Available online at www.sciencedirect.com

SciVerse ScienceDirect

Ann. I. H. Poincaré – AN 30 (2013) 1097–1126

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Optimal location of controllers for the one-dimensional wave equation

Yannick Privat [a], Emmanuel Trélat [b,*], Enrique Zuazua [c,d]

[a] IRMAR, ENS Cachan Bretagne, Univ. Rennes 1, CNRS, UEB, av. Robert Schuman, 35170 Bruz, France
[b] Université Pierre et Marie Curie (Univ. Paris 6) and Institut Universitaire de France, CNRS UMR 7598, Laboratoire Jacques-Louis Lions, F-75005 Paris, France
[c] BCAM – Basque Center for Applied Mathematics, Mazarredo, 14 E-48009 Bilbao, Basque Country, Spain
[d] Ikerbasque, Basque Foundation for Science, Alameda Urquijo 36-5, Plaza Bizkaia, E-48011 Bilbao, Basque Country, Spain

Received 30 March 2012; received in revised form 21 November 2012; accepted 24 November 2012

Available online 21 January 2013

Abstract

In this paper, we consider the homogeneous one-dimensional wave equation defined on $(0,\pi)$. For every subset $\omega \subset [0,\pi]$ of positive measure, every $T \geqslant 2\pi$, and all initial data, there exists a unique control of minimal norm in $L^2(0,T;L^2(\omega))$ steering the system exactly to zero. In this article we consider two optimal design problems. Let $L \in (0,1)$. The first problem is to determine the optimal shape and position of ω in order to minimize the norm of the control for given initial data, over all possible measurable subsets ω of $[0,\pi]$ of Lebesgue measure $L\pi$. The second problem is to minimize the norm of the control operator, over all such subsets. Considering a relaxed version of these optimal design problems, we show and characterize the emergence of different phenomena for the first problem depending on the choice of the initial data: existence of optimal sets having a finite or an infinite number of connected components, or nonexistence of an optimal set (relaxation phenomenon). The second problem does not admit any optimal solution except for $L = 1/2$. Moreover, we provide an interpretation of these problems in terms of a classical optimal control problem for an infinite number of controlled ordinary differential equations. This new interpretation permits in turn to study modal approximations of the two problems and leads to new numerical algorithms. Their efficiency will be exhibited by several experiments and simulations.

MSC: 49J20; 35L05; 49J15; 49Q10; 49K35

Keywords: Wave equation; Exact controllability; HUM method; Shape optimization; Relaxation; Optimal control; Pontryagin Maximum Principle

* Corresponding author.
E-mail addresses: Yannick.Privat@bretagne.ens-cachan.fr (Y. Privat), emmanuel.trelat@upmc.fr (E. Trélat), zuazua@bcamath.org (E. Zuazua).

0294-1449/$ – see front matter
http://dx.doi.org/10.1016/j.anihpc.2012.11.005

1. Introduction

1.1. The optimal design problems

Let T be a positive real number. Consider the one-dimensional controlled wave equation on $[0,\pi]$ with Dirichlet boundary conditions

$$\begin{aligned}
&\partial_{tt} y(t,x) - \partial_{xx} y(t,x) = h_\omega(t,x), \quad (t,x) \in (0,T)\times(0,\pi),\\
&y(t,0) = y(t,\pi) = 0, \quad t \in [0,T],\\
&y(0,x) = y^0(x), \qquad \partial_t y(0,x) = y^1(x), \quad x \in [0,\pi],
\end{aligned} \tag{1}$$

where h_ω is a control supported in $[0,T]\times\omega$ and ω is a measurable subset of $[0,\pi]$. For all initial data $(y^0,y^1) \in H_0^1(0,\pi)\times L^2(0,\pi)$ and every $h_\omega \in L^2((0,T)\times(0,\pi))$, there exists a unique solution $y \in \mathcal{C}^0(0,T;H_0^1(0,\pi)) \cap \mathcal{C}^1(0,T;L^2(0,\pi)) \cap \mathcal{C}^2(0,T;H^{-1}(0,\pi))$ of the Cauchy problem (1). The exact null controllability problem settled in these spaces consists of finding a control h_ω steering the control system (1) to

$$y(T,\cdot) = \partial_t y(T,\cdot) = 0. \tag{2}$$

It is well known that, for every subset ω of $[0,\pi]$ of positive Lebesgue measure, the exact null controllability problem has a solution whenever $T \geqslant 2\pi$ (see [23]). The Hilbert Uniqueness Method (HUM; see [12,13]) permits to design such a control, achieving moreover the null controllability in minimal $L^2((0,T)\times(0,\pi))$ norm. This (unique) control is referred to as the HUM control and is defined as follows. Using the observability inequality

$$C\left\|\left(\phi^0,\phi^1\right)\right\|^2_{L^2(0,\pi)\times H^{-1}(0,\pi)} \leqslant \int_0^T\int_\omega \phi(t,x)^2\,dx\,dt, \tag{3}$$

where C is a positive constant (only depending on T and ω), valuable for every solution ϕ of the adjoint system

$$\begin{aligned}
&\partial_{tt}\phi(t,x) - \partial_{xx}\phi(t,x) = 0, \quad (t,x)\in(0,T)\times(0,\pi),\\
&\phi(t,0) = \phi(t,\pi) = 0, \quad t\in[0,T],\\
&\phi(0,x) = \phi^0(x), \qquad \partial_t\phi(0,x) = \phi^1(x), \quad x\in[0,\pi],
\end{aligned} \tag{4}$$

and every $T \geqslant 2\pi$, the functional

$$J_\omega\left(\phi^0,\phi^1\right) = \frac{1}{2}\int_0^T\int_\omega \phi(t,x)^2\,dx\,dt - \left\langle\phi^1,y^0\right\rangle_{H^{-1},H_0^1} + \left\langle\phi^0,y^1\right\rangle_{L^2}, \tag{5}$$

has a unique minimizer (still denoted (ϕ^0,ϕ^1)) in the space $L^2(0,\pi)\times H^{-1}(0,\pi)$, for all $(y^0,y^1)\in H_0^1(0,\pi)\times L^2(0,\pi)$. In (5) the notation $\langle\cdot,\cdot\rangle_{H^{-1},H_0^1}$ stands for the duality bracket between $H^{-1}(0,\pi)$ and $H_0^1(0,\pi)$, and the notation $\langle\cdot,\cdot\rangle_{L^2}$ stands for the usual scalar product of $L^2(0,\pi)$. The HUM control h_ω steering (y^0,y^1) to $(0,0)$ in time T is then given by

$$h_\omega(t,x) = \chi_\omega(x)\phi(t,x), \tag{6}$$

for almost all $(t,x)\in(0,T)\times(0,\pi)$, where χ_ω denotes the characteristic function of the measurable set ω and ϕ is the solution of (4) with initial data (ϕ^0,ϕ^1) minimizing J_ω.

In this article we are interested in the problem of optimizing the shape and position of the control support ω, over all possible measurable subsets of $[0,\pi]$ of given Lebesgue measure. Throughout the article, let $L\in(0,1)$ be fixed. We define the set

$$\mathcal{U}_L = \left\{\chi_\omega \;\middle|\; \omega \text{ is a measurable subset of } [0,\pi] \text{ such that } |\omega| = L\pi\right\}, \tag{7}$$

where $|\omega|$ denotes the Lebesgue measure of ω. We consider the two following problems.

First problem. Let $(y^0, y^1) \in H_0^1(0,\pi) \times L^2(0,\pi)$ be some fixed initial data. We investigate the problem of minimizing

$$\|h_\omega\|^2_{L^2((0,T)\times(0,\pi))} = \int_0^T \int_\omega \phi(t,x)^2 \, dx \, dt, \tag{8}$$

where h_ω is the HUM control (6) steering (y^0, y^1) to $(0,0)$ in time T, over all possible subsets ω of $[0,\pi]$ such that $\chi_\omega \in \mathcal{U}_L$.

This problem is the most simple among all possible problems of optimal location of actuators for vibration models. In this minimization problem, the optimal set ω, whenever it exists, depends on the initial data (y^0, y^1). Although this problem has some mathematical interest and provides some insight in such shape optimization problems, it is not really relevant for practical purposes since the knowledge of the initial data of (1) is needed. To discard this dependence and improve the robustness of the cost function, we consider the following second problem. Recall that, for every subset ω of $[0,\pi]$ of positive measure, the so-called HUM operator Γ_ω is defined by

$$\Gamma_\omega : H_0^1(0,\pi) \times L^2(0,\pi) \to L^2\big((0,T)\times(0,\pi)\big),$$
$$\big(y^0, y^1\big) \mapsto h_\omega,$$

where h_ω is the HUM control (6) steering (y^0, y^1) to $(0,0)$ in time T.

Second problem. We investigate the problem of minimizing the norm of the operator Γ_ω

$$\|\Gamma_\omega\| = \sup\big\{\|h_\omega\|_{L^2((0,T)\times(0,\pi))} \bigm| \big(y^0, y^1\big) \in H_0^1(0,\pi) \times L^2(0,\pi),\ \big\|\big(y^0, y^1\big)\big\|_{H_0^1(0,\pi)\times L^2(0,\pi)} = 1\big\} \tag{9}$$

over the set $\mathcal{U}_L$.

1.2. A brief state of the art

Although the literature on such kinds of shape optimization problems is quite abundant in engineering applications (see e.g. [15,18,24,28]), very few theoretical results and mathematical proofs do exist. In [7,8], similar questions were investigated for optimal stabilization issues of the one-dimensional wave equation. These two articles are important because they were the first (and to our knowledge, up to now the only ones) to provide mathematical arguments and proofs to characterize the optimal set whenever it exists. They have inspired our own works such as the recent one [20] in which we investigate similar questions for the optimal observability of the wave equation (with practical issues for the optimal placement of sensors). Concerning the problem investigated in the present article of determining an optimal control domain, we quote the article [19] whose contribution is to prove the existence of a solution for a relaxed version of our first problem above. Note that we provide in Theorem 1 (Section 2.1) a more precise result with a new shorter proof of [19, Theorem 2.1] using the frequential analysis approach used throughout our paper. We also quote [16] in which the author investigates numerically this relaxed first problem (not only in dimension one), using shape and topological derivatives of the functional under consideration, and gives numerical simulations providing evidence that, for a specific initial data, the optimal solution of the relaxed problem is the characteristic function of some subset ω of $[0,\pi]$ and thus is solution of the (initial, non-relaxed) first problem. In [17] the authors investigate the optimal location of the support of the HUM control for the one-dimensional heat equation, with fixed initial data. They give a first-order optimality condition for a relaxed version and then provide some numerical simulations. We stress that in these contributions only the first problem is addressed, from the numerical point of view. As said above, this first problem leads to interesting mathematical properties but is of little practical interest for practical purposes since it depends on the initial data. Anyway the study of the first problem is interesting since it provides a better insight in such shape optimization problems and is a benchmark for studying other classes of such problems. But the second problem is still to be addressed in the context of the heat equation.

1.3. Short description of our contributions

In this article we provide a complete mathematical analysis of the two shape optimization problems settled above, and in turn we obtain efficient algorithms for computing the optimal domains. The article is structured as follows.

Section 2 is devoted to solving the first problem (8) (that is, the problem with fixed initial data). First of all, using a Fourier series expansion of the adjoint ϕ, we prove existence and uniqueness of a solution of a relaxed version of the first problem (Theorem 1). Here, the relaxation procedure consists of considering the convex closure of the set $\mathcal{U}_L$ in L^∞ weak star topology, that is

$$\overline{\mathcal{U}}_L=\left\{a\in L^\infty\big([0,\pi],[0,1]\big)\ \Big|\ \int_0^\pi a(x)\,dx=L\pi\right\}.$$

If the minimizer $a\in\overline{\mathcal{U}}_L$ of the relaxed first problem belongs to $\mathcal{U}_L$, then it is the characteristic function of a subset ω of $[0,\pi]$ of Lebesgue measure $L\pi$, and then ω is a solution of the (initial) first problem (8). At the opposite, if $a\in\overline{\mathcal{U}}_L\setminus\mathcal{U}_L$ then the first problem (8) does not have any solution, and we speak of a *relaxation phenomenon*. Throughout Section 2 we provide a complete characterization of these phenomena.

More precisely, in Section 2.4 we first provide several possible sufficient conditions ensuring that the first problem (8) has a solution. For instance we will prove the following result (see the more precise statement of Theorem 2).

Theorem. *Let $T\geqslant 2\pi$ and $L\in(0,1)$. Assume that the initial data (y^0,y^1) under consideration have a finite number of Fourier components. Then the first problem (8) has a unique solution $\chi_\omega\in\mathcal{U}_L$, which has a finite number of connected components.*

The uniqueness of the solution must be of course understood up to some subset of zero Lebesgue measure. In Theorem 3 we provide a variant of this result, valuable only when T is an integer multiple of 2π: we show that the above sufficient condition can then be weakened to an exponential decrease of the Fourier coefficients of the initial data. Note that this sufficient condition holds if the initial data are analytic or quasi-analytic.

In Section 2.2 we provide further comments on the relaxed version of the Hilbert Uniqueness Method and on its well-posedness, showing strict convexity properties.

Section 2.3 is another significant contribution of our work. In this section we provide an interpretation of the first problem (8) in terms of a classical (however infinite dimensional) optimal control problem, to which we can apply, up to some slight adaptations, the well-known Pontryagin Maximum Principle and derive necessary and sufficient optimality conditions for the relaxed first problem. The change of point of view consists in considering the functions $a\in\overline{\mathcal{U}}_L$ of the relaxed problem as controls. These results and this interpretation (which is new to the best of our knowledge) have two important consequences.

First, in the case where T is a multiple integer of 2π, these necessary and sufficient conditions enable us to provide a complete characterization of all initial data for which

- the first problem (8) has at least one solution,
- the first problem (8) has exactly one solution,
- the first problem (8) does not have any solution (in other words, the relaxation phenomenon occurs).

In turn, we establish some connections between the complexity of the optimal set of the first problem (8), whenever it exists, and the regularity of the initial data. As a particular case, we prove the following result, showing the sharpness of the previous sufficient conditions.

Theorem. *There exist initial data (y^0,y^1) of class C^∞ such that the first problem (8) has a unique solution ω, which is a fractal set and thus has an infinite number of connected components.*

There exist initial data (y^0,y^1) of class C^∞ such that the first problem (8) does not have any solution.

