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Conformal metrics on $\mathbb{R}^{2m}$ with constant Q-curvature and large volume

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Abstract

We study conformal metrics $g_u = e^{2u}|dx|^2$ on $\mathbb{R}^{2m}$ with constant Q-curvature $Q_{g_u} \equiv (2m-1)!$ (notice that $(2m-1)!$ is the Q-curvature of S^{2m}) and finite volume. When $m = 3$ we show that there exists V^* such that for any $V \in [V^*, \infty)$ there is a conformal metric $g_u = e^{2u}|dx|^2$ on $\mathbb{R}^6$ with $Q_{g_u} \equiv 5!$ and $\mathrm{vol}(g_u) = V$. This is in sharp contrast with the four-dimensional case, treated by C.-S. Lin. We also prove that when m is odd and greater than 1, there is a constant $V_m > \mathrm{vol}(S^{2m})$ such that for every $V \in (0, V_m]$ there is a conformal metric $g_u = e^{2u}|dx|^2$ on $\mathbb{R}^{2m}$ with $Q_{g_u} \equiv (2m-1)!$, $\mathrm{vol}(g) = V$. This extends a result of A. Chang and W.-X. Chen. When m is even we prove a similar result for conformal metrics of *negative* Q-curvature.

Keywords: Q-curvature; Paneitz operators; GMJS operators; Conformal geometry

1. Introduction and statement of the main theorems

We consider solutions to the equation

$$(-\Delta)^m u = (2m-1)! e^{2mu} \quad \text{in } \mathbb{R}^{2m}, \tag{1}$$

satisfying

$$V := \int_{\mathbb{R}^{2m}} e^{2mu(x)}\, dx < +\infty, \tag{2}$$

with particular emphasis on the role played by V.

Geometrically, if u solves (1) and (2), then the conformal metric $g_u := e^{2u}|dx|^2$ has Q-curvature $Q_{g_u} \equiv (2m-1)!$ and volume V (by $|dx|^2$ we denote the Euclidean metric). For the definition of Q-curvature and related remarks, we refer to Chapter 4 in [3] or to [8] and [9]. Notice that given a solution u to (1) and $\lambda > 0$, the function $v := u - \frac{1}{2m}\log\lambda$ solves

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$$(-\Delta)^m v = \lambda(2m-1)! e^{2mv} \quad \text{in } \mathbb{R}^{2m}, \qquad \int_{\mathbb{R}^{2m}} e^{2mv(x)}\,dx = \frac{V}{\lambda},$$

hence there is no loss of generality in the particular choice of the constant $(2m-1)!$ in (1). On the other hand this constant has the advantage of being the Q-curvature of the round sphere S^{2m}. This implies that the function $u_1(x) = \log \frac{2}{1+|x|^2}$, which satisfies $e^{2u_1}|dx|^2 = (\pi^{-1})^* g_{S^{2m}}$ (here $\pi : S^{2m} \to \mathbb{R}^{2m}$ is the stereographic projection) is a solution to (1)–(2) with $V = \mathrm{vol}(S^{2m})$. Translations and dilations (i.e. Möbius transformations) actually give us a large family of solutions to (1)–(2) with $V = \mathrm{vol}(S^{2m})$, namely

$$u_{x_0,\lambda}(x) := u_1\big(\lambda(x-x_0)\big) + \log\lambda = \log\frac{2\lambda}{1+\lambda^2|x-x_0|^2}, \quad x_0 \in \mathbb{R}^{2m},\ \lambda > 0. \tag{3}$$

We shall call the functions $u_{x_0,\lambda}$ standard or *spherical* solutions to (1)–(2).

The question whether the family of spherical solutions in (3) exhausts the set of solutions to (1)–(2) has raised a lot of interest and is by now well understood. W. Chen and C. Li [6] proved that on $\mathbb{R}^2$ ($m = 1$) every solution to (1)–(2) is spherical, while for every $m > 1$, i.e. in dimension 4 and higher, it was proven by A. Chang and W.-X. Chen [4] that problems (1)–(2) admit solutions which are non-spherical. In fact they proved

Theorem A. *(See A. Chang–W.-X. Chen, 2001, [4].) For every $m > 1$ and $V \in (0, \mathrm{vol}(S^{2m}))$ there exists a solution to* (1)–(2).

Several authors have tried to classify spherical solutions or, in other words, to give analytical and geometric conditions under which a solution to (1)–(2) is spherical (see [5,23,25]), and to understand some properties of non-spherical solutions, such as their asymptotic behavior, their volume and their symmetry (see [11,14,24]). In particular C.-S. Lin proved:

Theorem B. *(See C.-S. Lin, 1998, [11].) Let u solve* (1)–(2) *with $m = 2$. Then either u is spherical* (*i.e. as in* (3)) *or $V < \mathrm{vol}(S^4)$.*

Both spherical solutions and the solutions given by Theorem A are radially symmetric (i.e. of the form $u(|x-x_0|)$ for some $x_0 \in \mathbb{R}^{2m}$). On the other hand there also exist plenty of non-radial solutions to (1)–(2) when $m = 2$.

Theorem C. *(See J. Wei and D. Ye, 2006, [24].) For every $V \in (0, \mathrm{vol}(S^4))$ there exist* (*several*) *non-radial solutions to* (1)–(2) *for $m = 2$.*

Remark D. Probably the proof of Theorem C can be extended to higher dimension $2m \geqslant 2$, yielding several non-symmetric solutions to (1)–(2) for every $V \in (0, \mathrm{vol}(S^{2m}))$, but failing to produce non-symmetric solutions for $V \geqslant \mathrm{vol}(S^{2m})$. As in the proof of Theorem A, the condition $V < \mathrm{vol}(S^{2m})$ plays a crucial role.

Theorems A, B, C and Remark D strongly suggest that also in dimension 6 and higher all non-spherical solutions to (1)–(2) satisfy $V < \mathrm{vol}(S^{2m})$, i.e. (1)–(2) has no solution for $V > \mathrm{vol}(S^{2m})$ and the only solutions with $V = \mathrm{vol}(S^{2m})$ are the spherical ones. Quite surprisingly we found out that this is not at all the case. In fact in dimension 6 we found solutions to (1)–(2) with arbitrarily large V:

Theorem 1. *For $m = 3$ there exists $V^* > 0$ such that for every $V \geqslant V^*$ there is a solution u to* (1)–(2)*, i.e. there exists a metric on $\mathbb{R}^6$ of the form $g_u = e^{2u}|dx|^2$ satisfying $Q_{g_u} \equiv 5!$ and $\mathrm{vol}(g_u) = V$.*

