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Distributional Jacobian equal to $\mathcal{H}^1$ measure ☆

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Abstract

Let $1 \leqslant p < 2$. We construct a Hölder continuous $W^{1,p}$ mapping of a square into $\mathbb{R}^2$ such that the distributional Jacobian equals to one-dimensional Hausdorff measure on a line segment.

1. Introduction

In this paper we construct a planar example of a mapping whose distributional Jacobian is a singular measure uniformly distributed on a segment.

Let $\Omega \subset \mathbb{R}^n$ be open and let $f = (f^1, \dots, f^n) \in W^{1,p}(\Omega, \mathbb{R}^n)$. The Jacobian determinant

$$J_f(x) = \det \nabla f(x) = \det\big(\nabla f^1, \dots, \nabla f^n\big)$$

is basic tool in analysis of such a map useful in particular for a change of variables. The natural domain of definition for many integral identities is the space $W^{1,n}$ in $\mathbb{R}^n$ as this assumption easily implies that the Jacobian is integrable. However, in many applications the assumption $W^{1,n}$ is too strong.

Starting from the seminal works of C.B. Morrey [12], Yu. Reshetnyak [17] and J. Ball [1] we know that under weaker assumptions we can define the distributional Jacobian

$$\mathcal{J}_f(\varphi) = -\int_\Omega f^1(x) \det\big(\nabla\varphi, \nabla f^2, \dots, \nabla f^n\big)(x)\, dx$$

for $\varphi \in C_c^\infty(\Omega)$. It is easy to see that this distribution is well-defined for $f \in (W^{1,n-1} \cap L^\infty)(\Omega, \mathbb{R}^n)$ or we can use the Sobolev embedding theorem to show that it is well-defined also for $f \in W^{1,\frac{n^2}{n+1}}(\Omega, \mathbb{R}^n)$.

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On the other hand one can prove using integration by parts and interchangeability of second derivatives that for smooth function f we have

$$\mathcal{J}_f(\varphi) = \int_\Omega J_f(x)\varphi(x)\,dx \tag{1.1}$$

and by the approximation we can extend this for $f \in W^{1,n}$. The equality between the Jacobian and the distributional Jacobian was studied and extended to other situations, e.g. when the Jacobian is integrable (S. Müller [13]), or to fine scales of integrability of the gradient (T. Iwaniec and C. Sbordone [9], L. Greco [5], P. Koskela and X. Zhong [11]). For other studies see e.g. the works of V. Šverák [18], S. Müller, T. Qi and B.S. Yan [15], R.L. Jerrard and H.M. Soner [10], C. De Lellis [3], and C. De Lellis and F. Ghiraldin [4]. Finding new inspiration in Nonlinear Elasticity and other problems of Calculus of Variations, the distributional Jacobian is nowadays the basic tool in the development of Geometric Function Theory. For an overview of the field, discussion of interdisciplinary links and further references we recommend the monographs T. Iwaniec and G. Martin [8] and S. Hencl and P. Koskela [7].

To give some geometric intuition on the subject, already in 1993 S. Müller [14] started to study the examples of mappings with singular support of the distributional Jacobian. Using some ideas of the construction of S. Ponomarev [16] he showed that it is possible to construct a continuous mapping $f \in W^{1,p}([0,1]^n, \mathbb{R}^n)$, for every $1 \leqslant p < n$, whose distributional Jacobian is supported on a closed subset of Hausdorff dimension α, $\alpha \in (0, n)$. He also studied examples of mappings with distributional Jacobian somehow supported on the one-dimension line segment. However, he was not able to construct a mapping whose distributional Jacobian was a measure and its support essentially sits on the $(n-1)$-dimensional hyperplane and he conjectured that it is not possible.

