the continuity equation shows that $h_1 = \tilde{h}_0 e^{-\Delta t \operatorname{div}(u_1) - \delta \Delta t h_1} \geqslant 0$. Finally we obtain

$$\frac{1}{2} \|u_1\|_{L^2(\Omega_1)}^2 + \frac{1}{2} \|u_1 - \tilde{u}_0\|_{L^2(\Omega_1)}^2 + \mu \Delta t \|Du_1\|_{L^2(\Omega_1)}^2 + \|h_1\|_{L^1(\Omega_1)}$$

$$+ \delta \Delta t \|h_1\|_{L^2(\Omega_1)}^2 + \frac{1}{2} \left\| A^{\frac{1}{2}}(u_1 \circ \Gamma_1) \right\|_{L^2(\gamma_0)}^2 + \frac{1}{2} \left\| A^{\frac{1}{2}}(u_1 \circ \Gamma_1 - u_0 \circ \Gamma_0) \right\|_{L^2(\gamma_0)}^2$$

$$+ \Delta t^{1+\alpha} \|u_1\|_{H^3(\Omega_1)}^2 \leqslant D_0. \tag{19}$$

Since $\Omega_1 \subset \mathbb{R}^2$, then $H^3(\Omega_1) \hookrightarrow W^{1,\infty}(\Omega_1)$, thus there exists a constant $\mathbb{K}$ such that

$$\Delta t \|u_1\|_{W^{1,\infty}(\Omega_1)} \leqslant \mathbb{K} \Delta t \|u_1\|_{H^3(\Omega_1)} \leqslant \mathbb{K} \Delta t^{\frac{1-\alpha}{2}} \sqrt{2} \frac{\mu}{C_{GN}}.$$

This estimate shows that we can always choose Δt small enough such that $\det J_1 > 0$ and the transformation $x_2 = x_1 + u_1 \Delta t$ has a meaning.

Estimates for $k = 2$

In the same way, we have

$$\frac{1}{2} \|u_2\|_{L^2(\Omega_2)}^2 + \frac{1}{2} \|u_2 - \tilde{u}_1\|_{L^2(\Omega_1)}^2 + \mu \Delta t \|Du_2\|_{L^2(\Omega_1)}^2 - \Delta t \int_{\Omega_2} h_2 \operatorname{div} u_2 + \Delta t^{1+\alpha} \|u_2\|_{H^3(\Omega_2)}^2$$

$$+ \frac{1}{2} \left\| A^{\frac{1}{2}}(u_2 \circ \Gamma_2) \right\|_{L^2(\gamma_0)}^2 + \frac{1}{2} \left\| A^{\frac{1}{2}}(u_2 \circ \Gamma_2 - u_1 \circ \Gamma_1) \right\|_{L^2(\gamma_0)}^2$$

$$\leqslant \frac{1}{2} \int_{\Omega_1} |u_1|^2 |\det J_1| + \frac{1}{2} \left\| A^{\frac{1}{2}}(u_1 \circ \Gamma_1) \right\|_{L^2(\gamma_0)}^2. \tag{20}$$

Gagliardo–Niremberg inequality leads to

$$\frac{1}{2} \int_{\Omega_1} |u_1|^2 |\det J_1| \leqslant \frac{1}{2} \|u_1\|_{L^2(\Omega_1)}^2 + \frac{\Delta t}{2} C_{GN} \|u_1\|_{L^2(\Omega_1)} \|u_1\|_{H^1(\Omega_1)}^2 + \frac{K^2}{2} \Delta t^2 \|u_1\|_{L^2(\Omega_1)}^2 \|u_1\|_{H^3(\Omega_1)}^2, \tag{21}$$

since $H^3 \subset W^{1,\infty}$ and $\det J_1 = 1 + \Delta t \operatorname{div} u_1 + \Delta t^2 (\frac{\partial u_{21}}{\partial x_{21}} \frac{\partial u_{22}}{\partial x_{22}} - \frac{\partial u_{22}}{\partial x_{21}} \frac{\partial u_{21}}{\partial x_{22}})$. To handle the term $-\Delta t \int_{\Omega_2} h_2 \operatorname{div} u_2$, we write

$$-\Delta t \int_{\Omega_2} h_2 \operatorname{div} u_2 = \int_{\Omega_2} h_2 \log \frac{h_2}{\tilde{h}_1} - \delta \Delta t \int_{\Omega_2} h_2^2 = \int_{\Omega_2} h_2 \log \frac{h_2}{\tilde{h}_1} + \delta \Delta t \|h_2\|_{L^2(\Omega_2)}^2,$$

and we obtain

$$\int_{\Omega_2} h_2 \log \frac{h_2}{\tilde{h}_1} + \delta \Delta t \|h_2\|_{L^2(\Omega_2)}^2 \geqslant \int_{\Omega_2} (h_2 - \tilde{h}_1) + \delta \Delta t \|h_2\|_{L^2(\Omega_2)}^2,$$

moreover

$$\int_{\Omega_1} h_1 |\det J_1| \leqslant \int_{\Omega_1} h_1 + \Delta t \int_{\Omega_1} h_1 \operatorname{div} u_1 + K^2 \Delta t^2 \|h_1\|_{L^1(\Omega_1)} \|u_1\|_{H^3(\Omega_1)}^2, \qquad (22)$$

with

$$\Delta t \int_{\Omega_1} h_1 \operatorname{div} u_1 \leqslant \|h_1\|_{L_{\mathcal{A}'(\Omega_1)}} \|\Delta t \operatorname{div} u_1\|_{L_{\mathcal{A}(\Omega_1)}},$$

where $L_{\mathcal{A}(\Omega_1)}$ is the Orlicz space defined by the N-function $\mathcal{A}(t) = \exp(t^2) - 1$ and $L_{\mathcal{A}'(\Omega_1)}$ its dual defined by a N-function $\mathcal{A}'(t)$ equivalent to $t\sqrt{\log^+(t)}$. Since $\Delta t^{1+\alpha} \|\operatorname{div} u_1\|_{L^\infty(\Omega_1)}^2 \leqslant K D_0$, then if Δt is small enough

$$\int_{\Omega_1} \mathcal{A}(\Delta t \operatorname{div} u_1) = 1 + \Delta t^2 \|\operatorname{div} u_1\|_{L^2(\Omega_1)}^2 + \tau(\Delta t^{2-2\alpha}) - 1.$$