In order to prove Theorem 1 we will consider only rotationally symmetric solutions to (1)–(2), so that (1) reduces to and ODE. Precisely, given $a, b \in \mathbb{R}$ let $u = u_{a,b}(r)$ be the solution of

$$\begin{cases} \Delta^3 u = -e^{6u} \quad \text{in } \mathbb{R}^6, \\ u(0) = u'(0) = u'''(0) = u'''''(0) = 0, \\ u''(0) = \frac{\Delta u(0)}{6} = a, \\ u''''(0) = \frac{\Delta^2 u(0)}{16} = b. \end{cases} \tag{4}$$

Here and in the following we will always (by a little abuse of notation) see a rotationally symmetric function f both as a function of one variable $r \in [0, \infty)$ (when writing f', f'', etc.) and as a function of $x \in \mathbb{R}^6$ (when writing Δf, $\Delta^2 f$, etc.). We also used that

$$\Delta f(0) = 6f''(0), \qquad \Delta^2 f(0) = 16f''''(0),$$

see e.g. [14, Lemma 17]. Also notice that in (4) we replaced 5! by 1 to make the computations lighter. As we already noticed, this is not a problem.

Theorem 2. *Let $u = u_{a,3}$ solve (4) for a given $a < 0$ and $b = 3$.*[1] *Then*

$$\int_{\mathbb{R}^6} e^{6u_{a,3}}\,dx < \infty \quad \text{for } |a| \text{ sufficiently large}; \qquad \lim_{a\to-\infty} \int_{\mathbb{R}^6} e^{6u_{a,3}}\,dx = \infty. \tag{5}$$

In particular the conformal metric $g_{u_{a,3}} = e^{2u_{a,3}}|dx|^2$ of constant Q-curvature $Q_{g_{u_{a,3}}} \equiv 1$ satisfies

$$\lim_{a\to-\infty} \mathrm{vol}(g_{u_{a,3}}) = \infty.$$

Theorem 1 will follow from Theorem 2 and a continuity argument (Lemma 8 below).

Going through the proof of Theorem A it is clear that it does not extend to the case $V > \mathrm{vol}(S^{2m})$. With a different approach, we are able to prove that, at least when $m \geqslant 3$ is odd, one can extend Theorem A as follows.

Theorem 3. *For every $m \geqslant 3$ odd there exists $V_m > \mathrm{vol}(S^{2m})$ such that for every $V \in (0, V_m]$ there is a non-spherical solution u to (1)–(2), i.e. there exists a metric on $\mathbb{R}^{2m}$ of the form $g_u = e^{2u}|dx|^2$ satisfying $Q_{g_u} \equiv (2m-1)!$ and $\mathrm{vol}(g_u) = V$.*

The condition $m \geqslant 3$ odd is (at least in part) necessary in view of Theorem B and [6], but the case $m \geqslant 4$ even is open. Notice also that when $m = 3$, Theorems 1 and 3 guarantee the existence of solutions to (1)–(2) for

$$V \in (0, V_m] \cup \big[V^*, \infty\big),$$

but we cannot rule out that $V_m < V^*$ (the explicit value of V_m is given in (37) below) and the existence of solutions to (1)–(2) is unknown for $V \in (V_m, V^*)$. Could there be a gap phenomenon?

We now briefly investigate how large the volume of a metric $g_u = e^{2u}|dx|^2$ on $\mathbb{R}^{2m}$ can be when $Q_{g_u} \equiv \mathrm{const} < 0$. Again with no loss of generality we assume $Q_{g_u} \equiv -(2m-1)!$. In other words consider the problem

$$(-\Delta)^m u = -(2m-1)!e^{2mu} \quad \text{on } \mathbb{R}^{2m}. \tag{6}$$

Although for $m = 1$ it is easy to see that problems (6)–(2) admit no solutions for any $V > 0$, when $m \geqslant 2$ problems (6)–(2) have solutions for some $V > 0$, as shown in [15]. Then with the same proof of Theorem 3 we get:

Theorem 4. *For every $m \geqslant 2$ even there exists $V_m > \mathrm{vol}(S^{2m})$ such that for $V \in (0, V_m]$ there is a solution u to (6)–(2), i.e. there exists a metric on $\mathbb{R}^{2m}$ of the form $g_u = e^{2u}|dx|^2$ satisfying*

$$Q_{g_u} \equiv -(2m-1)!, \quad \mathrm{vol}(g_u) = V.$$

The cases of solutions to (1)–(2) with m even, or (6)–(2) and m odd seem more difficult to treat since the ODE corresponding to (1) or (6), in analogy with (4) becomes

$$\Delta^m u(r) = (2m-1)!e^{2mu(r)},$$

whose solutions can blow up in finite time (i.e. for finite r) if the initial data are not chosen carefully (contrary to Lemma 5 below).

[1] The choice $b = 3$ is convenient in the computations, but any other $b > 0$ would work.

2. Proof of Theorem 2

Set $\omega_{2m-1} := \mathrm{vol}(S^{2m-1})$ and let B_r denote the unit ball in $\mathbb{R}^{2m}$ centered at the origin. Given a smooth radial function $f = f(r)$ in $\mathbb{R}^{2m}$ we will often use the divergence theorem in the form

$$\int_{B_r} \Delta f\,dx = \int_{\partial B_r} \frac{\partial f}{\partial \nu}\,d\sigma = \omega_{2m-1} r^{2m-1} f'(r). \tag{7}$$

Dividing by $\omega_{2m-1} r^{2m-1}$ into (7) and integrating we also obtain

$$f(t) - f(s) = \int_s^t \frac{1}{\omega_{2m-1}\rho^{2m-1}} \int_{B_\rho} \Delta f\,dx\,d\rho, \quad 0 \leqslant s \leqslant t. \tag{8}$$

When no confusion can arise we will simply write u instead of $u_{a,3}$ or $u_{a,b}$ to denote the solution to (4). In what follows, also other quantities (e.g. R, r_0, r_1, r_2, r_3, ϕ, ξ_1, ξ_2) will depend on a and b, but this dependence will be omitted from the notation.

