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Erratum to: "The Schrödinger–Maxwell system with Dirac mass" [Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (5) (2007) 773–793] ☆

G.M. Coclite [a,b], H. Holden [a,c,*]

[a] *Centre of Mathematics for Applications, University of Oslo, P.O. Box 1053, Blindern, No-0316 Oslo, Norway*
[b] *Department of Mathematics, University of Bari, Via E. Orabona 4, I-70125 Bari, Italy*
[c] *Department of Mathematical Sciences, Norwegian University of Science and Technology, No-7491 Trondheim, Norway*

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Abstract

We correct the proof of [G.M. Coclite, H. Holden, The Schrödinger–Maxwell system with Dirac mass, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (5) (2007) 773–793, Lemma 4.1].

MSC: primary 35D05, 35Q40; secondary 35J65

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1. Introduction

The proof of [1, Lemma 4.1] is incorrect. We here present a new proof. We employ the notation and assumptions of [1].

Lemma 4.1. *Assume α and β are positive constants. There exists a subsequence $\{v_{n_k}\}_{k\in\mathbb{N}}$ and a map $v \in H^2(\Omega) \cap H_0^1(\Omega)$ such that*

$$v_{n_k} \rightharpoonup v \quad \text{weakly in } H^2(\Omega) \cap H_0^1(\Omega). \tag{1}$$

In particular,

$$v_{n_k} \longrightarrow v \quad \text{uniformly in } \Omega. \tag{2}$$

Proof. We split the proof in two steps.

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* Corresponding author.
E-mail addresses: coclitegm@dm.uniba.it (G.M. Coclite), holden@math.ntnu.no (H. Holden).
URLs: http://www.dm.uniba.it/Members/coclitegm, http://www.math.ntnu.no/~holden/.

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Step 1. We claim that the following inequality

$$J_n(v_n)=\min_{v\in H_0^1(\Omega)} J_n(v)\leqslant 0 \tag{3}$$

holds for infinitely many $n\in\mathbb{N}$.

Let φ_0 be the normalized positive first eigenfunction of $-\Delta$ on Ω, namely φ_0 is the unique smooth map satisfying the following conditions

$$\begin{cases} -\Delta\varphi_0=\omega_0\varphi_0, & \text{in } \Omega,\\ \varphi_0>0, & \text{in } \Omega,\\ \varphi_0=0, & \text{on } \partial\Omega,\\ \|\varphi_0\|_{L^2(\Omega)}=1. \end{cases} \tag{4}$$

Since

$$\int_\Omega \left|\nabla\varphi_0(x)\right|^2 \mathrm{d}x=\omega_0\int_\Omega \varphi_0^2(x)\,\mathrm{d}x=\omega_0,$$

we find, by evaluating J_n in[1] $\lambda\,\mathrm{sign}(v_{n-1}(x_0))\varphi_0$, $\lambda>0$, that (writing $\nu_{n-1}=\mathrm{sign}(v_{n-1}(x_0))$)

$$\begin{aligned} J_n(\lambda\nu_{n-1}\varphi_0)={}&\frac{\lambda^2}{2}\int_\Omega |\nabla\varphi_0|^2\,\mathrm{d}x+\lambda^4\frac{\alpha}{4}\int_{\Omega\times\Omega} G(x,y)\varphi_0^2(y)\varphi_0^2(x)\,\mathrm{d}x\,\mathrm{d}y\\ &-\left|v_{n-1}(x_0)\right|\lambda^3\frac{\alpha}{\beta}\int_{\Omega\times\Omega} G(x,y)G(y,x_0)\varphi_0(y)\varphi_0^2(x)\,\mathrm{d}x\,\mathrm{d}y\\ &+v_{n-1}^2(x_0)\lambda^2\frac{\alpha}{\beta^2}\int_{\Omega\times\Omega} G(x,y)G(y,x_0)G(x,x_0)\varphi_0(y)\varphi_0(x)\,\mathrm{d}x\,\mathrm{d}y\\ &+v_{n-1}^2(x_0)\lambda^2\frac{\alpha}{2\beta^2}\int_{\Omega\times\Omega} G(x,y)G^2(y,x_0)\varphi_0^2(x)\,\mathrm{d}x\,\mathrm{d}y\\ &-\left|v_{n-1}^3(x_0)\right|\lambda\frac{\alpha}{\beta^3}\int_{\Omega\times\Omega} G(x,y)G^2(y,x_0)G(x,x_0)\varphi_0(x)\,\mathrm{d}x\,\mathrm{d}y\\ &-\frac{\omega}{2}\lambda^2\int_\Omega \varphi_0^2(x)\,\mathrm{d}x-\left|v_{n-1}(x_0)\right|\lambda\frac{\omega}{\beta}\int_\Omega G(x,x_0)\varphi_0(x)\,\mathrm{d}x\\ ={}&\lambda^2\frac{\omega_0-\omega}{2}+\lambda^4\kappa_1-\left|v_{n-1}(x_0)\right|\lambda^3\kappa_2+v_{n-1}^2(x_0)\lambda^2\kappa_3-\left|v_{n-1}^3(x_0)\right|\lambda\kappa_4-\left|v_{n-1}(x_0)\right|\lambda\kappa_5. \end{aligned}$$