Thus since

$$\|h_1\|_{L_{\mathcal{A}'(\Omega_1)}} \leqslant \|h_1\|_{L^1(\Omega_1)}^{\frac{1}{2}} \left(\int_{\Omega_1} h_1 \log^+ h_1 \right)^{\frac{1}{2}} \leqslant \|h_1\|_{L^1(\Omega_1)} + \frac{1}{4} \int_{\Omega_1} h_1 \log^+ h_1$$

$$\leqslant \|h_1\|_{L^1(\Omega_1)} + \frac{1}{4} \|h_1\|_{L^2(\Omega_1)}^2,$$

then

$$\Delta t \int_{\Omega_1} h_1 \operatorname{div} u_1 \leqslant C_1 \Delta t^2 \|u_1\|_{H^3(\Omega_1)}^2 \|h_1\|_{L^2(\Omega_1)}^{\frac{3}{2}} + C_2 \Delta t^{2-2\alpha} \|h_1\|_{L^1(\Omega_1)} + C_3 \Delta t^{2-2\alpha} \|h_1\|_{L^2(\Omega_1)}^2. \qquad (23)$$

Thus, from (19) and (20)–(23), and by adding $\mu \Delta t \|u_1\|_{L^2(\Omega_1)}^2$, we deduce easily the following inequality

$$\frac{1}{2} \|u_2\|_{L^2(\Omega_2)}^2 + \frac{1}{2} \|u_1 - \tilde{u}_0\|_{L^2(\Omega_1)}^2 + \frac{1}{2} \|u_2 - \tilde{u}_1\|_{L^2(\Omega_1)}^2 + \Delta t \left(\mu - \frac{C_{GN}}{2} \|u_1\|_{L^2(\Omega_1)} \right) \|u_1\|_{H^1(\Omega_1)}^2$$

$$+ \mu \Delta t \|Du_2\|_{L^2(\Omega_2)}^2 + \|h_2\|_{L^1(\Omega_2)} + \Delta t (\delta - C_3 \Delta t^{1-2\alpha}) \|h_1\|_{L^2(\Omega_1)}^2 + \delta \Delta t \|h_2\|_{L^2(\Omega_2)}^2$$

$$+ \frac{1}{2} \|A^{\frac{1}{2}}(u_1 \circ \Gamma_1 - u_0 \circ \Gamma_0)\|_{L^2(\gamma_0)}^2 + \frac{1}{2} \|A^{\frac{1}{2}}(u_2 \circ \Gamma_2 - u_1 \circ \Gamma_1)\|_{L^2(\gamma_0)}^2 + \Delta t^{1+\alpha} A_1 \|u_1\|_{H^3(\Omega_1)}^2$$

$$+ \Delta t^{1+\alpha} \|u_2\|_{H^3(\Omega_2)}^2 + \frac{1}{2} \|A^{\frac{1}{2}}(u_2 \circ \Gamma_2)\|_{L^2(\gamma_0)}^2 \leqslant D_0 + 2\mu \Delta t D_0 + C_2 \Delta t^{1-2\alpha} \Delta t D_0 = M_2,$$

where

$$A_i = 1 - \Delta t^{1-\alpha} C_1 \|h_i\|_{L^2(\Omega_i)}^{\frac{3}{2}} - \Delta t^{1-\alpha} K^2 \|h_i\|_{L^1(\Omega_i)} - \Delta t^{1-\alpha} \frac{K^2}{2} \|u_i\|_{L^2(\Omega_i)}^2$$

$$- \Delta t^{1-\alpha} K^2 \|A^{\frac{1}{2}} u_i\|_{L^2(\gamma_i)}^2, \quad \text{with } i = 1, \dots, m-1.$$

Since we have the condition (18), $\mu - C_{GN}/2 \|u_1\|_{L^2(\Omega_1)} > 0$. Moreover for Δt small enough $\delta - C_3 \Delta t^{1-2\alpha} > 0$, $A_1 > 0$ and the term on the left hand side is positive. Notice also that the properties induced by the operator $\Delta t^\alpha B$ allow us to show that $\det J_2 > 0$.

Estimates for all k

In the same way, we obtain:

$$
\frac{1}{2}\|u_k\|^2_{L^2(\Omega_k)} + \frac{1}{2}\sum_{i=1}^{k}\|u_i - \tilde{u}_{i-1}\|^2_{L^2(\Omega_k)} + \sum_{i=1}^{k-1}\left(\mu - \frac{C_{GN}}{2}\|u_i\|_{L^2(\Omega_i)}\right)\Delta t\|u_i\|^2_{H^1(\Omega_i)}
$$

$$
+ \mu\Delta t\|Du_k\|^2_{L^2(\Omega_k)} + \|h_k\|_{L^1(\Omega_k)} + (\delta - C_3\Delta t^{1-2\alpha})\sum_{i=1}^{k-1}\Delta t\|h_i\|^2_{L^2(\Omega_i)} + \delta\Delta t\|h_k\|^2_{L^2(\Omega_k)}
$$

$$
+ \frac{1}{2}\sum_{i=1}^{k}\left\|A^{\frac{1}{2}}(u_i \circ \Gamma_i - u_{i-1} \circ \Gamma_{i-1})\right\|^2_{L^2(\gamma_0)} + \sum_{i=1}^{k-1}A_i\Delta t^{1+\alpha}\|u_i\|^2_{H^3(\Omega_i)}
$$

$$
+ \Delta t^{1+\alpha}\|u_k\|^2_{H^3(\Omega_k)} + \frac{1}{2}\left\|A^{\frac{1}{2}}(u_k \circ \Gamma_k)\right\|^2_{L^2(\gamma_0)} \leqslant M_k. \tag{24}
$$

Moreover, as for $k = 2$, the terms A_i and $\mu - C_{GN}/2\|u_i\|_{L^2(\Omega_i)}$ are positive. Thus for all $k \in \{1, 2, \ldots, m\}$, we obtain the announced estimates.