Lemma 5. *Given any $a, b \in \mathbb{R}$, the solution u to the ODE (4) exists for all times.*

Proof. Applying (8) to $f = \Delta^2 u$, and observing that $\Delta(\Delta^2 u) = -e^{6u} \leqslant 0$ we get

$$\Delta^2 u(t) \leqslant \Delta^2 u(s) \leqslant \Delta^2 u(0) = 16b, \quad 0 \leqslant s \leqslant t, \tag{9}$$

i.e. $\Delta^2 u(r)$ is monotone decreasing. This and (8) applied to Δu yield

$$\Delta u(r) \leqslant \Delta u(0) + \int_0^r \frac{1}{\omega_5 \rho^5} \int_{B_\rho} 16b\,dx\,d\rho = 6a + \int_0^r \frac{8}{3} b\rho\,d\rho = 6a + \frac{4}{3} b r^2.$$

A further application of (8) to u finally gives

$$u(r) \leqslant \int_0^r \frac{1}{\omega_5 \rho^5} \int_{B_\rho} \left(6a + \frac{4}{3} b|x|^2\right) dx\,d\rho = \int_0^r \left(a\rho + \frac{\rho^3 b}{6}\right) d\rho = \frac{a}{2} r^2 + \frac{b}{24} r^4 =: \phi(r). \tag{10}$$

Similar lower bounds can be obtained by observing that $-e^{6u} \geqslant -1$ for $u \leqslant 0$. This proves that $u(r)$ cannot blow-up in finite time and, by standard ODE theory, $u(r)$ exists for every $r \geqslant 0$. $\square$

Proof of (5) (completed). Fix $b = 3$ and take $a < 0$. The function $\phi(r) = \frac{a}{2} r^2 + \frac{1}{8} r^4$ vanishes for $r = R = R(a) := 2\sqrt{-a}$. In order to prove (5) we shall investigate the behavior of u in a neighborhood of R. The heuristic idea is that

$$u^{(j)}(0) = \phi^{(j)}(0), \quad \text{for } 0 \leqslant j \leqslant 5, \qquad \Delta^3 \phi \equiv 0,$$

and for every $\varepsilon > 0$ on $[\varepsilon, R - \varepsilon]$ we have $\phi \leqslant C_\varepsilon a \to -\infty$ and $|\Delta^3 u| \leqslant e^{C_\varepsilon a} \to 0$ as $a \to -\infty$, hence for $r \in [0, R - \varepsilon]$ we expect $u(r)$ to be very close to $\phi(r)$. On the other hand, u cannot stay close to ϕ for r much larger than R because eventually $-\Delta^3 u(r)$ will be large enough to make $\Delta^2 u$, Δu and u negative according to (8) (see Fig. 1). Then it is crucial to show that u stays close to ϕ for some $r > R$ (hence in a region where ϕ is positive and $\Delta^3 u$ is not necessarily small) and long enough to make the second integral in (5) blow up as $a \to -\infty$.

Step 1: *Estimates of $u(R)$, $\Delta u(R)$ and $\Delta^2 u(R)$.* From (10) we infer

$$\Delta^3 u = -e^{6u} \geqslant -e^{6\phi},$$

which, together with (8), gives

$$\Delta^2 u(r) = \Delta^2 u(0) + \int_0^r \frac{1}{\omega_5 \rho^5} \int_{B_\rho} \Delta^3 u\,dx\,d\rho \geqslant 48 - \int_0^r \frac{1}{\omega_5 \rho^5} \int_{B_\rho} e^{6\phi(|x|)}\,dx\,d\rho. \tag{11}$$

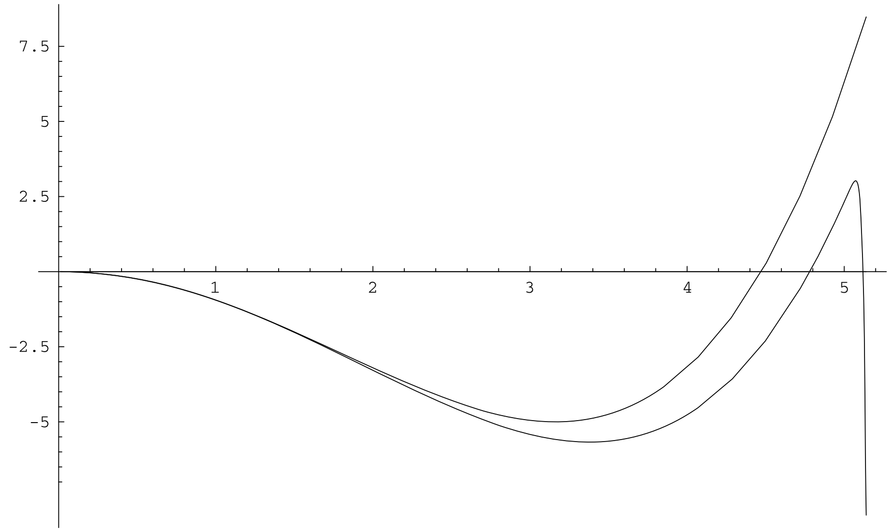

Fig. 1. The functions $\phi(r) = \frac{a}{2}r^2 + \frac{1}{8}r^4$ (above) and $u_{a,3}(r) \leqslant \phi(r)$.

We can explicitly compute (see Lemma 6 below and simplify (29) using that $\phi(R) = 0$ and $\int_{\sqrt{3}a}^{-\sqrt{3}a} e^{t^2}\,dt = 2\int_0^{-\sqrt{3}a} e^{t^2}\,dt$)

$$\int_0^R \frac{1}{\omega_5\rho^5}\int_{B_\rho} e^{6\phi(|x|)}\,dx\,d\rho = \frac{1}{48a} + \frac{(18a^2+1)\sqrt{3}}{144a^2}e^{-3a^2}\int_0^{-\sqrt{3}a} e^{t^2}\,dt.$$

Then by (9) and Lemma 7 below we conclude that

$$\Delta^2 u(r) \geqslant \Delta^2 u(R) \geqslant 48\big(1 + O\big(a^{-1}\big)\big) \quad \text{for } 0 \leqslant r \leqslant R = 2\sqrt{-a}. \tag{12}$$

Here and in the following $|a^k O(a^{-k})| \leqslant C = C(k)$ as $a \to -\infty$ for every $k \in \mathbb{R}$. Then applying (8) as before we also obtain

$$\Delta u(r) \geqslant 6a + 4\big(1 + O\big(a^{-1}\big)\big)r^2 \quad \text{for } 0 \leqslant r \leqslant R$$

and

$$u(r) \geqslant \frac{a}{2}r^2 + \frac{1 + O(a^{-1})}{8}r^4 = \phi(r) + O\big(a^{-1}\big)r^4 \quad \text{for } 0 \leqslant r \leqslant R.$$

At $r = R$ this reduces to

$$u(R) \geqslant O(a).$$

Step 2: *Behavior of* $u(r)$, $\Delta u(r)$, $\Delta^2 u(r)$ *for* $r \geqslant R$. Define r_0 (depending on $a < 0$) as

$$r_0 := \inf\big\{r > 0\colon\ u(r) = 0\big\} \in [R, \infty].$$

We first claim that $r_0 < \infty$. We have by Lemmas 6 and 7

$$\int_{B_R} e^{6\phi}\,dx = \omega_5\left(-\frac{4a}{3} + \frac{4(6a^2-1)\sqrt{3}}{9}e^{-3a^2}\int_0^{-\sqrt{3}a} e^{t^2}\,dt\right) = O(a). \tag{13}$$

