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Pointwise bounds and blow-up for nonlinear polyharmonic inequalities

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Abstract

We obtain results for the following question where $m \geqslant 1$ and $n \geqslant 2$ are integers.

Question. For which continuous functions $f:[0,\infty) \to [0,\infty)$ does there exist a continuous function $\varphi:(0,1) \to (0,\infty)$ such that every C^{2m} nonnegative solution $u(x)$ of

$$0 \leqslant -\Delta^m u \leqslant f(u) \quad \text{in } B_2(0) \setminus \{0\} \subset \mathbb{R}^n$$

satisfies

$$u(x) = O\big(\varphi\big(|x|\big)\big) \quad \text{as } x \to 0$$

and what is the optimal such φ when one exists?

Résumé

Nous obtenons des résultats pour la question suivante, avec $m \geqslant 1$ et $n \geqslant 2$ entiers.

Question. Pour quelles fonctions continues $f:[0,\infty) \to [0,\infty)$ existe-t-il une fonction continue $\varphi:(0,1) \to (0,\infty)$ telle que chaque solution C^{2m} non-negative $u(x)$ de

$$0 \leqslant -\Delta^m u \leqslant f(u) \quad \text{dans } B_2(0) \setminus \{0\} \subset \mathbb{R}^n$$

satisfasse à

$$u(x) = O\big(\varphi\big(|x|\big)\big) \quad \text{lorsque } x \to 0,$$

et quelle est la meilleure de ces fonctions φ quand elle existe ?

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1. Introduction

In this paper we consider the following question where $m \geqslant 1$ and $n \geqslant 2$ are integers.

Question 1. For which continuous functions $f : [0, \infty) \to [0, \infty)$ does there exist a continuous function $\varphi : (0, 1) \to (0, \infty)$ such that every C^{2m} nonnegative solution $u(x)$ of

$$0 \leqslant -\Delta^m u \leqslant f(u) \quad \text{in } B_2(0) \setminus \{0\} \subset \mathbb{R}^n \tag{1.1}$$

satisfies

$$u(x) = O\big(\varphi\big(|x|\big)\big) \quad \text{as } x \to 0 \tag{1.2}$$

and what is the optimal such φ when one exists?

We call a function φ with the above properties a pointwise a priori bound (as $x \to 0$) for C^{2m} nonnegative solutions $u(x)$ of (1.1).

As we shall see, when φ in Question 1 is optimal, the estimate (1.2) can sometimes be sharpened to

$$u(x) = o\big(\varphi\big(|x|\big)\big) \quad \text{as } x \to 0.$$

Remark 1.1. Let

$$\Gamma(r) = \begin{cases} r^{-(n-2)}, & \text{if } n \geqslant 3; \\ \log \frac{5}{r}, & \text{if } n = 2. \end{cases} \tag{1.3}$$

Since $u(x) = \Gamma(|x|)$ is a positive solution of $-\Delta^m u = 0$ in $B_2(0) \setminus \{0\}$, and hence a positive solution of (1.1), any pointwise a priori bound φ for C^{2m} nonnegative solutions $u(x)$ of (1.1) must be at least as large as Γ, and whenever $\varphi = \Gamma$ is such a bound it is necessarily an optimal bound.

Some of our results for Question 1 can be generalized to allow the function f in (1.1) to depend nontrivially on x and the partial derivatives of u up to order $2m - 1$. (See the second paragraph after Proposition 2.1.)

We also consider the following analog of Question 1 when the singularity is at ∞ instead of at the origin.

Question 2. For which continuous functions $f : [0, \infty) \to [0, \infty)$ does there exist a continuous function $\varphi : (1, \infty) \to (0, \infty)$ such that every C^{2m} nonnegative solution $v(y)$ of

$$0 \leqslant -\Delta^m v \leqslant f(v) \quad \text{in } \mathbb{R}^n \setminus B_{1/2}(0) \tag{1.4}$$

satisfies

$$v(y) = O\big(\varphi\big(|y|\big)\big) \quad \text{as } |y| \to \infty$$

and what is the optimal such φ when one exists?

The m-Kelvin transform of a function $u(x)$, $x \in \Omega \subset \mathbb{R}^n \setminus \{0\}$, is defined by

$$v(y) = |x|^{n-2m} u(x) \quad \text{where } x = y/|y|^2. \tag{1.5}$$

By direct computation, $v(y)$ satisfies

$$\Delta^m v(y) = |x|^{n+2m} \Delta^m u(x). \tag{1.6}$$

See [17, p. 221] or [18, p. 660]. Using this fact and some of our results for Question 1, we will obtain results for Question 2.

Nonnegative solutions in a punctured neighborhood of the origin in $\mathbb{R}^n$—or near $x=\infty$ via the m-Kelvin transform—of problems of the form

$$-\Delta^m u = f(x,u) \quad \text{or} \quad 0 \leqslant -\Delta^m u \leqslant f(x,u) \tag{1.7}$$

when f is a nonnegative function have been studied in [3,4,10–12,17,18] and elsewhere. These problems arise naturally in conformal geometry and in the study of the Sobolev embedding of H^{2m} into $L^{\frac{2n}{n-2m}}$.

Pointwise estimates at $x=\infty$ of solutions u of problems (1.7) can be crucial for proving existence results for entire solutions of (1.7) which in turn can be used to obtain, via scaling methods, existence and estimates of solutions of boundary value problems associated with (1.7), see e.g. [13,14]. An excellent reference for polyharmonic boundary value problems is [7].

Also, weak solutions of $\Delta^m u = \mu$, where μ is a measure on a subset of $\mathbb{R}^n$, have been studied in [2,5,6], and removable isolated singularities of $\Delta^m u = 0$ have been studied in [11].

Our proofs require Riesz potential estimates as stated, for example, in [9, Lemma 7.12] and a representation formula for C^{2m} nonnegative solutions of

$$-\Delta^m u \geqslant 0 \quad \text{in } B_2(0)\setminus\{0\}\subset\mathbb{R}^n, \tag{1.8}$$

which we state in Lemma 4.1.

2. Results for Question 1

In this section we state and discuss our results for Question 1. If $m\geqslant 1$ and $n\geqslant 2$ are integers then m and n satisfy one of the following five conditions:

(i) either m is even or $2m>n$;
(ii) $m=1$ and $n\geqslant 3$;
(iii) $m=1$ and $n=2$;
(iv) $m\geqslant 3$ is odd and $2m<n$;
(v) $m\geqslant 3$ is odd and $2m=n$.

The following three theorems, which we proved in [8,16,15], completely answer Question 1 when m and n satisfy either (i), (ii), or (iii). Consequently, in this paper, we will only prove results dealing with the case that m and n satisfy either (iv) or (v).

Theorem 2.1. *Suppose $m\geqslant 1$ and $n\geqslant 2$ are integers satisfying* (i) *and $f:[0,\infty)\to[0,\infty)$ is a continuous function. Let $u(x)$ be a C^{2m} nonnegative solution of* (1.1) *or, more generally, of* (1.8). *Then*

$$u(x)=O\bigl(\Gamma\bigl(|x|\bigr)\bigr) \quad \text{as } x\to 0, \tag{2.1}$$

where Γ is given by (1.3).

Theorem 2.2. *Let $u(x)$ be a C^2 nonnegative solution of* (1.1) *where the integers m and n satisfy* (ii) (*resp.* (iii)), *and $f:[0,\infty)\to[0,\infty)$ is a continuous function satisfying*

$$f(t)=O\bigl(t^{n/(n-2)}\bigr) \quad \bigl(\textit{resp. } \log\bigl(1+f(t)\bigr)=O(t)\bigr) \quad \text{as } t\to\infty. \tag{2.2}$$

Then u satisfies (2.1).

By Remark 1.1 the bound (2.1) for u in Theorems 2.1 and 2.2 is optimal.

By the following theorem, the condition (2.2) on f in Theorem 2.2 for the existence of a pointwise bound for u is essentially optimal.

Theorem 2.3. *Suppose m and n are integers satisfying* (ii) (*resp.* (iii)), *and $f:[0,\infty)\to[0,\infty)$ is a continuous function satisfying*

$$\lim_{t\to\infty}\frac{f(t)}{t^{n/(n-2)}}=\infty \quad \left(\textit{resp. } \lim_{t\to\infty}\frac{\log(1+f(t))}{t}=\infty\right). \tag{2.3}$$

Then for each continuous function $\varphi : (0, 1) \to (0, \infty)$ *there exists a* C^2 *positive solution* $u(x)$ *of* (1.1) *such that*

$$u(x) \neq O\big(\varphi\big(|x|\big)\big) \quad \text{as } x \to 0.$$

If m and n satisfy (i), (ii), or (iii), then according to Theorems 2.1, 2.2, and 2.3, either the optimal pointwise bound for u is given by (2.1) or there does not exist a pointwise bound for u (provided we don't allow the rather uninteresting and pathological possibility when m and n satisfy (ii) (resp. (iii)), that f satisfies neither (2.2) nor (2.3)).

The situation is very different and more interesting when m and n satisfy (iv) or (v). In this case, according to the following results, there are an infinite number of different optimal pointwise bounds for u depending on f.

The following three theorems deal with Question 1 when m and n satisfy (iv).

Theorem 2.4. *Let* $u(x)$ *be a* C^{2m} *nonnegative solution of* (1.1) *where the integers* m *and* n *satisfy* (iv) *and* $f : [0, \infty) \to [0, \infty)$ *is a continuous function satisfying*

$$f(t) = O\big(t^\lambda\big) \quad \text{as } t \to \infty$$

where

$$0 \leqslant \lambda \leqslant \frac{2m + n - 2}{n - 2} \quad \left(\text{resp. } \frac{2m + n - 2}{n - 2} < \lambda < \frac{n}{n - 2m}\right).$$

Then as $x \to 0$,

$$u(x) = O\big(|x|^{-(n-2)}\big) \tag{2.4}$$

$$\left(\text{resp. } u(x) = o\big(|x|^{-a}\big) \text{ where } a = \frac{4m(m-1)}{n - \lambda(n - 2m)}\right). \tag{2.5}$$

Since a in (2.5) is also given by

$$a = n - 2 + \frac{\lambda(n-2) - (2m + n - 2)}{n - \lambda(n - 2m)}(n - 2m) \tag{2.6}$$

we see that a increases from $n - 2$ to infinity as λ increases from $\frac{2m+n-2}{n-2}$ to $\frac{n}{n-2m}$.

By Remark 1.1, the bound (2.4) is optimal and by the following theorem so is the bound (2.5).

Theorem 2.5. *Suppose* m *and* n *are integers satisfying* (iv) *and* λ *and* a *are constants satisfying*

$$\frac{2m + n - 2}{n - 2} < \lambda < \frac{n}{n - 2m} \quad \text{and} \quad a = \frac{4m(m-1)}{n - \lambda(n - 2m)}. \tag{2.7}$$

Let $\varphi : (0, 1) \to (0, 1)$ *be a continuous function satisfying* $\lim_{r \to 0^+} \varphi(r) = 0$. *Then there exists a* C^∞ *positive solution* $u(x)$ *of*

$$0 \leqslant -\Delta^m u \leqslant u^\lambda \quad \text{in } \mathbb{R}^n \setminus \{0\} \tag{2.8}$$

such that

$$u(x) \neq O\big(\varphi\big(|x|\big)|x|^{-a}\big) \quad \text{as } x \to 0. \tag{2.9}$$

With regard to Theorem 2.4, it is natural to ask what happens when $\lambda \geqslant \frac{n}{n-2m}$. The answer, given by the following theorem, is that the solutions u can be arbitrarily large as $x \to 0$.

Theorem 2.6. *Suppose* m *and* n *are integers satisfying* (iv) *and* $\lambda \geqslant \frac{n}{n-2m}$ *is a constant. Let* $\varphi : (0, 1) \to (0, \infty)$ *be a continuous function satisfying* $\lim_{r \to 0^+} \varphi(r) = \infty$. *Then there exists a* C^∞ *positive solution* $u(x)$ *of* (2.8) *such that*

$$u(x) \neq O\big(\varphi\big(|x|\big)\big) \quad \text{as } x \to 0.$$

The following five theorems deal with Question 1 when m and n satisfy (v). This is the most interesting case.

Theorem 2.7. *Let $u(x)$ be a C^{2m} nonnegative solution of* (1.1) *where the integers m and n satisfy* (v) *and* $f:[0,\infty)\to[0,\infty)$ *is a continuous function satisfying*

$$f(t)=O\big(t^{\lambda}\big)\quad \text{as } t\to\infty$$

where

$$0\leqslant\lambda\leqslant\frac{2n-2}{n-2}\quad\left(\text{resp. } \lambda>\frac{2n-2}{n-2}\right).$$

Then as $x\to 0$,

$$u(x)=O\big(|x|^{-(n-2)}\big) \tag{2.10}$$

$$\left(\text{resp. } u(x)=o\left(|x|^{-(n-2)}\log\frac{5}{|x|}\right)\right). \tag{2.11}$$

By Remark 1.1, the bound (2.10) is optimal and by the following theorem so is the bound (2.11).