We stress that our results stated in Section 2.5 are stronger since we provide a characterization of all possible initial data for which such or such phenomenon occurs, by establishing a precise correspondence with some classes of Fourier series.

The second consequence of the interpretation in terms of an optimal control problem is the fact that we are able to design new and efficient numerical methods in order to compute an approximation of the optimal set whenever it exists. In Section 2.6 we define and study a modal approximation of the first problem consisting of truncating to N modes the infinite dimensional control system corresponding to our problem. We thus get a usual finite dimensional optimal control problem which can be easily analyzed. We prove in Proposition 5 that the truncated first problem has a unique solution (which is therefore a characteristic function). Moreover, we can apply to this finite dimensional optimal control problem the usual numerical methods of optimal control, which result into new numerical approaches to compute approximations of the optimal set. We provide several numerical simulations that show the efficiency of this new approach to optimal design.

We stress however in Section 2.6.2 on the limitations of numerics in optimal design. Indeed, according to our theoretical results, and although we have a nice Γ-convergence result (stated in Proposition 4), such numerical approaches cannot permit to guess a relaxation phenomenon. By the way note that the occurrence of the relaxation phenomenon for smooth initial data infirms a conjecture of [16] based on numerical observations.

The second problem (9), consisting of minimizing the norm of the HUM operator, is investigated in Section 3. First, in Section 3.1 we show that this problem can be reduced to the problem of maximizing the observability constant in the observability inequality (3), over all possible subsets of $[0,\pi]$ of Lebesgue measure $L\pi$. This problem was investigated and solved in [20], and we briefly report on the results that can be derived, obtaining in particular the following result.

Theorem. *Assume that T is an integer multiple of 2π. Then the optimal value of (9) is $\frac{2}{LT}$, and is reached if and only if $L=1/2$.*

Theorems 5 and 6 provide precise results and show that the natural modal truncation of the second problem has a unique solution sharing particular features such as the spillover phenomenon.

Section 4 contains a conclusion and some open problems. Appendix A is devoted to the proofs of the results.

2. Optimal location of controllers for fixed initial data

This section is devoted to solving the first problem, that is the problem of minimizing (8) on $\mathcal{U}_L$ for fixed initial data. Our objective is first to write the functional (8) in a more suitable way for the problem to be further interpreted in terms of an optimal control problem. For that purpose, we use a series expansion of ϕ in the Hilbertian basis of eigenfunctions of the Dirichlet Laplacian. This choice is also motivated by the fact that, in Section 2.6, we will take advantage of this optimal control formulation to derive efficient numerical methods in order to compute numerically the optimal domains.

Every solution $\phi\in C^0(0,2\pi;L^2(0,\pi))\cap C^1(0,2\pi;H^{-1}(0,\pi))$ of (4) can be expanded as

$$\phi(t,x)=\sum_{j=1}^{+\infty}\big(A_j\cos(jt)+B_j\sin(jt)\big)\sin(jx), \tag{10}$$

where $A=(A_j)_{j\in\mathbb{N}^*}$ and $B=(B_j)_{j\in\mathbb{N}^*}$ belong to $\ell^2(\mathbb{R})$. By the way, note that

$$\big\|\big(\phi^0,\phi^1\big)\big\|^2_{L^2\times H^{-1}}=\frac{\pi}{2}\sum_{j=1}^{+\infty}\big(A_j^2+B_j^2\big). \tag{11}$$

Let $(y^0,y^1)\in H^1_0(0,\pi)\times L^2(0,\pi)$ be fixed initial data. The functional (5) to be minimized in the HUM method, still denoted J_ω, is

$$J_\omega(A,B)=Q_\omega(A,B)-\sum_{j=1}^{+\infty}jB_j\big\langle\sin(j\cdot),y^0\big\rangle_{L^2}+\sum_{j=1}^{+\infty}A_j\big\langle\sin(j\cdot),y^1\big\rangle_{L^2}, \tag{12}$$

where

$$Q_\omega(A,B)=\frac{1}{2}\int_0^T\int_\omega\left(\sum_{j=1}^{+\infty}\big(A_j\cos(jt)+B_j\sin(jt)\big)\sin(jx)\right)^2dx\,dt. \tag{13}$$

In (12) the notation $\sin(j\cdot)$ stands for the function $x\mapsto\sin(jx)$. Note that both series in (12) converge, in particular the first one, since this term is the duality product $\langle\phi^1,y^0\rangle_{H^{-1},H_0^1}$, with $\phi^1\in H^{-1}(0,\pi)$ and $y^0\in H_0^1(0,\pi)$. It can be also seen from the Cauchy–Schwarz inequality, noting the fact that the series of general terms $j\langle\sin(j\cdot),y^0\rangle_{L^2}$ and $\langle\sin(j\cdot),y^1\rangle_{L^2}$ belong to $\ell^2(\mathbb{R})$ because $y^0\in H_0^1(0,\pi)$ and $y^1\in L^2(0,\pi)$, and moreover,

$$\|y^0\|^2_{H_0^1(0,\pi)}=\frac{2}{\pi}\sum_{j=1}^{+\infty}j^2\big\langle\sin(j\cdot),y^0\big\rangle^2_{L^2},\qquad \|y^1\|^2_{L^2(0,\pi)}=\frac{2}{\pi}\sum_{j=1}^{+\infty}\big\langle\sin(j\cdot),y^1\big\rangle^2_{L^2}. \tag{14}$$

2.1. Relaxation procedure, existence and uniqueness result

In this section we provide an existence and uniqueness result for a relaxed version of the first optimization problem (8). Recall that the HUM control h_ω is determined by (6), i.e.,

$$h_\omega(t,x)=\chi_\omega(x)\phi_{\chi_\omega}(t,x),$$

for almost all $(t,x)\in(0,T)\times(0,\pi)$, where ϕ is the solution of (4) with initial data (ϕ^0,ϕ^1) that is the unique minimizer of the HUM functional J_ω defined by (5) or equivalently by (12).

The first problem (8) consists of minimizing the functional F defined by

$$F(\chi_\omega)=\int_0^T\int_0^\pi\chi_\omega(x)\phi_{\chi_\omega}(t,x)^2\,dx\,dt, \tag{15}$$

over the set $\mathcal{U}_L$, where ϕ_{χ_ω} is the adjoint state associated with the HUM control, that is

$$\phi_{\chi_\omega}(t,x)=\sum_{j=1}^{+\infty}\big(A_j^\omega\cos(jt)+B_j^\omega\sin(jt)\big)\sin(jx),$$

where (A^ω,B^ω) is the unique minimizer of J_ω over $(\ell^2(\mathbb{R}))^2$. Since a minimizer of F over $\mathcal{U}_L$ defined by (5) does not necessarily exist, we carry out a relaxation procedure as in [19], consisting of convexifying the problem. This procedure is very usual in shape optimization problems (see e.g. [3]). Here, the closure of $\mathcal{U}_L$ for the weak star topology of L^∞ is the convex set

$$\overline{\mathcal{U}}_L=\left\{a\in L^\infty\big([0,\pi],[0,1]\big)\;\Big|\;\int_0^\pi a(x)\,dx=L\pi\right\}. \tag{16}$$

The relaxed version of the first problem (8) then consists of minimizing the functional F defined by

$$F(a)=\int_0^T\int_0^\pi a(x)\phi_a(t,x)^2\,dx\,dt, \tag{17}$$

over the set $\overline{\mathcal{U}}_L$, where ϕ_a is the adjoint state solution of (4) with $(\phi_a(0,\cdot),\partial_t\phi_a(0,\cdot))=(\phi_a^0,\phi_a^1)$, where (ϕ_a^0,ϕ_a^1) is the minimizer of the "relaxed" HUM functional J_a defined by

$$J_a\big(\phi^0,\phi^1\big)=\frac{1}{2}\int_0^T\int_0^\pi a(x)\phi(t,x)^2\,dx\,dt+\big\langle\phi^0,y^1\big\rangle_{L^2}-\big\langle\phi^1,y^0\big\rangle_{H^{-1},H_0^1}, \tag{18}$$

for every $(\phi^0, \phi^1) \in L^2(0,\pi) \times H^{-1}(0,\pi)$ and every $a \in \overline{\mathcal{U}}_L$ (this is the natural relaxation of (5)), where ϕ is the adjoint state solution of (4) with $(\phi(0,\cdot), \partial_t\phi(0,\cdot)) = (\phi^0, \phi^1)$. In this relaxed version of the first problem, the characteristic function of ω is replaced with a density function a. Note that the Hilbert Uniqueness Method can be easily extended to this relaxed case as underlined in the following proposition.

Proposition 1. *Let $a \in \overline{\mathcal{U}}_L$ and $T \geqslant 2\pi$. The functional J_a defined by* (18) *has a unique minimizer $(A^a, B^a) \in (\ell^2(\mathbb{R}))^2$.*

The proof of this proposition is done in Appendix A.2. This relaxation procedure permits to ensure existence and uniqueness results, as stated in the next theorem whose first part (existence) is the main result of [19] as already mentioned in the introduction.

Theorem 1. *The relaxed first problem of minimizing the functional F defined by* (17) *over the set $\overline{\mathcal{U}}_L$ defined by* (16) *has a unique solution.*

Proof. It follows from the HUM method (minimization of a quadratic functional) that, for every $a \in \overline{\mathcal{U}}_L$,

$$
\begin{aligned}
-\frac{1}{2}F(a) &= -\frac{1}{2}\int_0^T\int_0^\pi a(x)\phi_a^2(t,x)\,dx\,dt \\
&= \frac{1}{2}\left(\langle \phi_a^0, y^1\rangle_{L^2} - \langle \phi_a^1, y^0\rangle_{H^{-1},H_0^1}\right) \\
&= \frac{1}{2}\int_0^T\int_0^\pi a(x)\phi_a^2(t,x)\,dx\,dt + \langle \phi_a^0, y^1\rangle_{L^2} - \langle \phi_a^1, y^0\rangle_{H^{-1},H_0^1} \\
&= \min_{(A,B)\in(\ell^2(\mathbb{R}))^2} J_a(A,B).
\end{aligned}
$$

Therefore, J is convex (as the opposite of the minimum of affine functions) and lower semicontinuous. The existence of a minimizer follows since $\overline{\mathcal{U}}_L$ is compact for the L^∞ weak star topology. The uniqueness is due to the fact that F is actually strictly convex (see Lemma 4 of Section 2.2 further). □

Remark 1. If the minimizer $a \in \overline{\mathcal{U}}_L$ of the theorem actually belongs to the set $\mathcal{U}_L$ and thus is the characteristic function of a subset ω of $[0,\pi]$ of Lebesgue measure $L\pi$, then this set ω is a solution of the (initial, non-relaxed) first problem (8). Otherwise it means that the (initial) first problem (8) has no solution and we speak of a *relaxation phenomenon*. In what follows we will characterize all initial data for which such a relaxation phenomenon occurs.

In the proof of Theorem 1, we only use the lower semicontinuity of F, but note that F is actually continuous for the weak star topology of L^∞, as mentioned in [19].

Lemma 1. *The map $a \mapsto F(a)$ defined on $\overline{\mathcal{U}}_L$ is continuous for the L^∞ weak star topology.*

In the sequel, we will prove that there exist some initial data $(y^0, y^1) \in H_0^1(0,\pi) \times L^2(0,\pi)$ for which a relaxation phenomenon occurs, in other words, the unique solution of the problem of minimizing F defined by (17) belongs to $\overline{\mathcal{U}}_L \setminus \mathcal{U}_L$. Thus, we will be led to consider a sequence $(\chi_{\omega_n})_{n\in\mathbb{N}}$ of $\mathcal{U}_L$ such that $F(\chi_{\omega_n}) \to F(a)$ as $n \to +\infty$, that we will call a minimizing sequence. The following question is then natural: denote by $(y_n)_{n\in\mathbb{N}}$ a sequence of solutions of system (1) associated with the controls

$$
h_{\omega_n}(x,t) = \chi_{\omega_n}(x)\phi_{\chi_{\omega_n}}(t,x),
$$

where $\phi_{\chi_{\omega_n}}$ denotes the adjoint state provided by the Hilbert Uniqueness Method (with the fixed initial data (y^0, y^1)) and control domain ω_n. Denote also by y_a the solution of (1) associated with the control $h_a(x,t) = a(x)\phi_a(t,x)$. Does the sequence $(y_n)_{n\in\mathbb{N}}$ converge in some sense to y_a? The following proposition provides a positive answer to this question.

Proposition 2. *The sequence* $(\phi_{\chi_{\omega_n}})_{n\in\mathbb{N}}$ *converges to* ϕ_a *strongly in* $L^2(0,T;H^{-1}(0,\pi))$ *and weakly in* $L^2(0,T;L^2(0,\pi))\cap H^1(0,T;H^{-1}(0,\pi))$*, and the sequence* $(y_n)_{n\in\mathbb{N}}$ *converges to* y_a *strongly in* $L^2(0,T;L^2(0,\pi))$ *and weakly in* $L^2(0,T;H_0^1(0,\pi))\cap H^1(0,T;L^2(0,\pi))$.

The proof of this proposition is done in Appendix A.3.