Remark 6. From the discretized continuity equation $(\mathcal{P}^\delta_k)_1$ and the previous Eq. (24) we have

$$
\sum_{i=1}^{m}\int_{\Omega_k}\left|\log\left(\frac{h_i}{h_{i-1}}\right)\right| \leqslant \mathbb{C}.
$$

Remark 7. From classical results on Sobolev's spaces we deduce

$$
\forall x \in \Omega_k, \ \forall y \in \partial\Omega_k, \quad \left|\Delta t^{\frac{1+\alpha}{2}}u_k(x) - \Delta t^{\frac{1+\alpha}{2}}u_k(y)\right| \leqslant \Delta t^{\frac{1+\alpha}{2}}\|u_k\|_{W^{1,\infty}(\Omega_k)}|x - y|.
$$

However, inequality (24) shows that $\Delta^{1+\alpha}\|u_k\|^2_{H^3(\Omega_k)}$ is bounded independently of k. In dimension 2, $H^3(\Omega_k) \hookrightarrow W^{1,\infty}(\Omega_k)$, thus

$$
\forall x \in \Omega_k, \ \forall y \in \partial\Omega_k, \quad \left|\Delta t u_k(x) - \Delta t u_k(y)\right| \leqslant \Delta t^{\frac{1-\alpha}{2}}\mathbb{C}|x - y|.
$$

If we choose Δt such that $\Delta^{(1-\alpha)/2}\mathbb{C} \leqslant 1$, this inequality shows that the distance variation from a point to one on the boundary between two consecutive time steps is lower than the same distance at the previous time step. Thus there are no point leaving the domain from a time step to the next one. □

5. Passage to the limit on Δt

Here, we show that the approximate solutions have the necessary compactness to pass to the limit in the time-discretized domain and inside the equations.

5.1. In the time-discretized domain

We introduce the following notations for all $k \in \{0, \ldots, m-1\}$, $t \in [k\Delta t, (k+1)\Delta t[$

$$
\hat{d}(X, t) = d_k(X), \quad \text{for all } t \in \left[k\Delta t, (k+1)\Delta t\right[\tag{25}
$$

and

$$
\tilde{d}(X, t) = d_k(X) + u_k\big(X + d_k(X)\big)(t - k\Delta t) \quad \text{for all } t \in \left[k\Delta t, (k+1)\Delta t\right[. \tag{26}
$$

We have

$$
\|d_1\|_{H^2(\gamma_0)} \leqslant \Delta t\|u_0 \circ \Gamma_0\|_{H^2(\gamma_0)},
$$

and

$$\|d_k\|_{H^2(\gamma_0)} \leqslant \|d_{k-1}\|_{H^2(\gamma_0)} + \Delta t \|u_{k-1} \circ \Gamma_{k-1}\|_{H^2(\gamma_0)},$$

then

$$\|d_k\|_{H^2(\gamma_0)} \leqslant \sum_{i=0}^{k-1} \Delta t \|u_i \circ \Gamma_i\|_{H^2(\gamma_0)}.$$

Thus, from Lemma 5, we deduce that

$$\sup_{0 \leqslant k \leqslant m} \|d_k\|_{H^2(\gamma_0)} \leqslant \sqrt{2K_\alpha} T \leqslant \frac{2\mu T}{C_{GN}}.$$

Then we obtain

$$\|\hat{d}\|_{L^\infty(H^2(\gamma_0))} \leqslant \sqrt{2K_\alpha} T \leqslant \frac{2\mu T}{C_{GN}},$$

$$\|\tilde{d}\|_{L^\infty(H^2(\gamma_0))} \leqslant \sup_{0 \leqslant k \leqslant m} \|d_k\|_{H^2(\gamma_0)} + \Delta t \sup_{0 \leqslant k \leqslant m} \|u_k \circ \Gamma_k\|_{H^2(\gamma_0)}$$

$$\leqslant \sqrt{2K_\alpha} T + \Delta t \sqrt{2K_\alpha} \leqslant \frac{2\mu T}{C_{GN}} + \Delta t \frac{2\mu}{C_{GN}}$$

and since $\frac{\partial \tilde{d}}{\partial t}(X, t) = u_k(X + d_k(X))$ for all $t \in]k\Delta t, (k+1)\Delta t[$,

$$\left\|\frac{\partial \tilde{d}}{\partial t}\right\|_{L^\infty(H^2(\gamma_0))} \leqslant \sqrt{2K_\alpha} \leqslant \frac{2\mu}{C_{GN}}.$$

Thus, we deduce that $\tilde{d}$ is bounded in $W^{1,\infty}(0, T; H^2(\gamma_0))$. So, there exists $d \in W^{1,\infty}(0, T; H^2(\gamma_0)) \subset W^{1,\infty}(0, T; C^1(\gamma_0))$ such that

$$\tilde{d} \xrightarrow{\Delta t \to 0} d \quad \text{in } W^{1,\infty}\big(0, T; H^2(\gamma_0)\big) \text{ weak star.}$$