Due to the positivity and the boundedness of φ_0, we have $\kappa_1,\dots,\kappa_5>0$, hence

$$J_n(\lambda\nu_{n-1}\varphi_0)\leqslant\lambda^2\frac{\omega_0-\omega}{2}+\lambda^4\kappa_1+v_{n-1}^2(x_0)\lambda^2\kappa_3-\left|v_{n-1}(x_0)\right|\lambda\kappa_5. \tag{5}$$

We have the following two cases.

(i) If

$$\liminf_n\left|v_{n-1}(x_0)\right|=0,$$

then there exists an n_0 such that, by passing to a subsequence and using, e.g., [1, (2.6)], we find

$$J_n(v_n)=\min_{v\in H_0^1(\Omega)} J_n(v)\leqslant\min_{\lambda>0} J_n(\lambda\nu_{n-1}\varphi_0)\leqslant\min_{\lambda>0}\left(\lambda^2\frac{\omega_0-\omega}{2}+\lambda^4\kappa_1\right)<0,\quad n>n_0. \tag{6}$$

[1] The point x_0 is where the point interaction is located, cf. [1].

(ii) If

$$0 < \liminf_n \left| v_{n-1}(x_0) \right| \leqslant \infty,$$

then there exists n_0 and $c_1 \in (0, |v_{n-1}(x_0)|)$ for $n > n_0$ such that

$$\begin{aligned} J_n(v_n) &= \min_{v \in H_0^1(\Omega)} J_n(v) \leqslant \min_{\lambda > 0} J_n(\lambda v_{n-1} \varphi_0) \\ &\leqslant \min_{\lambda > 0} \left(\lambda^2 \frac{\omega_0 - \omega}{2} + \lambda^4 \kappa_1 + \left| v_{n-1}(x_0) \right|^2 \lambda^2 \kappa_3 - c_1 \lambda \kappa_5 \right) < 0, \quad n > n_0. \end{aligned} \tag{7}$$

Clearly, (6) and (7) prove (3). So *Step* 1 is concluded.

Step 2. We prove (1) and (2). Clearly it suffices to prove that

$$\text{the sequence } \{v_n\}_{n \in \mathbb{N}} \text{ is bounded in } H^2(\Omega) \cap H_0^1(\Omega). \tag{8}$$

Due to [1, Lemmas 2.1 and 3.5] we only have to show that

$$\text{the sequence } \{v_n\}_{n \in \mathbb{N}} \text{ is bounded in } H_0^1(\Omega), \tag{9}$$

$$\text{the sequence } \left\{v_n(x_0)\right\}_{n \in \mathbb{N}} \text{ is bounded in } \mathbb{R}. \tag{10}$$

If, by contradiction, (9) does not hold, we have that

$$\limsup_n \|v_n\|_{H_0^1(\Omega)} = \infty. \tag{11}$$

Therefore, [1, Lemma 3.3] and a diagonal argument guarantee

$$\limsup_n J_n(v_n) = \infty. \tag{12}$$

Since, by construction, v_n is a minimizer for J_n, Eq. (3) says

$$J_n(v_n) = \min_{f \in H_0^1(\Omega)} J_n(f) \leqslant 0, \quad n \in \mathbb{N}, \tag{13}$$

which contradicts (12). This proves (9).

We conclude by proving (10). Assume by contradiction that $\{v_n(x_0)\}_{n \in \mathbb{N}}$ is not bounded, namely (passing to a subsequence)

$$\lim_n \left| v_n(x_0) \right| = \infty. \tag{14}$$

Multiplying [1, (2.21)] by u_n, which is defined by [1, (2.20)], and integrating over Ω we get

$$\int_\Omega \left| \nabla v_n(x) \right|^2 \mathrm{d}x + \frac{v_{n-1}(x_0)}{\beta} \int_\Omega G(x, x_0) \Delta v_n(x)\, \mathrm{d}x + \alpha \int_{\Omega \times \Omega} G(x, y) u_n^2(y) u_n^2(x)\, \mathrm{d}x\, \mathrm{d}y = \omega \int_\Omega u_n^2(x)\, \mathrm{d}x.$$