Theorem 2.8. *Suppose m and n are integers satisfying* (v) *and λ is a constant satisfying*

$$\lambda>\frac{2n-2}{n-2}. \tag{2.12}$$

Let $\varphi:(0,1)\to(0,1)$ be a continuous function satisfying $\lim_{r\to 0^+}\varphi(r)=0$. *Then there exists a C^∞ positive solution $u(x)$ of* (2.8) *such that*

$$u(x)\neq O\left(\varphi\big(|x|\big)|x|^{-(n-2)}\log\frac{5}{|x|}\right)\quad \text{as } x\to 0. \tag{2.13}$$

By the following theorem $u(x)$ may satisfy a pointwise a priori bound even when $f(t)$ grows, as $t\to\infty$, faster than any power of t.

Theorem 2.9. *Let $u(x)$ be a C^{2m} nonnegative solution of* (1.1) *where the integers m and n satisfy* (v) *and* $f:[0,\infty)\to[0,\infty)$ *is a continuous function satisfying*

$$\log\big(1+f(t)\big)=O\big(t^{\lambda}\big)\quad \text{as } t\to\infty$$

where

$$0<\lambda<1. \tag{2.14}$$

Then

$$u(x)=o\big(|x|^{\frac{-(n-2)}{1-\lambda}}\big)\quad \text{as } x\to 0. \tag{2.15}$$

By the following theorem, the estimate (2.15) in Theorem 2.9 is optimal.

Theorem 2.10. *Suppose m and n are integers satisfying* (v) *and λ is a constant satisfying* (2.14). *Let $\varphi:(0,1)\to(0,1)$ be a continuous function satisfying* $\lim_{r\to 0^+}\varphi(r)=0$. *Then there exists a C^∞ positive solution $u(x)$ of*

$$0\leqslant-\Delta^m u\leqslant e^{u^{\lambda}}\quad \text{in } \mathbb{R}^n\setminus\{0\} \tag{2.16}$$

such that

$$u(x)\neq O\big(\varphi\big(|x|\big)|x|^{\frac{-(n-2)}{1-\lambda}}\big)\quad \text{as } x\to 0. \tag{2.17}$$

With regard to Theorem 2.9, it is natural to ask what happens when $\lambda\geqslant 1$. The answer, given by the following theorem, is that the solutions u can be arbitrarily large as $x\to 0$.

Theorem 2.11. *Suppose m and n are integers satisfying* (v) *and $\lambda \geqslant 1$ is a constant. Let $\varphi : (0,1) \to (0,\infty)$ be a continuous function satisfying $\lim_{r\to 0^+} \varphi(r) = \infty$. Then there exists a C^∞ positive solution of* (2.16) *such that*

$$u(x) \neq O\big(\varphi\big(|x|\big)\big) \quad \text{as } x \to 0. \tag{2.18}$$

Theorems 2.3–2.11 are "nonradial". By this we mean that if one requires the solutions $u(x)$ in Question 1 to be radial then, according to the following proposition, the complete answer to Question 1 is very different.

Proposition 2.1. *Suppose $m \geqslant 1$ and $n \geqslant 2$ are integers and $f : [0,\infty) \to [0,\infty)$ is a continuous function. Let $u(x)$ be a C^{2m} nonnegative* ***radial*** *solution of* (1.1) *or, more generally, of* (1.8)*. Then u satisfies* (2.1)*.*

By Remark 1.1, the bound (2.1) for u in Proposition 2.1 is optimal.

Theorems 2.4 and 2.7 are special cases of much more general results, in which, instead of obtaining pointwise upper bounds (when they exist) for u where u is a nonnegative solution of

$$0 \leqslant -\Delta^m u \leqslant (u+1)^\lambda \quad \text{in } B_2(0) \setminus \{0\},$$

we obtain pointwise upper bounds (when they exist) for $|D^i u|$, $i = 0, 1, 2, \ldots, 2m-1$, where u is a solution of

$$0 \leqslant -\Delta^m u \leqslant \sum_{k=0}^{2m-1} |x|^{-\alpha_k} \big(\big|D^k u\big| + g_k(x)\big)^{\lambda_k} \quad \text{in } B_2(0) \setminus \{0\} \subset \mathbb{R}^n$$

such that

$$|x|^{n-2} u(x) \quad \text{is bounded below in } B_2(0) \setminus \{0\},$$

where the functions $g_k(x)$ tend to infinity as $x \to 0$. See Theorems 5.1 and 5.2 in Section 5 for the precise statements of these more general results.

Estimates for some derivatives of nonnegative solutions of (1.1) when m and n satisfy (i) were obtained in [8].

If $m \geqslant 1$ and $n \geqslant 2$ are integers satisfying (i) then, according to Theorem 2.1, u satisfies a pointwise upper bound as $x \to 0$ without imposing an upper bound $f(u)$ on $-\Delta^m u$. On the other hand, if m and n do not satisfy (i) then according to Theorems 2.2–2.11, u satisfies a pointwise upper bound as $x \to 0$ if and only if an appropriate upper bound $f(u)$ is placed on $-\Delta^m u$. This is due to the following two reasons.

1. According to formulas (4.1)–(4.3) for the fundamental solution Φ of Δ^m in $\mathbb{R}^n$, Φ is bounded below in $B_1(0) \setminus \{0\}$ if and only if m and n satisfy (i).
2. There is a term in a decomposed version of the representation formula (4.5) for nonnegative solutions u of (1.8) which is bounded above when Φ is bounded below. However, when Φ is not bounded below, one needs an upper bound on $-\Delta^m u$ to estimate this term. The crux of many of the proofs consists of obtaining this estimate.

The term referred to in 2 can be thought of as the convolution

$$\int_{|y|<1} \Phi(x-y)\Delta^m u(y)\, dy. \tag{2.19}$$

However it may happen when $m \geqslant 2$ that $-\Delta^m u \notin L^1(B_1(0))$, in which case this convolution is not finite for every $x \in \mathbb{R}^n$. This difficulty is overcome in Lemma 4.1 by replacing $\Phi(x-y)$ in (2.19) with the difference of $\Phi(x-y)$ and a partial sum of the Taylor series of Φ at x.

3. Results for Question 2

In this section we state our results for Question 2.

As noted in [8], by applying the m-Kelvin transform (1.5) to the function u in Theorem 2.1, we immediately obtain the following result concerning Question 2 when m and n satisfy condition (i) at the beginning of Section 2.

Theorem 3.1. *Suppose $m \geqslant 1$ and $n \geqslant 2$ are integers satisfying* (i) *and $f : [0,\infty) \to [0,\infty)$ is a continuous function. Let $v(y)$ be a C^{2m} nonnegative solution of* (1.4) *or, more generally, of*

$$-\Delta^m v \geqslant 0 \quad \text{in } \mathbb{R}^n \setminus B_{1/2}(0).$$

Then

$$v(y) = O\big(\Gamma_\infty(|y|)\big) \quad \text{as } |y| \to \infty, \tag{3.1}$$

where

$$\Gamma_\infty(r) = \begin{cases} r^{2m-2}, & \text{if } n \geqslant 3; \\ r^{2m-2}\log 5r, & \text{if } n = 2. \end{cases}$$

The estimate (3.1) is optimal because $\Delta^m \Gamma_\infty(|y|) = 0$ in $\mathbb{R}^n \setminus \{0\}$.

Using the m-Kelvin transform and Theorems 5.1, 5.2, and 5.3 in Section 5 we will prove in Section 6 the following three theorems dealing with Question 2, the first of which deals with the case that m and n satisfy condition (iv) at the beginning of Section 2.

Theorem 3.2. *Let $v(y)$ be a C^{2m} nonnegative solution of* (1.4) *where the integers m and n satisfy* (iv) *and $f : [0,\infty) \to [0,\infty)$ is a continuous function satisfying*

$$f(t) = O\big(t^\lambda\big) \quad \text{as } t \to \infty$$

where

$$0 < \lambda < \frac{n}{n-2m}.$$

Then

$$v(y) = o\big(|y|^a\big) \quad \text{as } |y| \to \infty \tag{3.2}$$

where

$$a = \frac{2m(n-2)}{n-\lambda(n-2m)} = 2m-2+\frac{2(1+\lambda(m-1))}{n-\lambda(n-2m)}(n-2m).$$

The next two theorems deal with Question 2 when m and n satisfy condition (v) at the beginning of Section 2.

Theorem 3.3. *Let $v(y)$ be a C^{2m} nonnegative solution of* (1.4) *where the integers m and n satisfy* (v) *and $f : [0,\infty) \to [0,\infty)$ is a continuous function satisfying*

$$f(t) = O\big(t^\lambda\big) \quad \text{as } t \to \infty$$

where $\lambda > 0$. Then

$$v(y) = o\big(|y|^{n-2}\log 5|y|\big) \quad \text{as } |y| \to \infty. \tag{3.3}$$

Theorem 3.4. *Let $v(y)$ be a C^{2m} nonnegative solution of* (1.4) *where the integers m on n satisfy* (v) *and $f : [0,\infty) \to [0,\infty)$ is a continuous function satisfying*

$$\log\big(1+f(t)\big) = O\big(t^\lambda\big) \quad \text{as } t \to \infty$$

where $0 < \lambda < 1$. Then

$$v(y) = o\big(|y|^{\frac{n-2}{1-\lambda}}\big) \quad \text{as } |y| \to \infty.$$

Theorems 3.2–3.4 are optimal for Question 2 in the same way that Theorems 2.4, 2.7, and 2.9 are optimal for Question 1. For example, according to the following theorem, the bound (3.2) in Theorem 3.2 is optimal. We will omit the precise statements and proofs of the other optimality results for Theorems 3.2–3.4.

Theorem 3.5. *Suppose m and n are integers satisfying* (iv) *and λ and a are constants satisfying*

$$0<\lambda<\frac{n}{n-2m} \quad \text{and} \quad a=\frac{2m(n-2)}{n-\lambda(n-2m)}. \tag{3.4}$$

Let $\varphi:(1,\infty)\to(0,1)$ be a continuous function satisfying $\lim_{r\to\infty}\varphi(r)=0$. Then there exists a C^∞ positive solution $v(y)$ of

$$0\leqslant -\Delta^m v\leqslant v^\lambda \quad \text{in } \mathbb{R}^n\setminus\{0\}$$

such that

$$v(y)\neq O\big(\varphi\big(|y|\big)|y|^a\big) \quad \text{as } |y|\to\infty.$$

See [16, Corollary 2.5] for the optimal result concerning Question 2 when m and n satisfy (iii). We have no results for Question 2 when m and n satisfy (ii), but see [1] for some related results.

4. Preliminary results

A fundamental solution of Δ^m in $\mathbb{R}^n$, where $m\geqslant 1$ and $n\geqslant 2$ are integers, is given by

$$\Phi(x):=A\begin{cases}(-1)^m|x|^{2m-n}, & \text{if } 2\leqslant 2m<n; \quad (4.1)\\ (-1)^{\frac{n-1}{2}}|x|^{2m-n}, & \text{if } 3\leqslant n<2m \text{ and } n \text{ is odd}; \quad (4.2)\\ (-1)^{\frac{n}{2}}|x|^{2m-n}\log\frac{5}{|x|}, & \text{if } 2\leqslant n\leqslant 2m \text{ and } n \text{ is even}; \quad (4.3)\end{cases}$$

where $A=A(m,n)$ is a *positive* constant whose value may change from line to line throughout this entire paper. In the sense of distributions, $\Delta^m\Phi=\delta$, where δ is the Dirac mass at the origin in $\mathbb{R}^n$. For $x\neq 0$ and $y\neq x$, let

$$\Psi(x,y)=\Phi(x-y)-\sum_{|\alpha|\leqslant 2m-3}\frac{(-y)^\alpha}{\alpha!}D^\alpha\Phi(x) \tag{4.4}$$

be the error in approximating $\Phi(x-y)$ with the partial sum of degree $2m-3$ of the Taylor series of Φ at x.

The following lemma, which we proved in [8], gives representation formula (4.5) for nonnegative solutions of inequality (1.8). See [5,6] for similar results.