Recently it was shown by H. Brezis and H.M. Nguyen [2, Section 2.3] that it is possible to construct a discontinuous mapping $g \in W^{1,p}((-1,1)^n, \mathbb{R}^n)$, $p < n$, whose distributional Jacobian equals to

$$\mathcal{J}_g = \sum_{n=1}^{\infty} \frac{1}{2^n}(\delta_{P_n} - \delta_{D_n}),$$

where δ denotes the Dirac measure and the segment $[P_n, D_n]$ is perpendicular to $H := (-1,1)^{n-1} \times \{0\}$, $|P_n - D_n| \to 0$ and the centers of $[P_n, D_n]$ are dense in H. This shows that the support of $\mathcal{J}_g$ essentially contains H and the claim of S. Müller as formally stated in [14] is not true. However they point out that S. Müller's conjecture still might be true if one assumes that g is in addition continuous. In Remark 9 they also ask if there is an example such that the singular part of $\mathcal{J}_g$ restricted to H is truly $(n-1)$-dimensional, say absolutely continuous with respect to $\mathcal{H}^{n-1}$.

The aim of this paper is to show that this is indeed possible and we construct the following example. In this example, $\Omega = \{x \in \mathbb{R}^2\colon |x_1| + |x_2| < 1\}$.

Theorem 1.1. *Let $1 \leqslant p < 2$ and $\beta < \frac{1}{2}$. Then there exists a β-Hölder continuous mapping $f : \overline{\Omega} \to \mathbb{R}^2$ such that $f \in W^{1,p}(\Omega, \mathbb{R}^2)$, $J_f = 0$ a.e. in Ω, but the distributional Jacobian $\mathcal{J}_f$ equals the one-dimensional Hausdorff measure on the line segment $[-1,1] \times \{0\}$, i.e.*

$$\mathcal{J}_f(\varphi) = -\int_\Omega f^1(x) \det\bigl(\nabla\varphi, \nabla f^2\bigr)(x)\,dx = \int_{-1}^{1} \varphi(t,0)\,dt \quad \textit{for every } \varphi \in C_c^\infty(\Omega).$$

Our construction is purely two-dimensional. We do not see any obstruction for the existence of a similar example in higher dimension in $W^{1,p}$, $n-1 \leqslant p < n$. However, it seems to be much more difficult to find some efficient pattern, so that the existence of such a construction is only conjectured.

Of course, we can define on $\Omega^* := \Omega \times [-1,1]^{n-2}$ the mapping

$$f\bigl([x_1, \dots, x_n]\bigr) = \bigl[f_1(x_1,x_2), f_2(x_1,x_2), x_3, \dots, x_n\bigr]$$

with f_1 and f_2 as in the previous theorem and we obtain a Hölder continuous mapping such that $f \in W^{1,p}$, $p < 2$, and

$$-\int_{\Omega^*} f^1(x)\det(\nabla\varphi, \nabla f^2, \dots, \nabla f^n)(x)\,dx = \int_{(-1,1)^{n-1}} \varphi(x_1, 0, x_3, \dots, x_n)\,d\mathcal{L}^{n-1} \tag{1.2}$$

for every $\varphi \in C_c^\infty(\Omega)$. However, this example is not satisfactory as p is too small and (1.2) does not allow us to interpret the expression on the left as the action of the distributional Jacobian. Namely, the genuine distributional Jacobian should satisfy

$$\mathcal{J}(\varphi) = -\int_{\Omega^*} f^i(x)\det(\nabla f^1, \dots, \nabla f^{i-1}, \nabla\varphi, \nabla f^{i+1}, \dots, \nabla f^n)(x)\,dx$$

for each $i = 1, \dots, n$, which is not the case for the above example.

Let us also note that it is possible to construct even a homeomorphism in $W^{1,p}$, $p < n$, whose distributional Jacobian is a (purely) singular measure with respect to the Lebesgue measure [6]. Our example of the mapping $f : \Omega \to \mathbb{R}^2$ as stated in Theorem 1.1 is the uniform limit of a sequence of Lipschitz mappings $f_k : \overline{\Omega} \to \mathbb{R}^2$ with the properties that $f_k \to f$ in $W^{1,p}$, $p < 2$. Our construction has a self-similar structure obtained by iteration. The leading pattern is visible already in the first step of the construction. Therefore, it is important for us to understand f_1 first.