2.2. Further comments on the relaxed control problem

Recall that

$$F(a)=\int_0^T\int_0^\pi a(x)\phi_a(t,x)^2\,dx\,dt, \tag{19}$$

with

$$\phi_a(t,x)=\sum_{j=1}^{+\infty}\big(A_j^a\cos(jt)+B_j^a\sin(jt)\big)\sin(jx), \tag{20}$$

where $(A^a,B^a)\in(\ell^2(\mathbb{R}))^2$ is the minimizer of the quadratic functional J_a defined by (18). Since J_a is convex, the first-order optimality conditions for the problem of minimizing J_a over $(\ell^2(\mathbb{R}))^2$ are necessary and sufficient. They are written as

$$\Lambda_a(A,B)=C, \tag{21}$$

where $\Lambda_a:(\ell^2(\mathbb{R}))^2\to(\ell^2(\mathbb{R}))^2$ is defined by

$$\begin{aligned}\Lambda_a(A,B)_j&=\int_0^T\int_0^\pi a(x)\sum_{k=1}^{+\infty}\big(A_k\cos(kt)+B_k\sin(kt)\big)\sin(kx)\sin(jx)\begin{pmatrix}\cos(jt)\\ \sin(jt)\end{pmatrix}dx\,dt\\&=\sum_{k=1}^{+\infty}\int_0^T\big(A_k\cos(kt)+B_k\sin(kt)\big)\int_0^\pi a(x)\sin(kx)\sin(jx)\,dx\begin{pmatrix}\cos(jt)\\ \sin(jt)\end{pmatrix}dt\end{aligned}$$

for every $j\in\mathbb{N}^*$, with the notation $\Lambda_a(A,B)=(\Lambda_a(A,B)_j)_{j\in\mathbb{N}^*}$ and $\Lambda_a(A,B)_j\in\mathbb{R}^2$, and where

$$C_j=\begin{pmatrix}-\langle\sin(j\cdot),y^1\rangle_{L^2,L^2}\\ \langle j\sin(j\cdot),y^0\rangle_{H^{-1},H_0^1}\end{pmatrix} \tag{22}$$

for every $j\in\mathbb{N}^*$. The fact that $\Lambda_a(A,B)\in(\ell^2(\mathbb{R}))^2$ for every $(A,B)\in(\ell^2(\mathbb{R}))^2$ can be seen as follows. There holds

$$\big\langle\Lambda_a(A,B),(A,B)\big\rangle_{(\ell^2(\mathbb{R}))^2}=\int_0^T\int_0^\pi a(x)\left(\sum_{k=1}^{+\infty}\big(A_k\cos(kt)+B_k\sin(kt)\big)\sin(kx)\sin(jx)\right)^2dx\,dt,$$

for every $(A,B)\in(\ell^2(\mathbb{R}))^2$. Using Lemmas 6 and 7, we get

$$\left[\frac{T}{2\pi}\right]\frac{(\pi L-\sin(\pi L))}{2}\leqslant\frac{\langle\Lambda_a(A,B),(A,B)\rangle_{(\ell^2(\mathbb{R}))^2}}{\|(A,B)\|^2_{(\ell^2(\mathbb{R}))^2}}\geqslant\left(\left[\frac{T}{2\pi}\right]+1\right)\frac{(\pi L+\sin(\pi L))}{2}, \tag{23}$$

which ensures that $\langle\Lambda_a(\cdot),\cdot\rangle_{(\ell^2(\mathbb{R}))^2}$ is a coercive quadratic form in $(\ell^2(\mathbb{R}))^2$. By duality, we immediately deduce the following lemma.

Lemma 2. *The operator* Λ_a *is a continuous invertible (and symmetric) operator from* $(\ell^2(\mathbb{R}))^2$ *to* $(\ell^2(\mathbb{R}))^2$.

Note that Λ_a is nothing else but the frequential representation of the Gramian of the relaxed Hilbert Uniqueness Method.

As a result, denoting by (A^a, B^a) the unique solution of (21), one gets

$$F(a) = \frac{1}{2}\big\langle \Lambda_a\big(A^a, B^a\big), \big(A^a, B^a\big)\big\rangle_{(\ell^2(\mathbb{R}))^2} = \frac{1}{2}\big\langle C, \Lambda_a^{-1}(C)\big\rangle_{(\ell^2(\mathbb{R}))^2}.$$

An interesting consequence of this expression is that it becomes easier to compute the derivative of F in an admissible direction h. We have indeed the following result. For every $a \in \overline{\mathcal{U}}_L$, we denote by $\mathcal{T}_{a,\overline{\mathcal{U}}_L}$ the tangent cone to the set $\overline{\mathcal{U}}_L$ at a, that is the set of functions $h \in L^\infty(0,\pi)$ such that, for any sequence of positive real numbers ε_n decreasing to 0, there exists a sequence of functions $h_n \in L^\infty(0,\pi)$ converging to h as $n \to +\infty$, and $a + \varepsilon_n h_n \in \overline{\mathcal{U}}_L$ for every $n \in \mathbb{N}$ (see for instance [9, Chapter 7]).

Lemma 3. *Let $a \in \overline{\mathcal{U}}_L$ and $h \in \mathcal{T}_{a,\overline{\mathcal{U}}_L}$. The function F is two times Fréchet-differentiable at a in the direction h and one has*

$$dF(a).h = -\frac{1}{2}\big\langle \Lambda_h\big(A^a, B^a\big), \big(A^a, B^a\big)\big\rangle_{(\ell^2(\mathbb{R}))^2}, \tag{24}$$

$$d^2F(a).(h,h) = \big\langle \Lambda_a^{-1}\big(\Lambda_h\big(A^a, B^a\big)\big), \big(\Lambda_h\big(A^a, B^a\big)\big)\big\rangle_{(\ell^2(\mathbb{R}))^2}. \tag{25}$$

Proof. Note that the mapping $a \in \overline{\mathcal{U}}_L \mapsto \Lambda_a \in \mathcal{L}((\ell^2(\mathbb{R}))^2, (\ell^2(\mathbb{R}))^2)$ is linear and hence obviously differentiable. Since the quadratic form $\langle \Lambda_a(\cdot), \cdot\rangle_{(\ell^2(\mathbb{R}))^2}$ is coercive in $(\ell^2(\mathbb{R}))^2$, with a constant that is uniform with respect to a, it follows that the mapping $a \mapsto \Lambda_a^{-1} \in \mathcal{L}((\ell^2(\mathbb{R}))^2, (\ell^2(\mathbb{R}))^2)$ is differentiable. By composition of differentiable functions, F is differentiable with respect to a, and one has

$$\begin{aligned}
\big\langle dF(a), h\big\rangle &= -\frac{1}{2}\big\langle \Lambda_a^{-1}\Lambda_h\Lambda_a^{-1}(C), C\big\rangle_{(\ell^2(\mathbb{R}))^2} \\
&= -\frac{1}{2}\big\langle \Lambda_h\big(A^a, B^a\big), \big(A^a, B^a\big)\big\rangle_{(\ell^2(\mathbb{R}))^2} \\
&= -\frac{1}{2}\int_0^\pi h(x)\int_0^T \phi_a(t,x)^2\,dt\,dx,
\end{aligned}$$

for every $h \in L^\infty(0,\pi)$. The second derivative is easily obtained as well from a similar computation. □

This computation will be important to apply the Pontryagin Maximum Principle in the sequel. The following lemma is a direct consequence of all previous remarks.

Lemma 4. *The function F is strictly convex on $\overline{\mathcal{U}}_L$.*

Proof. Using Lemmas 6 and 7 stated in Appendix A.1, we get, for every $(A,B) \in (\ell^2(\mathbb{R}))^2$,

$$\big\langle \Lambda_a(A,B), (A,B)\big\rangle_{(\ell^2(\mathbb{R}))^2} \leqslant \left(\left[\frac{T}{2\pi}\right] + 1\right)\frac{(\pi L + \sin(\pi L))}{2}\big\|(A,B)\big\|^2_{(\ell^2(\mathbb{R}))^2},$$

and hence it follows that Λ_a^{-1} is coercive as well in $(\ell^2(\mathbb{R}))^2$. Thus, using Lemma 3, the Hessian of F is coercive and hence F is strictly convex. □

2.3. Interpretation in terms of optimal control

We now give an interpretation of the relaxed first problem (17) in terms of a classical optimal control problem, to which we will apply the Pontryagin Maximum Principle. For that purpose, functions a of $\overline{\mathcal{U}}_L$ are now considered as controls, and we consider the control system

$$z_0'(x) = a(x),$$
$$z_{j,k}'(x) = a(x)\sin(jx)\sin(kx), \quad (j,k) \in (\mathbb{N}^*)^2, \tag{26}$$

for almost every $x \in \pi$, with initial conditions

$$z_0(0) = 0, \qquad z_{j,k}(0) = 0, \quad (j,k) \in (\mathbb{N}^*)^2. \tag{27}$$

Note the important change of point of view in this presentation. Indeed, the spatial profile of the control of the wave equation becomes a dynamical control of an infinite dimensional differential system where the space variable of the wave equation becomes a pseudo-time.

The relaxed first problem of minimizing (17) over $\overline{\mathcal{U}}_L$ is then equivalent to the optimal control problem of determining a control $a \in \overline{\mathcal{U}}_L$ steering the infinite dimensional control system (26) from the initial conditions (27) to the final condition

$$z_0(\pi) = L\pi, \tag{28}$$

and minimizing the functional F defined by (17), written here as

$$F(a) = \frac{1}{2}\langle C, \Lambda_a^{-1}(C)\rangle_{(\ell^2(\mathbb{R}))^2}, \tag{29}$$

where $\Lambda_a : (\ell^2(\mathbb{R}))^2 \to (\ell^2(\mathbb{R}))^2$ is the mapping defined as before by $\Lambda_a(A,B) = (\Lambda_a(A,B)_j)_{j\in\mathbb{N}^*}$ and $\Lambda_a(A,B)_j \in \mathbb{R}^2$, with

$$\Lambda_a(A,B)_j = \int_0^T \sum_{k=1}^{+\infty} \big(A_k\cos(kt) + B_k\sin(kt)\big) z_{j,k}(\pi) \begin{pmatrix}\cos(jt)\\ \sin(jt)\end{pmatrix} dt,$$

for every $j \in \mathbb{N}^*$. This change of point of view happens to be relevant to solve the problem, both theoretically and numerically. For the theoretical part, we will next see that the application of the Pontryagin Maximum Principle to this optimal control problem leads to a complete characterization of minimizers and of all cases for which the relaxation phenomenon occurs. The optimal control point of view will also permit to derive time-efficient algorithms of numerical computation.

Versions of the Pontryagin Maximum Principle in the infinite dimensional setting can be found e.g. in [11]. These versions suffer however from two severe limitations: the first of which is that the functional state space is required to be a strictly convex Banach space, and the second is that the final state must satisfy a finite codimension assumption. In our case here, the second requirement is obviously fulfilled, however the first one is not satisfied a priori. We are however able to find an appropriate equivalent formulation of our problem, which is suitable for the application of the Pontryagin Maximum Principle. This leads to the following result for the unique solution $a \in \overline{\mathcal{U}}_L$ of the relaxed first problem (stated in Theorem 1), interpreted here as well as the unique solution of the optimal control problem (26)–(29).

Proposition 3. *Let $a \in \overline{\mathcal{U}}_L$ be the (unique) solution of the relaxed first problem (as stated in Theorem 1). Then there exists a real number p_0 such that*

$$a(x) = \begin{cases} 1 & \text{if } \varphi(x) > 0,\\ 0 & \text{if } \varphi(x) < 0, \end{cases} \tag{30}$$

for almost every $x \in [0,\pi]$, where the function φ, called switching function, is defined by

$$\varphi(x) = p_0 + \sum_{j,k=1}^{+\infty} p_{j,k}\sin(jx)\sin(kx), \tag{31}$$

and

$$p_{j,k} = -\int_0^T \big(A_j^a\cos(jt) + B_j^a\sin(jt)\big)\big(A_k^a\cos(kt) + B_k^a\sin(kt)\big)\,dt, \tag{32}$$

for every $(j,k) \in (\mathbb{N}^*)^2$.

The proof of Proposition 3 is given in Appendix A.4.

Remark 2. Note that the switching function φ defined by (31) can be as well written as

$$\varphi(x) = p_0 - \int_0^T \phi_a(t,x)^2\,dt. \tag{33}$$

Since $(A^a, B^a) \in (\ell^2(\mathbb{R}))^2$, it follows easily from (32) (or directly from (20) and (33)) that $(p_{j,k})_{j\in\mathbb{N}^*} \in \ell^1(\mathbb{R})$. Therefore the switching function φ is continuous on $[0,\pi]$.

Remark 3. If the switching function φ vanishes identically on a subset of positive measure then the value of $a(x)$ remains undetermined on this subset (this is a usual fact in the application of the Pontryagin Maximum Principle in optimal control). If this situation does not occur then a is completely determined by (30), and in this case, it can be noted that, due to the strict convexity of F proved in Lemma 4, the first-order necessary conditions (30)–(32) inferred from the Pontryagin Maximum Principle are also sufficient.

At this step, we realize that the application of the Pontryagin Maximum Principle leads to a simple characterization of the minimizer a of Theorem 1 in terms of the switching function φ defined by (31) (or (33)). If φ does not vanish identically on any subset of $[0,\pi]$ of positive measure then the undetermined case does not occur, the optimal control a (which is said bang-bang) is completely determined by (30), and in that case a is the characteristic function of some subset ω, which is the (unique) optimal solution of the (non-relaxed) first problem (15). In the next subsection we provide simple sufficient conditions on the coefficients implying this conclusion. At the opposite, if the switching function vanishes identically on some subset of $[0,\pi]$ of positive measure (degenerate case), then the optimal control a is not bang-bang and thus is not a characteristic function. This means that in that case the (non-relaxed) first problem (15) does not admit any optimal solution. This is what we call a relaxation phenomenon. In Section 2.4.2 we characterize all possible initial data for which the relaxation phenomenon occurs.

2.4. *Sufficient conditions for the first problem (8)*

In this section we prove that, under suitable assumptions on the regularity of the initial data (y^0, y^1), the first problem has a unique solution. The main idea is to prove that if the initial data are regular enough then the (unique) minimizer of the relaxed first problem (17) characterized previously is the characteristic function of a subset ω and therefore ω is the solution of the initial (non-relaxed) first problem (8).

2.4.1. *A first sufficient condition*

Theorem 2. *Assume that the initial data $y^0 \in H_0^1(0,\pi)$ and $y^1 \in L^2(0,\pi)$ satisfy the following property: there exists $N \in \mathbb{N}^*$ such that*

$$\int_0^\pi y^0(x)\sin(jx)\,dx = \int_0^\pi y^1(x)\sin(jx)\,dx = 0,$$

for every $j > N$. Then the first problem (8) has a unique[1] solution χ_ω, where ω is a measurable subset of $[0,\pi]$ of Lebesgue measure $L\pi$. Moreover,

- *ω has at most N connected components,*
- *there exists $\eta > 0$ such that $\omega \subset [\eta, \pi - \eta]$.*

[1] Similarly to the definition of elements of L^∞, the subset ω is unique within the class of all measurable subsets of $[0,\pi]$ quotiented by sets of zero measure. It means that if ω is optimal, thus any set $\omega \cup N$ or $\omega \setminus N$ is also optimal, for every measurable subset N of $[0,\pi]$ of zero Lebesgue measure.