Since on B_{r_0} we have $u \leqslant 0$, hence $\Delta^3 u \geqslant -1$, using (7)–(8) and (13) we get for $r \in [R, r_0]$

$$\Delta^2 u(r) \geqslant \Delta^2 u(R) - \int_R^r \frac{1}{\omega_5 \rho^5}\left(\int_{B_R} e^{6\phi}\,dx + \int_{B_\rho \setminus B_R} 1\,dx\right) d\rho$$
$$\geqslant 48 + O(a)\left[\frac{1}{R^4} - \frac{1}{r^4}\right] - \int_R^r \frac{\rho^6 - R^6}{6\rho^5}\,d\rho. \tag{14}$$

Assuming $r \in [R, 2R]$ we can now bound with a Taylor expansion

$$\frac{1}{R^4} - \frac{1}{r^4} = R^{-4}\tilde{O}\left(\frac{r-R}{R}\right) \tag{15}$$

and

$$\rho^6 - R^6 \leqslant r^6 - R^6 = R^6 \tilde{O}\left(\frac{r-R}{R}\right), \quad \text{for } \rho \in [R, r],$$

which together with (15) yields

$$\int_R^r \frac{\rho^6 - R^6}{6\rho^5}\,d\rho \leqslant \int_R^r \frac{r^6 - R^6}{6\rho^5}\,d\rho \leqslant R^2 \tilde{O}\left(\left(\frac{r-R}{R}\right)^2\right), \tag{16}$$

where for any $k \in \mathbb{R}$ we have $|t^{-k}\tilde{O}(t^k)| \leqslant C = C(k)$ uniformly for $0 \leqslant t \leqslant 1$. Using (15) and (16) we bound in (14)

$$\Delta^2 u(r) \geqslant 48 + O(a^{-1})\tilde{O}\left(\frac{r-R}{R}\right) + R^2\tilde{O}\left(\left(\frac{r-R}{R}\right)^2\right), \quad r \in \left[R, \min\{r_0, 2R\}\right],$$

whence

$$\Delta^2 u(r) \geqslant 48 + O(a^{-1}) + R^2\tilde{O}\left(\left(\frac{r-R}{R}\right)^2\right)\chi_{(R,\infty)}(r), \quad r \in \left[0, \min\{r_0, 2R\}\right],$$

where $\chi_{(R,\infty)}(r) = 0$ for $r \in [0, R]$ and $\chi_{(R,\infty)}(r) = 1$ for $r > R$. Then with (8) we estimate for $r \in [0, \min\{r_0, 2R\}]$

$$\Delta u(r) \geqslant 6a + 4\left(1 + O(a^{-1})\right)r^2 + \chi_{(R,\infty)}(r)\int_R^r \frac{1}{\omega_5\rho^5}\int_{B_\rho\setminus B_R} R^2\tilde{O}\left(\left(\frac{|x|-R}{R}\right)^2\right)dx\,d\rho$$
$$= 6a + 4\left(1 + O(a^{-1})\right)r^2 + R^4\tilde{O}\left(\left(\frac{r-R}{R}\right)^4\right)\chi_{(R,\infty)}(r) \tag{17}$$

and

$$u(r) \geqslant \frac{a}{2}r^2 + \frac{1 + O(a^{-1})}{8}r^4 + \chi_{(R,\infty)}(r)\int_R^r \frac{1}{\omega_5\rho^5}\int_{B_\rho\setminus B_R} R^4\tilde{O}\left(\left(\frac{|x|-R}{R}\right)^4\right)dx\,d\rho$$
$$= \phi(r) + O(a^{-1})r^4 + R^6\tilde{O}\left(\left(\frac{r-R}{R}\right)^6\right)\chi_{(R,\infty)}(r), \tag{18}$$

where the integrals in (17) and (18) are easily estimated bounding $|x|$ with r and applying (16).

Making a Taylor expansion of $\phi(r)$ at $r = R$ and using that $\phi(R) = 0$, we can further estimate the right-hand side of (18) for $r \in [R, \min\{r_0, 2R\}]$ as

$$u(r) \geqslant \phi'(R)(r-R) + R^2\tilde{O}\left(\left(\frac{r-R}{R}\right)^2\right) + O(a^{-1})r^4 + R^6\tilde{O}\left(\left(\frac{r-R}{R}\right)^6\right)$$
$$= -aR(r-R) + O(a^{-1})R^4 + R^2\tilde{O}\left(\left(\frac{r-R}{R}\right)^2\right) + R^6\tilde{O}\left(\left(\frac{r-R}{R}\right)^6\right) =: \psi_a(r).$$

Now choosing $r = R(1 + 1/\sqrt{-a}\,)$, so that $(r - R)/R \to 0$ as $a \to -\infty$, we get

$$\lim_{a\to-\infty} \psi_a\big(R(1 + 1/\sqrt{-a}\,)\big) \geqslant \lim_{a\to-\infty}\big(4(-a)^{\frac{3}{2}} + O(a) - C\big) = \infty.$$

In particular

$$r_0 \in \big[R, R(1 + 1/\sqrt{-a}\,)\big].$$

We now claim that

$$\lim_{a\to-\infty} \Delta u(r_0) = \infty. \tag{19}$$

Indeed we infer from (17)

$$\Delta u(r_0) \geqslant 6a + 4\big(1 + O\big(a^{-1}\big)\big)r_0^2 - C \geqslant 6a + 4\big(1 + O\big(a^{-1}\big)\big)R^2 - C \geqslant -10a - C,$$

for $-a$ large enough, whence (19). Set

$$r_1 = r_1(a) := \inf\big\{r > r_0 \colon u(r) = 0\big\}.$$

Applying (7) to (17), and recalling that $\frac{r_0-R}{R} \leqslant \frac{1}{\sqrt{a}}$, similar to (18) we obtain

$$u'(r_0) \geqslant ar_0 + \frac{1 + O(a^{-1})}{2}r_0^3 - C \geqslant ar_0 + \frac{1 + O(a^{-1})}{2}r_0R^2 - C \geqslant -ar_0 - C.$$

In particular for $-a$ large enough we have $u'(r_0) > 0$, which implies $r_1 > r_0$. Using (7)–(8) and that $\Delta^3 u(r) \leqslant -1$ for $r \in [r_0, r_1]$, it is not difficult to see that $r_1 < \infty$. Moreover there exists at least a point $r_2 = r_2(a) \in (r_0, r_1]$ such that $u'(r_2) \leqslant 0$, which in turn implies that

$$\Delta u(r_3) < 0 \quad \text{for some } r_3 = r_3(a) \in (r_0, r_2], \tag{20}$$

since otherwise we would have by (7)

$$u'(r_2) = \frac{1}{\omega_5 r_2^5}\int_{B_{r_0}} \Delta u\,dx + \frac{1}{\omega_5 r_2^5}\int_{B_{r_2}\setminus B_{r_0}} \Delta u\,dx \geqslant \frac{r_0^5}{r_2^5}u'(r_0) > 0,$$

contradiction.