2. Construction of f_1

Recall that $\Omega = \{x \in \mathbb{R}^2\colon\ |x_1| + |x_2| < 1\}$ and denote $A = \overline{\Omega}$.

First we define a linear map

$$S = \begin{pmatrix} \frac{1}{9} & 0 \\ 0 & \frac{\alpha}{9} \end{pmatrix} \tag{2.1}$$

with $\alpha \in (0, 1)$ to be determined later, and points

$$\mathbf{a}_i = \frac{2i-11}{9}\mathbf{e}_1, \quad i = 1, \dots, 10, \qquad \mathbf{b}_i = \frac{2i-10}{9}\mathbf{e}_1 + \frac{\alpha}{9}\mathbf{e}_2, \quad i = 1, \dots, 9,$$
$$\mathbf{c} = \frac{\mathbf{e}_2 - 2\mathbf{e}_1}{3}, \qquad \mathbf{d} = \frac{\mathbf{e}_2 + 2\mathbf{e}_1}{3}.$$

Then we consider the affine maps

$$g_i(x) = S(x + \mathbf{e}_1) + \mathbf{a}_i, \quad i = 1, \dots, 9.$$

To construct f_1 we divide A into ten parts: nine inner rhombuses

$$A_i = g_i(A), \quad i = 1, \dots, 9,$$

and the remaining part $A_0 = A \setminus \bigcup_i A_i$. (See Fig. 1.) Now, we define the group of rotations by right angles

$$\mathcal{R} = \{R^k,\ k \in \mathbb{Z}\}, \quad \text{where } R = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

We set

$$h_i(x) = \frac{1}{3}R^{p_i}(x + \mathbf{e}_1) + \mathbf{a}'_i, \quad i = 1, \dots, 9,$$

where

$$\begin{aligned}
&p_1 = 0, && \mathbf{a}'_1 = -\mathbf{e}_1, && p_2 = 1, && \mathbf{a}'_2 = -\frac{1}{3}\mathbf{e}_1,\\
&p_3 = 0, && \mathbf{a}'_3 = \frac{-\mathbf{e}_1 + 2\mathbf{e}_2}{3}, && p_4 = 3, && \mathbf{a}'_4 = \frac{\mathbf{e}_1 + 2\mathbf{e}_2}{3},\\
&p_5 = 2, && \mathbf{a}'_5 = \frac{1}{3}\mathbf{e}_1, && p_6 = 3, && \mathbf{a}'_6 = -\tfrac{1}{3}\mathbf{e}_1,\\
&p_7 = 0, && \mathbf{a}'_7 = \frac{-\mathbf{e}_1 - 2\mathbf{e}_2}{3}, && p_8 = 1, && \mathbf{a}'_8 = \frac{\mathbf{e}_1 - 2\mathbf{e}_2}{3},\\
&p_9 = 0, && \mathbf{a}'_9 = \frac{1}{3}\mathbf{e}_1.
\end{aligned}$$

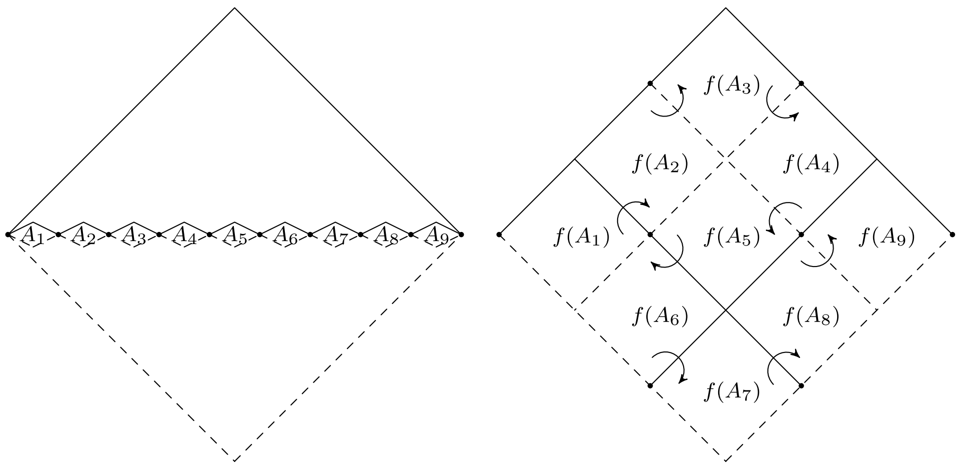

Fig. 1. Images of $A_1, \dots, A_9$. The round arrows show the direction of mapping of the inner rhombuses.