Proof. Since y^0 and y^1 have only a finite number of nonzero components in their Fourier series, it follows that $C_j = 0$ for $j > N$, where C_j is defined by (22). Then, clearly, the unique solution (A^a, B^a) of (21) is as well of compact support. Hence, only a finite number of the coefficients $p_{j,k}$ are nonzero. Therefore, the function φ is analytic and hence cannot vanish identically on a subset of positive measure. Then the optimal control a is bang-bang, determined by (30), and is the characteristic function of some subset ω of $[0, \pi]$. Moreover, the switching function $\varphi(x)$ is written as

$$\begin{aligned}\varphi(x) &= p_0 + \sum_{j,k=1}^{N} p_{j,k} \sin(jx)\sin(kx)\\ &= p_0 + \frac{1}{2}\sum_{u=1-N}^{N-1}\left(\sum_{j=u+1}^{u+N} p_{j,j-u}\right)\cos(ux) - \frac{1}{2}\sum_{u=2}^{2N}\left(\sum_{j=u-N}^{u-1} p_{j,u-j}\right)\cos(ux),\end{aligned}$$

for every $x \in [0, \pi]$. Since φ can be written as a linear combination of the $2N$ first Tchebychev polynomials, it follows that φ has at most $2N$ zeros. Finally, since $\varphi(0) = \varphi(\pi) = p_0$ and $\varphi(x) \leqslant p_0$ for every $x \in [0, \pi]$, the existence of $\eta > 0$ follows easily. □

A natural question arises when considering modal approximations $(y^{0,N}, y^{1,N})$ of the initial data (y^0, y^1), defined by

$$\begin{pmatrix} y^{0,N}(x) \\ y^{1,N}(x) \end{pmatrix} = \sum_{k=1}^{N} \begin{pmatrix} \langle y^0, \sin(j\cdot)\rangle_{L^2} \\ \langle y^1, \sin(j\cdot)\rangle_{L^2} \end{pmatrix} \sin(jx).$$

Consider the optimal set $\omega^N \in \mathcal{U}_L$ solution of the first problem with the initial data $(y^{0,N}, y^{1,N})$ (as claimed in Theorem 2). Let $a \in \overline{\mathcal{U}}_L$ be the solution of the relaxed first problem with the initial data (y^0, y^1). According to Proposition 3, let p_0^N and $p_{j,k}^N$, for $(j,k) \in (\mathbb{N}^*)^2$, the coefficients of the switching function φ^N for the truncated optimal control problem (26)–(29) with the initial data $(y^{0,N}, y^{1,N})$. Note that $p_{j,k}^N = 0$ for $j > N$ or $k > N$. The following Γ-convergence result follows easily from the previous optimal control considerations.

Proposition 4. *The sequence of finite dimensional optimal control problem* (26)–(29) *with truncated initial data* $(y^{0,N}, y^{1,N})$ *Γ-converges to the infinite dimensional optimal control problem* (26)–(29) *with initial data* (y^0, y^1)*, in the sense that, when N tends to $+\infty$, χ_{ω^N} converges to a for the weak star topology of L^∞, p_0^N converges to p_0, $p_{j,k}^N$ converges to $p_{j,k}$ for every $(j,k) \in (\mathbb{N}^*)^2$, and $F^N(\chi_{\omega^N})$ converges to $F(a)$.*

2.4.2. Sufficient condition when T is an integer multiple of 2π

Throughout this section we assume that $T = 2\pi$, the case $T = 2p\pi$ with $p \in \mathbb{N}^*$ being obviously deduced from this case. Note that $T = 2\pi$ is a very particular time since the orthogonality property leads to a strong simplification of the expression of the norm of the control as explained next. Let us expand $\phi(t,x)$ as in (10). Then the "relaxed" HUM functional defined by (18) is given by

$$J_a(\phi^0, \phi^1) = \sum_{j=1}^{+\infty}\left(\frac{\pi}{2}\left(A_j^2 + B_j^2\right)\int_0^\pi a(x)\sin^2(jx)\,dx - A_j\int_0^\pi y^1(x)\sin(jx)\,dx + jB_j\int_0^\pi y^0(x)\sin(jx)\,dx\right),$$

and for given initial data (y^0, y^1), the minimization of J_a obviously leads to

$$A_j^a = \frac{1}{\pi}\frac{\int_0^\pi y^1(x)\sin(jx)\,dx}{\int_0^\pi a(x)\sin^2(jx)\,dx}, \quad \text{and} \quad B_j^a = -\frac{j}{\pi}\frac{\int_0^\pi y^0(x)\sin(jx)\,dx}{\int_0^\pi a(x)\sin^2(jx)\,dx}. \tag{34}$$

Denote

$$\rho_j = \left(\left(\int_0^\pi y^1(x)\sin(jx)\,dx\right)^2 + \left(j\int_0^\pi y^0(x)\sin(jx)\,dx\right)^2\right)^{1/2}, \tag{35}$$

for every $j \in \mathbb{N}^*$. Note that, since $(y^0, y^1) \in H_0^1(0,\pi) \times L^2(0,\pi)$, it follows that the sequence $(\rho_j)_{j\in\mathbb{N}^*}$ belongs to $\ell^2(\mathbb{R})$.

Using these particular and more precise expressions for $T = 2\pi$, we immediately derive the following generalization of Theorem 2.

Theorem 3. *Assume that $T = 2\pi$, and that there exist $M > 0$ and $\delta > 0$ such that*

$$|\rho_j| \leqslant M e^{-\delta j}, \tag{36}$$

for every $j \in \mathbb{N}^*$. Then the first problem (15) has a unique solution χ_ω, where ω is a measurable subset of $[0,\pi]$ of Lebesgue measure $L\pi$. Moreover,

- *ω is symmetric with respect to $\pi/2$,*
- *there exists $\eta > 0$ such that $\omega \subset [\eta, \pi - \eta]$,*
- *ω has a finite number of connected components.*

Remark 4. The assumption (36) holds for instance as soon as y^0 and y^1 are analytic functions on $[0,\pi]$. Actually, the assumption (36) guarantees the analyticity of the function φ, and can be slightly weakened into

$$|\rho_j| \leqslant M e^{-w(j)},$$

for every $j \in \mathbb{N}^*$, where w is a positive differentiable function such that $t \mapsto t w'(t)$ increases to $+\infty$ and such that $\int_1^{+\infty} \frac{w(t)}{t^2}\,dt = +\infty$. Indeed, under this weakened assumption the function φ enjoys the following unique continuation property (see [14]): if φ is constant on a subset of $[0,\pi]$ of positive measure then φ is constant on $[0,\pi]$. Hence, the statement of the theorem holds under this slightened property. More generally this property is related to quasi-analyticity but up to our knowledge there is no simple necessary and sufficient condition on the coefficients α_j ensuring that unique continuation property.

Remark 5. Given a sequence $(\rho_j)_{j\in\mathbb{N}^*} \in \ell^2(\mathbb{R})$ there exist an infinite number of possible initial data $(y^0, y^1) \in H_0^1(0,\pi) \times L^2(0,\pi)$ satisfying (35), and all of them lead to the same solution of the first problem (it is indeed a consequence from the formula (39) in the proof).

Proof of Theorem 3. For this particular case where $T = 2\pi$, all previously considered expansions do not involve any crossed terms. In particular, when interpreting the first problem as an optimal control problem, the control system (26) reduces here to

$$\begin{aligned} &z_0'(x) = a(x), \\ &z_j'(x) = a(x)\sin^2(jx), \quad j \in \mathbb{N}^*, \end{aligned} \tag{37}$$

for almost every $x \in \pi$, with initial conditions

$$z_0(0) = 0, \qquad z_j(0) = 0, \quad j \in \mathbb{N}^*. \tag{38}$$

Using the Parseval identity, we get

$$\begin{aligned} F(a) &= \int_0^{2\pi}\int_0^{\pi} a(x)\phi_a^2(t,x)\,dx\,dt \\ &= \frac{\pi}{2}\sum_{j=1}^{+\infty}\big(A_j^{a2} + B_j^{a2}\big)\int_0^{\pi} a(x)\sin^2(jx)\,dx \\ &= \frac{1}{2\pi}\sum_{j=1}^{+\infty}\frac{\rho_j^2}{\int_0^{\pi} a(x)\sin^2(jx)\,dx} \end{aligned}$$

$$= \frac{1}{2\pi} \sum_{j=1}^{+\infty} \frac{\rho_j^2}{z_j(\pi)}. \tag{39}$$

The optimal control problem under consideration consists of steering the control system (37) from the initial conditions (38) to the final condition $z_0(\pi) = L\pi$, minimizing the functional F defined by (39) with controls $a \in \overline{\mathcal{U}}_L$. Note that in order to apply the Pontryagin Maximum Principle in a strict convex Banach space one has to use the identity $\sin^2(jx) = \frac{1}{2} - \frac{1}{2}\cos(2jx)$ and hence to consider the control system given by

$$\begin{aligned} &w_0'(x) = a(x), \\ &w_j'(x) = \frac{1}{2} a(x)\cos(2jx), \quad j \in \mathbb{N}^*, \end{aligned} \tag{40}$$

with initial conditions

$$w_0(0) = 0, \qquad w_j(0) = 0, \quad j \in \mathbb{N}^*. \tag{41}$$

The control system (37) is then inferred from the above one by setting

$$z_0(x) = w_0(x), \qquad z_{j,k}(x) = \frac{1}{2}\big(w_{|j-k|}(x) - w_{j+k}(x)\big), \tag{42}$$

for almost every $x \in [0, \pi]$ and for all integers j and k.

With these notations, one has

$$F(a) = \frac{1}{2\pi} \sum_{j=1}^{+\infty} \frac{\rho_j^2}{\frac{L\pi}{2} - w_j(\pi)},$$

and, skipping the details, the application of the Pontryagin Maximum Principle leads to the characterization of the (unique) optimal control

$$a(x) = \begin{cases} 1 & \text{if } \varphi(x) > 0, \\ 0 & \text{if } \varphi(x) < 0, \end{cases} \tag{43}$$

for almost every $x \in [0, \pi]$, where the switching function φ is defined by

$$\varphi(x) = \lambda_0 + \frac{1}{2} \sum_{j=1}^{+\infty} \lambda_j \cos(2jx), \tag{44}$$

with

$$\lambda_j = -\frac{1}{2\pi} \frac{\rho_j^2}{(\frac{L\pi}{2} - w_j(\pi))^2} = -\frac{1}{2\pi} \frac{\rho_j^2}{(\int_0^\pi a(x)\sin^2(jx)\,dx)^2}, \tag{45}$$

for every $j \in \mathbb{N}^*$. Note that

$$\varphi(x) = \lambda_0 - \int_0^T \phi_a(t,x)^2 dt = \lambda_0 - \frac{1}{2\pi} \sum_{j=1}^{+\infty} \frac{\rho_j^2}{(\int_0^\pi a(s)\sin^2(js)\,ds)^2} \sin^2(jx), \tag{46}$$

for every $x \in [0, \pi]$. Under the assumption (36), it follows from (45) and from Lemma 6 that the decay of the coefficients λ_j is exponential, and hence that the switching function φ defined by (44) is analytic (see e.g. [1, Chapter 11, §63]) and thus φ cannot vanish identically on a subset of positive measure. Therefore, the optimal control a is bang-bang, determined by (43), and thus is the characteristic function of some subset ω of $[0, \pi]$. The symmetry property follows from the fact that $\varphi(\pi - x) = \varphi(x)$ for every $x \in [0, \pi]$. The existence of η comes from the fact that $\varphi(0) = \varphi(\pi)$ is the minimum of φ (since $\lambda_j < 0$ for every $j \in \mathbb{N}^*$). □

2.5. Necessary and sufficient conditions when T is an integer multiple of 2π

Theorems 2 and 3 stated previously provide sufficient conditions for the first problem (8) to have a (unique) solution. In the case where T is an integer multiple of 2π, we are actually able to provide a complete characterization (however not so tractable in practice) of all initial data for which this problem has a (unique) solution, that is, for which the optimal control solution of the relaxed first problem is actually a solution of the non-relaxed one. This characterization is based on the fact that the necessary conditions for optimality are also sufficient because of the strict convexity of F on $\overline{\mathcal{U}}_L$, as already mentioned above. The following result is a particular consequence of the more precise considerations further.

Theorem 4. *Assume that T is an integer multiple of 2π and let $L \in (0,1)$ be fixed.*

1. *There exist initial data $(y^0, y^1) \in (C^\infty(0,\pi))^2$ for which the first problem (8) has a solution $\chi_\omega \in \mathcal{U}_L$, with ω a fractal domain of Cantor type.*
2. *There exist initial data $(y^0, y^1) \in (C^\infty(0,\pi))^2$ for which the first problem (8) has no solution, but its relaxed version has a solution $a \in \overline{\mathcal{U}}_L \setminus \mathcal{U}_L$.*

Let us assume without loss of generality that $T = 2\pi$. Theorem 4 is a direct consequence of the following characterization of all initial data for which either the first problem has a solution or it has no solution (relaxation phenomenon).

Note first that $(\lambda_j)_{j\in\mathbb{N}^*} \in \ell^1(\mathbb{R})$, where λ_j is defined by (45). Therefore, more precisely than in Remark 2, the switching function φ defined by (44) belongs to the set $\mathcal{A}(0,\pi)$ of all functions ϕ integrable on $[0,\pi]$ and having a Fourier series expansion of the form

$$\phi(x) = a_0 + \frac{1}{2}\sum_{j=1}^{+\infty} a_j \cos(2jx), \tag{47}$$

with $a_0 \in \mathbb{R}$ and nonpositive coefficients a_j satisfying $(a_j)_{j\in\mathbb{N}^*} \in \ell^1(\mathbb{R})$. Such considerations are classical in harmonic analysis. In particular, φ is continuous, but the above property is stronger.

We are now in a position to derive the following characterizations, mainly based on the fact that the optimal solution a of the above optimal control problem is the solution of the initial first problem (15) if and only if the switching function φ defined by (44) does not vanish identically on any subset of $[0,\pi]$ of positive measure.