Moreover since the embedding of $W^{1,\infty}(0, T; H^2(\gamma_0))$ into $C^{0,\alpha}(0, T; C^1(\gamma_0))$, $0 < \alpha < 1$, is compact

$$\tilde{d} \xrightarrow{\Delta t \to 0} d \quad \text{in } C^{0,\alpha}\big(0, T; C^1(\gamma_0)\big) \text{ strong.}$$

Moreover, since $\|d\|_{W^{1,\infty}(0,T;W^{1,\infty}(\gamma_0))} \leqslant \liminf_{\Delta t \to 0} \|\tilde{d}\|_{W^{1,\infty}(0,T;H^2(\gamma_0))} \leqslant (1 + T)\sqrt{2K_\alpha}$, we deduce that $\det \mathcal{J}(X, t) \neq 0$, $X \in \gamma_0$, $t \in [0, T]$, for small data. Moreover, from Lemma 5, we have

$$\|\tilde{d} - \hat{d}\|_{H^2(\gamma_0)} \leqslant \Delta t \sup_k \|u_k \circ \Gamma_k\|_{H^2(\gamma_0)} \leqslant \mathbb{C}\Delta t$$

thus

$$\tilde{d} - \hat{d} \xrightarrow{\Delta t \to 0} 0 \quad \text{in } L^\infty\big(0, T; C^1(\gamma_0)\big) \text{ strong,}$$

then we have $\widehat{Q}_{\Delta t} \xrightarrow{\Delta t \to 0} Q$.

5.2. In the time-discretized problem

In this section, we give some elements for the passage to the limit in the time-discretized equations. We introduce the following notations, for all $k \in \{1, \dots, m-1\}$, $t \in [k\Delta t, (k+1)\Delta t[$

$$\hat{x}(t) = x_k,$$

$$\hat{u}\big(\hat{x}(t), t\big) = u(x_k, t_k = k\Delta t) = u_k(x_k), \qquad \hat{h}\big(\hat{x}(t), t\big) = h(x_k, t_k = k\Delta t) = h_k(x_k),$$

$$\tilde{u}\big(\hat{x}(t), t\big) = \hat{u}\big(\hat{x}(t - \Delta t), t - \Delta t\big) + \frac{\hat{u}(\hat{x}(t), t) - \hat{u}(\hat{x}(t - \Delta t), t - \Delta t)}{\Delta t}(t - k\Delta t),$$

$$\tilde{h}\big(\hat{x}(t), t\big) = \hat{h}\big(\hat{x}(t - \Delta t), t - \Delta t\big) + \frac{\hat{h}(\hat{x}(t), t) - \hat{h}(\hat{x}(t - \Delta t), t - \Delta t)}{\Delta t}(t - k\Delta t).$$

In $\widehat{Q}_{\Delta t}$, the time-discretized solutions verify the problem

$$(\widetilde{\mathcal{P}}) \quad \begin{cases} \frac{\partial \tilde{u}(\hat{x}(t),t)}{\partial t} - \mu \Delta \hat{u}(\hat{x}(t),t) + \Delta t^{\alpha} B \hat{u}(\hat{x}(t),t) + \nabla \hat{h}(\hat{x}(t),t) = 0, \\ \frac{\partial \tilde{h}(\hat{x}(t),t)}{\partial t} + \hat{h}(\hat{x}(t),t) \operatorname{div} \hat{u}(\hat{x}(t),t) + \delta \hat{h}(\hat{x}(t),t) = 0, \end{cases}$$

with the boundary conditions

$$\begin{aligned} &\hat{\sigma}\big(X + \hat{d}(X,t),t\big)\hat{n}\big(X + \hat{d}(X,t),t\big)|\det \widehat{\mathcal{J}}|(X,t) \\ &\quad + \Delta t^{\alpha} \operatorname{Tr}(B\hat{u})\big(X + \hat{d}(X,t),t\big) = -A\big(\partial \widetilde{U}(X,t)/\partial t\big) \quad \text{on } \gamma_0, \end{aligned} \tag{27}$$

where $\widetilde{U}(X,t) = \tilde{u}(X + \hat{d}(X,t),t)$. We set $\Omega_{t,\Delta t} = \widehat{Q}_{\Delta t} \cap \{(x,t); x \in \mathbb{R}^2\}$. The compactness results obtained in Lemma 5 allow to deduce that

$$\big\|\hat{u}\big(\hat{x}(t),t\big)\big\|_{L^{\infty}(L^2(\Omega_{t,\Delta t}))} \leqslant \sup_{0 \leqslant i \leqslant k} \|u_i\|_{L^2(\Omega_i)} \leqslant 2\frac{\mu}{C_{GN}}$$

and

$$\big\|\hat{u}\big(\hat{x}(t),t\big)\big\|^2_{L^2(H^1(\Omega_{t,\Delta t}))} = \int\limits_{\Delta t}^{T} \big\|\hat{u}\big(\hat{x}(t),t\big)\big\|^2_{H^1(\Omega_{t,\Delta t})} = \sum_{k=1}^{m} \Delta t \|u_k\|^2_{H^1(\Omega_k)} \leqslant \mathbb{C}' \Delta t.$$

Then, there exists $u \in L^{\infty}(0,T;L^2(\Omega_t)) \cap L^2(0,T;H^1(\Omega_t))$ such that

$$\begin{cases} \hat{u} \to u & \text{in } L^{\infty}(0,T;L^2(\Omega_t)) \text{ weak star}, \\ \hat{u} \to u & \text{in } L^2(0,T;H^1(\Omega_t)) \text{ weak}. \end{cases}$$

In addition

$$\int\limits_{\Delta t}^{T} \big\|\hat{u}\big(\hat{x}(t),t\big) - \hat{u}\big(\hat{x}(t-\Delta t),t-\Delta t\big)\big\|^2_{L^2(\Omega_{t,\Delta t})} = C \sum_{k=1}^{m} \Delta t \|u_k - u_{k-1}\|^2_{L^2(\Omega_k)} \leqslant \mathbb{C}\Delta t.$$