Lemma 4.1. *Let $u(x)$ be a C^{2m} nonnegative solution of* (1.8) *where $m\geqslant 1$ and $n\geqslant 2$ are integers. Then $\int_{|y|<1}|y|^{2m-2}(-\Delta^m u(y))\,dy<\infty$ and*

$$u=N+h+\sum_{|\alpha|\leqslant 2m-2}a_\alpha D^\alpha\Phi \quad \text{in } B_1(0)\setminus\{0\} \tag{4.5}$$

where a_α, $|\alpha|\leqslant 2m-2$, are constants, $h\in C^\infty(B_1(0))$ is a solution of

$$\Delta^m h=0 \quad \text{in } B_1(0),$$

and

$$N(x)=\int_{|y|\leqslant 1}\Psi(x,y)\Delta^m u(y)\,dy \quad \text{for } x\neq 0.$$

Lemma 4.2. *Suppose f is locally bounded, nonnegative, and measurable in $\overline{B_1(0)}\setminus\{0\}\subset\mathbb{R}^n$ and*

$$\int_{|y|<1}|y|^{2m-2}f(y)\,dy<\infty \tag{4.6}$$

where $m\geqslant 2$ and $n\geqslant 2$ are integers, m is odd, and $2m\leqslant n$. Let

$$N(x)=\int_{|y|<1}-\Psi(x,y)f(y)\,dy \quad \text{for } x\in\mathbb{R}^n\setminus\{0\} \tag{4.7}$$

where Ψ is given by (4.4). Then $N \in C^{2m-1}(\mathbb{R}^n \setminus \{0\})$. Moreover when $|\beta| < 2m$ and either $2m = n$ and $|\beta| \neq 0$ or $2m < n$ we have

$$\big(D^\beta N\big)(x) = \int\limits_{\substack{|y|<1\\ |y-x|<|x|/2}} -\big(D^\beta \Phi\big)(x-y) f(y)\,dy + O\big(|x|^{2-n-|\beta|}\big) \quad \text{for } x \neq 0 \tag{4.8}$$

and when $2m = n$ we have

$$N(x) = A \int\limits_{\substack{|y|<1\\ |y-x|<|x|/2}} \left(\log \frac{|x|}{|x-y|}\right) f(y)\,dy + O\big(|x|^{2-n}\big) \quad \text{for } x \neq 0. \tag{4.9}$$

Proof. Differentiating (4.4) with respect to x we get

$$D_x^\beta \Psi(x,y) = \big(D^\beta \Phi\big)(x-y) - \sum_{|\alpha| \leqslant 2m-3} \frac{(-y)^\alpha}{\alpha!} \big(D^{\alpha+\beta}\Phi\big)(x) \quad \text{for } x \neq 0 \text{ and } y \neq x$$

and so by Taylor's theorem applied to $D^\beta \Phi$ we have

$$\big|D_x^\beta \Psi(x,y)\big| \leqslant C|y|^{2m-2}|x|^{2-n-|\beta|} \quad \text{for } |y| < \frac{|x|}{2} \tag{4.10}$$

where in this proof $C = C(m,n,\beta)$ is a positive constant whose value may change from line to line.

Let $\varepsilon \in (0,1)$ be fixed. Then $N = N_1 + N_2$ in $\mathbb{R}^n \setminus \{0\}$ where

$$N_1(x) = \int\limits_{|y|<\varepsilon} -\Psi(x,y) f(y)\,dy \quad \text{and} \quad N_2(x) = \int\limits_{\varepsilon<|y|<1} -\Psi(x,y) f(y)\,dy.$$

It follows from (4.6) and (4.10) that $N_1 \in C^\infty(\mathbb{R}^n \setminus \overline{B_{2\varepsilon}(0)})$ and

$$\big(D^\beta N_1\big)(x) = \int\limits_{|y|<\varepsilon} -D^\beta \Psi(x,y) f(y)\,dy \quad \text{for } |x| > 2\varepsilon.$$

Also, by the boundedness of f in $B_1(0) \setminus B_\varepsilon(0)$, $N_2 \in C^{2m-1}(\mathbb{R}^n \setminus \overline{B_{2\varepsilon}(0)})$ and for $|\beta| < 2m$ we have

$$\big(D^\beta N_2\big)(x) = \int\limits_{\varepsilon<|y|<1} -D^\beta \Psi(x,y) f(y)\,dy \quad \text{for } |x| > 2\varepsilon.$$

Thus since $\varepsilon \in (0,1)$ was arbitrary, we have $N \in C^{2m-1}(\mathbb{R}^n \setminus \{0\})$ and for $|\beta| < 2m$ we have

$$\big(D^\beta N\big)(x) = \int\limits_{|y|<1} -D_x^\beta \Psi(x,y) f(y)\,dy \quad \text{for } x \neq 0. \tag{4.11}$$

Case 1. Suppose $|\beta| < 2m$ and either $2m = n$ and $|\beta| \neq 0$ or $2m < n$. Then for $0 < |x|/2 < |y|$ we have

$$\left|\sum_{|\alpha| \leqslant 2m-3} \frac{(-y)^\alpha}{\alpha!} D^{\alpha+\beta}\Phi(x)\right| \leqslant C \sum_{|\alpha| \leqslant 2m-3} |y|^{|\alpha|}|x|^{2m-n-|\alpha|-|\beta|} \leqslant C|y|^{2m-2}|x|^{2-n-|\beta|}$$

and for $0 < |x|/2 < |y|$ and $|y-x| > |x|/2$ we have

$$\big|\big(D^\beta \Phi\big)(x-y)\big| \leqslant C|x-y|^{2m-n-|\beta|} \leqslant C|x|^{2m-n-|\beta|} \leqslant C|y|^{2m-2}|x|^{2-n-|\beta|}.$$

Thus (4.6), (4.10) and (4.11) imply (4.8).

Case 2. Suppose $2m = n$. Then for $0 < |x|/2 < |y|$ we have

$$\left|\sum_{1 \leqslant |\alpha| \leqslant 2m-3} \frac{(-y)^\alpha}{\alpha!} D^\alpha \Phi(x)\right| \leqslant C \sum_{1 \leqslant |\alpha| \leqslant 2m-3} |y|^{|\alpha|}|x|^{2m-n-|\alpha|} \leqslant C|y|^{2m-2}|x|^{2-n}$$

and if $0 < |x|/2 < |y|$ and $|y - x| > |x|/2$ then using the fact that $|\log z| \leqslant \log 4z$ for $z \geqslant 1/2$ we have

$$\begin{aligned}
\left|-\Phi(x-y)+\Phi(x)\right| &= A\left|\log\frac{|x-y|}{|x|}\right| \leqslant A\log 4\frac{|x-y|}{|x|}\\
&\leqslant A\frac{|y|^{n-2}}{|x|^{n-2}}\left(\frac{|x|}{|y|}\right)^{n-2}\log 4\left(1+\frac{|y|}{|x|}\right)\\
&\leqslant A\frac{|y|^{n-2}}{|x|^{n-2}}\max_{r\geqslant 1/2} r^{2-n}\log 4(1+r).
\end{aligned}$$

Thus (4.9) follows from (4.6), (4.7), and (4.10). □

Lemma 4.3. *Suppose $u(x)$ is a C^{2m} nonnegative solution of (1.8), where $m \geqslant 2$ and $n \geqslant 2$ are integers, m is odd, and $2m \leqslant n$. Let $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}^n$ and $\{r_j\}_{j=1}^{\infty} \subset \mathbb{R}$ be sequences such that*

$$0 < 4|x_{j+1}| \leqslant |x_j| \leqslant 1/2 \quad \text{and} \quad 0 < r_j \leqslant |x_j|/4. \tag{4.12}$$

Define $f_j : B_2(0) \to [0,\infty)$ by

$$f_j(\eta) = |x_j|^{2m-2} r_j^n f(y) \quad \text{where } y = x_j + r_j\eta \text{ and } f = -\Delta^m u. \tag{4.13}$$

Then

$$\int_{|\eta|<2} f_j(\eta)\,d\eta \to 0 \quad \text{as } j \to \infty \tag{4.14}$$

and when $|\beta| < 2m$ and either $2m = n$ and $|\beta| \neq 0$ or $2m < n$ we have for $|\xi| < 1$ that

$$\left(\frac{r_j}{|x_j|}\right)^{n-2m+|\beta|} |x_j|^{n-2+|\beta|}\left|\left(D^\beta u\right)(x_j + r_j\xi)\right| \leqslant C\left(\frac{r_j}{|x_j|}\right)^{n-2m+|\beta|} + \varepsilon_j + \int_{|\eta|<2} \frac{A f_j(\eta)\,d\eta}{|\xi-\eta|^{n-2m+|\beta|}} \tag{4.15}$$

and when $2m = n$ we have for $|\xi| < 1$ that

$$\frac{|x_j|^{n-2}}{\log\frac{|x_j|}{r_j}} u(x_j + r_j\xi) \leqslant \frac{C}{\log\frac{|x_j|}{r_j}} + \varepsilon_j + \frac{1}{\log\frac{|x_j|}{r_j}} \int_{|\eta|<2} A\left(\log\frac{5}{|\xi-\eta|}\right) f_j(\eta)\,d\eta \tag{4.16}$$

where in (4.15) and (4.16) the constant A depends only on m and n, the constant C is independent of ξ and j, the constants ε_j are independent of ξ, and $\varepsilon_j \to 0$ as $j \to \infty$.

Proof. By Lemma 4.1, f satisfies (4.6) and for $|\beta| < 2m$ we have

$$\left(D^\beta u\right)(x) = \left(D^\beta N\right)(x) + O\left(|x|^{2-n-|\beta|}\right) \quad \text{for } 0 < |x| \leqslant 3/4 \tag{4.17}$$

where N is given by (4.7).

If

$$|y - x| < |x|/2, \qquad |y - x_j| > 2r_j, \quad \text{and} \quad |x - x_j| < r_j$$

then

$$|x - y| > r_j \quad \text{and} \quad 2|y| > |x| > |x_j| - r_j > |x_j|/2$$

and thus when $|\beta| < 2m$ and either $2m = n$ and $|\beta| \neq 0$ or $2m < n$ we have

$$\left|\left(D^\beta\Phi\right)(x-y)\right| \leqslant \frac{A}{|x-y|^{n-2m+|\beta|}} \leqslant \frac{A}{r_j^{n-2m+|\beta|}} \leqslant \frac{A|y|^{2m-2}}{r_j^{n-2m+|\beta|}|x_j|^{2m-2}}$$

and when $2m = n$ we have

$$\log\frac{|x|}{|x-y|} \leqslant \log\frac{\frac{5}{4}|x_j|}{r_j} \leqslant 2\cdot 4^{n-2}\frac{|y|^{n-2}}{|x_j|^{n-2}}\log\frac{|x_j|}{r_j}.$$

Thus by (4.6) and Lemma 4.2, when $|\beta| < 2m$ and either $2m = n$ and $|\beta| \neq 0$ or $2m < n$ we have

$$\begin{aligned} \left|\left(D^\beta N\right)(x)\right| &\leqslant \int\limits_{|y-x_j|<2r_j} \frac{Af(y)\,dy}{|x-y|^{n-2m+|\beta|}} + A\frac{\int_{|y-x|<|x|/2}|y|^{2m-2}f(y)\,dy}{r_j^{n-2m+|\beta|}|x_j|^{2m-2}} + \frac{C}{|x_j|^{n-2+|\beta|}} \\ &\leqslant \int\limits_{|y-x_j|<2r_j} \frac{Af(y)\,dy}{|x-y|^{n-2m+|\beta|}} + \frac{\varepsilon_j}{r_j^{n-2m+|\beta|}|x_j|^{2m-2}} + \frac{C}{|x_j|^{n-2+|\beta|}} \quad \text{for } |x-x_j|<r_j \end{aligned} \tag{4.18}$$

and when $2m = n$ we have

$$\begin{aligned} N(x) &\leqslant A \int\limits_{|y-x_j|<2r_j} \left(\log\frac{|x|}{|x-y|}\right) f(y)\,dy + 2A4^{n-2}\left(\int\limits_{|y-x|<|x|/2} |y|^{n-2}f(y)\,dy\right)\frac{\log\frac{|x_j|}{r_j}}{|x_j|^{n-2}} + \frac{C}{|x_j|^{n-2}} \\ &\leqslant A \int\limits_{|y-x_j|<2r_j} \left(\log\frac{|x|}{|x-y|}\right) f(y)\,dy + \varepsilon_j\frac{\log\frac{|x_j|}{r_j}}{|x_j|^{n-2}} + \frac{C}{|x_j|^{n-2}} \quad \text{for } |x-x_j|<r_j \end{aligned} \tag{4.19}$$

where in (4.18) and (4.19) the constant A depends only on m and n, the constant C is independent of x and j, the constants ε_j are independent of x, and $\varepsilon_j \to 0$ as $j \to \infty$.

For $|\eta| < 2$ and y given by (4.13) we have $|x_j| < 2|y|$. Thus

$$\begin{aligned} \int\limits_{|\eta|<2} f_j(\eta)\,d\eta &= \int\limits_{|y-x_j|<2r_j} |x_j|^{2m-2} f(y)\,dy \\ &\leqslant 2^{2m-2} \int\limits_{|y-x_j|<|x_j|/2} |y|^{2m-2} f(y)\,dy \to 0 \quad \text{as } j\to\infty \end{aligned} \tag{4.20}$$

because f satisfies (4.6).