Characterization of all initial data for which the first problem has a solution. Consider any function $\varphi \in \mathcal{A}(0,\pi)$ such that $\varphi(x) \neq 0$ for almost every $x \in [0,\pi]$, written as

$$\varphi(x) = \lambda_0 + \frac{1}{2}\sum_{j=1}^{+\infty} \lambda_j \cos(2jx),$$

with nonpositive coefficients λ_j satisfying $(\lambda_j)_{j\in\mathbb{N}^*} \in \ell^1(\mathbb{R})$. Let a be the bang-bang control defined by (30), and let $\rho_j \geqslant 0$ be defined by (45) for every $j \in \mathbb{N}^*$, that is,

$$\rho_j = \sqrt{-2\pi\lambda_j}\int_0^\pi a(x)\sin^2(jx)\,dx.$$

Using Lemma 6, it is clear that $(\rho_j)_{j\in\mathbb{N}^*} \in \ell^2(\mathbb{R})$. Such a sequence $(\rho_j)_{j\in\mathbb{N}^*}$ characterizes a set of initial data $(y^0, y^1) \in H_0^1(0,\pi) \times L^2(0,\pi)$ (see Remark 5): these are all initial data for which (35) holds for every $j \in \mathbb{N}^*$. The control a, defined in such a way, satisfies all necessary and sufficient conditions of the Pontryagin Maximum Principle and hence is the optimal control of the first problem associated with the initial data (y^0, y^1), and with the constant L defined by $\int_0^\pi a(x)\,dx = L\pi$ (note that L may be equal to 0 or 1, depending on the value of λ_0, but these cases are of course not interesting). Since a is bang-bang, it is the characteristic function of a subset ω of $[0,\pi]$, which is the solution of the first problem. Note that, in order to realize a specific value of L, it suffices to tune adequately the constant λ_0 of the Fourier expansion of φ so that $\int_0^\pi a(x)\,dx = L\pi$.

The construction above establishes the correspondence between the optimal sets of the first problem and the zero level sets of functions of $\mathcal{A}(0,\pi)$ that do not vanish almost everywhere. Such level sets may have an intricate structure, for instance may have a fractal structure. We refer to [21] for the construction of an explicit example where ω is of Cantor type.

We are able as well to characterize precisely the relaxation phenomenon.

Characterization of the relaxation phenomenon. The data (y^0, y^1) for which the relaxation phenomenon occurs are those whose associated coefficients ρ_j defined by (35) are such that the corresponding switching function φ vanishes identically on a subset I of positive measure (note that this set must be symmetric with respect to $\frac{\pi}{2}$). The more precise construction goes as follows.

Consider any nontrivial function $\varphi \in \mathcal{A}(0,\pi)$ vanishing identically on a subset I of $[0,\pi]$ of positive measure, with Fourier coefficients $(\lambda_j)_{j\in\mathbb{N}^*}$ as before. Let a be a control function defined by (30) on $[0,\pi]\setminus I$ and taking arbitrary values on I, and satisfying however $\int_0^\pi a(x)\,dx = L\pi$. Let $(\rho_j)_{j\in\mathbb{N}^*} \in \ell^2(\mathbb{R})$ be defined by (45), as previously. Such a sequence characterizes a set of initial data $(y^0, y^1) \in H_0^1(0,\pi)\times L^2(0,\pi)$. As before the control a defined in such a way is the optimal control of the relaxed version of the first problem associated with the initial data (y^0, y^1). Since a is not bang-bang (it is not a characteristic function on I, by construction), this means that the first problem does not have any solution and that the relaxation phenomenon occurs.

Remark 6. This set is indeed non-empty, and is even, in some sense, a very large set. To get convinced, it suffices to note the following fact. Consider any nontrivial function ψ of class C^∞ on $[0,\pi]$, symmetric with respect to $\frac{\pi}{2}$, and whose support is contained in $[\frac{\pi}{2}-\alpha, \frac{\pi}{2}+\alpha]$ for some $\alpha > 0$ small. The C^1 regularity ensures that its Fourier coefficients are summable. To ensure the nonpositivity of its Fourier coefficients, it suffices to consider the function φ defined by the convolution

$$\varphi(x) = \int_{\mathbb{R}} \psi\left(y+\frac{\pi}{2}\right)\psi(x-y)\,dy.$$

Indeed, the function φ defined in such a way is of class C^∞ on $[0,\pi]$, of support contained in $[\frac{\pi}{2}-2\alpha, \frac{\pi}{2}+2\alpha]$, and all its Fourier coefficients are nonpositive (by the way note that all functions whose Fourier coefficients are nonpositive are of this type).

Another explicit example is easily built as follows. Consider a triangle function[2] defined on $[\frac{\pi}{2}-\alpha, \frac{\pi}{2}+\alpha]$, for some $\alpha > 0$, equal to 0 outside. Its Fourier coefficients are the values on integers of the Fourier transform of the triangle function, hence are positive and summable. The rest of the construction is obvious.

To illustrate this correspondence, we yield a concrete example of construction of initial data for which we observe a relaxation phenomenon.

Example 1. We set $T = 2\pi$ (for example). Consider the π-periodic switching function ϕ, symmetric with respect to $\pi/2$ defined by

$$\phi(x) = \begin{cases} -\frac{4}{\pi}x + 1 & \text{on } [0,\pi/4], \\ 0 & \text{on } [\pi/4,\pi/2]. \end{cases}$$

Thus, ϕ has a Fourier series expansion of the form (47) with $a_0 = \frac{1}{4}$ and $a_j = \frac{4(1-(-1)^j)}{j^2} \geqslant 0$ for every $j \geqslant 1$. Introduce now the function $a(\cdot)$, π-periodic and symmetric with respect to $\pi/2$, such that for every $x \in [0,\pi]$,

$$a(x) = \begin{cases} 0 & \text{on } [0,\pi/4], \\ 2L + \frac{8L}{\pi}(x-\frac{\pi}{2}) & \text{on } [\pi/4,\pi/2]. \end{cases}$$

[2] By triangle function, we mean a function whose graph is an equilateral triangle.

Using the correspondence detailed before, we obtain that a is the unique solution of problem (17), provided that the initial data (y^0, y^1) are such that the coefficients $(\rho_j)_{j\in\mathbb{N}^*}$ defined by (35) verify

$$\rho_j = \sqrt{-2\pi a_j}\int_0^\pi a(x)\sin^2(jx)\,dx$$

for every $j \in \mathbb{N}^*$.

Remark 7. It is not clear whether the above characterizations can be generalized or not to the case where T is not an integer multiple of 2π, in particular because the relation between the coefficients $(p_{j,k})_{j\in\mathbb{N}^*}$ of the switching function and the coefficients A_j^a, B_j^a is not easily invertible (note however that the infinite dimensional symmetric matrix whose coefficients are the $p_{j,k}$ is a Gramian).

2.6. *Modal approximation and numerical simulations*

2.6.1. *Case $T = 2\pi$*

We propose in this section some numerical simulations in the case where $T = 2\pi$. The method we use here works exactly in the same way in the case where $T \geqslant 2\pi$ is not an integer multiple of 2π except that it is then required to invert the Gramian operator Λ_a defined in Section 2.3. In view of numerical simulations, it is natural to truncate the control system (40) at some order $N \in \mathbb{N}^*$, by considering the optimal control problem of steering the finite dimensional control system

$$\begin{aligned} &w_0'(x) = a(x),\\ &w_j'(x) = \frac{1}{2}a(x)\cos(2jx), \quad j \in \{1,\dots,N\}, \end{aligned} \tag{48}$$

from the initial conditions

$$w_0(0) = 0, \qquad w_j(0) = 0, \quad j \in \{1,\dots,N\} \tag{49}$$

to the final condition (28) ($y(\pi) = L\pi$), and minimizing

$$F^N(a) = \frac{1}{2\pi}\sum_{j=1}^N \frac{\rho_j^2}{\frac{L\pi}{2} - w_j(\pi)}. \tag{50}$$

As in Theorem 1, this optimal control problem has a unique solution $a^N \in \overline{\mathcal{U}}_L$. The usual Pontryagin Maximum Principle of [22], applied to this finite dimensional optimal control problem, implies that there exist $\lambda^{0N} \leqslant 0$ and $(\lambda_0^N, \lambda_1^N, \dots, \lambda_N^N) \in \mathbb{R}^{N+1}$, called costates,[3] with $(\lambda^0, \lambda_0^N, \lambda_1^N, \dots, \lambda_N^N) \neq (0,\dots,0)$, such that

$$a^N(x) = \begin{cases} 1 & \text{if } \varphi^N(x) > 0,\\ 0 & \text{if } \varphi^N(x) < 0, \end{cases} \tag{51}$$

for almost every $x \in [0,\pi]$, where the function φ^N, called switching function, is defined by

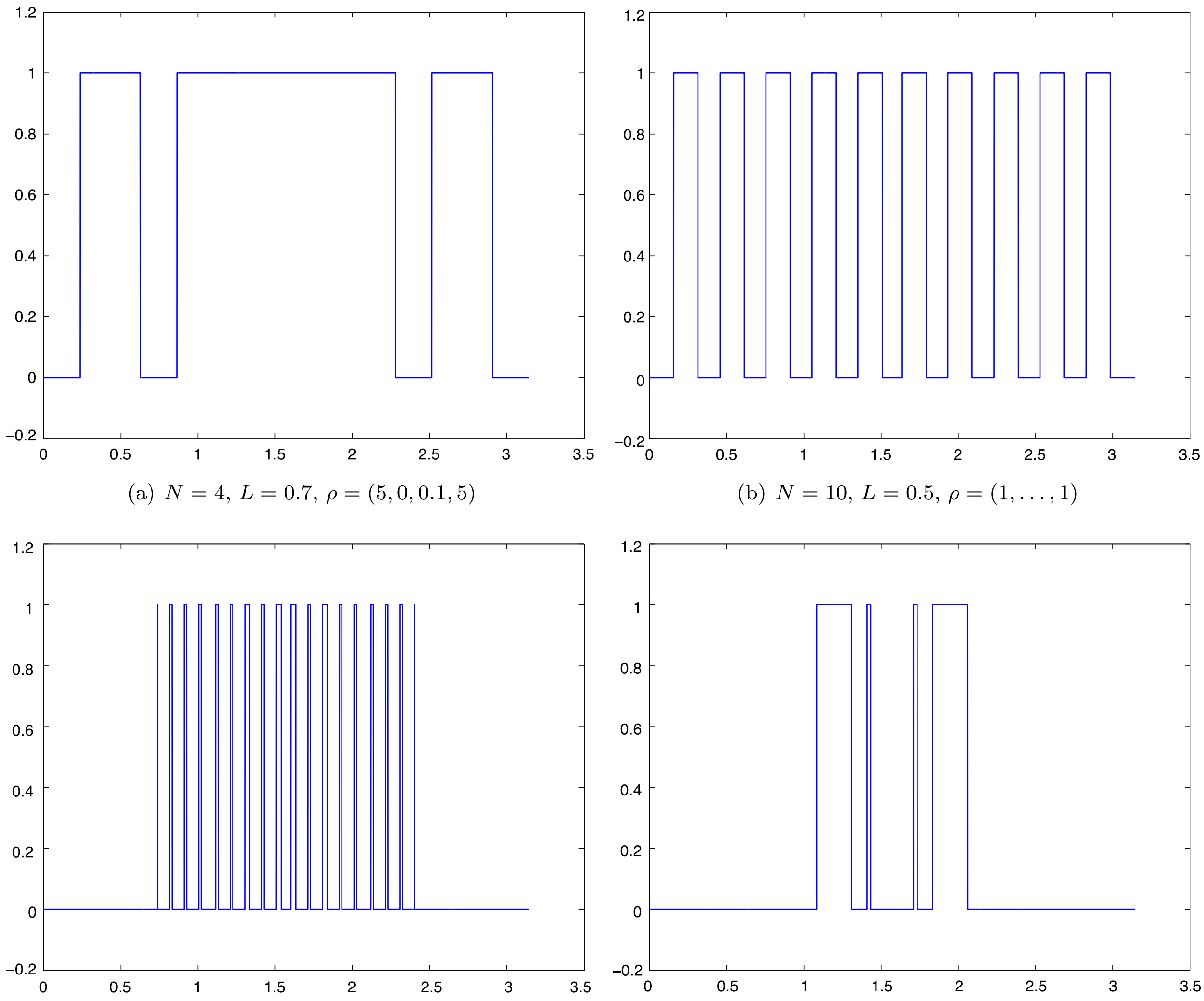
$$\varphi^N(x) = \lambda_0^N + \frac{1}{2}\sum_{j=1}^N \lambda_j^N \cos(2jx), \tag{52}$$

the control a^N being undetermined whenever ϕ^N vanishes identically on some subinterval. Moreover, one has the transversality assumptions

$$\lambda_j^N = \lambda^{0N}\frac{\rho_j^2}{(\frac{L\pi}{2} - z_j^N(\pi))^2}, \tag{53}$$

for every $j \in \{1,\dots,N\}$. As before, there must hold $\lambda^{0N} \neq 0$ and we assume next that $\lambda^{0N} = -1$.

[3] Note that, since the dynamics of (48) do not depend on the state, it follows that the costates are constant.



(a) $N = 4,\ L = 0.7,\ \rho = (5, 0, 0.1, 5)$

(b) $N = 10,\ L = 0.5,\ \rho = (1, \dots, 1)$

(c) $N = 30,\ L = 0.1,\ \rho_j = 1/j,\ j = 1, \dots, 30$

(d) $N = 50,\ L = 0.2,\ \rho_j = 1/j^2,\ j = 1, \dots, 50$

Fig. 1. Several numerical simulations.

Since φ^N can be written as a linear combination of the $2N$ first Tchebychev polynomials, it follows that φ^N has at most $2N$ zeros and cannot vanish identically on any subinterval. We finally get the following result.

Proposition 5. *The optimal control a^N is the characteristic function of a subset ω^N of $[0, \pi]$ of Lebesgue measure $L\pi$, which has at most N connected components, is symmetric with respect to $\pi/2$, and is such that there exists $\eta > 0$ such that $\omega^N \subset [\eta, \pi - \eta]$.*

We provide hereafter several numerical simulations based on this modal approximation. This new numerical approach is based on the interpretation as an optimal control problem. Some results are provided on Fig. 1.

These simulations were obtained with a direct method applied to the optimal control problem described previously (see e.g. [26] for a description of possible numerical approaches), consisting in discretizing the underlying differential equations, the control, and to reduce the optimal control problem to some minimization problem with constraints. We used the code `IPOPT` (see [29]) combined with `AMPL` (see [6]) on a standard desktop machine. The resulting code works out the solution very quickly (for instance, within 3 seconds for $N = 50$) and is far more efficient than methods based on gradient methods using topological derivatives. It must however be noted that the problem is one-dimensional and that we benefit of the interpretation as an optimal control problem, using spectral considerations.