So we deduce that $\hat{u}(\hat{x}(t),t) - \hat{u}(\hat{x}(t-\Delta t),t-\Delta t) \xrightarrow{\Delta t \to 0} 0$ in $L^2(Q)$ strong. Moreover, according to the estimates on u_k, we show that

$$\big\|\tilde{u}\big(\hat{x}(t),t\big)\big\|_{L^2(Q_{\Delta t})} \leqslant \mathbb{C}$$

and

$$\big\|\tilde{u}\big(\hat{x}(t),t\big) - \hat{u}\big(\hat{x}(t-\Delta t),t-\Delta t\big)\big\|^2_{L^2(Q_{\Delta t})} \leqslant \sum_{k=1}^{m} \Delta t \|u_k - u_{k-1}\|^2_{L^2(\Omega_k)} \leqslant \mathbb{C}\Delta t.$$

We thus deduce the following convergence results:

$$\begin{cases} \tilde{u} - \hat{u} \xrightarrow{\Delta t \to 0} 0 & \text{in } L^2(Q) \text{ strong}, \\ \tilde{u} \xrightarrow{\Delta t \to 0} u & \text{in } L^2(Q) \text{ weak}. \end{cases} \tag{28}$$

In the same way there exists $h \in L^2(Q)$ such that

$$\hat{h} \xrightarrow{\Delta t \to 0} h \quad \text{in } L^2(0,T;L^2(\Omega_t)) \text{ weak}. \tag{29}$$

Moreover from Remark 6 we deduce

$$\left(\int\limits_{\Delta t}^{T} \int\limits_{\Omega_{t,\Delta t}} \left|\log\left(\frac{\hat{h}(\hat{x}(t),t)}{\hat{h}(\hat{x}(t-\Delta t),t-\Delta t)}\right)\right|\right) = \Delta t \sum_{i=1}^{m-1} \int\limits_{\Omega_k} \left|\log\left(\frac{h_i}{h_{i-1}}\right)\right| \leqslant \mathbb{C}\Delta t$$

thus

$$\log\left(\frac{\hat{h}(\hat{x}(t),t)}{\hat{h}(\hat{x}(t-\Delta t),t-\Delta t)}\right) \xrightarrow{\Delta t \to 0} 0 \quad \text{in } L^1(Q) \text{ strong}.$$

As $\hat{h}$ is bounded in $L^2(Q)$, we deduce from this strong convergence that

$$\hat{h}\big(\hat{x}(t), t\big) - \hat{h}\big(\hat{x}(t - \Delta t), t - \Delta t\big) \xrightarrow{\Delta t \to 0} 0 \quad \text{in } L^p(Q) \text{ strong, } p < 2,$$

and so consequently that

$$\tilde{h} - \hat{h} \xrightarrow{\Delta t \to 0} 0 \quad \text{in } L^p(Q) \text{ strong, } p < 2. \tag{30}$$

We show now that $\partial \tilde{u}/\partial t \xrightarrow{\Delta t \to 0} \partial u/\partial t + u \cdot \nabla u$ in the sense of distributions. Since $\widehat{Q}_{\Delta t} \xrightarrow{\Delta t \to 0} Q$, for all $\phi \in D(Q)$, there exists $\underline{\Delta t}$ such that $\forall \Delta t \leqslant \underline{\Delta t}$, supp $\phi \Subset \widehat{Q}_{\Delta t}$. Considering a time step $\Delta t \leqslant \underline{\Delta t}$, we multiply $\partial \tilde{u}/\partial t$ by $\phi \in D(Q)$, we obtain

$$\int\limits_{\widehat{Q}_{\Delta t}} \frac{\hat{u}(\hat{x}(t), t) - \hat{u}(\hat{x}(t - \Delta t), t - \Delta t)}{\Delta t} \cdot \phi\big(\hat{x}(t), t\big)$$

$$= \int\limits_{\widehat{Q}_{\Delta t}} \frac{\hat{u}(\hat{x}(t), t)}{\Delta t} \cdot \phi\big(\hat{x}(t), t\big) - \int\limits_{\widehat{Q}_{\Delta t}} \frac{\hat{u}(\hat{x}(t - \Delta t), t - \Delta t)}{\Delta t} \cdot \phi\big(\hat{x}(t), t\big). \tag{31}$$

In the last term we introduce the following variable change: $t^* = t - \Delta t$. We notice J the Jacobian matrix associated to this variable change and $\det J$ its determinant

$$\det J = 1 + \Delta t \operatorname{div} \hat{u}\big(\hat{x}(t^*), t^*\big) + \Delta t^2 \big(D\hat{u}\big(\hat{x}(t^*), t^*\big)\big)^2$$

where $(D\zeta)^2 = \frac{\partial \zeta_1}{\partial x_1} \frac{\partial \zeta_2}{\partial x_2} - \frac{\partial \zeta_1}{\partial x_2} \frac{\partial \zeta_2}{\partial x_1}$. This determinant is strictly positive and bounded if Δt is small enough according to the compactness of Lemma 5. Thus we obtain, using that $\int_{\widehat{Q}_{\Delta t}} = \int_{\Delta t}^T \int_{\Omega_{t, \Delta t}}$,

$$(31) = \int\limits_{\widehat{Q}_{\Delta t}} \frac{\hat{u}(\hat{x}(t), t)}{\Delta t} \cdot \phi\big(\hat{x}(t), t\big) - \int\limits_{\Delta t}^{T - \Delta t} \int\limits_{\Omega_{t^*, \Delta t}} \frac{\hat{u}(\hat{x}(t^*), t^*)}{\Delta t} \cdot \phi\big(\hat{x}(t^* + \Delta t), t^* + \Delta t\big) \det J$$