If $|\beta| < 2m$ and either $2m = n$ and $|\beta| \neq 0$ or $2m < n$ then by (4.18) and (4.13) we have for $|\xi| < 1$ that

$$\begin{aligned} &\left(\frac{r_j}{|x_j|}\right)^{n-2m+|\beta|} |x_j|^{n-2+|\beta|}\left|\left(D^\beta N\right)(x_j + r_j\xi)\right| \\ &\leqslant C\left(\frac{r_j}{|x_j|}\right)^{n-2m+|\beta|} + \varepsilon_j + r_j^{n-2m+|\beta|}|x_j|^{2m-2} \int\limits_{|\eta|<2} \frac{Af(y)r_j^n\,d\eta}{r_j^{n-2m+|\beta|}|\xi-\eta|^{n-2m+|\beta|}} \\ &= C\left(\frac{r_j}{|x_j|}\right)^{n-2m+|\beta|} + \varepsilon_j + \int\limits_{|\eta|<2} \frac{Af_j(\eta)\,d\eta}{|\xi-\eta|^{n-2m+|\beta|}}. \end{aligned} \tag{4.21}$$

If $2m = n$ and $|\xi| < 1$ then by (4.19), (4.13), and (4.20) we have

$$\begin{aligned} \frac{|x_j|^{n-2}}{\log\frac{|x_j|}{r_j}} N(x_j + r_j\xi) &\leqslant \frac{C}{\log\frac{|x_j|}{r_j}} + \varepsilon_j + \frac{|x_j|^{n-2}}{\log\frac{|x_j|}{r_j}} A \int\limits_{|\eta|<2} \left(\log\frac{5|x_j|}{r_j|\xi-\eta|}\right)|x_j|^{2-n} f_j(\eta)\,d\eta \\ &\leqslant \frac{C}{\log\frac{|x_j|}{r_j}} + \varepsilon_j + \frac{1}{\log\frac{|x_j|}{r_j}} \int\limits_{|\eta|<2} A\left(\log\frac{5}{|\xi-\eta|}\right) f_j(\eta)\,d\eta. \end{aligned} \tag{4.22}$$

Inequalities (4.15) and (4.16) now follow from (4.21), (4.22), and (4.17). □

Lemma 4.4. *Suppose $m \geqslant 2$ and $n \geqslant 2$ are integers, m is odd, and $2m \leqslant n$. Let $\psi : (0,1) \to (0,1)$ be a continuous function such that $\lim_{r\to 0^+} \psi(r) = 0$. Let $\{x_j\}_{j=1}^\infty \subset \mathbb{R}^n$ be a sequence such that*

$$0 < 4|x_{j+1}| \leqslant |x_j| \leqslant 1/2 \tag{4.23}$$

and

$$\sum_{j=1}^{\infty} \varepsilon_j < \infty \quad \text{where } \varepsilon_j = \psi\big(|x_j|\big). \tag{4.24}$$

Let $\{r_j\}_{j=1}^{\infty} \subset \mathbb{R}$ *be a sequence satisfying*

$$0 < r_j \leqslant |x_j|/5. \tag{4.25}$$

Then there exist a positive function $u \in C^\infty(\mathbb{R}^n \setminus \{0\})$ *and a positive constant* $A = A(m,n)$ *such that*

$$20 \leqslant -\Delta^m u \leqslant \frac{\varepsilon_j}{|x_j|^{2m-2} r_j^n} \quad \text{in } B_{r_j}(x_j), \tag{4.26}$$

$$-\Delta^m u(x) = 0 \quad \text{in } \mathbb{R}^n \setminus \left(\{0\} \cup \bigcup_{j=1}^{\infty} B_{r_j}(x_j)\right), \tag{4.27}$$

and

$$u \geqslant \begin{cases} \frac{A\varepsilon_j}{|x_j|^{2m-2} r_j^{n-2m}} & \text{in } B_{r_j}(x_j) \text{ if } 2m < n, \\ \frac{A\varepsilon_j}{|x_j|^{n-2}} \log \frac{|x_j|}{r_j} & \text{in } B_{r_j}(x_j) \text{ if } 2m = n. \end{cases} \tag{4.28}$$

Proof. Let $\varphi : \mathbb{R}^n \to [0,1]$ be a C^∞ function whose support is $\overline{B_1(0)}$. Define $\varphi_j : \mathbb{R}^n \to [0,1]$ by

$$\varphi_j(y) = \varphi(\eta) \quad \text{where } y = x_j + r_j \eta.$$

Then

$$\int_{\mathbb{R}^n} \varphi_j(y)\,dy = \int_{\mathbb{R}^n} \varphi(\eta) r_j^n\,d\eta = r_j^n I \tag{4.29}$$

where $I = \int_{\mathbb{R}^n} \varphi(\eta)\,d\eta > 0$. Let

$$f = \sum_{j=1}^{\infty} M_j \varphi_j \quad \text{where } M_j = \frac{\varepsilon_j}{|x_j|^{2m-2} r_j^n}. \tag{4.30}$$

Since the functions φ_j have disjoint supports, $f \in C^\infty(\mathbb{R}^n \setminus \{0\})$ and by (4.25), (4.29), (4.30), and (4.24) we have

$$\begin{aligned} \int_{\mathbb{R}^n} |y|^{2m-2} f(y)\,dy &= \sum_{j=1}^{\infty} M_j \int_{|y-x_j|<r_j} |y|^{2m-2} \varphi_j(y)\,dy \\ &\leqslant 2^{2m-2} I \sum_{j=1}^{\infty} M_j |x_j|^{2m-2} r_j^n \\ &= 2^{2m-2} I \sum_{j=1}^{\infty} \varepsilon_j < \infty. \end{aligned} \tag{4.31}$$

Using the fact that

$$|x - x_j| < r_j \leqslant |x_j|/5 \quad \text{implies} \quad B_{r_j}(x_j) \subset B_{\frac{|x|}{2}}(x), \tag{4.32}$$

we have for $2m < n$, $x = x_j + r_j \xi$, and $|\xi| < 1$ that

$$\int_{|y-x|<|x|/2} \frac{1}{|x-y|^{n-2m}} f(y)\,dy \geqslant \int_{|y-x_j|<r_j} \frac{1}{|x-y|^{n-2m}} M_j \varphi_j(y)\,dy$$

$$= \int_{|\eta|<1} \frac{1}{r_j^{n-2m}} \frac{M_j}{|\xi - \eta|^{n-2m}} \varphi(\eta) r_j^n \, d\eta$$
$$= \frac{\varepsilon_j}{|x_j|^{2m-2} r_j^{n-2m}} \int_{|\eta|<1} \frac{\varphi(\eta)}{|\xi - \eta|^{n-2m}} \, d\eta$$
$$\geqslant \frac{J \varepsilon_j}{|x_j|^{2m-2} r_j^{n-2m}} \quad \text{where } J = \min_{|\xi| \leqslant 1} \int_{|\eta|<1} \frac{\varphi(\eta)\, d\eta}{|\xi - \eta|^{n-2m}}.$$

Similarly, using (4.32) we have for $2m = n$, $x = x_j + r_j \xi$, and $|\xi| < 1$ that

$$\int_{|y-x|<|x|/2} \left(\log \frac{|x|}{|x-y|}\right) f(y)\, dy \geqslant \int_{|y-x_j|<r_j} \left(\log \frac{|x|}{|x-y|}\right) M_j \varphi_j(y)\, dy$$
$$\geqslant \int_{|\eta|<1} \left(\log \frac{\frac{4}{5}|x_j|}{r_j |\xi - \eta|}\right) M_j \varphi(\eta) r_j^n \, d\eta$$
$$= \frac{\varepsilon_j}{|x_j|^{n-2}} \int_{|\eta|<1} \left(\log \frac{2}{|\xi - \eta|} + \log \frac{|x_j|}{r_j} - \log \frac{5}{2}\right) \varphi(\eta)\, d\eta$$
$$\geqslant \frac{I \varepsilon_j}{|x_j|^{n-2}} \log \frac{|x_j|}{r_j} - \frac{I}{|x_j|^{n-2}} \log \frac{5}{2}.$$

Thus defining N by (4.7), where f is given by (4.30), it follows from (4.31) and Lemma 4.2 that there exists a positive constant C independent of ξ and j such that if we define $u : \mathbb{R}^n \setminus \{0\} \to \mathbb{R}$ by

$$u(x) = N(x) + C|x|^{-(n-2)}$$

then u is a C^∞ positive solution of

$$-\Delta^m u = f \quad \text{in } \mathbb{R}^n \setminus \{0\} \tag{4.33}$$

and for some positive constant $A = A(m, n)$, u satisfies (4.28).

Also, (4.33) and (4.30) imply that u satisfies (4.26) and (4.27). □

Remark 4.1. Suppose the hypotheses of Lemma 4.4 hold and u is as in Lemma 4.4.

Case 1. Suppose $2m < n$. Then it follows from (4.26), (4.27), and (4.28) that u is a C^∞ positive solution of

$$0 \leqslant -\Delta^m u \leqslant |x|^\tau u^\lambda \quad \text{in } \mathbb{R}^n \setminus \{0\},\ \lambda > 0,\ \tau \in \mathbb{R},$$

provided

$$\frac{\psi(|x_j|)}{|x_j|^{2m-2} r_j^n} \leqslant 2^{-|\tau|} |x_j|^\tau \left(\frac{A\psi(|x_j|)}{|x_j|^{2m-2} r_j^{n-2m}}\right)^\lambda$$

which holds if and only if

$$r_j^{n-\lambda(n-2m)} \geqslant \frac{2^{|\tau|}}{A^\lambda} \frac{|x_j|^{(\lambda-1)(2m-2)-\tau}}{\psi(|x_j|)^{\lambda-1}}. \tag{4.34}$$

Case 2. Suppose $2m = n$. Then it follows from (4.26), (4.27), and (4.28) that u is a C^∞ positive solution of

$$0 \leqslant -\Delta^m u \leqslant f(u) \quad \text{in } \mathbb{R}^n \setminus \{0\},$$

where $f : [0, \infty) \to [0, \infty)$ is a nondecreasing continuous function, provided

$$\frac{\psi(|x_j|)}{|x_j|^{n-2} r_j^n} \leqslant f\left(\frac{A\psi(|x_j|)}{|x_j|^{n-2}} \log \frac{|x_j|}{r_j}\right). \tag{4.35}$$

If $f(u)=u^\lambda$, $\lambda>1$, then (4.35) holds if and only if

$$\log\frac{|x_j|}{r_j}\geqslant\left(\frac{|x_j|}{r_j}\right)^{\frac{n}{\lambda}}\frac{|x_j|^a}{A\psi(|x_j|)^{\frac{\lambda-1}{\lambda}}}\quad\text{where } a=\frac{(n-2)(\lambda-1)-n}{\lambda}.$$

If $f(u)=e^{u^\lambda}$, $\lambda>0$, then (4.35) holds if and only if

$$\log\frac{\psi(|x_j|)}{|x_j|^{2n-2}}+n\log\frac{|x_j|}{r_j}\leqslant\left(\frac{A\psi(|x_j|)}{|x_j|^{n-2}}\log\frac{|x_j|}{r_j}\right)^\lambda.$$

Lemma 4.5. *Suppose $p>1$ and $R\in(0,2)$ are constants and $g:\mathbb{R}^n\to\mathbb{R}$ is defined by*

$$g(\xi)=\int\limits_{|\eta|<R}\left(\log\frac{5}{|\xi-\eta|}\right)f(\eta)\,d\eta$$

where $f\in L^1(B_R(0))$ (resp. $f\in L^p(B_R(0))$). Then

$$\|g\|_{L^p(B_R(0))}\leqslant C\|f\|_{L^1(B_R(0))}\quad\bigl(\textit{resp. } \|g\|_{L^\infty(B_R(0))}\leqslant C\|f\|_{L^p(B_R(0))}\bigr),$$

where $C=C(n,p,R)$ is a positive constant.

Proof. Define p' by $\frac{1}{p}+\frac{1}{p'}=1$. Then by Hölder's inequality we have

$$\begin{aligned}\int\limits_{|\xi|<R}\bigl|g(\xi)\bigr|^p\,d\xi&\leqslant\int\limits_{|\xi|<R}\left[\int\limits_{|\eta|<R}\left(\log\frac{5}{|\xi-\eta|}\right)\bigl|f(\eta)\bigr|^{1/p}\bigl|f(\eta)\bigr|^{1/p'}\,d\eta\right]^p d\xi\\&\leqslant\int\limits_{|\xi|<R}\left[\left(\int\limits_{|\eta|<R}\left(\log\frac{5}{|\xi-\eta|}\right)^p\bigl|f(\eta)\bigr|\,d\eta\right)^{1/p}\left(\int\limits_{|\eta|<R}\bigl|f(\eta)\bigr|\,d\eta\right)^{1/p'}\right]^p d\xi\\&=\left(\int\limits_{|\eta|<R}\bigl|f(\eta)\bigr|\,d\eta\right)^{p/p'}\int\limits_{|\eta|<R}\left(\int\limits_{|\xi|<R}\left(\log\frac{5}{|\xi-\eta|}\right)^p d\xi\right)\bigl|f(\eta)\bigr|\,d\eta\\&\leqslant C(n,p,R)\left(\int\limits_{|\eta|<R}\bigl|f(\eta)\bigr|\,d\eta\right)^p.\end{aligned}$$

The parenthetical part follows from Hölder's inequality. □

5. Proofs when the singularity is at the origin

In this section we prove Theorems 2.4–2.11 and Proposition 2.1 which deal with the case that the singularity is at the origin. Theorem 2.4 will follow easily from the following more general result.