Another possible method is to implement a shooting method, based on the optimality conditions provided by the Pontryagin Maximum Principle. Here, it consists of numerically determining the costates $(\lambda_0^N, \lambda_1^N, \dots, \lambda_N^N)$ such that (28) and (32) are satisfied. The shooting method is a combination of a Newton type method and of some integration method of differential equations. The main difficulty is the initialization of this method but it can be done using a direct method. We used the code COTCOT (see [2]). The convergence is then obtained instantaneously and can permit, when combined with a continuation method, to reach very large values of N (see [27] for a survey on these numerical approaches).

2.6.2. Complexity of the optimal set and limitations of numerics

The graphs presented in the last paragraph represent the optimal set for fixed initial data that have a finite number of Fourier components. It follows from Theorem 4 that numerical simulations cannot permit in general to guess what happens when the initial data have an infinite number of components in their Fourier series expansion. Indeed, note that in [16,17] the authors conjecture the existence of a relaxation phenomenon with smooth initial data, based on numerical simulations. Although such an intuition seems reasonable, according to our previous theoretical results this is not the case. In Section 2.5 we have shown a precise correspondence between the regularity of the optimal set and the regularity of the initial data. We have also characterized all initial data for which the relaxation phenomenon occurs, leading to Theorem 4. Using Theorem 4, there exist smooth initial data for which the optimal set has a fractal structure, and thus an infinite number of connected components but for which relaxation does not occur. Of course this particular feature cannot be guessed from numerical simulations. To summarize, considering as in the previous section a truncated version of the first problem (8) with the N first Fourier components, two different phenomena arise at the limit, depending on the initial data under consideration: a relaxation phenomenon, or the emergence of complex (possibly fractal) optimal sets. In particular the occurrence of relaxation cannot be illustrated on numerical simulations.

2.6.3. Additional comments on the case $T = 2\pi$: study of the case $N = 2$

In this section, we study the particular case $N = 2$ of the modal truncature of the first problem. This means that the only nonzero terms of the sequence $(\rho_j)_{j\in\mathbb{N}^*}$ are ρ_1 and ρ_2. In this case, the functional (50) is written as

$$F^2(\chi_\omega) = \frac{\rho_1^2}{\int_0^\pi \chi_\omega(x)\sin^2 x\,dx} + \frac{\rho_2^2}{\int_0^\pi \chi_\omega(x)\sin^2(2x)\,dx}.$$

Setting $t = \frac{\rho_1^2}{\rho_1^2+\rho_2^2}$, the minimization problem reduces to the problem of minimizing

$$\frac{t}{\int_0^\pi \chi_\omega(x)\sin^2 x\,dx} + \frac{1-t}{\int_0^\pi \chi_\omega(x)\sin^2(2x)\,dx} \tag{54}$$

over the set $\mathcal{U}_L$. The following proposition yields a precise characterization of the minimizers in the case $N = 2$ and describes the variations of the optimal set with respect to L.

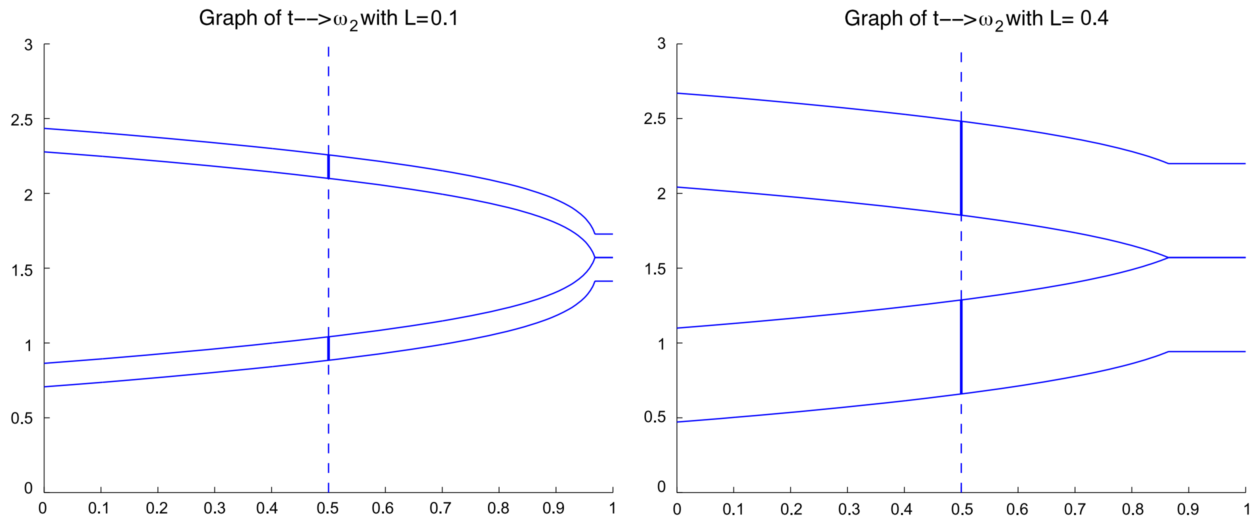
Proposition 6. *Let χ_{ω^2} be the solution of the minimization problem (54). Then,*

- *ω^2 is symmetric with respect to $\pi/2$;*
- *the set $\omega^2 \cap [0, \pi/2]$ has only one connected component $[\alpha_t, \alpha_t + \frac{\pi L}{2}]$;*
- *$\alpha_0 = \frac{\pi}{4}(1-L)$ and $\alpha_1 = \frac{\pi}{2}(1-L)$;*
- *there exists $t_0(L) \in (0,1)$ such that $t \mapsto \alpha_t$ is increasing on $[0, t_0(L)]$, and is constant equal to α_1 on $[t_0(L), 1]$.*

The proof of this proposition, based on lengthy and tedious computations, is provided in Appendix A.5. Fig. 2 illustrates this proposition.

3. Solving of the second problem (9)

This section is devoted to solving the second problem (9), that is the problem of minimizing the norm of the HUM operator over $\mathcal{U}_L$.



(a) $L = 0.1$ (b) $L = 0.4$

Fig. 2. Graph of the optimal domain ω^2 in function of t.

3.1. Reduction of the problem

For every measurable subset ω of $[0, \pi]$, set

$$W(\chi_\omega) = \sup\left\{ \frac{\|h_\omega\|^2_{L^2((0,T)\times(0,\pi))}}{\|(y^0, y^1)\|^2_{H^1_0(0,\pi)\times L^2(0,\pi)}} \;\middle|\; \left(y^0, y^1\right) \in H^1_0(0,\pi) \times L^2(0,\pi) \setminus \left\{(0,0)\right\} \right\}.$$

Recall that the second problem consists of minimizing $W(\chi_\omega)$ over all possible subsets ω of $[0, \pi]$ of measure equal to $L\pi$.

Lemma 5. *For every measurable subset ω of $[0, \pi]$, there holds*

$$W(\chi_\omega) = \frac{1}{C_T(\chi_\omega)},$$

where $C_T(\chi_\omega)$ is the largest observability constant in the inequality (3), associated with the wave equation (4), that is

$$C_T(\chi_\omega) = \inf\left\{ \frac{\int_0^T \int_\omega \phi(t,x)^2\,dx\,dt}{\|(\phi^0, \phi^1)\|^2_{L^2(0,\pi)\times H^{-1}(0,\pi)}} \;\middle|\; \left(\phi^0, \phi^1\right) \in L^2(0,\pi) \times H^{-1}(0,\pi) \setminus \left\{(0,0)\right\} \right\},$$

where ϕ is the solution of (4).

Remark 8. Expanding ϕ in Fourier series as in (10), it is easy to see that

$$C_T(\chi_\omega) = \inf_{\substack{(A,B)\in(\ell^2(\mathbb{R}))^2 \\ \sum_{j=1}^{+\infty}(A_j^2+B_j^2)=1}} \left\{ \int_0^T \int_\omega \left(\sum_{j=1}^{+\infty} \big(A_j \cos(jt) + B_j \sin(jt)\big) \sin(jx) \right)^2 dx\,dt \right\}.$$

This lemma illustrates the classical duality between controllability and observability (see [23]), however we provide a short proof hereafter.

Proof. Let us expand ϕ in Fourier series as in (10), and use the operator Λ_a defined by (21) to rewrite $W(\chi_\omega)$. One has

$$W(\chi_\omega) = \sup_{C\in(\ell^2(\mathbb{R}))^2} \frac{\langle \Lambda_{\chi_\omega}^{-1}(C), C\rangle_{(\ell^2(\mathbb{R}))^2}}{\|C\|^2_{(\ell^2(\mathbb{R}))^2}} = \sup_{C\in(\ell^2(\mathbb{R}))^2} \frac{\|\Lambda_{\chi_\omega}^{-1/2}(C)\|^2_{(\ell^2(\mathbb{R}))^2}}{\|C\|^2_{(\ell^2(\mathbb{R}))^2}},$$

where $\Lambda_{\chi_\omega}^{-1/2}$ denotes the square root of the operator $\Lambda_{\chi_\omega}^{-1}$. Introducing $\varphi = \Lambda_{\chi_\omega}^{-1/2}(C)$, one computes

$$\begin{aligned} W(\chi_\omega) &= \sup_{\varphi\in(\ell^2(\mathbb{R}))^2} \frac{\|\varphi\|^2_{(\ell^2(\mathbb{R}))^2}}{\|\Lambda_{\chi_\omega}^{1/2}(\varphi)\|^2_{(\ell^2(\mathbb{R}))^2}} \\ &= \frac{1}{\inf\{\frac{\|\Lambda_{\chi_\omega}^{1/2}(\varphi)\|^2_{(\ell^2(\mathbb{R}))^2}}{\|\varphi\|^2_{(\ell^2(\mathbb{R}))^2}} \mid \varphi\in(\ell^2(\mathbb{R}))^2\}} \\ &= \frac{1}{\inf\{\frac{\langle\Lambda_{\chi_\omega}(\varphi),\varphi\rangle_{(\ell^2(\mathbb{R}))^2}}{\|\varphi\|^2_{(\ell^2(\mathbb{R}))^2}} \mid \varphi\in(\ell^2(\mathbb{R}))^2\}} = \frac{1}{C_T(\chi_\omega)}. \end{aligned}$$

The lemma is proved. □

As a consequence, we have

$$\inf_{\chi_\omega\in\mathcal{U}_L} W(\chi) = \inf_{\chi_\omega\in\mathcal{U}_L} \frac{1}{C_T(\chi_\omega)} = \frac{1}{\sup_{\chi_\omega\in\mathcal{U}_L} C_T(\chi_\omega)},$$

and therefore the second problem (9) is equivalent to the new optimization problem

$$\sup_{\chi_\omega\in\mathcal{U}_L} C_T(\chi_\omega), \tag{55}$$

that is, the problem of maximizing the observability constant over all possible subsets of $[0,\pi]$ of Lebesgue measure $L\pi$. This problem was investigated in [20], and solved in the case where T is an integer multiple of 2π (the general case being still open). The problem of maximizing the functional $\lim_{T\to+\infty} C_T(\chi_\omega)/T$ over $\mathcal{U}_L$ was also solved.

3.2. Case where T is an integer multiple of 2π

In this case, according to the computations of Section 2.4.2, the second problem (9) is exactly equivalent to

$$\sup_{\chi_\omega\in\mathcal{U}_L} \inf_{j\in\mathbb{N}^*} \int_\omega \sin^2(jx)\,dx. \tag{56}$$

This kind of problem was addressed in [7,8], where a different optimal design problem was studied, concerning stabilization issues for the one-dimensional wave equation. More precisely, they consider the damped wave equation

$$\begin{aligned} &\partial_{tt}y - \partial_{xx}y + 2k\chi_\omega\partial_t y = 0, \quad (t,x)\in(0,T)\times(0,\pi), \\ &y(t,0) = y(t,\pi) = 0, \quad t\in[0,T], \\ &y(0,x) = y^0(x), \qquad \partial_t y(0,x) = y^1(x), \quad x\in[0,\pi], \end{aligned}$$

where $k>0$, and investigate the problem of determining the best possible subset ω of $[0,\pi]$ maximizing the decay rate of the total energy of the system, among all possible subsets of $[0,\pi]$ of Lebesgue measure $L\pi$ and having a finite number of connected components (note that this last restriction is discarded in [20]). This corresponds to choose in (1) the control $h_\omega = -2k\chi_\omega\partial_t y$, which does not coincide with the HUM control. The overdamping phenomenon is underlined in [7] (see also [5]), meaning that if k is too large then the decay rate tends to zero. It is moreover explained that, if k is small enough then the decay rate is equivalent to $k\inf_{j\in\mathbb{N}^*}\int_\omega \sin^2(jx)\,dx$. This explains their motivation to investigate the optimization problem (56).

It is proved in [7,8,20] that the optimization problem (56) does not have any solution except for $L = 1/2$. In other words, the supremum is not reached. In particular, as an immediate consequence of [20, Theorem 1], we have the following result.

Theorem 5. *For every $L \in (0, 1)$, there holds*

$$\inf_{\chi_\omega \in \mathcal{U}_L} W(\chi_\omega) = \frac{2}{LT}, \tag{57}$$

and the infimum is reached if and only if $L = 1/2$. Moreover, if $L = 1/2$ then the problem has an infinite number of solutions, consisting of all measurable subsets $\omega \subset [0, \pi]$ of measure $\pi/2$ such that ω and its symmetric $\omega' = \pi - \omega$ are disjoint and complementary in $[0, \pi]$.

In particular if $L \neq 1/2$ then the second problem (9) does not have any optimal set. Since it is more realistic from an engineering point of view to take into consideration only a finite number of modes, the authors of [8] consider a truncated version of (56) involving only the first N modes, for a given $N \in \mathbb{N}^*$, and investigate the optimization problem

$$\sup_{\chi_\omega \in \mathcal{U}_L} \min_{1 \leqslant j \leqslant N} \int_0^\pi \chi_\omega(x) \sin^2(jx)\, dx. \tag{58}$$

The following theorem was proved in [8] but with an erroneous proof, corrected in [20].

Theorem 6. *For every $N \in \mathbb{N}^*$, the problem (58) has a unique*[4] *solution χ_{ω^N}, where ω^N is a subset of $[0, \pi]$ of measure $L\pi$ that is the union of at most N intervals and is symmetric with respect to $\pi/2$. Moreover there exists $L_N \in (0, 1]$ such that, for every $L \in (0, L_N]$, the optimal domain ω^N satisfies*

$$\int_{\omega^N} \sin^2 x\, dx = \int_{\omega^N} \sin^2(2x)\, dx = \cdots = \int_{\omega^N} \sin^2(Nx)\, dx. \tag{59}$$

It is explained in [8] that (59) permits to show that the optimal domain ω^N concentrates around the nodes $\frac{k\pi}{N+1}$, $k = 1, \ldots, N$. This implies the well-known spillover phenomenon, according to which the optimal domain ω^N solution of (58) with the N first modes is the worst possible domain for the problem with the $N + 1$ first modes.