$$= \int\limits_{\widehat{Q}_{\Delta t}} \frac{\hat{u}(\hat{x}(t), t)}{\Delta t} \cdot \phi\big(\hat{x}(t), t\big) - \int\limits_{\Delta t}^{T - \Delta t} \int\limits_{\Omega_{t^*, \Delta t}} \frac{\hat{u}(\hat{x}(t^*), t^*)}{\Delta t} \cdot \phi\big(\hat{x}(t^* + \Delta t), t^* + \Delta t\big)$$

$$- \int\limits_{\Delta t}^{T - \Delta t} \int\limits_{\Omega_{t^*, \Delta t}} \operatorname{div} \hat{u}\big(\hat{x}(t^*), t^*\big) \hat{u}\big(\hat{x}(t^*), t^*\big) \cdot \phi\big(\hat{x}(t^* + \Delta t), t^* + \Delta t\big)$$

$$- \Delta t \int\limits_{\Delta t}^{T - \Delta t} \int\limits_{\Omega_{t^*, \Delta t}} \big|D\hat{u}\big(\hat{x}(t^*), t^*\big)\big|^2 \hat{u}\big(\hat{x}(t^*), t^*\big) \cdot \phi\big(\hat{x}(t^* + \Delta t), t^* + \Delta t\big).$$

Using that $\int_{\Delta t}^{T - \Delta t} \cdot = \int_{\Delta t}^T \cdot - \int_{T - \Delta t}^T \cdot$, we find

$$(31) = \int\limits_{\widehat{Q}_{\Delta t}} \hat{u}\big(\hat{x}(t), t\big) \cdot \frac{\phi(\hat{x}(t), t) - \phi(\hat{x}(t + \Delta t), t + \Delta t)}{\Delta t}$$

$$+ \underbrace{\int\limits_{T - \Delta t}^T \int\limits_{\Omega_{t, \Delta t}} \frac{\hat{u}(\hat{x}(t), t)}{\Delta t} \cdot \phi\big(\hat{x}(t + \Delta t), t + \Delta t\big)}_{\xrightarrow{\Delta t \to 0} 0 \text{ see Remark (8)}}$$

$$- \int\limits_{\Delta t}^{T - \Delta t} \int\limits_{\Omega_{t, \Delta t}} \operatorname{div} \hat{u}\big(\hat{x}(t), t\big) \hat{u}\big(\hat{x}(t), t\big) \cdot \phi\big(\hat{x}(t + \Delta t), t + \Delta t\big)$$

$$- \Delta t \int\limits_{0}^{T-\Delta t} \int\limits_{\Omega_{t,\Delta t}} \big|D\hat{u}\big(\hat{x}(t),t\big)\big|^2 \hat{u}\big(\hat{x}(t),t\big) \cdot \phi\big(\hat{x}(t+\Delta t),t+\Delta t\big).$$

$$\underbrace{}_{\xrightarrow{\Delta t \to 0} \ 0 \ \text{see Remark (9)}}$$

To simplify we do not make appear the terms going to 0 as Δt goes to 0^+, so (31) can be written under the form

$$(31) = \int\limits_{\widehat{Q}_{\Delta t}} \hat{u}\big(\hat{x}(t),t\big) \cdot \frac{\phi(\hat{x}(t),t) - \phi(\hat{x}(t),t+\Delta t)}{\Delta t}$$

$$+ \int\limits_{\widehat{Q}_{\Delta t}} \hat{u}\big(\hat{x}(t),t\big) \cdot \frac{\phi(\hat{x}(t),t+\Delta t) - \phi(\hat{x}(t+\Delta t),t+\Delta t)}{\Delta t}$$

$$- \int\limits_{\Delta t}^{T-\Delta t} \int\limits_{\Omega_{t,\Delta t}} \operatorname{div} u\big(\hat{x}(t),t\big)\hat{u}\big(\hat{x}(t),t\big) \cdot \phi\big(\hat{x}(t+\Delta t),t+\Delta t\big)$$

$$= - \int\limits_{\widehat{Q}_{\Delta t}} \hat{u}\big(\hat{x}(t),t\big) \cdot \frac{\phi(\hat{x}(t),t+\Delta t) - \phi(\hat{x}(t),t)}{\Delta t}$$

$$- \int\limits_{\widehat{Q}_{\Delta t}} \hat{u}\big(\hat{x}(t),t\big) \cdot \frac{\phi(\hat{x}(t)+\Delta t\hat{u}(\hat{x}(t),t),t+\Delta t) - \phi(\hat{x}(t),t+\Delta t)}{\Delta t}$$

$$- \int\limits_{\Delta t}^{T-\Delta t} \int\limits_{\Omega_{t,\Delta t}} \operatorname{div} \hat{u}\big(\hat{x}(t),t\big)\hat{u}\big(\hat{x}(t),t\big) \cdot \phi\big(\hat{x}(t+\Delta t),t+\Delta t\big).$$