Theorem 5.1. *Suppose $u(x)$ is a C^{2m} solution of*

$$0\leqslant-\Delta^m u\leqslant K\sum_{k=0}^{2m-1}|x|^{-\alpha_k}\bigl(\bigl|D^k u\bigr|+g_k(x)\bigr)^{\lambda_k}\quad\textit{in } B_2(0)\setminus\{0\}\subset\mathbb{R}^n\tag{5.1}$$

such that

$$|x|^{n-2}u(x)\quad\textit{is bounded below in } B_2(0)\setminus\{0\},\tag{5.2}$$

where $K>0$, λ_k, and α_k are constants, $m\geqslant2$ and $n\geqslant2$ are integers, m is odd, $2m<n$, and for $k=0,1,\dots,2m-1$ we have

$$0\leqslant\lambda_k<\frac{n}{n-2m+k}\tag{5.3}$$

and $g_k : B_2(0) \setminus \{0\} \to [1,\infty)$ *is a continuous function. Let*

$$a_k = (n-2+k) + b(n-2m+k)$$

where

$$b = \max\left\{0, \max_{0\leqslant k\leqslant 2m-1} \frac{\alpha_k + \lambda_k(n-2+k) - (2m+n-2)}{n-\lambda_k(n-2m+k)}\right\}. \tag{5.4}$$

(i) *If for* $k = 0,1,\dots,2m-1$ *we have*

$$g_k(x) = O\big(|x|^{-a_k}\big) \quad \text{as } x \to 0 \tag{5.5}$$

then for $i = 0,1,\dots,2m-1$ *we have*

$$\big|D^i u(x)\big| = O\big(|x|^{-a_i}\big) \quad \text{as } x \to 0.$$

(ii) *If* $b > 0$ *and for* $k = 0,1,\dots,2m-1$ *we have*

$$g_k(x) = o\big(|x|^{-a_k}\big) \quad \text{as } x \to 0 \tag{5.6}$$

and

$$\lambda_k > 0 \tag{5.7}$$

then for $i = 0,1,\dots,2m-1$ *we have*

$$\big|D^i u(x)\big| = o\big(|x|^{-a_i}\big) \quad \text{as } x \to 0.$$

Proof. It suffices to prove Theorem 5.1 when u is nonnegative. To see this choose $M > 0$ such that

$$v(x) := u(x) + M|x|^{-(n-2)} > 0 \quad \text{for } 0 < |x| < 2,$$

which is possible by (5.2), and then apply Theorem 5.1 to v after noting that $-\Delta^m v = -\Delta^m u$ and

$$\big|D^k u(x) - D^k v(x)\big| = O\big(|x|^{-(n-2+k)}\big) = \begin{cases} O(|x|^{-a_k}) & \text{if } b = 0, \\ o(|x|^{-a_k}) & \text{if } b > 0. \end{cases}$$

Suppose for contradiction that part (i) (resp. part (ii)) is false. Then there exist $i \in \{0,1,2,\dots,2m-1\}$ and a sequence $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}^n$ such that

$$0 < 4|x_{j+1}| < |x_j| < 1/2,$$

and

$$|x_j|^{a_i}\big|D^i u(x_j)\big| \to \infty \quad \text{as } j \to \infty \tag{5.8}$$

$$\Big(\text{resp. } \liminf_{j\to\infty} |x_j|^{a_i}\big|D^i u(x_j)\big| > 0\Big). \tag{5.9}$$

Let $r_j = |x_j|^{b+1}/4$. Then x_j and r_j satisfy (4.12). Let f_j be as in Lemma 4.3. Since

$$\frac{r_j}{|x_j|} = \frac{|x_j|^b}{4} \tag{5.10}$$

it follows from (4.15) with $|\beta| = i$ and $\xi = 0$ that

$$\frac{|x_j|^{n-2+i+b(n-2m+i)}}{4^{n-2m+i}}\big|D^i u(x_j)\big| \leqslant C|x_j|^{(n-2m+i)b} + \varepsilon_j + \int_{|\eta|<2} \frac{Af_j(\eta)\,d\eta}{|\eta|^{n-2m+i}}.$$

Hence (5.8) (resp. (5.9)) implies

$$\int_{|\eta|<2} \frac{f_j(\eta)\,d\eta}{|\eta|^{n-2m+i}} \to \infty \quad \text{as } j \to \infty \tag{5.11}$$

$$\Big(\text{resp. } \liminf_{j\to\infty} \int_{|\eta|<2} \frac{f_j(\eta)\,d\eta}{|\eta|^{n-2m+i}} > 0\Big). \tag{5.12}$$

On the other hand, (4.13), (5.1), and (4.15) imply for $|\xi| < 1$ that

$$\begin{aligned} f_j(\xi) &\leqslant |x_j|^{2m+n-2}\left(\frac{r_j}{|x_j|}\right)^n K \sum_{k=0}^{2m-1} |x_j + r_j\xi|^{-\alpha_k}\left(\left|D^k u(x_j + r_j\xi)\right| + g_k(x_j + r_j\xi)\right)^{\lambda_k} \\ &\leqslant C \sum_{k=0}^{2m-1} \frac{|x_j|^{2m+n-2}(\frac{r_j}{|x_j|})^n |x_j|^{-\alpha_k}}{(|x_j|^{n-2+k}(\frac{r_j}{|x_j|})^{n-2m+k})^{\lambda_k}}\left(\left(\frac{r_j}{|x_j|}\right)^{n-2m+k} + \varepsilon_j + \int\limits_{|\eta|<2} \frac{f_j(\eta)\,d\eta}{|\xi-\eta|^{n-2m+k}} \right. \\ &\quad \left. + |x_j|^{n-2+k}\left(\frac{r_j}{|x_j|}\right)^{n-2m+k} g_k(x_j + r_j\xi)\right)^{\lambda_k}, \end{aligned} \tag{5.13}$$

where C is a constant independent of ξ and j whose value may change from line to line. But (5.10) and (5.4) imply

$$\begin{aligned} \frac{|x_j|^{2m+n-2}(\frac{r_j}{|x_j|})^n |x_j|^{-\alpha_k}}{(|x_j|^{n-2+k}(\frac{r_j}{|x_j|})^{n-2m+k})^{\lambda_k}} &= |x_j|^{(2m+n-2)-\lambda_k(n-2+k)-\alpha_k}\left(\frac{r_j}{|x_j|}\right)^{n-\lambda_k(n-2m+k)} \\ &\leqslant |x_j|^{(2m+n-2)-\lambda_k(n-2+k)-\alpha_k+(n-\lambda_k(n-2m+k))b} \\ &\leqslant 1, \end{aligned}$$

$$\begin{aligned} |x_j|^{n-2+k}\left(\frac{r_j}{|x_j|}\right)^{n-2m+k} &\leqslant |x_j|^{n-2+k+b(n-2m+k)} \\ &= |x_j|^{a_k}, \end{aligned}$$

and

$$\left(\frac{r_j}{|x_j|}\right)^{n-2m+k} \leqslant |x_j|^{b(n-2m+k)}.$$

Hence by (5.5) (resp. (5.6)) and (5.13) we have

$$f_j(\xi) \leqslant C \sum_{k=0}^{2m-1}\left(1 + \int\limits_{|\eta|<2} \frac{f_j(\eta)\,d\eta}{|\xi-\eta|^{n-2m+k}}\right)^{\lambda_k} \quad \text{for } |\xi| < 1 \tag{5.14}$$

$$\left(\text{resp. } f_j(\xi) \leqslant C \sum_{k=0}^{2m-1}\left(\varepsilon_j + \int\limits_{|\eta|<2} \frac{f_j(\eta)\,d\eta}{|\xi-\eta|^{n-2m+k}}\right)^{\lambda_k} \text{ for } |\xi| < 1\right). \tag{5.15}$$

Since

$$\int\limits_{2R\leqslant|\eta|<2} \frac{f_j(\eta)\,d\eta}{|\xi-\eta|^{n-2m+k}} \leqslant \frac{1}{R^{n-2m+k}} \int\limits_{|\eta|<2} f_j(\eta)\,d\eta \quad \text{for } |\xi| < R < 1$$

we have by (5.14) (resp. (5.15)) and (4.14) that

$$f_j(\xi) \leqslant C \sum_{k=0}^{2m-1}\left(\frac{1}{R^{n-2m+k}} + \int\limits_{|\eta|<2R} \frac{f_j(\eta)\,d\eta}{|\xi-\eta|^{n-2m+k}}\right)^{\lambda_k} \quad \text{for } |\xi| < R \leqslant 1 \tag{5.16}$$

where C is independent of ξ, j, and R (resp.

$$f_j(\xi) \leqslant C \sum_{k=0}^{2m-1}\left(\frac{\varepsilon_j}{R^{n-2m+k}} + \int\limits_{|\eta|<2R} \frac{f_j(\eta)\,d\eta}{|\xi-\eta|^{n-2m+k}}\right)^{\lambda_k} \quad \text{for } |\xi| < R \leqslant 1 \tag{5.17}$$

where ε_j is independent of ξ and R and $\varepsilon_j \to 0$ as $j \to \infty$).

We can assume the λ_k in (5.16) satisfy, instead of (5.3), the stronger condition

$$0 < \lambda_k < \frac{n}{n-2m+k} \tag{5.18}$$

because slightly increasing those λ_k in (5.16) which are zero will increase the right side of (5.16). By (5.3) and (5.7) the λ_k in (5.17) already satisfy (5.18).

It follows from (5.16) (resp. (5.17)) and Riesz potential estimates (see [9, Lemma 7.12]) that if the functions f_j are bounded (resp. tend to zero) in $L^p(B_{2R}(0))$ for some $p \geqslant 1$ and $R \in (0,1]$ then the functions f_j are bounded (resp. tend to zero) in $L^q(B_R(0))$, $0 < q \leqslant \infty$, provided

$$\frac{1}{p} - \frac{1}{q\lambda_k} < \frac{2m-k}{n} \quad \text{for } k = 0, 1, \dots, 2m-1,$$

which holds if and only if

$$\frac{1}{p} - \frac{1}{q} < \min_{0 \leqslant k \leqslant 2m-1} \left(\frac{(2m-k)\lambda_k}{n} - \frac{\lambda_k - 1}{p} \right).$$

However,

$$\inf_{p \geqslant 1} \frac{(2m-k)\lambda_k}{n} - \frac{\lambda_k - 1}{p} = \min\left\{ \frac{n - \lambda_k(n-2m+k)}{n}, \frac{\lambda_k(2m-k)}{n} \right\}$$

which, by (5.18), is bounded below by some positive constant independent of k. So starting with (4.14) and iterating the above L^p to L^q comment a finite number of times, we see that there exists $R_0 \in (0,1)$ such that the functions f_j are bounded (resp. tend to zero) in $L^\infty(B_{R_0}(0))$ which together with (4.14) contradicts (5.11) (resp. (5.12)) and thereby completes the proof of Theorem 5.1. □

Proof of Theorem 2.4. For some constant $K > 0$, u satisfies

$$0 \leqslant \Delta^m u \leqslant K\left(u^\lambda + 1\right) \quad \text{in } B_2(0) \setminus \{0\} \subset \mathbb{R}^n. \tag{5.19}$$

Thus u satisfies (5.1) with $\lambda_0 = \lambda$, $\alpha_0 = 0$, $g_0(x) \equiv 1$, and $\lambda_k = 1$, $\alpha_k = -(n-2+k)$, $g_k(x) \equiv 1$ for $k = 1, 2, \dots, 2m-1$.

Let b and a_k be as in Theorem 5.1. Then

$$b = 0 \quad \left(\text{resp. } b = \frac{\lambda(n-2) - (2m+n-2)}{n - \lambda(n-2m)} > 0 \right)$$

and so $a_0 = n-2$ (resp. $a_0 = a$, where a is given by (2.6)). Hence (2.4) (resp. (2.5)) follows from part (i) (resp. part (ii)) of Theorem 5.1. □

Proof of Theorem 2.5. Define $\psi : (0,1) \to (0,1)$ by

$$\psi(r) = \max\left\{ \varphi(r)^p, r^{\frac{n-\lambda(n-2m)}{\lambda-1} \frac{b}{2}} \right\} \tag{5.20}$$

where

$$b := \frac{\lambda(n-2) - (2m+n-2)}{n - \lambda(n-2m)} \quad \text{and} \quad p := \frac{n - \lambda(n-2m)}{4m}$$

are positive by (2.7). Let $\{x_j\}_{j=1}^\infty \subset \mathbb{R}^n$ be a sequence satisfying (4.23) and (4.24). Define $r_j > 0$ by (4.34) with the greater than sign replaced with an equal sign and with $\tau = 0$. Then by (5.20)

$$\begin{aligned} r_j &= A^{\frac{-\lambda}{n-\lambda(n-2m)}} \frac{|x_j|^{1+b}}{\psi(|x_j|)^{\frac{\lambda-1}{n-\lambda(n-2m)}}} \\ &\leqslant A^{\frac{-\lambda}{n-\lambda(n-2m)}} |x_j|^{1+b/2}. \end{aligned} \tag{5.21}$$

Thus by taking a subsequence of j, r_j will satisfy (4.25).