Note that, although (56) has no solution, the optimal solution χ_{ω^N} of (58) converges for the weak star topology of L^∞ to the function $a \in \overline{\mathcal{U}}_L$ that is the solution of the relaxed problem

$$\max_{a \in \overline{\mathcal{U}}_L} \inf_{j \in \mathbb{N}^*} \int_0^\pi a(x) \sin^2(jx)\, dx, \tag{60}$$

as expected, with a convergence of the optimal values.

4. Conclusion and open problems

In this article we have described a frequential approach based on Fourier series expansions, permitting to derive some existence and uniqueness results and to interpret these problems in terms of optimal control. The use of the Pontryagin Maximum Principle led to a complete characterization of optimal solutions permitting in particular to characterize all initial data for which the first problem has a solution or not. Efficient numerical algorithms have been derived from this approach and numerical simulations were given on a modal approximation of the problem. The second problem was shown to be equivalent to the problem of

[4] Here the uniqueness must be understood up to some subset of zero Lebesgue measure. In other words if ω is optimal then the union of ω with any subset of zero measure is also a solution.

maximizing the observability constant over the class of subsets under consideration. The optimal value of the second problem was given, although, except for $L = 1/2$, there does not exist any optimal set. In accordance with this nonexistence result, the modal approximation of that problem leads to the spillover phenomenon.

We end this article indicating several open directions for future investigation.

In the present article we restricted the study of the second problem to the case where T is an integer multiple of 2π. Indeed for more general values of T we do not know how to handle the general expressions of quadratic functional like (13). The problem happens to be related to the one of determining what are the best constants in Ingham's inequality (see [20] for detailed comments on this issue).

For every subset ω of $[0, \pi]$ of positive measure and $T \geqslant 2\pi$, the observability inequality (3) is satisfied. However 2π is not the smallest possible time for a specific choice of ω. For instance if ω is a subinterval of $[0, \pi]$ then the smallest such time is $2\,\mathrm{diam}((0, \pi) \setminus \omega)$. This question is nontrivial if, instead of an interval, the set ω is chosen to be a fractal measurable set, for example of Cantor kind. To our knowledge, given $L \in (0, 1)$, the question of determining the existence or not of a time $T_L \in (0, 2\pi)$ such that for every ω of measure $L\pi$, the system (1) is controllable in any time $T \geqslant T_L$, is open.

In the same spirit, another interesting possibility consists of maximizing the criterion

$$(\chi_\omega, T) \mapsto \inf_{(y^0, y^1) \in H_0^1(0,\pi) \times L^2(0,\pi)} \frac{1}{T} \int_0^T \int_0^\pi \frac{h_\omega(t, x)^2}{\|(y^0, y^1)\|^2_{H_0^1 \times L^2}} \, dx \, dt$$

over the set $\mathcal{U}_L \times (0, +\infty)$. The same questions arise when the control domain belongs to a larger class in which it is not necessarily cylindrical but is rather a measurable space–time set of measure $2\pi L$.

For $T \geqslant 2\pi$ that is not an integer multiple of 2π, the problem of determining the set of all initial data (y^0, y^1) for which the HUM control h_ω is analytic, is open. The answer to this question would permit to make Theorem 2 more precise since the analyticity of the switching function implies the existence of a solution of the first problem (8) having moreover a finite number of connected components.

In the present article truncations of the functionals were considered with the N first modes. It would be interesting to consider similar optimal design problems for other classes of initial data, for instance initial data whose Fourier coefficients satisfy a uniform exponential decreasing property. We could also truncate the Fourier series and keep only the modes whose index is between two integers N and M.

Our analysis can be extended to the multi-dimensional case, and for wave equations with more general non-constant coefficients, but such an analysis requires other considerations related to ergodicity features (see [21]) that are beyond the scope of the present article. Here for the one-dimensional wave equation we used many times the fact that the Hilbertian basis of eigenfunctions of the Dirichlet Laplacian consists of sine functions, but our approach may be led with more general eigenfunctions, although it may be more technical and not so precise. For example the derivation of such a precise identity like (59) is probably not easily provable in general.

Finally, it is an open question to investigate the shape and position of the support of the HUM control of the one-dimensional heat equation or more general parabolic systems. This question will be investigated in forthcoming works.

Acknowledgements

The authors wish to thank Institut Henri Poincaré (Paris, France) for providing a very stimulating environment during the "Control of Partial and Differential Equations and Applications" program in the Fall 2010.

The first author was partially supported by the ANR project GAOS "Geometric analysis of optimal shapes".

The third author was partially supported by the Grant MTM2011-29306-C02-00 of the MICINN (Spain), project PI2010-04 of the Basque Government, the ERC Advanced Grant FP7-246775 NUMERIWAVES, the ESF Research Networking Program OPTPDE.

Appendix A. Proofs of Section 2

A.1. Preliminaries

We start with some elementary lemmas that play an important role. The following very simple lemma, noticed as well in [19], is useful.

Lemma 6. *Let $j \in \mathbb{N}^*$.*

1. *The problem of minimizing the functional*

$$K_j(a) = \int_0^\pi a(x)\sin^2(jx)\,dx \tag{61}$$

over the set $\overline{\mathcal{U}}_L$ has a unique solution $\chi_{\omega_j^{\inf}} \in \mathcal{U}_L$ that is the characteristic function of the measurable set $\omega_j^{\inf}$ given by

$$\omega_j^{\inf} = \left(0, \frac{L\pi}{2j}\right) \cup \bigcup_{k=1}^{j-1}\left(\frac{k\pi}{j} - \frac{L\pi}{2j}, \frac{k\pi}{j} + \frac{L\pi}{2j}\right) \cup \left(\pi - \frac{L\pi}{2j}, \pi\right),$$

and the value of the minimum is $\frac{L\pi}{2} - \frac{\sin L\pi}{2} > 0$.

2. *The problem of maximizing the functional K_j over the set $\overline{\mathcal{U}}_L$ has a unique solution $\chi_{\omega_j^{\sup}} \in \mathcal{U}_L$ that is the characteristic function of the measurable set $\omega_j^{\sup}$ given by*

$$\omega_j^{\sup} = \bigcup_{k=1}^{j}\left(\frac{(2k-1)\pi}{2j} - \frac{L\pi}{2j}, \frac{(2k-1)\pi}{2j} + \frac{L\pi}{2j}\right),$$

and the value of the maximum is $\frac{L\pi}{2} + \frac{\sin L\pi}{2} > 0$.

Remark 9. Note that the extremal values of K_j, given in this lemma, do not depend on j.

Proof of Lemma 6. It suffices to note that optimal sets exist and are clearly characterized in terms of level sets of the function $x \mapsto \sin^2(jx)$, for every $j \in \mathbb{N}^*$. Using the symmetry properties of this function and the fact that its minimum on $[0, \pi]$ is equal to 0 and is reached at every $x_k = \frac{k\pi}{j}$, $k \in \{0, 1, \ldots, j\}$, the formula on $\omega_j^{\inf}$ follows, and the value of the minimum is given by

$$\int_{\omega_j^{\inf}} \sin^2(jx)\,dx = 2j\int_0^{\pi L/2j} \sin^2 jx\,dx = 2\int_0^{\pi L/2} \sin^2 u\,du = \frac{1}{2}\big(L\pi - \sin(L\pi)\big).$$

The proof of the second part is similar. □

The next lemma makes precise a well-known result due to Ingham (see [10]) in the very particular context of harmonic Fourier series. The bracket notation stands for the integer floor value.

Lemma 7. *(See [10].) For every 2π-periodic complex valued function f on $\mathbb{R}$ such that $f \in L^2(0, 2\pi; \mathbb{C})$, for every $T \geqslant 2\pi$, there holds*

$$M_1(T)\int_0^{2\pi} \left|f(t)\right|^2 dt \leqslant \int_0^T \left|f(t)\right|^2 dt \leqslant M_2(T)\int_0^{2\pi} \left|f(t)\right|^2 dt,$$

with $M_1(T) = [\frac{T}{2\pi}]$ and $M_2(T) = M_1(T) + 1$. Moreover, these constants are sharp.

Proof. The existence of two positive constants $M_1(T)$ and $M_2(T)$ is a direct consequence of Ingham's Lemma (see [10, Theorems 1 and 2]). We define

$$M_1(T)=\inf\left\{\frac{\int_0^T |f(t)|^2\,dt}{\int_0^{2\pi} |f(t)|^2\,dt}\;\middle|\; f\in L^2(0,2\pi;\mathbb{C})\setminus\{0\}\right\},$$

and $M_2(T)$ is defined in the same way as a supremum. Notice that given $f\in L^2(0,2\pi;\mathbb{C})$, one has

$$\begin{aligned}\int_0^T \left|f(t)\right|^2 dt &= \int_0^{2\pi[\frac{T}{2\pi}]} \left|f(t)\right|^2 dt + \int_0^{T-2\pi[\frac{T}{2\pi}]} \left|f(t)\right|^2 dt \\ &= \left[\frac{T}{2\pi}\right]\int_0^{2\pi} \left|f(t)\right|^2 dt + \int_0^{T-2\pi[\frac{T}{2\pi}]} \left|f(t)\right|^2 dt.\end{aligned}$$

The conclusion of the lemma is then obvious. The sharpness of the constants is obtained by considering either the function g vanishing on $(0, T-2\pi[\frac{T}{2\pi}])$ and equal to 1 on $(T-2\pi[\frac{T}{2\pi}], 2\pi)$ or the function $1-g$. □

A.2. *Proof of Proposition 1*

Using Lemmas 6 and 7, (14) and the Cauchy–Schwarz inequality, one has

$$\begin{aligned}J_a(A,B) \geqslant{} & \frac{\pi}{2}\left[\frac{T}{2\pi}\right]\frac{(\pi L-\sin(\pi L))}{2}\left(\|A\|^2_{\ell^2(\mathbb{R})}+\|B\|^2_{\ell^2(\mathbb{R})}\right)\\ & -\sqrt{\frac{\pi}{2}}\left\|y^0\right\|_{H_0^1(0,\pi)}\|B\|_{\ell^2(\mathbb{R})}-\sqrt{\frac{\pi}{2}}\left\|y^1\right\|_{L^2(0,\pi)}\|A\|_{\ell^2(\mathbb{R})},\end{aligned}$$

for every $(A,B)\in(\ell^2(\mathbb{R}))^2$. It follows easily that $\delta=\inf\{J_a(A,B)\mid (A,B)\in(\ell^2(\mathbb{R}))^2\}$ is finite. Consider now a minimizing sequence $(A^n,B^n)_{n\in\mathbb{N}^*}$ of J_ω such that

$$J_a\left(A^n,B^n\right)\leqslant \delta+\frac{1}{n^2},$$

for every $n\in\mathbb{N}^*$. Fix temporarily n and m, two nonzero integers. Any convex combination $t(A^n,B^n)+(1-t)(A^m,B^m)$, with $t\in[0,1]$ and $(m,n)\in(\mathbb{N}^*)^2$ is admissible and thus satisfies $J_a(t(A^n,B^n)+(1-t)(A^m,B^m))\geqslant\delta$. Moreover, easy computations show that

$$\begin{aligned}&\left(J_a\left(t\left(A^n,B^n\right)+(1-t)\left(A^m,B^m\right)\right)-\delta\right)+t(1-t)Q_a\left(\left(A^n,B^n\right)-\left(A^m,B^m\right)\right)\\ &\quad=(1-t)\left(J_a\left(A^m,B^m\right)-\delta\right)+t\left(J_a\left(A^n,B^n\right)-\delta\right).\end{aligned}\tag{62}$$

This implies

$$0\leqslant t(1-t)Q_a\left(\left(A^n,B^n\right)-\left(A^m,B^m\right)\right)\leqslant\frac{1-t}{m^2}+\frac{t}{n^2}.$$

From Lemmas 6 and 7, and using the observability inequality (3), the quadratic form Q_a induces a norm on $\ell^2(\mathbb{R})$ that is equivalent to the standard Hilbertian norm of $\ell^2(\mathbb{R})$. Hence, $(A^n,B^n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence in $(\ell^2(\mathbb{R}))^2$ and converges to some $(A^a,B^a)\in(\ell^2(\mathbb{R}))^2$, which is clearly a minimizer of J_a. The uniqueness of the minimizer follows from the strict convexity of J_a which is proved in Lemma 4.

A.3. *Proof of Proposition 2*

By weak compactness of $\overline{\mathcal{U}}_L$, up to a subsequence the sequence $(\chi_{\omega_n})_{n\in\mathbb{N}}$ converges to some $\widetilde{a}\in\overline{\mathcal{U}}_L$ for the L^∞ weak star topology. Since F is lower semicontinuous for the L^∞ weak star topology (see Theorem 1), we have

$$F(\widetilde{a})\leqslant\lim_{n\to+\infty}F(\chi_{\omega_n})=F(a),$$

and necessarily $a=\widetilde{a}$ by uniqueness of the minimizer (see Theorem 1). Using Lemmas 6 and 7, one has

$$F(\chi_{\omega_n}) \geqslant M_1(T)\frac{(\pi L-\sin(\pi L))}{2}\left\|\left(\phi_{\chi_{\omega_n}}(0,\cdot),\partial_t\phi_{\chi_{\omega_n}}(0,\cdot)\right)\right\|^2_{L^2\times H^{-1}},$$

for every $n\in\mathbb{N}$ and since the sequence $(F(\chi_{\omega_n}))_{n\in\mathbb{N}^*}$ is bounded, it follows that the sequence $(\phi_{\chi_{\omega_n}}(0,\cdot),\partial_t\phi_{\chi_{\omega_n}}(0,\cdot))_{n\in\mathbb{N}^*}$ is bounded in $L^2(0,\pi)\times H^{-1}(0,\pi)$. Since $(\phi_{\chi_{\omega_n}})_{n\in\mathbb{N}}$ is a solution of the wave equation (4), it follows immediately that the sequence $(\phi_{\chi_{\omega_n}})_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;L^2(0,\pi))$ and $(\partial_t\phi_{\chi_{\omega_n}})_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;H^{-1}(0,\pi))$. Then by Aubin's Lemma (see e.g. [25]), it follows that, up to a subsequence, the sequence $(\phi_{\chi_{\omega_n}})_{n\in\mathbb{N}}$ converges weakly in $L^2(0,T;L^2(0,\pi))\cap H^1(0,T;H^{-1}(0,\pi))$ and strongly in $L^2(0,T;L^2(0,\pi))$ to some $\widetilde{\phi}$. Since $(\chi_{\omega_n})_{n\in\mathbb{N}}$ converges L^∞ weakly star to a, this sequence converges also weakly in L^2. From this fact and the uniqueness of the minimizer of the HUM functional J_a, we deduce that the sequence $(\phi_{\chi_{\omega_n}})_{n\in\mathbb{N}}$ converges weakly in $L^2(0,T;L^2(0,\pi))\cap H^1(0,T;H^{-1}(0,\pi))$ and strongly in $L^2(0,T;L^2(0,\pi))$ to ϕ_a.