Step 3: *Conclusion*. We now use the estimates obtained in Steps 1 and 2 to prove (5).

From (8), (19) and (20) we infer

$$\lim_{a\to-\infty} \int_{r_0}^{r_3} \frac{1}{\omega_5 r^5}\int_{B_r} \Delta^2 u\,dx\,dr = \lim_{a\to-\infty}\big(\Delta u(r_3) - \Delta u(r_0)\big) = -\infty, \tag{21}$$

hence by the monotonicity of $\Delta^2 u(r)$ (see (9))

$$\lim_{a\to-\infty} \Delta^2 u(r_3)\big(r_3^2 - r_0^2\big) = -\infty. \tag{22}$$

We now claim that

$$\lim_{a\to-\infty} \int_{B_{r_3}} e^{6u}\,dx = \infty. \tag{23}$$

Indeed consider on the contrary an arbitrary sequence a_k with $\lim_{k\to\infty} a_k = -\infty$ and

$$\lim_{k\to\to\infty} \int_{B_{r_3}} e^{6u}\,dx < \infty, \tag{24}$$

where here r_3 and u depend on a_k instead of a of course. Since $u \geqslant 0$ in $B_{r_3} \setminus B_{r_0}$ we have

$$\int_{B_{r_3}} e^{6u}\,dx \geqslant \int_{B_{r_3}\setminus B_{r_0}} 1\,dx = \frac{\omega_5}{6}\big(r_3^6 - r_0^6\big).$$

Now observe that $(r_3^6 - r_0^6) \geqslant (r_3^2 - r_0^2)r_0^4$ to conclude that (24) implies

$$\lim_{k\to\infty}\big(r_3^2 - r_0^2\big) \leqslant \lim_{k\to\infty}\frac{r_3^6 - r_0^6}{r_0^4} = 0. \tag{25}$$

Then (8), (12) and (22) yield

$$\begin{aligned}\big(r_3^2 - r_0^2\big)\int_R^{r_3}\frac{1}{\omega_5 r^5}\int_{B_r} e^{6u}\,dx\,dr &= \big(r_3^2 - r_0^2\big)\big(\Delta^2 u(R) - \Delta^2 u(r_3)\big)\\ &\geqslant -\Delta^2 u(r_3)\big(r_3^2 - r_0^2\big) \to \infty \quad \text{as } k \to \infty.\end{aligned}$$

By (25) we also have

$$\lim_{k\to\infty}\int_R^{r_3}\frac{1}{\omega_5 r^5}\int_{B_r} e^{6u}\,dx\,dr = \infty,$$

which implies at once

$$\lim_{k\to\infty}\int_{B_{r_3}} e^{6u}\,dx \geqslant \lim_{k\to\infty} 4R^4\omega_5\int_R^{r_3}\frac{1}{\omega_5 r^5}\int_{B_{r_3}} e^{6u}\,dx\,dr = \infty,$$

contradicting (24). Then (23) is proven.

It remains to show that

$$\int_{\mathbb{R}^6} e^{6u}\,dx < \infty,$$

at least for $-a$ large enough. It follows from (22) and the monotonicity of $\Delta^2 u$ that for $-a$ large enough we have

$$\Delta^2 u(r) < B < 0, \quad \text{for } r \geqslant r_3, \tag{26}$$

and, using (7)–(8) as already done several times, we can find $r_a \geqslant r_3$ such that

$$(\Delta u)'(r) < \frac{B}{6}r, \qquad \Delta u(r) < \frac{B}{12}r^2, \qquad u'(r) < \frac{B}{96}r^3, \qquad u(r) < \frac{B}{384}r^4, \quad \text{for } r \geqslant r_a. \tag{27}$$

Then

$$\int_{\mathbb{R}^6} e^{6u}\,dx \leqslant \int_{B_{r_a}} e^{6u}\,dx + \int_{\mathbb{R}^6\setminus B_{r_a}} e^{\frac{B}{64}|x|^2}\,dx < \infty,$$

as wished. □

2.1. Two useful lemmas

We now state and prove two lemmas used in the proof of Theorem 2. Their proof is based on elementary calculus, but we provide it for completeness and because Lemma 6 in particular was crucial for the estimates of the previous section.

Lemma 6. *For $\phi(r)=\frac{a}{2}r^2+\frac{1}{8}r^4$, $a\leqslant 0$, we have*

$$\int_{B_r} e^{6\phi(|x|)}\,dx=\omega_5\left[\frac{2}{3}a+\frac{1}{3}e^{6\phi(r)}\left(-2a+r^2\right)+\frac{(12a^2-2)\sqrt{3}}{9}e^{-3a^2}\int_{-\sqrt{3}(a+r^2/2)}^{-\sqrt{3}a} e^{t^2}\,dt\right]=:\xi_1(r) \tag{28}$$

and

$$\int_0^r \frac{1}{\omega_5\rho^5}\int_{B_\rho} e^{6\phi(|x|)}\,dx\,d\rho$$
$$=\frac{-2a-e^{6\phi(r)}(-2a+r^2)}{12r^4}+\frac{(2-12a^2+3r^4)\sqrt{3}}{36r^4}e^{-3a^2}\int_{-\sqrt{3}(a+r^2/2)}^{-\sqrt{3}a} e^{t^2}\,dt:=\xi_2(r). \tag{29}$$

Proof. Patiently differentiating, using that $e^{-3a^2}\frac{d}{dr}\int_{-\sqrt{3}(a+r^2/2)}^{-\sqrt{3}a} e^{t^2}\,dt=\sqrt{3}re^{6\phi(r)}$, one sees that

$$\xi_1'(r)=\omega_5 r^5 e^{6\phi(r)},\qquad \xi_2'(r)=\frac{\xi_1(r)}{\omega_5 r^5}.$$

Using that $\phi(0)=0$ it is also easy to see that $\xi_1(0)=0$.