Let u be as in Lemma 4.4. Then by Case 1 of Remark 4.1, u is a C^∞ positive solution of (2.8) and by (4.28), (5.21), (5.20), and (2.6) we have

$$u(x_j) \geqslant \frac{A\psi(|x_j|)}{|x_j|^{2m-2}} \frac{A^{\frac{\lambda(n-2m)}{n-\lambda(n-2m)}} \psi(|x_j|)^{\frac{(\lambda-1)(n-2m)}{n-\lambda(n-2m)}}}{|x_j|^{(n-2m)(1+b)}}$$
$$= C(m,n,\lambda)\frac{\psi(|x_j|)^{\frac{2m}{n-\lambda(n-2m)}}}{|x_j|^{n-2+(n-2m)b}}$$
$$\geqslant C(m,n,\lambda)\frac{\varphi(|x_j|)^{1/2}}{|x_j|^a}$$

which implies (2.9). □

Proof of Theorem 2.6. Define $\psi : (0,1) \to (0,1)$ by $\psi(r) = r^{m-1}$. Let $\{x_j\}_{j=1}^\infty \subset \mathbb{R}^n$ be a sequence satisfying (4.23), (4.24), and

$$\frac{1}{A^\lambda}\frac{|x_j|^{(\lambda-1)(2m-2)}}{\psi(|x_j|)^{\lambda-1}} = A^{-\lambda}|x_j|^{(\lambda-1)(m-1)} < 1 \tag{5.22}$$

where $A = A(m,n)$ is as in Lemma 4.4. Let $\{r_j\}_{j=1}^\infty \subset \mathbb{R}$ be a sequence satisfying (4.25) and

$$\frac{A\psi(|x_j|)}{|x_j|^{2m-2}r_j^{n-2m}} > \varphi\big(|x_j|\big)^2. \tag{5.23}$$

Since $r_j < 1$ and $n - \lambda(n-2m) \leqslant 0$, we see that the left side of (4.34) is greater than or equal to one. Thus (5.22) implies (4.34) with $\tau = 0$. Let u be as in Lemma 4.4. Then by (4.28) and (5.23)

$$\frac{u(x_j)}{\varphi(|x_j|)} \geqslant \varphi\big(|x_j|\big) \to \infty \quad \text{as } j \to \infty$$

and by Case 1 of Remark 4.1, u is a C^∞ positive solution of (2.8). □

Theorem 2.7 will follow easily from the following more general result.

Theorem 5.2. *Suppose u is a C^{2m} solution of* (5.1) *satisfying* (5.2) *where $K > 0$, λ_k, and α_k are constants*; *$m \geqslant 2$, and $n \geqslant 2$ are integers, m is odd, $2m = n$*;

$$\lambda_0 \geqslant 0 \quad \textit{and} \quad 0 \leqslant \lambda_k < n/k \quad \textit{for } k = 1,2,\dots,n-1; \tag{5.24}$$

and $g_k : B_2(0) \setminus \{0\} \to [1,\infty)$ is a continuous function. Let

$$b = \max\{0, b_0, b_1, \dots, b_{n-1}\} \quad \textit{where } b_k = \frac{\alpha_k + \lambda_k(n-2+k) - (2n-2)}{n - k\lambda_k}.$$

(i) *Suppose as $x \to 0$ we have*

$$g_0(x) = \begin{cases} O(|x|^{-(n-2)}) & \textit{if } b = 0, \\ O(|x|^{-(n-2)} \log \frac{5}{|x|}) & \textit{if } b > 0 \end{cases} \tag{5.25}$$

and for $k = 1,2,\dots,n-1$ we have

$$g_k(x) = \begin{cases} O(|x|^{-(n-2+k)}) & \textit{if } b = 0, \\ O(|x|^{-(n-2+k)}a(x)^{-k}) & \textit{if } b > 0, \end{cases} \tag{5.26}$$

where

$$a(x) = \min\left\{\frac{|x|^{b_0}}{(\log \frac{5}{|x|})^{\lambda_0/n}}, |x|^{b_1}, \dots, |x|^{b_{n-1}}\right\}. \tag{5.27}$$

Then as $x \to 0$ we have

$$u(x) = \begin{cases} O(|x|^{-(n-2)}) & \text{if } b = 0, \\ O(|x|^{-(n-2)} \log \frac{5}{|x|}) & \text{if } b > 0 \end{cases} \tag{5.28}$$

and for $i = 1, 2, \dots, n-1$ we have

$$|D^i u(x)| = \begin{cases} O(|x|^{-(n-2+i)}) & \text{if } b = 0, \\ O(|x|^{-(n-2+i)} a(x)^{-i}) & \text{if } b > 0. \end{cases} \tag{5.29}$$

(ii) *Suppose $b > 0$,*

$$\lambda_k > 0 \quad \text{for } k = 0, 1, \dots, n-1, \tag{5.30}$$

and as $x \to 0$ we have

$$g_0(x) = o\left(|x|^{-(n-2)} \log \frac{5}{|x|}\right) \tag{5.31}$$

and

$$g_k(x) = o\big(|x|^{-(n-2+k)} a(x)^{-k}\big) \quad \text{for } k = 1, 2, \dots, n-1, \tag{5.32}$$

where $a(x)$ is defined by (5.27). Then as $x \to 0$ we have

$$u(x) = o\left(|x|^{-(n-2)} \log \frac{5}{|x|}\right) \tag{5.33}$$

and

$$|D^i u(x)| = o\big(|x|^{-(n-2+i)} a(x)^{-i}\big) \quad \text{for } i = 1, 2, \dots, n-1. \tag{5.34}$$

Proof. As in the proof of Theorem 5.1, it suffices to prove Theorem 5.2 when u is nonnegative.

For $b \geqslant 0$ and $0 < |x| < 2$ we define

$$a_b(x) = \begin{cases} \frac{1}{4} & \text{if } b = 0, \\ a(x) & \text{if } b > 0 \end{cases}$$

where $a(x)$ is given by (5.27). Then

$$\log \frac{1}{a_0(x)} = \log 4 \quad \text{and} \quad \lim_{x \to 0} \frac{\log \frac{1}{a_b(x)}}{\log \frac{5}{|x|}} = b \quad \text{when } b > 0. \tag{5.35}$$

Thus (5.28), (5.29), (5.33), and (5.34) are equivalent to

$$u(x) = O\left(|x|^{-(n-2)} \log \frac{1}{a_b(x)}\right), \tag{5.36}$$

$$|D^i u(x)| = O\big(|x|^{-(n-2+i)} a_b(x)^{-i}\big), \tag{5.37}$$

$$u(x) = o\left(|x|^{-(n-2)} \log \frac{1}{a_b(x)}\right), \tag{5.38}$$

$$|D^i u(x)| = o\big(|x|^{-(n-2+i)} a_b(x)^{-i}\big) \tag{5.39}$$

and similarly for (5.25), (5.26), (5.31), and (5.32).

Suppose for contradiction that part (i) (resp. part (ii)) of Theorem 5.2 is false. Then there exists $i_0 \in \{0, 1, \dots, n-1\}$ such that the estimates (5.36), (5.37) (resp. (5.38), (5.39)) for $D^i u$ do not hold when $i = i_0$. Thus there is a sequence $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}^n$ satisfying

$$0 < 4|x_{j+1}| < |x_j| < 1/2 \quad \text{and} \quad a_b(x_j) \leqslant \frac{1}{4} \tag{5.40}$$

such that

$$\liminf_{j\to\infty} \frac{|x_j|^{n-2}}{\log\frac{1}{a_b(x_j)}} u(x_j) = \infty \quad (\text{resp. } > 0) \quad \text{if } i_0 = 0, \tag{5.41}$$

$$\liminf_{j\to\infty} |x_j|^{n-2+i_0} a_b(x_j)^{i_0} \left|D^{i_0} u(x_j)\right| = \infty \quad (\text{resp. } > 0) \quad \text{if } i_0 \in \{1, 2, \ldots, n-1\}. \tag{5.42}$$

Let

$$r_j = |x_j| a_b(x_j). \tag{5.43}$$

Then by (5.40), the sequences x_j and r_j satisfy (4.12). Let f_j be as in Lemma 4.3.

Using (5.43), (5.41), and (5.42), it follows from (4.16) and (4.15) with $\xi = 0$ and $|\beta| = i_0$ that

$$\liminf_{j\to\infty} \int\limits_{|\eta|<2} \left(\log\frac{5}{|\eta|}\right) f_j(\eta)\, d\eta = \infty \quad (\text{resp. } > 0) \quad \text{if } i_0 = 0 \tag{5.44}$$

and

$$\liminf_{j\to\infty} \int\limits_{|\eta|<2} \frac{f_j(\eta)}{|\eta|^{i_0}}\, d\eta = \infty \quad (\text{resp. } > 0) \quad \text{if } i_0 \in \{1, 2, \ldots, n-1\}. \tag{5.45}$$

On the other hand, (4.13), (5.1), (4.16), and (4.15) imply for $|\xi| < 1$ that

$$\begin{aligned} f_j(\xi) &\leqslant |x_j|^{2n-2}\left(\frac{r_j}{|x_j|}\right)^n C \sum_{k=0}^{n-1} |x_j|^{-\alpha_k}\left(\left|D^k u(x_j + r_j\xi)\right| + g_k(x_j + r_j\xi)\right)^{\lambda_k} \\ &\leqslant C B_{0j}\left(\frac{1}{\log\frac{|x_j|}{r_j}} + \varepsilon_j + \frac{1}{\log\frac{|x_j|}{r_j}} \int\limits_{|\eta|<2} \left(\log\frac{5}{|\xi-\eta|}\right) f_j(\eta)\, d\eta + G_{0j}(\xi)\right)^{\lambda_0} \\ &\quad + C\sum_{k=1}^{n-1} B_{kj}\left(\left(\frac{r_j}{|x_j|}\right)^k + \varepsilon_j + \int\limits_{|\eta|<2} \frac{f_j(\eta)}{|\xi-\eta|^k}\, d\eta + G_{kj}(\xi)\right)^{\lambda_k} \end{aligned}$$

where C is a constant independent of ξ and j whose value may change from line to line,

$$B_{0j} := \frac{|x_j|^{2n-2}(\frac{r_j}{|x_j|})^n |x_j|^{-\alpha_0}}{(\frac{|x_j|^{n-2}}{\log\frac{|x_j|}{r_j}})^{\lambda_0}} = \left(\frac{a_b(x_j)(\log\frac{1}{a_b(x_j)})^{\lambda_0/n}}{|x_j|^{b_0}}\right)^n \leqslant C,$$

$$B_{kj} := \frac{|x_j|^{2n-2}(\frac{r_j}{|x_j|})^n |x_j|^{-\alpha_k}}{((\frac{r_j}{|x_j|})^k |x_j|^{n-2+k})^{\lambda_k}} = \left(\frac{a_b(x_j)}{|x_j|^{b_k}}\right)^{n-k\lambda_k} \leqslant 1,$$

$$G_{0j}(\xi) := \frac{|x_j|^{n-2}}{\log\frac{|x_j|}{r_j}} g_0(x_j + r_j\xi) = \frac{|x_j|^{n-2}}{\log\frac{1}{a_b(x_j)}} g_0(x_j + r_j\xi) \leqslant C,$$

$$G_{kj}(\xi) := \left(\frac{r_j}{|x_j|}\right)^k |x_j|^{n-2+k} g_k(x_j + r_j\xi) = a_b(x_j)^k |x_j|^{n-2+k} g_k(x_j + r_j\xi) \leqslant C,$$

where we have used (5.35) and (5.27). Therefore for $|\xi| < 1$ we have

$$f_j(\xi) \leqslant C\left[\left(1 + \int\limits_{|\eta|<2} \left(\log\frac{5}{|\xi-\eta|}\right) f_j(\eta)\, d\eta\right)^{\lambda_0} + \sum_{k=1}^{n-1}\left(1 + \int\limits_{|\eta|<2} \frac{f_j(\eta)\, d\eta}{|\xi-\eta|^k}\right)^{\lambda_k}\right] \tag{5.46}$$

$$\left(\text{resp. } f_j(\xi) \leqslant C\left[\left(\varepsilon_j + \int\limits_{|\eta|<2} \left(\log\frac{5}{|\xi-\eta|}\right) f_j(\eta)\, d\eta\right)^{\lambda_0} + \sum_{k=1}^{n-1}\left(\varepsilon_j + \int\limits_{|\eta|<2} \frac{f_j(\eta)\, d\eta}{|\xi-\eta|^k}\right)^{\lambda_k}\right]\right), \tag{5.47}$$

where ε_j is independent of ξ, and $\varepsilon_j \to 0$ as $j \to \infty$.

We can assume the λ_k in (5.46) satisfy, instead of (5.24), the stronger condition

$$\lambda_0 > 0 \quad \text{and} \quad 0 < \lambda_k < n/k \quad \text{for } k = 1, 2, \dots, n-1 \tag{5.48}$$

because slightly increasing those λ_k in (5.46) which are zero will increase the right side of (5.46). By (5.24) and (5.30) the λ_k in (5.47) already satisfy (5.48).