We are ready to define f_1 on the sets A_i, namely

$$f_1 = h_i \circ g_i^{-1} \quad \text{on } A_i, \quad i = 1, \dots, 9. \tag{2.2}$$

If we set

$$f_1(x) = x, \quad x \in \partial\Omega, \tag{2.3}$$

we have defined f_1 on the whole boundary of A_0. Now, our aim is to extend f_1 to A_0 to make f_1 piecewise affine and rank-one thereon. The triangulation we use is the following:

$$(\mathbf{a}_1, \mathbf{b}_1, \mathbf{c}),\ (\mathbf{b}_1, \mathbf{b}_2, \mathbf{c}),\ (\mathbf{b}_3, \mathbf{e}_2, \mathbf{c}),\ (\mathbf{b}_3, \mathbf{b}_4, \mathbf{e}_2),\ (\mathbf{b}_4, \mathbf{d}, \mathbf{e}_2),\ (\mathbf{b}_4, \mathbf{b}_5, \mathbf{d}),\ (\mathbf{b}_5, \mathbf{b}_6, \mathbf{d}),\ (\mathbf{b}_6, \mathbf{b}_7, \mathbf{d}),$$
$$(\mathbf{b}_7, \mathbf{b}_8, \mathbf{d}),\ (\mathbf{b}_8, \mathbf{b}_9, \mathbf{d}),\ (\mathbf{b}_9, \mathbf{a}_{10}, \mathbf{d}),$$

and also

$$(\mathbf{b}_i, \mathbf{a}_{i+1}, \mathbf{b}_{i+1}), \quad i = 1, \dots, 8.$$

It is easy to see (consult Fig. 2) that if we extend f_1 from vertices of selected triangles (where it is already defined) to the triangles as affine maps, then the extension will be compatible with previously defined values on parts of boundaries of these triangles. Moreover, this extension to the triangles is always rank-one since the vertices of each triangle are mapped to colinear points see e.g. $f(\mathbf{b}_2)$, $f(\mathbf{a}_3)$, $f(\mathbf{b}_3)$; in fact, in most cases two vertices of the triangle are mapped to the same point, e.g. $f(\mathbf{b}_1) = f(\mathbf{c})$, $f(\mathbf{b}_3) = f(\mathbf{e}_2)$. We have already defined f_1 on $A_0 \cap \{x_2 > 0\}$ and extend it as an odd function to the whole A_0. (Note that on $A_1 \cup \dots \cup A_9$ it has been already defined as an odd function.)

3. Construction of the sequence $\{f_k\}$

Inside each rhomboid $A_1, \dots, A_9$ we find nine smaller rhomboids and we continue in similar pattern. If $s \in \{1, \dots, 9\}^k$ is a multiindex, we define