We pass to the limit on Δt in each term $\forall \phi \in D(Q)$

$$\lim_{\Delta t \to 0} \int\limits_{\widehat{Q}_{\Delta t}} \hat{u}\big(\hat{x}(t),t\big) \cdot \frac{\phi(\hat{x}(t),t+\Delta t) - \phi(\hat{x}(t),t)}{\Delta t} = \left\langle u\big(x(t),t\big), \frac{\partial \phi(x(t),t)}{\partial t} \right\rangle_{D'(Q),D(Q)},$$

$$\lim_{\Delta t \to 0} \int\limits_{\widehat{Q}_{\Delta t}} \hat{u}\big(\hat{x}(t),t\big) \cdot \frac{\phi(\hat{x}(t)+\Delta t\hat{u}(\hat{x}(t),t),t+\Delta t) - \phi(\hat{x}(t),t+\Delta t)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \sum_{i=1}^{2} \int\limits_{\widehat{Q}_{\Delta t}} \hat{u}_i\big(\hat{x}(t),t\big) \frac{\phi_i(\hat{x}(t)+\Delta t\hat{u}(\hat{x}(t),t),t+\Delta t) - \phi_i(\hat{x}(t),t+\Delta t)}{\Delta t}$$

$$= \sum_{i=1}^{2} \sum_{j=1}^{2} \left\langle u_i\big(x(t),t\big)u_j\big(x(t),t\big), \frac{\partial \phi_i(x(t),t)}{\partial x_j} \right\rangle_{D'(Q),D(Q)}$$

and

$$\lim_{\Delta t \to 0} \int\limits_{\Delta t}^{T-\Delta t} \int\limits_{\Omega_{t,\Delta t}} \operatorname{div} \hat{u}\big(\hat{x}(t),t\big)\hat{u}\big(\hat{x}(t),t\big) \cdot \phi\big(\hat{x}(t+\Delta t),t+\Delta t\big) = \big\langle u\big(x(t),t\big) \operatorname{div} u\big(x(t),t\big), \phi\big(x(t),t\big)\big\rangle_{D'(Q),D(Q)}.$$

Finally we obtain

$$\lim_{\Delta t \to 0} \int\limits_{\widehat{Q}_{\Delta t}} \frac{\hat{u}(\hat{x}(t),t) - \hat{u}(\hat{x}(t-\Delta t),t-\Delta t)}{\Delta t} \cdot \phi\big(\hat{x}(t),t\big)$$

$$
\begin{aligned}
&= -\left\langle u\big(x(t),t\big), \frac{\partial \phi(x(t),t)}{\partial t}\right\rangle_{D'(Q),D(Q)} \\
&\quad - \sum_{i=1}^{2}\sum_{j=1}^{2}\left\langle u_i\big(x(t),t\big)u_j\big(x(t),t\big), \frac{\partial \phi_i(x(t),t)}{\partial x_j}\right\rangle_{D'(Q),D(Q)} \\
&\quad - \left\langle u\big(x(t),t\big)\operatorname{div} u\big(x(t),t\big), \phi\big(x(t),t\big)\right\rangle_{D'(Q),D(Q)} \\
&= \left\langle \frac{\partial u(x(t),t)}{\partial t}, \phi\big(x(t),t\big)\right\rangle_{D'(Q),D(Q)} \\
&\quad + \left\langle \big(u\big(x(t),t\big).\nabla u\big(x(t),t\big)\big), \phi\big(x(t),t\big)\right\rangle_{D'(Q),D(Q)} \\
&\quad + \left\langle u\big(x(t),t\big)\operatorname{div} u\big(x(t),t\big), \phi\big(x(t),t\big)\right\rangle_{D'(Q),D(Q)} \\
&\quad - \left\langle u\big(x(t),t\big)\operatorname{div} u\big(x(t),t\big), \phi\big(x(t),t\big)\right\rangle_{D'(Q),D(Q)},
\end{aligned}
\tag{32}
$$

thus

$$
\lim_{\Delta t \to 0} \int_{\widehat{Q}_{\Delta t}} \frac{\partial \tilde{u}(\hat{x}(t),t)}{\partial t}\cdot \phi\big(\hat{x}(t),t\big) = \left\langle \frac{\partial u(x(t),t)}{\partial t} + \big(u\big(x(t),t\big)\cdot \nabla\big)u\big(x(t),t\big), \phi\big(x(t),t\big)\right\rangle_{D'(Q),D(Q)}.
\tag{33}
$$

Remark 8. According to the definition of the support of ϕ, there exists $\widetilde{\Delta t}$ such that

$$
\left[\bigcup_{t\in[T-\Delta t,T]} \Omega_{t,\Delta t}\times\{t\}\right] \cap \operatorname{supp}\phi = \varnothing, \quad \text{for all } \Delta t \leqslant \widetilde{\Delta t}.
$$

Remark 9. We have

$$
\Delta t \int_{\Delta t}^{T-\Delta t}\int_{\Omega_{t,\Delta t}} \big|\big|D\hat{u}\big(\hat{x}(t),t\big)\big|^2\hat{u}\big(\hat{x}(t),t\big)\cdot\phi\big(\hat{x}(t),t\big)\big|
$$

$$
\leqslant 2\Delta t \sup_{k}\|u_k\|_{L^2(\Omega_k)}\left(\sum_{k=1}^{m-1}\Delta t\|u_k\|_{H^3(\Omega_k)}^2\right)\sup_{k}\|\phi\|_{L^\infty(\Omega_k)},
$$

according to estimates obtained in Lemma 5

$$
\Delta t \int_{0}^{T-\Delta t}\int_{\Omega_{t,\Delta t}} \big|\big|D\hat{u}\big(\hat{x}(t),t\big)\big|^2\hat{u}\big(\hat{x}(t),t\big)\cdot\phi\big(\hat{x}(t),t\big)\big| \leqslant \mathbb{C}'\Delta t^{1-\alpha}
$$

thus since $0 < \alpha < 1$

$$
\lim_{\Delta t \to 0}\Delta t \int_{\Delta t}^{T-\Delta t}\int_{\Omega_{t,\Delta t}} \big|D\hat{u}\big(\hat{x}(t),t\big)\big|^2\hat{u}\big(\hat{x}(t),t\big)\cdot\phi\big(\hat{x}(t),t\big) = 0.
$$

We use the same method to pass to the limit in

$$
\int_{\widehat{Q}_{\Delta t}} \frac{\partial \log\tilde{h}(\hat{x}(t),t)}{\partial t}\phi\big(\hat{x}(t),t\big),
$$

we obtain

$$
\left\langle \frac{\partial \log h(x(t),t)}{\partial t} + u\big(x(t),t\big)\cdot\nabla \log h\big(x(t),t\big), \phi\big(x(t),t\big)\right\rangle_{D'(Q),D(Q)}.
\tag{34}
$$

Finally, in the sense of distributions, when Δt goes to 0^+, $(\widetilde{\mathcal{P}})$ leads to $(\mathcal{P}^\delta)$.