Using an argument very similar to the one used at the end of the proof of Theorem 5.1 to show that (5.14) (resp. (5.15)) leads to a contradiction of (5.11) (resp. (5.12)), one can show that (5.46) (resp. (5.47)) leads to a contradiction of (5.44) (resp. (5.45))—the only significant difference being that where we used Riesz potential estimates in the proof of Theorem 5.1, we must now use Riesz potential estimates and Lemma 4.5. □

Proof of Theorem 2.7. For some constant $K > 0$, u satisfies (5.19). Thus u satisfies (5.1) with λ_k, α_k, and $g_k(x)$ as in the proof of Theorem 2.4. Let b be as in Theorem 5.2. Then

$$b = 0 \quad \left(\text{resp. } b = \frac{\lambda(n-2) - (2n-2)}{n} > 0\right).$$

Hence (2.10) (resp. (2.11)) follows from part (i) (resp. part (ii)) of Theorem 5.2. □

Proof of Theorem 2.8. It follows from (2.12) that

$$a := \frac{(n-2)(\lambda - 1) - n}{\lambda} > 0.$$

Define $\psi : (0, 1) \to (0, 1)$ and $\rho : (0, 1) \to (0, \infty)$ by

$$\psi(r) = \max\left\{\sqrt{\varphi(r)}, r^{\frac{a\lambda}{2(\lambda-1)}}\right\} \tag{5.49}$$

and

$$\rho(r) = \frac{n}{\lambda A} \frac{r^a}{\psi(r)^{\frac{\lambda-1}{\lambda}}} \tag{5.50}$$

where $A = A(m, n)$ is as in Lemma 4.4. By (5.49)

$$\rho(r) \leqslant \frac{n}{\lambda A} r^{a/2}. \tag{5.51}$$

Thus there exists a sequence $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}^n$ satisfying (4.23), (4.24), and

$$e^{-1} > \rho_j := \rho\big(|x_j|\big) \to 0 \quad \text{as } j \to \infty \tag{5.52}$$

such that if we define the sequence $\{r_j\}_{j=1}^{\infty}$ by

$$\left(\frac{|x_j|}{r_j}\right)^{n/\lambda} = \frac{1}{\rho_j} \log \frac{1}{\rho_j} \tag{5.53}$$

then r_j satisfies (4.25). By (5.53), (5.52), and (5.50) we have

$$\begin{aligned}
\log \frac{|x_j|}{r_j} &= \frac{\lambda}{n} \log\left(\frac{1}{\rho_j} \log \frac{1}{\rho_j}\right) \\
&\geqslant \frac{\lambda}{n} \log \frac{1}{\rho_j} \qquad (5.54) \\
&= \frac{\lambda}{n} \rho_j \left(\frac{|x_j|}{r_j}\right)^{n/\lambda} \\
&= \frac{1}{A} \frac{|x_j|^a}{\psi(|x_j|)^{\frac{\lambda-1}{\lambda}}} \left(\frac{|x_j|}{r_j}\right)^{n/\lambda}. \qquad (5.55)
\end{aligned}$$

Let u be as in Lemma 4.4. Then by (5.55) and Case 2 of Remark 4.1, u is a C^∞ positive solution of (2.8) and by Lemma 4.4 we have

$$u(x_j) \geqslant \frac{A\psi(|x_j|)}{|x_j|^{n-2}} \log \frac{|x_j|}{r_j}$$

and by (5.54) and (5.51),

$$\log \frac{|x_j|}{r_j} \geqslant \frac{\lambda}{n} \log \frac{1}{\rho_j} \geqslant \frac{\lambda}{n} \log\left(\frac{\lambda A}{n}|x_j|^{-a/2}\right) = \frac{\lambda}{n}\left(\frac{a}{2}\log\frac{1}{|x_j|} + \log\frac{\lambda A}{n}\right).$$

Thus by (5.49) we have

$$\liminf_{j\to\infty} \frac{u(x_j)}{\sqrt{\varphi(|x_j|)}|x_j|^{-(n-2)}\log\frac{1}{|x_j|}} \geqslant A\frac{\lambda a}{2n} > 0$$

from which we obtain (2.13). □

By scaling u in Theorem 2.9, the following theorem implies Theorem 2.9.

Theorem 5.3. *Let $u(x)$ be a C^{2m} nonnegative solution of*

$$0 \leqslant -\Delta^m u \leqslant e^{u^\lambda + g^\lambda} \quad \text{in } B_2(0)\setminus\{0\} \subset \mathbb{R}^n \tag{5.56}$$

where $n \geqslant 2$ and $m \geqslant 2$ are integers, m is odd, $2m = n$, $0 < \lambda < 1$, and $g : B_2(0)\setminus\{0\} \to [0,\infty)$ is a continuous function such that

$$g(x) = o\left(|x|^{\frac{-(n-2)}{1-\lambda}}\right) \quad \text{as } x \to 0. \tag{5.57}$$

Then

$$u(x) = o\left(|x|^{\frac{-(n-2)}{1-\lambda}}\right) \quad \text{as } x \to 0. \tag{5.58}$$

Proof. Suppose for contradiction that (5.58) does not hold. Then there exists a sequence $\{x_j\}_{j=1}^\infty \subset \mathbb{R}^n$ such that

$$0 < 4|x_{j+1}| < |x_j| < 1/2$$

and

$$\liminf_{j\to\infty} |x_j|^{\frac{n-2}{1-\lambda}} u(x_j) > 0. \tag{5.59}$$

Define $r_j > 0$ by

$$\log\frac{1}{r_j} = |x_j|^{\frac{-(n-2)\lambda}{1-\lambda}}. \tag{5.60}$$

Then

$$\log\frac{|x_j|}{r_j} = \log\frac{1}{r_j} - \log\frac{1}{|x_j|} = |x_j|^{\frac{-(n-2)\lambda}{1-\lambda}}\left[1 - |x_j|^{\frac{(n-2)\lambda}{1-\lambda}}\log\frac{1}{|x_j|}\right] = |x_j|^{\frac{-(n-2)\lambda}{1-\lambda}}\big(1+o(1)\big) \quad \text{as } j\to\infty. \tag{5.61}$$

So, by taking a subsequence of j if necessary, we can assume $r_j < |x_j|/4$.

Let f_j be as in Lemma 4.3. Multiplying (4.16) by $|x_j|^{\frac{(n-2)\lambda}{1-\lambda}} \log\frac{|x_j|}{r_j}$ and using (5.61) we get for $|\xi| < 1$ that

$$|x_j|^{\frac{n-2}{1-\lambda}} u(x_j + r_j\xi) \leqslant \varepsilon_j + |x_j|^{\frac{(n-2)}{1-\lambda}} \int_{|\eta|<2} A\left(\log\frac{5}{|\xi-\eta|}\right)\frac{f_j(\eta)}{|x_j|^{n-2}}\,d\eta \tag{5.62}$$

where the constant A depends only on m and n, the constants ε_j are independent of ξ, and $\varepsilon_j \to 0$ as $j \to \infty$. Substituting $\xi = 0$ in (5.62) and using (5.59) and (4.14) we get

$$\liminf_{j\to\infty} |x_j|^{\frac{n-2}{1-\lambda}} \int\limits_{|\eta|<1} \left(\log \frac{5}{|\eta|}\right) \frac{f_j(\eta)}{|x_j|^{n-2}}\, d\eta > 0. \tag{5.63}$$

By (4.13), (5.56), (5.62), and (5.57) we have

$$\frac{f_j(\xi)}{|x_j|^{n-2} r_j^n} \leqslant e^{u_j(\xi)^\lambda + M_j^\lambda} \quad \text{for } |\xi| < 1 \tag{5.64}$$

where

$$u_j(\xi) = \int\limits_{|\eta|<2} A\left(\log \frac{5}{|\xi - \eta|}\right) \frac{f_j(\eta)}{|x_j|^{n-2}}\, d\eta$$

and the positive constants M_j satisfy

$$M_j |x_j|^{\frac{n-2}{1-\lambda}} \to 0 \quad \text{and} \quad M_j \to \infty \quad \text{as } j \to \infty. \tag{5.65}$$

Let $\Omega_j = \{\xi \in B_1(0):\ u_j(\xi) > M_j\}$. Then for $\xi \in \Omega_j$ it follows from (5.64) that

$$\begin{aligned}\frac{f_j(\xi)^2}{(|x_j|^{n-2} r_j^n)^2} &\leqslant e^{4u_j(\xi)^\lambda} \\ &\leqslant \exp\left[\left(\int\limits_{|\eta|<2} b_j \left(\log \frac{5}{|\xi - \eta|}\right) \frac{f_j(\eta)}{\int_{B_2} f_j}\, d\eta\right)^\lambda\right]\end{aligned} \tag{5.66}$$

where

$$b_j = 4^{1/\lambda} A |x_j|^{-(n-2)} \max\left\{\int\limits_{B_2} f_j,\ |x_j|^{\frac{n-2}{2}}\right\}.$$

By (4.14),

$$b_j |x_j|^{n-2} \to 0 \quad \text{and} \quad b_j \to \infty \quad \text{as } j \to \infty. \tag{5.67}$$

Hence by (5.66), Jensen's inequality, and the fact that $\exp(t^\lambda)$ is concave up for t large we have for $\xi \in \Omega_j$ that

$$\frac{f_j(\xi)^2}{(|x_j|^{n-2} r_j^n)^2} \leqslant \int\limits_{|\eta|<2} \exp\left(b_j^\lambda \left(\log \frac{5}{|\xi - \eta|}\right)^\lambda\right) \frac{f_j(\eta)}{\int_{B_2} f_j}\, d\eta$$

and consequently

$$\begin{aligned}\int\limits_{\Omega_j} \frac{f_j(\xi)^2}{(|x_j|^{n-2} r_j^n)^2}\, d\xi &\leqslant \int\limits_{|\eta|<2} \left(\int\limits_{|\xi|<1} \exp\left(b_j^\lambda \left(\log \frac{5}{|\xi - \eta|}\right)^\lambda\right) d\xi\right) \frac{f_j(\eta)}{\int_{B_2} f_j}\, d\eta \\ &\leqslant \max_{|\eta|\leqslant 2} \int\limits_{|\xi|<1} \exp\left(b_j^\lambda \left(\log \frac{5}{|\xi - \eta|}\right)^\lambda\right) d\xi \\ &= \int\limits_{|\xi|<1} \exp\left(b_j^\lambda \left(\log \frac{5}{|\xi|}\right)^\lambda\right) d\xi \\ &= I_1 + I_2\end{aligned} \tag{5.68}$$

where

$$I_1 = \int\limits_{\substack{|\xi|<1 \\ \log\frac{5}{|\xi|}<(b_j^\lambda\lambda)^{\frac{1}{1-\lambda}}}} \exp\left(b_j^\lambda\left(\log\frac{5}{|\xi|}\right)^\lambda\right)d\xi \quad \text{and} \quad I_2 = \int\limits_{\log\frac{5}{|\xi|}>(b_j^\lambda\lambda)^{\frac{1}{1-\lambda}}} \exp\left(b_j^\lambda\left(\log\frac{5}{|\xi|}\right)^\lambda\right)d\xi.$$

Clearly

$$\frac{I_1}{|B_1(0)|} \leqslant \exp\left(\left(b_j\left(b_j^\lambda\lambda\right)^{\frac{1}{1-\lambda}}\right)^\lambda\right) = \exp\left((b_j\lambda)^{\frac{\lambda}{1-\lambda}}\right) \leqslant \exp\left(b_j^{\frac{\lambda}{1-\lambda}}\right)$$

and using Jensen's inequality and the fact that $e^{b_j^\lambda(\log t)^\lambda}$ is concave down as a function of t for $\log t > (b_j^\lambda\lambda)^{\frac{1}{1-\lambda}}$ one can show that

$$I_2 \leqslant \exp\left(Cb_j^{\frac{\lambda}{1-\lambda}}\right)$$

where C depends only on n. Therefore by (5.68) and (5.60),

$$\begin{aligned}\int\limits_{\Omega_j}\left(\frac{f_j(\xi)}{|x_j|^{n-2}}\right)^2 d\xi &\leqslant \exp\left(-2n|x_j|^{\frac{-(n-2)\lambda}{1-\lambda}}\right)\exp\left(Cb_j^{\frac{\lambda}{1-\lambda}}\right) \\ &= \exp\left(Cb_j^{\frac{\lambda}{1-\lambda}} - 2n|x_j|^{\frac{-(n-2)\lambda}{1-\lambda}}\right) \to 0 \quad \text{as } j \to \infty\end{aligned}$$

by (5.67). Thus by Hölder's inequality

$$\lim_{j\to\infty}\int\limits_{\Omega_j}\left(\log\frac{5}{|\eta|}\right)\frac{f_j(\eta)}{|x_j|^{n-2}}\,d\eta = 0.$$

Hence defining $g_j : B_1(0) \to [0,\infty)$ by

$$g_j(\xi) := \begin{cases} f_j(\xi) & \text{for } \xi \in B_1 \setminus \Omega_j, \\ 0 & \text{for } \xi \in \Omega_j, \end{cases}$$

it follows from (5.63) and (5.65) that

$$\frac{1}{M_j^\lambda}\int\limits_{|\eta|<1}\left(\log\frac{5}{|\eta|}\right)g_j(\eta)\,d\eta \to \infty \quad \text{as } j \to \infty. \tag{5.69}$$

By (4.14), we have

$$\int\limits_{|\eta|<1} g_j(\eta)\,d\eta \to 0 \quad \text{as } j \to \infty \tag{5.70}$$

and by (5.64) we have

$$g_j(\xi) \leqslant e^{2M_j^\lambda} \quad \text{in } B_1(0). \tag{5.71}$$

For fixed j, think of $g_j(\eta)$ as the density of a distribution of mass in B_1 satisfying (5.69), (5.70), and (5.71). By moving small pieces of this mass nearer to the origin in such a way that the new density (which we again denote by $g_j(\eta)$) does not violate (5.71), we will not change the total mass $\int_{B_1} g_j(\eta)\,d\eta$ but $\int_{B_1}(\log 5/|\eta|)g_j(\eta)\,d\eta$ will increase. Thus for some $\rho_j \in (0,1)$ the functions

$$g_j(\eta) = \begin{cases} e^{2M_j^\lambda} & \text{for } |\eta| < \rho_j, \\ 0 & \text{for } \rho_j < |\eta| < 1 \end{cases}$$

satisfy (5.69), (5.70), and (5.71), which, as elementary and explicit calculations show, is impossible because $M_j \to \infty$ as $j \to \infty$. This contradiction proves Theorem 5.3. □