$$A_s = g_{s_1} \circ \dots \circ g_{s_k}(A).$$

Denote

$$A^k = \bigcup_{s \in \{1,\dots,9\}^k} A_s.$$

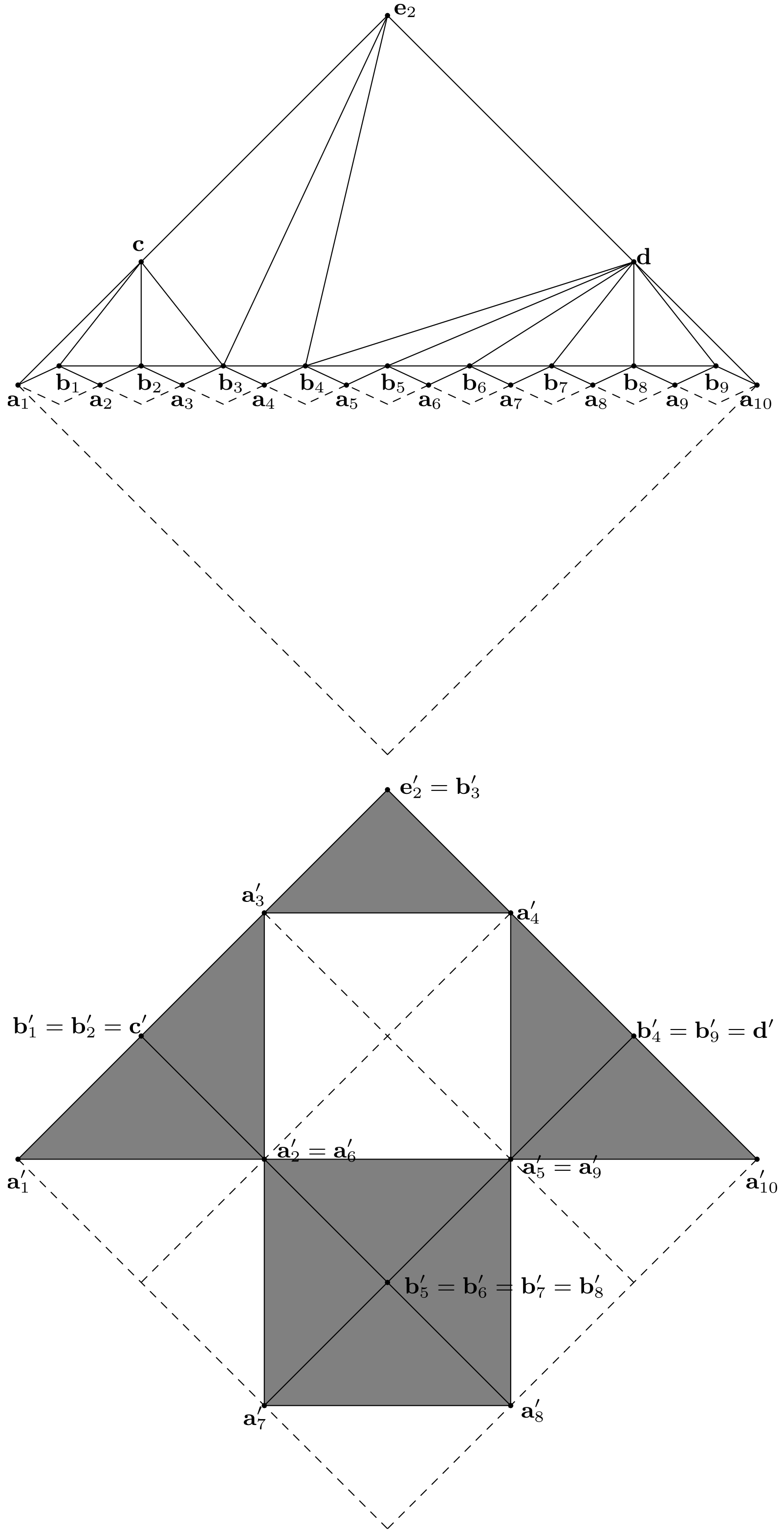

Fig. 2. Above: The partition of the domain. Below: How the vertices are mapped into the target space. The shaded part is the image of the part above the x_1-axis.

We construct f_k by induction. If f_{k-1} is already constructed, we define

$$f_k(x)=\begin{cases} f_{k-1}(x), & x\in A\setminus A^{k-1},\\ h_{s_1}\circ\cdots\circ h_{s_{k-1}}\circ f_1\circ g_{s_{k-1}}^{-1}\circ\cdots\circ g_{s_1}^{-1}(x), & x\in A_s,\ s\in\{1,\dots,9\}^{k-1}.\end{cases}$$

By iteration of identities (2.2) and (2.3) we check that

$$f_k(x)=h_{s_1}\circ\cdots\circ h_{s_k}\circ g_{s_k}^{-1}\circ\cdots\circ g_{s_1}^{-1}(x),\quad x\in A_s,\ s\in\{1,\dots,9\}^k$$

and that f_k is continuous on A.