Since $\xi_2(0)$ is not defined, we will compute the limit of $\xi_2(r)$ as $r\to 0$. We first compute the Taylor expansions

$$e^{6\phi(r)}=1+3ar^2+\frac{3}{4}\left(1+6a^2\right)r^4+r^4o(1),$$

and

$$\sqrt{3}e^{-3a^2}\int_{-\sqrt{3}(a+r^2/2)}^{-\sqrt{3}a} e^{t^2}\,dt=\frac{3}{2}r^2+\frac{9}{4}ar^4+r^4o(1),$$

with errors $o(1)\to 0$ as $r\to 0$. Then

$$\frac{-2a-e^{6\phi(r)}(-2a+r^2)}{12r^4}=\frac{(1-6a^2)r^2+(\frac{3}{2}a-9a^3)r^4}{12r^4}+o(1)$$
$$=-\frac{(2-12a^2+3r^4)\sqrt{3}}{36r^4}e^{-3a^2}\int_{-\sqrt{3}(a+r^2/2)}^{-\sqrt{3}a} e^{t^2}\,dt+o(1),$$

with $o(1)\to 0$ as $r\to 0$. Hence $\lim_{r\to 0}\xi_2(r)=0$. □

Lemma 7. *We have*

$$\lim_{r\to\infty} re^{-r^2}\int_0^r e^{t^2}\,dt=\frac{1}{2}. \tag{30}$$

Proof. This is a simple calculus exercise. For instance one notices that (30) is equivalent to

$$\lim_{r\to\infty} re^{-r^2}\int_2^r e^{t^2}\,dt=\frac{1}{2}, \tag{31}$$

which follows integrating by parts twice and noticing that $re^{-r^2}\int_2^r t^{-4}e^{t^2}\,dt\to 0$ as $r\to\infty$. □

3. Proof of Theorem 1

We start with the following lemma.

Lemma 8. *Set*

$$V(a)=\frac{1}{5!}\int_{\mathbb{R}^6} e^{6u_{a,3}}\,dx$$

where $u=u_{a,3}$ is the solution to (4) for given $a<0$ and $b=3$. Then there exists $a^*<0$ such that V is continuous on $(-\infty,a^*]$.

Proof. It follows from (21) and the monotonicity of $\Delta^2 u$ that we can fix $-a^*$ so large that

$$\lim_{r\to\infty}\Delta^2 u_{a,3}(r)<0,\quad \text{for every } a\leqslant a^*.$$

Fix now $\varepsilon>0$. Given $a\leqslant a^*$ it is not difficult to find $r_a>0$ and $B=B(a)<0$ such that

$$\Delta^2 u_{a,3}(r)<B<0,\quad \text{for } r\geqslant r_a \tag{32}$$

and, possibly choosing r_a larger, using (7)–(8) as already done in the proof of Theorem 2, we get

$$(\Delta u_{a,3})'(r)<\frac{B}{6}r,\qquad \Delta u_{a,3}(r)<\frac{B}{12}r^2,\qquad u'_{a,3}(r)<\frac{B}{96}r^3,\qquad u_{a,3}(r)<\frac{B}{384}r^4,\quad \text{for } r\geqslant r_a. \tag{33}$$

By possibly choosing r_a even larger we can also assume that

$$\int_{\mathbb{R}^6\setminus B_{r_a}} e^{\frac{B}{64}|x|^4}\,dx<\frac{\varepsilon}{2}. \tag{34}$$

By ODE theory the solution $u_{a,3}$ to (4) is continuous with respect to a in $C^k_{\mathrm{loc}}(\mathbb{R}^6)$ for every $k\geqslant 0$, in the sense that for any $r'>0$, $u_{a',3}\to u_{a,3}$ in $C^k(B_{r'})$ as $a'\to a$. In particular we can find $\delta>0$ (depending on ε) such that if $|a-a'|<\delta$ then (32)–(33) with a replaced by a' are still satisfied for $r=r_a$ (not $r_{a'}$) and (32) holds also for every $r>r_a$ since $\Delta^2 u_{a',3}(r)$ is decreasing in r (see (9)). Then, with (7)–(8) we can also get the bounds in (33) for every $r\geqslant r_a$ (and $u_{a',3}$ instead of $u_{a,3}$). For instance

$$\begin{aligned}(\Delta u_{a',3})'(r)&=\frac{1}{\omega_5 r^5}\int_{B_r}\Delta^3 u_{a',3}\,dx=\left(\frac{r_a}{r}\right)^5(\Delta u_{a',3})'(r_a)+\frac{1}{\omega_5 r^5}\int_{B_r\setminus B_{r_a}}\Delta^2 u_{a',3}\,dx\\&<\left(\frac{r_a}{r}\right)^5\frac{Br_a}{6}+\frac{B(r^6-r_a^6)}{6r^5}=\frac{B}{6}r.\end{aligned}$$

Furthermore, up to taking $\delta>0$ even smaller, we can assume that

$$\left|\int_{B_{r_a}} e^{6u_{a',3}}\,dx-\int_{B_{r_a}} e^{6u_{a,3}}\,dx\right|<\frac{\varepsilon}{2}. \tag{35}$$

Finally, the last bound in (33) and (34) imply at once

$$\left|\int_{\mathbb{R}^6\setminus B_{r_a}} e^{6u_{a',3}}\,dx-\int_{\mathbb{R}^6\setminus B_{r_a}} e^{6u_{a,3}}\,dx\right|<\frac{\varepsilon}{2},$$

which together with (35) completes the proof. □

Proof of Theorem 1 (completed). Set $V^*=V(a^*)$, where a^* is given by Lemma 8. By Lemma 8, Theorem 2 and the intermediate value theorem, for every $V\geqslant V^*$ there exists $a\leqslant a^*$ such that

$$\frac{1}{5!}\int\limits_{\mathbb{R}^6} e^{6u_{a,3}}\,dx = V,$$

hence the metric $g_{u_{a,3}} = e^{2u_{a,3}}|dx|^2$ has constant Q-curvature equal to 1 and $\mathrm{vol}(g_{u_{a,3}}) = 5!V$. Applying the transformation

$$u = u_{a,3} - \frac{1}{6}\log 5!$$

it follows at once that the metric $g_u = e^{2u}|dx|^2$ satisfies $\mathrm{vol}(g_u) = V$ and $Q_{g_u} \equiv 5!$, hence u solves (1)–(2). □

4. Proof of Theorems 3 and 4

When $f:\mathbb{R}^n \to \mathbb{R}$ is radially symmetric we have $\Delta f(x) = f''(|x|) + \frac{n-1}{|x|}f'(|x|)$. In particular we have

$$\Delta^m r^{2m} = 2^{2m} m(2m-1)! \quad \text{in } \mathbb{R}^{2m}. \tag{36}$$

For $m \geqslant 2$ and $b \leqslant 0$ let u_b solve

$$\begin{cases} \Delta^m u_b = -(2m-1)!e^{2mu_b} & \text{in } \mathbb{R}^{2m}, \\ u_b^{(j)}(0) = 0 & \text{for } 0 \leqslant j \leqslant 2m-1,\ j \neq 2m-2, \\ u_b^{(2m-2)} = b. \end{cases}$$

From (7)–(8) it follows that $u_0 \leqslant 0$, hence $\Delta^m u_0 \geqslant -(2m-1)!$. We claim that

$$u_0(r) \geqslant \psi(r) := -\frac{r^{2m}}{2^{2m}m}.$$

Indeed according to (36) ψ solves

$$\Delta^m \psi = -(2m-1)! \leqslant \Delta^m u_0 \quad \text{in } \mathbb{R}^{2m}$$

and

$$\psi^{(j)}(0) = 0 = u_0^{(j)}(0) \quad \text{for } 0 \leqslant j \leqslant 2m-1,$$

which implies

$$\Delta^j \psi(0) = 0 = \Delta^j u_0(0) \quad \text{for } 0 \leqslant j \leqslant m-1,$$

see [14, Lemma 17]. Then the claim follows from (7)–(8) and a simple induction.