Integration by parts gives

$$-\frac{v_{n-1}(x_0)}{\beta} \int_\Omega G(x, x_0) \Delta v_n(x)\, \mathrm{d}x = -\frac{v_{n-1}(x_0)}{\beta} \int_\Omega v_n(x) \Delta G(x, x_0)\, \mathrm{d}x = \frac{v_{n-1}(x_0) v_n(x_0)}{\beta},$$

thus

$$\int_\Omega \left| \nabla v_n(x) \right|^2 \mathrm{d}x + \alpha \int_{\Omega \times \Omega} G(x, y) u_n^2(y) u_n^2(x)\, \mathrm{d}x\, \mathrm{d}y = \omega \int_\Omega u_n^2(x)\, \mathrm{d}x + \frac{v_n(x_0) v_{n-1}(x_0)}{\beta}. \tag{15}$$

Introducing the notation

$$a_n = \alpha \int_{\Omega \times \Omega} G(x, y) \left(\frac{v_n(y)}{v_{n-1}(x_0)} - \frac{G(y, x_0)}{\beta} \right)^2 \left(\frac{v_n(x)}{v_{n-1}(x_0)} - \frac{G(x, x_0)}{\beta} \right)^2 \mathrm{d}x\, \mathrm{d}y,$$

$$b_n = \omega \int_\Omega \left(\frac{v_n(x)}{v_{n-1}(x_0)} - \frac{G(x, x_0)}{\beta} \right)^2 \mathrm{d}x,$$

Eq. (15) reads (cf. [1, (2.20)])

$$\int_{\Omega} |\nabla v_n|^2 \,\mathrm{d}x + v_{n-1}^4(x_0) a_n = v_{n-1}^2(x_0) b_n + \frac{v_n(x_0) v_{n-1}(x_0)}{\beta}. \tag{16}$$

Due to (9) and (14)

$$a_n \to \alpha \int_{\Omega\times\Omega} G(x,y) \frac{G^2(y,x_0)}{\beta^2} \frac{G^2(x,x_0)}{\beta^2} \,\mathrm{d}x\,\mathrm{d}y, \qquad b_n \to \omega \int_{\Omega} \frac{G^2(x,x_0)}{\beta^2} \,\mathrm{d}x. \tag{17}$$

Passing to a subsequence we can assume that the sequence $\{v_n(x_0)/v_{n-1}^3(x_0)\}$ has a limit as $n \to \infty$. We have the following three cases.

(i) If

$$\lim_n \frac{v_n(x_0)}{v_{n-1}^3(x_0)} = \infty, \tag{18}$$

we divide (16) by $v_{n-1}^4(x_0)$

$$\int_{\Omega} \left| \frac{\nabla v_n}{v_{n-1}^2(x_0)} \right|^2 \mathrm{d}x + a_n = \frac{b_n}{v_{n-1}^2(x_0)} + \frac{v_n(x_0)}{v_{n-1}^3(x_0)\beta}. \tag{19}$$

Using (9), (14), and (17) in (19), we have

$$\alpha \int_{\Omega\times\Omega} G(x,y) \frac{G^2(y,x_0)}{\beta^2} \frac{G^2(x,x_0)}{\beta^2} \,\mathrm{d}x\,\mathrm{d}y = \infty,$$

which is a contradiction.

(ii) If

$$\lim_n \frac{v_n(x_0)}{v_{n-1}^3(x_0)} = 0, \tag{20}$$

we divide (16) by $v_{n-1}(x_0) v_n(x_0)$

$$\int_{\Omega} \frac{|\nabla v_n|^2}{v_{n-1}(x_0) v_n(x_0)} \,\mathrm{d}x + \frac{a_n v_{n-1}^3(x_0) - b_n v_{n-1}(x_0)}{v_n(x_0)} = \frac{1}{\beta}. \tag{21}$$

Since by (9), (14), and (17),

$$\lim_n \frac{a_n v_{n-1}^3(x_0) - b_n v_{n-1}(x_0)}{v_n(x_0)} = \lim_n \frac{v_{n-1}^3(x_0)}{v_n(x_0)} \left(a_n - \frac{b_n}{v_{n-1}^2(x_0)} \right) = \infty,$$

which implies, using (21), that $\infty = \frac{1}{\beta}$, which is a contradiction.

(iii) Finally, if

$$\lim_n \frac{v_n(x_0)}{v_{n-1}^3(x_0)} = \ell \in (0,\infty),$$

we observe that (see (14))

$$\lim_n \frac{v_{n+1}(x_0)}{v_{n-1}^3(x_0)} = \lim_n \frac{v_{n+1}(x_0)}{v_n^3(x_0)} \frac{v_n(x_0)}{v_{n-1}^3(x_0)} v_n^2(x_0) = \ell^2 \lim_n v_n^2(x_0) = \infty,$$

therefore the subsequence $\{v_{2n}(x_0)\}_{n\in\mathbb{N}}$ satisfies (18) and we get a contradiction.

Therefore (10) is proved. □

References

[1] G.M. Coclite, H. Holden, The Schrödinger–Maxwell system with Dirac mass, Ann. Inst. H. Poincaré Anal. Non Linéaire 24 (5) (2007) 773–793.