4. Passage to the limit and estimates

We define our function f as $f=\lim_{k\to\infty} f_k$. This is allowed by the following lemma.

Lemma 4.1. *The sequence f_k converges uniformly to a continuous function.*

Proof. Choose $x\in A$. Let us estimate $|f_k(x)-f_{k-1}(x)|$, $k\geqslant 2$. The estimate is trivial if $x\notin A^{k-1}$ as $f_k(x)=f_{k-1}(x)$ in this case. If $x\in A^{k-1}$, then there exists $s\in\{1,\dots,9\}^{k-1}$ such that $x\in A_s$. Set $A'_s=f_{k-1}(A_s)$. Then also $A'_s=f_k(A_s)$. Since $\operatorname{diam} A'_s=2\cdot 3^{1-k}$, we have

$$\left|f_k(x)-f_{k-1}(x)\right|\leqslant 2\cdot 3^{1-k},\quad x\in A. \tag{4.1}$$

This is enough to deduce the uniform convergence. As the functions f_k are continuous, the limit is also continuous. □

4.1. Derivative and Lipschitz estimates

Notice that f_1 is piecewise affine and there exists a finite set $\mathcal{T}\subset\mathbb{R}^{2\times 2}$ of rank-one matrices such that $\nabla f_1\in\mathcal{A}$ a.e. in A_0.

Lemma 4.2. *Let $k\in\mathbb{N}$. Then the function f_k is piecewise affine. If $x\in A^{k-1}$ and f_k is differentiable at x, then there exist $R\in\mathcal{R}$ and $T\in\mathcal{T}$ (needed if $x\notin A^k$) such that*

$$\nabla f_k(x)=\begin{cases} 3^{-k+1}RTS^{-k+1}, & \text{if } x\in A^{k-1}\setminus A^k,\\ 3^{-k}RS^{-k}, & \text{if } x\in A^k.\end{cases}$$

In particular, $J_{f_k}=9^k/\alpha^k$ on A^k, $J_{f_k}=0$ on $A^{k-1}\setminus A^k$ and

$$|\nabla f_k|\leqslant C\frac{3^k}{\alpha^k}\quad \text{on } A \tag{4.2}$$

(which controls the Lipschitz constant of f_k).

Proof. The computation of derivative is straightforward once we know that $\nabla g_i^{-1}=S^{-1}$, $i=1,\dots,9$ and $3^k\nabla h_i\in\mathcal{R}$, $i=1,\dots,9$. We use that $\mathcal{R}$ is closed under multiplication. The computation of Jacobian follows by the product rule for determinants as $\det R=1$ for $R\in\mathcal{R}$ and $\det S=3^{-4}\alpha$. The gradient estimate is proved by induction. On $A\setminus A^{k-1}$ we have even a better estimate by the induction hypothesis, as $f_k=f_{k-1}$ there, on A^{k-1} we use that $|\nabla S^{-1}|\leqslant 9/\alpha$. □

4.2. Hölder estimates

Lemma 4.3. *Let $0<\beta<\frac{1}{2}$ and*

$$3^{2\beta-1}\leqslant\alpha<1. \tag{4.3}$$

Then there exists C such that

$$\left|f(x')-f(x)\right|\leqslant C\left|x-x'\right|^\beta,\quad x,x'\in A.$$

Proof. Choose $x, x' \in A$. It is enough to consider the case $|x - x'| < 1/9$. We find $j \in \mathbb{N}$ such that $9^{-j-1} \leqslant |x - x'| < 9^{-j}$. We distinguish three cases.