Now integrating we get

$$\int\limits_{\mathbb{R}^{2m}} e^{2mu_0}\,dx \geqslant \int\limits_{\mathbb{R}^{2m}} e^{2m\psi}\,dx = \omega_{2m-1}\int\limits_0^\infty r^{2m-1}\exp\left(-\frac{r^{2m}}{2^{2m-1}}\right)dr = \frac{2^{2m-2}\omega_{2m-1}}{m} =: V_m.$$

Using the formulas

$$\omega_{2m-1} = \mathrm{vol}\big(S^{2m-1}\big) = \frac{2\pi^m}{(m-1)!}, \qquad \omega_{2m} = \mathrm{vol}\big(S^{2m}\big) = \frac{2^{2m}(m-1)!\pi^m}{(2m-1)!}, \quad m \geqslant 1$$

we verify

$$V_m = \frac{(2m)!}{4(m!)^2}\omega_{2m}, \qquad \frac{V_2}{\omega_4} = \frac{3}{2} > 1, \qquad \frac{V_{m+1}}{\omega_{2m+2}}\left(\frac{V_m}{\omega_{2m}}\right)^{-1} = \frac{(2m+2)(2m+1)}{(m+1)^2} > 1, \tag{37}$$

hence by induction

$$V_m > \mathrm{vol}\big(S^{2m}\big) \quad \text{for } m \geqslant 2. \tag{38}$$

With the same argument used to prove Lemma 8 we can show that the function

$$V(b) := \int_{\mathbb{R}^{2m}} e^{6u_b}\,dx, \quad b \in (-\infty, 0]$$

is finite and continuous. Indeed it is enough to replace (32) with

$$\Delta^{m-1}u_b(r) \leqslant B < 0 \quad \text{for } r \geqslant r_b,$$

and (33) with

$$\left(\Delta^{m-1-j}u_b\right)'(r) < C_{m,j}Br^{2j-1}, \qquad \Delta^{m-1-j}u_b(r) < D_{m,j}Br^{2j}, \quad \text{for } r \geqslant r_b,\ 1 \leqslant j \leqslant m-1$$

where r_b is chosen large enough and

$$C_{m,1} = \frac{1}{2m}, \qquad D_{m,j} = \frac{C_{m,j}}{2j}, \qquad C_{m,j+1} = \frac{D_{m,j}}{2m+2j},$$

whence

$$C_{m,j} = \frac{(m-1)!}{2^{2j-1}(j-1)!(m+j-1)!}, \qquad D_{m,j} = \frac{(m-1)!}{2^{2j}j!(m+j-1)!}.$$

Moreover, using that $\Delta^{m-1}u_b(0) = C_m b$ for some constant $C_m > 0$, $\Delta^m u_b(r) \leqslant 0$ for $r \geqslant 0$ and (7)–(8) as before, we easily obtain

$$u_b(r) \leqslant E_m b r^{2m-2}, \tag{39}$$

where $E_m := C_m C_{m,m-1} > 0$, hence

$$\lim_{b\to-\infty} V(b) \leqslant \lim_{b\to-\infty} \int_{\mathbb{R}^6} e^{6E_m b|x|^{2m-2}}\,dx = 0.$$

By continuity we conclude that for every $V \in (0, V_m]$ there exists $b \leqslant 0$ such that $u = u_b$ solves (1)–(2) if m is odd or (6)–(2) if m is even. Taking (38) into account it only remains to prove that the solutions u_b corresponding to $V = \mathrm{vol}(S^{2m})$ is not a spherical one. This follows immediately from (39), which is not compatible with (3). □

5. Applications and open questions

5.1. Possible gap phenomenon

Theorems 1 and 3 guarantee that for $m = 3$ there exists a solution to (1)–(2) for every $V \in (0, V_3] \cup [V^*, \infty)$, with possibly $V_3 < V^*$. Could it be that for some $V \in (V_3, V^*)$ problems (1)–(2) admit no solution?

If we restrict to rotationally symmetric solutions, some heuristic arguments show that the volume of a solution to (4), i.e. the function

$$V(a,b) := \int_{\mathbb{R}^6} e^{6u_{a,b}(|x|)}\,dx$$

need not be continuous for all $(a,b) \in \mathbb{R}^2$, hence the image of the function V might not be connected.

5.2. Higher dimensions and negative curvature

It is natural to ask whether Theorems 1 and 2 generalize to the case $m > 3$ or whether an analogous statement holds when $m \geqslant 2$ and (6) is considered instead of (1). Since the sign on the right-hand side of the ODE (4) plays a crucial role, we would expect that part of the proof of Theorem 2 can be recycled for (1) when $m \geqslant 5$ is odd, or for (6) when m is even.

For instance let $u_a = u_a(r)$ be the solution in $\mathbb{R}^4$ of

$$\begin{cases} \Delta^2 u_a = -6e^{4u_a}, \\ u_a(0) = u_a'(0) = u_a'''(0) = 0, \\ u_a''(0) = a. \end{cases}$$

It should not be difficult to see that $u_a(r)$ exists for all $r \geqslant 0$ and that $\int_{\mathbb{R}^4} e^{4u_a(|x|)}\,dx < \infty$. Do we also have

$$\lim_{a\to+\infty} \int_{\mathbb{R}^4} e^{4u_a(|x|)}\,dx = \infty?$$

5.3. *Non-radial solutions*

The proof of Theorem C cannot be extended to provide non-radial solutions to (1)–(2) for $m \geqslant 3$ and $V \geqslant \mathrm{vol}(S^{2m})$, but it is natural to conjecture that they do exist.

5.4. *Concentration phenomena*

The classification results of the solutions to (1)–(2), [6,11,25] and [14], have been used to understand the asymptotic behavior of unbounded sequences of solutions to the prescribed Gaussian curvature problem on 2-dimensional domains (see e.g. [2] and [10]), on S^2 (see [21]) and to the prescribed Q-curvature equation in dimension $2m$ (see e.g. [7,12,13,18–20,16,17]).