5.3. Passage to the limit in the boundary conditions

To obtain estimates of the first section and pass to the limit in the regularized problem $(\mathcal{P}^\delta)$, we have to show the boundary condition (2). To do this, we formulate the time-discretized problem under the following variational form

$$
\int_{\widehat{Q}_{\Delta t}} \frac{\partial \widetilde{u}}{\partial t} \cdot \phi + \int_{\widehat{Q}_{\Delta t}} \nabla \hat{u} \cdot \nabla \phi - \int_{\widehat{Q}_{\Delta t}} \hat{h} \operatorname{div} \phi + \Delta t^\alpha \int_{\widehat{Q}_{\Delta t}} B^{\frac{1}{2}}(\hat{u}) \cdot B^{\frac{1}{2}}(\phi)
$$
$$
+ \left\langle \frac{\partial}{\partial t} A^{1/2}(\widetilde{U}), A^{1/2}(\Phi) \right\rangle_{D'(0,T;L^2(\gamma_0))} = 0, \quad \forall \phi \in D\big(0, T; C^\infty(\mathbb{R}^2)\big), \tag{35}
$$

where $\Phi(X, t) = \phi(X + \hat{d}(X, t), t)$. The relation (32) is still valid if we take $\phi \in D(0, T; C^\infty(\mathbb{R}^2))$, but we cannot at this point apply directly the Green formula and write the relation (33) for $\phi \in D(0, T; C^\infty(\mathbb{R}^2))$. Thus, by noticing that $\hat{d} \xrightarrow{\Delta t \to 0} d$ in $L^\infty(C^1)$ and $\widetilde{U}$ is bounded in $L^\infty(H^2(\gamma_0))$, we only deduce that, at the limit,

$$
\int_Q u \cdot \frac{\partial \phi}{\partial t} - \sum_{i=1}^2 \sum_{j=1}^2 \int_Q u_i u_j \frac{\partial \phi_i}{\partial x_j} - \int_Q \operatorname{div} u u \cdot \phi + \int_Q \nabla u \cdot \nabla \phi - \int_Q h \operatorname{div} \phi
$$
$$
+ \left\langle \frac{\partial}{\partial t} A^{1/2}(U), A^{1/2}(\Phi) \right\rangle_{D'(0,T;L^2(\gamma_0))} = 0, \quad \forall \phi \in D\big(0, T; C^\infty(\mathbb{R}^2)\big). \tag{36}
$$

Moreover, in the previous section we have shown that

$$
\frac{\partial u}{\partial t} + u \nabla u - \Delta u + \nabla h = 0 \quad \text{in } D'(Q).
$$

Since $u \nabla u \in L^{4/3}(Q)$, then $\frac{\partial u}{\partial t} - \Delta u + \nabla h \in L^{4/3}(Q)$. So considering a function $\phi \in D(0, T; C^\infty(\Omega_t))$ we have

$$
\int_Q \left(\frac{\partial u}{\partial t} - \Delta u + \nabla h \right) \cdot \phi + \int_Q \big((u \cdot \nabla) u \big) \cdot \phi = 0. \tag{37}
$$

To apply the Green formula, we introduce $\Theta_1 = (-\frac{\partial u_1}{\partial x_1} + h, 0, u_1)$ and $\Theta_2 = (0, -\frac{\partial u_2}{\partial x_2} + h, u_2)$. Thus, (37) can be formulated under the form

$$
\sum_{i=1}^2 \int_Q \overline{\operatorname{div}} \Theta_i \phi_i + \int_Q \big((u \cdot \nabla) u \big) \cdot \phi = 0,
$$

where $\overline{\operatorname{div}}$ represents the divergence operator in Q. Then we can apply the Green formula in Q and we obtain

$$
\int_Q u \cdot \frac{\partial \phi}{\partial t} + \int_Q \nabla u \cdot \nabla \phi + \sum_{i=1}^2 \int_\Sigma u_i \phi_i N_t - \sum_{i=1}^2 \int_\Sigma \phi_i \nabla u_i \cdot N_x + \sum_{i=1}^2 \int_\Sigma \phi_i h_i N_{x_i}
$$
$$
- \sum_{i=1}^2 \sum_{j=1}^2 \int_Q u_i u_j \frac{\partial \phi_i}{\partial x_j} - \int_Q \operatorname{div} u u \cdot \phi + \sum_{i=1}^2 \int_\Sigma u_i \phi_i u \cdot N_x = 0, \tag{38}
$$

where $N = (N_{x_1}, N_{x_2}, N_t)$ is the unitary outward normal of Q. So, noticing that $u \cdot N_x = -N_t$ and $\int_\Sigma \psi N_{x_i} = \int_0^T \int_{\gamma_t} \psi n_i$, we obtain the relation (2) in $W^{-1,\infty}(0, T; H^{-2}(\gamma_0))$ by combining (36) and (38).

6. Concluding remark

This survey follows a series of papers [9–12] dealing with fluid structure interaction problems in which a thin structure (plate or shell) surrounds a domain occupied by a compressible fluid. In these papers, we give existence results in which the proofs are based on approximate solutions constructed by using a fixed point method which

connects the fluid problem and the structure equation. Numerically, the method proposed in the present survey allows to avoid the use of such a fixed point and consequently to decrease the computational time. Notice also the essential role of the boundary operator *A* since it ensures some physical properties and the regularity of the boundary, which is necessary to pass to the limit inside the equations.

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