Case 1. If $x \in A_s$, $x' \in A_{s'}$ with $s, s' \in \{1, \dots, 9\}^j$, then the inequality $|x - x'| \leqslant 9^{-j}$ implies that either $s = s'$ or at least $A_s \cap A_{s'} \neq \emptyset$, namely A_s and $A_{s'}$ are "neighbors". Since $\operatorname{diam} f(A_s) = 2 \cdot 3^{-j}$, we have

$$\left|f(x) - f\left(x'\right)\right| \leqslant 4 \cdot 3^{-j} \leqslant 12 \cdot 3^{-j-1} \leqslant 12\left|x - x'\right|^{1/2} \leqslant 12\left|x - x'\right|^{\beta}.$$

Case 2. If $x, x' \notin A^j$, then we use the equality $f = f_j$ on $A \setminus A^j$ and Lipschitz estimate (4.2) of f_j on A and deduce that

$$\left|f(x) - f\left(x'\right)\right| \leqslant C\left(\frac{3}{\alpha}\right)^j \left|x - x'\right| \leqslant C 9^{(1-\beta)j}\left|x - x'\right| \leqslant C\left|x - x'\right|^{\beta-1}\left|x - x'\right| = C\left|x - x'\right|^{\beta}.$$

Case 3. If $x \in A^j$, $x' \in A \setminus A^j$, then we find $x'' \in \partial A^j$ on the line segment connecting x and x'. Then we apply Case 1 to the pair (x, x'') and Case 2 to the pair (x'', x'). □

4.3. Sobolev estimates

Lemma 4.4. *The sequence f_k converges to f in $W^{1,p}(\Omega)$ if*

$$3^{\frac{p-2}{p-1}} < \alpha < 1. \tag{4.4}$$

Proof. The strategy of the proof is the following: we prove that $\|f_k - f_{k-1}\|_{1,p}$ is estimated by a geometric series, in particular, $\{f_k\}$ is a fundamental sequence in $W^{1,p}(\Omega, \mathbb{R}^n)$. By completeness, there exists a limit $\tilde{f}$ of f_k in $W^{1,p}(\Omega, \mathbb{R}^n)$. However, since we know that $f_k \to f$ uniformly, it follows that $f = \tilde{f}$.

If $s \in \{1, \dots, 9\}^k$, then

$$|A_s| = (\det S)^k |A| = \left(\frac{\alpha}{9^2}\right)^k |A| = 2\frac{\alpha^k}{9^{2k}}. \tag{4.5}$$

The number of multiindices $s \in \{1, \dots, 9\}^k$ is 9^k, so that

$$\left|A^k\right| = 2\frac{\alpha^k}{9^k}. \tag{4.6}$$

Since $|\nabla f_k| \leqslant C(\frac{3}{\alpha})^k$ a.e. (by Lemma 4.2; in fact, up to a multiplicative constant we may use the same estimate for ∇f_k and ∇f_{k-1}) and $f_k = f_{k-1}$ on $A \setminus A_{k-1}$, we have (with a generic constant C which change at each occurrence)

$$\begin{aligned}\int_A |\nabla f_{k-1} - \nabla f_k|^p &\leqslant 2^{p-1}\left(\int_{A_{k-1}} |\nabla f_{k-1}|^p + \int_{A_{k-1}} |\nabla f_k|^p\right)\\ &\leqslant C\frac{\alpha^{k-1}}{9^{k-1}}\left(\frac{3}{\alpha}\right)^{kp} \leqslant C\left(3^{p-2}\alpha^{1-p}\right)^k.\end{aligned}$$

By (4.4), $3^{p-2}\alpha^{1-p} < 1$, so that we have obtained the desired estimate of the first order derivatives. Of course, the estimate of $\|f_{j-1} - f_j\|_{L^p}$ causes no difficulty because of (4.1). □

5. Distributional Jacobian

Now, we are ready to prove our main theorem.