Proof of Theorem 2.10. Define $\psi:(0,1)\to(0,1)$ by

$$\psi(r)=\max\bigl\{\varphi(r)^{\frac{1-\lambda}{2}},r^{\frac{n-2}{2}}\bigr\}.$$

Since $\psi(r)\geqslant r^{\frac{n-2}{2}}$ there exists a sequence $\{x_j\}_{j=1}^\infty\subset\mathbb{R}^n$ satisfying (4.23) and (4.24) such that if we define the sequence $\{r_j\}_{j=1}^\infty\subset(0,\infty)$ by

$$\log\frac{|x_j|}{r_j}=\left[\frac{1}{2n}\left(\frac{A\psi(|x_j|)}{|x_j|^{n-2}}\right)^\lambda\right]^{\frac{1}{1-\lambda}},\tag{5.72}$$

where $A=A(m,n)$ is as in Lemma 4.4, then r_j will satisfy (4.25) and

$$\log\frac{1}{|x_j|^{2n-2}}<\log\frac{|x_j|}{r_j}.$$

Thus

$$\begin{aligned}\frac{\log\frac{\psi(|x_j|)}{|x_j|^{2n-2}}+n\log\frac{|x_j|}{r_j}}{(\frac{A\psi(|x_j|)}{|x_j|^{n-2}}\log\frac{|x_j|}{r_j})^\lambda}&\leqslant\frac{2n\log\frac{|x_j|}{r_j}}{(\frac{A\psi(|x_j|)}{|x_j|^{n-2}}\log\frac{|x_j|}{r_j})^\lambda}\\&=\frac{(\log\frac{|x_j|}{r_j})^{1-\lambda}}{\frac{1}{2n}(\frac{A\psi(|x_j|)}{|x_j|^{n-2}})^\lambda}=1.\end{aligned}\tag{5.73}$$

Let u be as in Lemma 4.4. Then by (5.73) and Case 2 of Remark 4.1, u is a C^∞ positive solution of (2.16) and by Lemma 4.4 and (5.72) we have

$$\begin{aligned}u(x_j)&\geqslant\frac{A\psi(|x_j|)}{|x_j|^{n-2}}\left[\frac{1}{2n}\left(\frac{A\psi(|x_j|)}{|x_j|^{n-2}}\right)^\lambda\right]^{\frac{1}{1-\lambda}}\\&=\left(\frac{A\psi(|x_j|)}{2n}\right)^{\frac{1}{1-\lambda}}\frac{1}{|x_j|^{\frac{n-2}{1-\lambda}}}\\&\geqslant\left(\frac{A}{2n}\right)^{\frac{1}{1-\lambda}}\sqrt{\varphi\bigl(|x_j|\bigr)}|x_j|^{-\frac{n-2}{1-\lambda}}\end{aligned}$$

which implies (2.17). □

Proof of Theorem 2.11. Define $\psi:(0,1)\to(0,1)$ by $\psi(r)=r^{\frac{n-2}{2}}$. Choose a sequence $\{x_j\}_{j=1}^\infty\subset\mathbb{R}^n$ satisfying (4.23), (4.24), and

$$\frac{A\psi(|x_j|)}{|x_j|^{n-2}}>n+1\tag{5.74}$$

where $A=A(m,n)$ is as in Lemma 4.4. Choose a sequence $\{r_j\}_{j=1}^\infty\subset\mathbb{R}$ satisfying (4.25),

$$\log\frac{1}{|x_j|^{2n-2}}<\log\frac{|x_j|}{r_j}\tag{5.75}$$

and

$$(n+1)\log\frac{|x_j|}{r_j}>\varphi\bigl(|x_j|\bigr)^2.\tag{5.76}$$

Then, since $\psi(|x_j|)<1$, it follows from (5.75) and (5.74) that

$$\log\frac{\psi(|x_j|)}{|x_j|^{2n-2}}+n\log\frac{|x_j|}{r_j}\leqslant(n+1)\log\frac{|x_j|}{r_j}\leqslant\left((n+1)\log\frac{|x_j|}{r_j}\right)^\lambda\leqslant\left(\frac{A\psi(|x_j|)}{|x_j|^{n-2}}\log\frac{|x_j|}{r_j}\right)^\lambda.\tag{5.77}$$

Let u be as in Lemma 4.4. Then by (5.77) and Case 2 of Remark 4.1, u is a C^∞ positive solution of (2.16) and by Lemma 4.4, (5.74) and (5.76) we have

$$u(x_j) \geqslant \varphi(x_j)^2$$

which implies (2.18). □

We now prove Proposition 2.1. It follows immediately from the following more general proposition, which is easier to prove than Proposition 2.1.

Proposition 5.1. *Suppose u is a C^{2m} radial solution of*

$$\Delta^m u \leqslant A\Gamma \quad \text{in } \overline{B_1(0)} \setminus \{0\} \subset \mathbb{R}^n, \tag{5.78}$$

where $m \geqslant 1$ and $n \geqslant 2$ are integers, Γ is given by (1.3), and A is a positive constant. Then

$$u \leqslant C\Gamma \quad \text{in } \overline{B_1(0)} \setminus \{0\} \tag{5.79}$$

for some positive constant C.

Proof. We use induction. Suppose $m = 1$. Let $g = \Delta u$. Then for $0 < r \leqslant 1$ we have

$$u(r) = u(1) - u'(1)\int_r^1 \rho^{1-n}\,d\rho + \int_r^1 s^{1-n}\int_s^1 g(\rho)\rho^{n-1}\,d\rho\,ds. \tag{5.80}$$

Since $g \leqslant A\Gamma$ we have

$$\int_s^1 g(\rho)\rho^{n-1}\,d\rho \leqslant A\int_0^1 \Gamma(\rho)\rho^{n-1}\,d\rho < \infty.$$

Hence (5.79) follows from (5.80).

Suppose inductively that the proposition is true for $m-1$ where $m \geqslant 2$ and u is a C^{2m} radial solution of (5.78). Then by the $m = 1$ case, $\Delta^{m-1}u \leqslant A_1\Gamma$ in $\overline{B_1(0)} \setminus \{0\}$ for some positive constant A_1. Thus (5.79) follows from the inductive assumption. □

6. Proofs when the singularity is at infinity

In this section we prove Theorems 3.2–3.5, which deal with the case that the singularity is at infinity.

Since the proofs of Theorems 3.2 and 3.3 are similar, we prove them together.

Proof of Theorem 3.2 (resp. 3.3). For some constant $K > 0$, $v(y)$ satisfies

$$0 \leqslant -\Delta^m v \leqslant K(v+1)^\lambda \quad \text{in } \mathbb{R}^n \setminus B_{1/2}(0).$$

Let $u(x)$ be defined by (1.5). Then, by (1.6), $u(x)$ is a nonnegative solution of

$$0 \leqslant -\Delta^m u \leqslant K|x|^{-\alpha_0}\big(u + g_0(x)\big)^\lambda \quad \text{in } B_2(0) \setminus \{0\},$$

where $\alpha_0 = 2m + n - (n-2m)\lambda$ and

$$g_0(x) = |x|^{-(n-2m)} = o\big(|x|^{-(n-2)}\big) \quad \text{as } x \to 0.$$

Thus u satisfies (5.1) and (5.2) with $\lambda_0 = \lambda$ and $\lambda_k = 1$, $\alpha_k = -(n-2+k)$, $g_k(x) \equiv 1$ for $k = 1, 2, \ldots, 2m-1$. Using these values of λ_k and α_k in b, as defined in Theorem 5.1 (resp. 5.2), we get

$$b = \frac{\lambda(2m-2)+2}{n - \lambda(n-2m)} > 0.$$

It therefore follows from part (ii) of Theorem 5.1 (resp. 5.2), that as $x \to 0$ we have

$$u(x) = o\big(|x|^{-a_0}\big) \quad \left(\text{resp. } u(x) = o\left(|x|^{-(n-2)} \log \frac{5}{|x|}\right)\right),$$

where $a_0 = (n-2m)b + n - 2$. This estimate for $u(x)$ is equivalent, via (1.5), to (3.2) (resp. (3.3)). □

By scaling and translating v in Theorem 3.4, we see that the following theorem implies Theorem 3.4.

Theorem 6.1. *Let $v(y)$ be a C^{2m} nonnegative solution of*

$$0 \leqslant -\Delta^m v \leqslant e^{v^\lambda + g^\lambda} \quad \text{in } \mathbb{R}^n \setminus B_{1/2}(0) \tag{6.1}$$

where $m \geqslant 2$ and $n \geqslant 2$ are integers, m is odd, $2m = n$, $0 < \lambda < 1$, and $g : \mathbb{R}^n \setminus B_{1/2}(0) \to [1, \infty)$ is a continuous function satisfying

$$g(y) = o\big(|y|^{\frac{n-2}{1-\lambda}}\big) \quad \text{as } |y| \to \infty. \tag{6.2}$$

Then

$$v(y) = o\big(|y|^{\frac{n-2}{1-\lambda}}\big) \quad \text{as } |y| \to \infty. \tag{6.3}$$

Proof. Let $u(x)$ be defined by (1.5). Then by (6.1) and (1.6) we have

$$0 \leqslant -|x|^{2n} \Delta^m u(x) \leqslant \exp\left(u(x)^\lambda + g\left(\frac{x}{|x|^2}\right)^\lambda\right)$$

and thus by (6.2),

$$0 \leqslant -\Delta^m u(x) \leqslant \exp\left(u(x)^\lambda + o\left(\left(\frac{1}{|x|}\right)\right)^{\frac{\lambda(n-2)}{1-\lambda}}\right) \quad \text{in } B_2(0) \setminus \{0\}.$$

Hence Theorem 5.3 implies

$$u(x) = o\big(|x|^{-\frac{(n-2)}{1-\lambda}}\big) \quad \text{as } x \to 0$$

and so (6.3) holds. □

Proof of Theorem 3.5. By using the m-Kelvin transform (1.5), we see that to prove Theorem 3.5 it suffices to prove that there exists a C^∞ positive solution $u(x)$ of

$$0 \leqslant -\Delta^m u \leqslant |x|^\tau u^\lambda \quad \text{in } \mathbb{R}^n \setminus \{0\}, \tag{6.4}$$

where

$$\tau = \lambda(n-2m) - n - 2m$$

such that

$$u(x) \neq O\big(\varphi\big(|x|^{-1}\big)|x|^{-(a+n-2m)}\big) \quad \text{as } x \to 0. \tag{6.5}$$

Define $\psi : (0,1) \to (0,1)$ by

$$\psi(r) = \max\big\{\varphi\big(r^{-1}\big)^p, r^{b\frac{n-\lambda(n-2m)}{\lambda}}\big\} \tag{6.6}$$

where

$$b := \frac{\lambda(m-1)+1}{n-\lambda(n-2m)} \quad \text{and} \quad p := \frac{n-\lambda(n-2m)}{2n}.$$

By (3.4), b and p are positive. Also

$$1 + 2b = \frac{\lambda(2m-2) - 2m + 2 - \tau}{n - \lambda(n-2m)} \quad \text{and} \quad a = 2m - 2 + (n-2m)2b. \tag{6.7}$$

Let $\{x_j\}_{j=1}^{\infty} \subset \mathbb{R}^n$ be a sequence satisfying (4.23) and (4.24). Define $r_j > 0$ by

$$r_j^{n-\lambda(n-2m)} = \frac{2^{|\tau|}}{A^\lambda} \frac{|x_j|^{\lambda(2m-2)-2m+2-\tau}}{\psi(|x_j|)^\lambda}$$

where $A = A(m,n)$ is as in Lemma 4.4. Then r_j satisfies (4.34) and by (6.6) and (6.7),

$$\begin{aligned} r_j &= C(m,n,\lambda) \frac{|x_j|^{1+2b}}{\psi(|x_j|)^{\frac{\lambda}{n-\lambda(n-2m)}}} \\ &\leqslant C(m,n,\lambda)|x_j|^{1+b}. \end{aligned} \tag{6.8}$$

Thus by taking a subsequence of j, r_j will satisfy (4.25). Let u be as in Lemma 4.4. Then by Case 1 of Remark 4.1, u is a C^∞ positive solution of (6.4) and by (4.28), (6.6), (6.7), and (6.8) we have

$$\begin{aligned} u(x_j) &\geqslant \frac{C(m,n,\lambda)\psi(|x_j|)}{|x_j|^{2m-2}} \frac{\psi(|x_j|)^{\frac{\lambda(n-2m)}{n-\lambda(n-2m)}}}{|x_j|^{(1+2b)(n-2m)}} \\ &= \frac{C(m,n,\lambda)\psi(|x_j|)^{\frac{n}{n-\lambda(n-2m)}}}{|x_j|^{(n-2m)+(2m-2)+(n-2m)2b}} \\ &\geqslant C(m,n,\lambda)\frac{\varphi(|x_j|^{-1})^{1/2}}{|x_j|^{a+n-2m}} \end{aligned}$$

which implies (6.5). □

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