For instance consider the following model problem. Let $\Omega \subset \mathbb{R}^{2m}$ be a connected open set and consider a sequence (u_k) of solutions to the equation

$$(-\Delta)^m u_k = Q_k e^{2mu_k} \quad \text{in } \Omega, \tag{40}$$

where

$$Q_k \to Q_0 \quad \text{in } C^1_{\mathrm{loc}}(\Omega), \qquad \limsup_{k\to\infty} \int_\Omega e^{2mu_k}\,dx < \infty, \tag{41}$$

with the following interpretation: $g_k := e^{2u_k}|dx|^2$ is a sequence of conformal metrics on Ω with Q-curvatures $Q_{g_k} = Q_k$ and equibounded volumes.

As shown in [1] unbounded sequences of solutions to (40)–(41) can exhibit pathological behaviors in dimension 4 (and higher), contrary to the elegant results of [2] and [10] in dimension 2. This is partly due to Theorem A. In fact for $m \geqslant 2$ and $\alpha \in (0, (2m-1)!\,\mathrm{vol}(S^{2m})]$ one can found a sequence (u_k) of solutions to (40)–(41) with $Q_0 > 0$ and

$$\lim_{R\to0}\lim_{k\to\infty} \int_{B_R(x_0)} |Q_k|e^{2mu_k}\,dx = \alpha \quad \text{for some } x_0 \in \Omega. \tag{42}$$

For $m = 2$ this was made very precise by F. Robert [19] in the radially symmetric case. In higher dimension or when Q_0 is not necessarily positive, thanks to Theorems 1–4 we see that α can take values larger than $(2m-1)!\,\mathrm{vol}(S^{2m})$. Indeed if u is a solution to (1)–(2) or (6)–(2), then $u_k := u(kx) + \log k$ satisfies (40)–(41) with $\Omega = \mathbb{R}^{2m}$, $Q_k \equiv \pm(2m-1)!$ and

$$|Q_k|e^{2mu_k}\,dx \rightharpoonup (2m-1)!\,V\delta_0, \quad \text{weakly as measures.}$$

When $m = 2$, $Q_0 > 0$ (say $Q_0 \equiv 6$) it is unclear whether one could have concentration points carrying more Q-curvature than $6\,\mathrm{vol}(S^4)$, i.e. whether one can take $\alpha > 6\,\mathrm{vol}(S^4)$ in (42). Theorem B suggests that if the answer is affirmative, this should be due to the convergence to the same blow-up point of two or more blow-ups. Such a phenomenon is unknown in dimension 4 and higher, but was shown in dimension 2 by Wang [22] with a technique which, based on the abundance of conformal transformations of $\mathbb{C}$ into itself, does not extend to higher dimensions.

References

[1] Adimurthi, F. Robert, M. Struwe, Concentration phenomena for Liouville's equation in dimension 4, J. Eur. Math. Soc. 8 (2006) 171–180.
[2] H. Brézis, F. Merle, Uniform estimates and blow-up behaviour for solutions of $-\Delta u = V(x)e^u$ in two dimensions, Comm. Partial Differential Equations 16 (1991) 1223–1253.
[3] S.-Y.A. Chang, Non-linear Elliptic Equations in Conformal Geometry, Zur. Lect. Notes Adv. Math., EMS, 2004.
[4] S.-Y.A. Chang, W. Chen, A note on a class of higher order conformally covariant equations, Discrete Contin. Dyn. Syst. 63 (2001) 275–281.
[5] S.-Y.A. Chang, P. Yang, On uniqueness of solutions of n-th order differential equations in conformal geometry, Math. Res. Lett. 4 (1997) 91–102.
[6] W. Chen, C. Li, Classification of solutions of some nonlinear elliptic equations, Duke Math. J. 63 (3) (1991) 615–622.
[7] O. Druet, F. Robert, Bubbling phenomena for fourth-order four-dimensional PDEs with exponential growth, Proc. Amer. Math. Soc. 3 (2006) 897–908.
[8] C. Fefferman, C.R. Graham, Q-curvature and Poincaré metrics, Math. Res. Lett. 9 (2002) 139–151.
[9] C. Fefferman, K. Hirachi, Ambient metric construction of Q-curvature in conformal and CR geometry, Math. Res. Lett. 10 (2003) 819–831.
[10] Y. Li, I. Shafrir, Blow-up analysis for solutions of $-\Delta u = Ve^u$ in dimension two, Indiana Univ. Math. J. 43 (1994) 1255–1270.
[11] C.S. Lin, A classification of solutions of conformally invariant fourth order equations in $\mathbb{R}^n$, Comment. Math. Helv. 73 (1998) 206–231.
[12] A. Malchiodi, Compactness of solutions to some geometric fourth-order equations, J. Reine Angew. Math. 594 (2006) 137–174.
[13] A. Malchiodi, M. Struwe, Q-curvature flow on S^4, J. Differential Geom. 73 (2006) 1–44.
[14] L. Martinazzi, Classification of solutions to the higher order Liouville's equation on $\mathbb{R}^{2m}$, Math. Z. 263 (2009) 307–329.
[15] L. Martinazzi, Conformal metrics on $\mathbb{R}^{2m}$ with constant Q-curvature, Rend. Lincei. Mat. Appl. 19 (2008) 279–292.
[16] L. Martinazzi, Concentration-compactness phenomena in higher order Liouville's equation, J. Funct. Anal. 256 (2009) 3743–3771.
[17] L. Martinazzi, Quantization for the prescribed Q-curvature equation on open domains, Commun. Contemp. Math. 13 (2011) 533–551.
[18] C.B. Ndiaye, Ndiaye constant Q-curvature metrics in arbitrary dimension, J. Funct. Anal. 251 (1) (2007) 1–58.
[19] F. Robert, Concentration phenomena for a fourth order equation with exponential growth: The radial case, J. Differential Equations 231 (2006) 135–164.
[20] F. Robert, Quantization effects for a fourth order equation of exponential growth in dimension four, Proc. Roy. Soc. Edinburgh Sect. A 137 (2007) 531–553.
[21] M. Struwe, A flow approach to Nirenberg's problem, Duke Math. J. 128 (1) (2005) 19–64.
[22] S. Wang, An example of a blow-up sequence for $-\Delta u = V(x)e^u$, Differential Integral Equations 5 (1992) 1111–1114.
[23] J. Wei, X.-W. Xu, Classification of solutions of higher order conformally invariant equations, Math. Ann. 313 (1999) 207–228.
[24] J. Wei, D. Ye, Nonradial solutions for a conformally invariant fourth order equation in $\mathbb{R}^4$, preprint, 2006.
[25] X.-W. Xu, Uniqueness theorems for integral equations and its application, J. Funct. Anal. 247 (1) (2007) 95–109.