Proof of Theorem 1.1. We use f as constructed in the preceding sections. Given $1 < p < 2$ and $0 < \beta < \frac{1}{2}$, we find $\alpha < 1$ such that both conditions (4.3) and (4.4) are satisfied. Then f is β-Hölder continuous on A and belongs to $W^{1,p}(\Omega, \mathbb{R}^n)$. By the definition of f_k and Lemma 4.2 we have

$$J_f = J_{f_k} = 0 \quad \text{on } A \setminus A^{k-1},\ k = 2, 3, \dots.$$

Since, by (4.6), $|A_k| \to 0$, we obtain that $J_f = 0$ a.e.

Denote

$$\ell = [-1, 1] \times \{0\}.$$

Choose $\varphi \in C_c^\infty(\Omega)$. For each $k \in \mathbb{N}$ and $s \in \{1, \dots, 9\}^k$ select a point $\mathbf{a}_s \in A_s$, for example

$$\mathbf{a}_s = g_{s_1} \circ \cdots \circ g_{s_k}(-\mathbf{e}_1) = \left(-1 + 2\sum_{j=1}^{k} 9^{-j}(s_j - 1)\right)\mathbf{e}_1 \in A_s,$$

and consider the line segment

$$\ell_s = \ell \cap A_s.$$

Then

$$\mathcal{H}^1(\ell_s) = 2 \cdot 9^{-k} = \left|f(A_s)\right|.$$

We estimate

$$\begin{aligned}
&\left|\int_{A_s} J_{f_k}(x)\varphi(x)\,dx - \int_{\ell_s} \varphi(t)\,d\mathcal{H}^1(t)\right| \\
&\quad\leqslant \left|\int_{A_s} J_{f_k}(x)\varphi(x)\,dx - 2\cdot 9^{-k}\varphi(\mathbf{a}_s)\right| + \left|2\cdot 9^{-k}\varphi(\mathbf{a}_s) - \int_{\ell_s} \varphi(t)\,d\mathcal{H}^1(t)\right| \\
&\quad\leqslant \int_{A_s} \left|J_{f_k}(x)\big(\varphi(x) - \varphi(\mathbf{a}_s)\big)\right|dx + \int_{\ell_s} \left|\varphi(t) - \varphi(\mathbf{a}_s)\right|d\mathcal{H}^1(t) \\
&\quad\leqslant 4\cdot 9^{-k} \operatorname{osc}_{A_s}\varphi \leqslant 2\cdot 2\cdot 9^{-k} \operatorname{diam}(A_s)\|\nabla\varphi\|_\infty.
\end{aligned}$$

Since $J_{f_k} = 0$ outside A^k and

$$\sum_{s\in\{1,\dots,9\}^k} \operatorname{diam}(A_s) = 2,$$

summing over $s \in \{1, \dots, 9\}^k$ we obtain

$$\left|\int_{A} J_{f_k}(x)\varphi(x)\,dx - \int_{\ell} \varphi(t)\,d\mathcal{H}^1(t)\right| \leqslant 8\cdot 9^{-k}\|\nabla\varphi\|_\infty,$$

so that

$$\int_{\ell} \varphi(t)\,d\mathcal{H}^1(t)\,dt = \lim_{k\to\infty} \int_{A} J_{f_k}(x)\varphi(x)\,dx. \tag{5.1}$$

On the other hand, since $f_k \to f$ uniformly and $\nabla f_k \to \nabla f$ in L^p, the passage to limit

$$\mathcal{J}_f(\varphi) = -\int_{A} f^1(x)\det\big(\nabla\varphi, \nabla f^2\big)(x)\,dx = -\lim_{k\to\infty}\int_{A} f_k^1(x)\det\big(\nabla\varphi, \nabla f_k^2\big)(x)\,dx$$

is legitimate. We can integrate by parts when dealing the Lipschitz functions f_k (cf. (1.1)), hence

$$\mathcal{J}_f(\varphi) = \lim_{k\to\infty}\int_{A} \varphi(x)\det\big(\nabla f_k^1, \nabla f_k^2\big)(x)\,dx = \lim_{k\to\infty}\int_{A} J_{f_k}(x)\varphi(x)\,dx. \tag{5.2}$$

Comparing (5.1) and (5.2) we conclude the proof. □

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