Since for $n\in\mathbb{N}$, y_n is solution of (1) with right hand side $h_{\chi_{\omega_n}}$, it follows that $(y_n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;H^1_0(0,\pi))$ and $(\partial_t y_n)_{n\in\mathbb{N}}$ is bounded in $L^2(0,T;L^2(0,\pi))$. Then by Aubin's Lemma it follows that, up to a subsequence, $(y_n)_{n\in\mathbb{N}}$ converges strongly to y_a in $L^2(0,T;L^2(0,\pi))$ and weakly in $L^2(0,T;H^1_0(0,\pi))\cap H^1(0,T;L^2(0,\pi))$.

In the previous reasonings, since there is only one closure point at the limit, then the convergence is not only up to subsequence, but for the whole sequence. The proposition follows.

A.4. *Proof of Proposition 3*

Due to technical requirements, the application of the Pontryagin Maximum Principle in the infinite dimensional setting requires to work within a strictly convex Banach space (see [11]). The optimal control problem settled above does not fit exactly (a priori) in this framework since the diagonal coefficients $z_{j,j}(\pi)$ are bounded below and above by positive constants (see Lemma 6), and hence a priori the adapted state space is $\ell^\infty(\mathbb{R})$, which is not strictly convex. To overcome this difficulty, it suffices to write the control system (26) in another way so that the new control system is settled in a strictly convex Banach space. This reduction relies on the identity

$$\sin(jx)\sin(kx)=\frac{1}{2}\left(\cos\left((j-k)x\right)-\cos\left((j+k)x\right)\right). \tag{63}$$

We consider the control system

$$w_m'(x)=a(x)\cos(mx),\qquad m\in\mathbb{N}, \tag{64}$$

with initial conditions $w_m(0)=0$ for every $m\in\mathbb{N}$. The control system (26) is then inferred from the above one by setting

$$z_0(x)=w_0(x),\qquad z_{j,k}(x)=\frac{1}{2}\left(w_{|j-k|}(x)-w_{j+k}(x)\right), \tag{65}$$

for almost every $x\in[0,\pi]$ and for all integers j and k. Then the optimal control problem (26)–(29) is equivalent to the reduced problem of determining a control $a\in\overline{\mathcal{U}}_L$ steering the infinite dimensional control system (64) from the zero initial conditions to the final condition $w_0(\pi)=L\pi$ (this is (28)), and minimizing the criterion $F(a)$ (easily rewritten in terms of the w_p's). This reduced optimal control problem now perfectly fits the requirement of having a strictly convex Banach space. Indeed, for every $x\in[0,\pi]$ and every $a\in\overline{\mathcal{U}}_L$, the sequence $(w_m(x))_{m\in\mathbb{N}^*}$ belongs[5] to the Hilbert space $\ell^2(\mathbb{R})$, and hence $\ell^2(\mathbb{R})$ can be chosen as the functional state space for this reduced optimal control problem. It enjoys the strict convexity property.

The second main assumption of [11] required to apply the Pontryagin Maximum Principle is, as mentioned above, a finite codimension assumption, which is obvious here since the final states $z_{j,k}(\pi)$ are not fixed (there is only one constraint, on $z_0(\pi)=w_0(\pi)$).

[5] Indeed, writing $w_m(x)=\int_0^{2\pi}\chi_{[0,x]}(s)a(s)\cos(ms)\,ds$, we see that $w_m(x)$ is the scalar product of an element of the usual Fourier basis with the function $s\mapsto\chi_{[0,x]}(s)a(s)$ which belongs to $L^2(0,2\pi)$ since $a(\cdot)\in\overline{\mathcal{U}}_L$.

Therefore, the infinite dimensional version of the Pontryagin Maximum Principle stated in [11] can be applied to our problem, and implies the following necessary conditions. There exist $p^0 \leqslant 0$ and $(q_m)_{m\in\mathbb{N}} \in \ell^2(\mathbb{R})$ called costates,[6] with $(p^0, (q_m)_{m\in\mathbb{N}}) \neq 0$, such that

$$a(x) = \begin{cases} 1 & \text{if } \varphi(x) > 0, \\ 0 & \text{if } \varphi(x) < 0, \end{cases}$$

for almost every $x \in [0, \pi]$, where the function φ, called switching function, is defined by

$$\varphi(x) = \sum_{m=0}^{+\infty} q_m \cos(mx).$$

The control a is however undetermined from the maximization condition of the Pontryagin Maximum Principle in the case where the switching function φ vanishes identically on a subset of positive measure.

To transpose these results to the original optimal control problem (26)–(29), we use the formulas (65), which indicate how the state coordinates must be changed. To get the corresponding change of costate coordinates, we apply the result of [4, Appendix], yielding to the relations

$$q_0 = p_0 + \frac{1}{2}\sum_{j=1}^{+\infty} p_{j,j}, \qquad q_1 = \sum_{j\in\mathbb{N}^*} p_{j,j+1}, \qquad q_m = \sum_{\substack{j,k\in\mathbb{N}^* \\ j-k=m}} p_{j,k} - \frac{1}{2}\sum_{\substack{j,k\in\mathbb{N}^* \\ j+k=m}} p_{j,k},$$

where p_0 and the $p_{j,k}$ denote the costates of the original optimal control problem. This change of coordinates leads exactly to the expression (31), as expected.

Moreover, since the coefficients $z_j(\pi)$ are not fixed, the transversality conditions of the Pontryagin Maximum Principle (see e.g. [26] for a detailed discussion) imply, using a computation similar to (24), that

$$p_{j,k} = p^0 \int_0^T \big(A_j^a \cos(jt) + B_j^a \sin(jt)\big)\big(A_k^a \cos(kt) + B_k^a \sin(kt)\big)\, dt, \tag{66}$$

for every $(j,k) \in (\mathbb{N}^*)^2$. Clearly, the constant p^0 cannot be equal to 0, otherwise (66) would imply that $p_{j,k} = 0$ for every $(j,k) \in (\mathbb{N}^*)^2$, and necessarily $p_0 \neq 0$, so that the switching function φ would be constant, and the optimal control would be constant on $[0,\pi]$, either equal to 1 or to 0. This would raise a contradiction with (28) since $L \in (0,1)$. Therefore $p^0 < 0$, and since $(p^0, p_0, (p_{j,k})_{(j,k)\in(\mathbb{N}^*)^2})$ is defined up to a multiplicative scalar, it is then usual to choose a normalization of the costates so that $p^0 = -1$. This leads finally to (32). The proposition is proved.

Remark 10. Everything works as if the Pontryagin Maximum Principle could be applied directly to the optimal control problem (26)–(29). However as explained previously for technical reasons we had to consider an equivalent formulation with a change of coordinates.

A.5. Proof of Proposition 6

First of all, observe that Theorem 5 implies that the set $\omega_2 \cap [0, \pi/2]$ has only one connected component. According to the symmetry property proved in Theorem 5 and since $\{\chi_{\omega_2} = 1\} \cap [0, \pi/2]$ has only one component, we write $\omega_2 = [\alpha, \alpha + \frac{\pi L}{2}]$ and rewrite the minimization problem (54) as a minimization problem with respect to the variable α. The functional to minimize is then

$$\alpha \in \left[0, \frac{\pi}{2}(1-L)\right] \mapsto \left(\frac{t}{\int_{[\alpha,\alpha+\frac{\pi L}{2}]} \sin^2 x\, dx} + \frac{1-t}{\int_{[\alpha,\alpha+\frac{\pi L}{2}]} \sin^2(2x)\, dx}\right).$$

To solve this minimization problem, it is more convenient to use the change of variable $\xi = \cos(2\alpha + \frac{\pi L}{2})$ so that, after computation of the integrals, the minimization problem reduces to

[6] Note that, since the dynamics of (64) do not depend on the state, it follows that the costates are constant.

$$\min J_t(\xi) = \frac{4t}{\pi L - \sin(\frac{\pi L}{2})\xi} + \frac{4(1-t)}{\pi L - \frac{\sin(\pi L)}{2}(2\xi^2-1)},$$
$$\xi \in [-\xi_1, \xi_1] \quad \text{where } \xi_1 = \cos\left(\frac{\pi L}{2}\right) > 0. \tag{67}$$

Let us study the variations of J_t on $[-\xi_1, \xi_1]$. One has

$$J_t'(\xi) = \frac{4t\sin(\frac{\pi L}{2})}{(\pi L - \sin(\frac{\pi L}{2})\xi)^2} + \frac{8(1-t)\sin(\pi L)\xi}{(\pi L - \frac{\sin(\pi L)}{2}(2\xi^2-1))^2},$$

hence the sign of $J_t'(\xi)$ is positive on $[0, \xi_1]$ and is equal to the sign of $G_t(\xi)$ on $[-\xi_1, 0]$, where

$$G_t(\xi) = 4t\sin\left(\frac{\pi L}{2}\right)\left(\pi L - \frac{\sin(\pi L)}{2}\left(2\xi^2-1\right)\right)^2 + 8(1-t)\sin(\pi L)\xi\left(\pi L - \sin\left(\frac{\pi L}{2}\right)\xi\right)^2.$$

The first and second derivative of G_t with respect to ξ are

$$G_t'(\xi) = 8\sin(\pi L)\left(2\sin\left(\frac{\pi L}{2}\right)\sin(\pi L)\xi^3 + 3(1-t)\sin^2\left(\frac{\pi L}{2}\right)\xi^2 + \sin\left(\frac{\pi L}{2}\right)\left(\left(2\pi L - \sin(\pi L)\right)t - 4\pi L\right)\xi + (1-t)\pi^2L^2\right),$$
$$G_t''(\xi) = 8\sin(\pi L)\sin\left(\frac{\pi L}{2}\right)\left(6\sin(\pi L)\xi^2 + 6(1-t)\sin\left(\frac{\pi L}{2}\right)\xi + \left(2\pi L - \sin(\pi L)\right)t - 4\pi L\right).$$

Let us investigate the sign of $G_t''(\xi)$ on $[-\xi_1, 0]$. Since $(2\pi L - \sin(\pi L)) \in [0, 2\pi L]$ whenever $t \in [0,1]$, there holds

$$G_t''(0) = 8\sin(\pi L)\sin\left(\frac{\pi L}{2}\right)\left(\left(2\pi L - \sin(\pi L)\right)t - 4\pi L\right) \leqslant 0.$$

Hence,

$$G_t''(-\xi_1) = 6\sin(\pi L)\cos^2\left(\frac{\pi L}{2}\right) - 3(1-t)\sin(\pi L) + \left(2\pi L - \sin(\pi L)\right)t - 4\pi L$$
$$= \frac{3}{2}\sin(2\pi L) + \left(2\pi L - 4\sin(\pi L)\right)t - 4\pi L \leqslant 0.$$

Indeed, the sign of $G_t''(-\xi_1)$ can be obtained in the following way. Define the function f on $[0,\pi]$ by $f(u) = \frac{3}{2}\sin(2u) - 4t\sin u + 2(t-2)u$, where u plays the role of πL. One has $f'(u) = 6\cos^2 u - 4t\cos u + 2t - 7$ and $f''(u) = 4\sin u(t - 3\cos u)$. Hence f' is decreasing on $[0, \arccos(t/3)]$ and increasing on $[\arccos(t/3), \pi]$. Moreover, $f'(0) = -2t - 1 < 0$ and $f'(\arccos(t/3)) < 0$ since $t \in [0,1]$. Since $f'(\pi) = 6t - 1$, distinguishing between the two cases $t \leqslant \frac{1}{6}$ and $t > \frac{1}{6}$, we obtain in the first case that f is decreasing on $[0,\pi]$ and in the second case the existence of $u_1 \in (0,\pi)$ such that f is decreasing on $[0, u_1]$ and then increasing on $[u_1, \pi]$. Since $f(0) = 0$ and $f(\pi) = 2\pi(t-2) < 0$, in the two cases f is negative, which proves that $G_t''(-\xi_1) \leqslant 0$.

Since G_t'' is a convex polynomial of degree 2 in the variable ξ, the set $\{G_t'' \leqslant 0\}$ is connected and hence, $G_t''(\xi) \leqslant 0$ for every $\xi \in [-\xi_1, 0]$.

Let us next investigate the sign of $G_t'(\xi)$ and of $G_t(\xi)$ on $[-\xi_1, 0]$. First, we deduce from the sign of $G_t''(\xi)$ on $[-\xi_1, 0]$ that G_t' is decreasing on $[-\xi_1, 0]$ and since $G_t'(0) = 8\sin(\pi L)(1-t)\pi^2L^2$, it follows that $G_t'(\xi) \geqslant 0$ for every $\xi \in [-\xi_1, 0]$. Hence, G_t is increasing on $[-\xi_1, 0]$. Using the facts that $G_t(0) = 8\sin(\pi L)(1-t)\pi^2L^2 \geqslant 0$, that $t \mapsto G_t(-\xi_1)$ is continuous, affine, has a positive limit when t tends to 1 and a negative one when t tends to 0, we deduce that

- if $t < t_0$ then there exists $-\xi_t \in [-\xi_1, 0]$ such that $G_t(\xi)$ is negative on $[-\xi_1, -\xi_t]$ and positive on $[-\xi_t, 0]$; note moreover that $-\xi_t$ is solution of the equation $G_t(\xi) = 0$;
- if $t \geqslant t_0$ then $G_t(\xi) \geqslant 0$ for every $\xi \in [-\xi_1, 0]$,

where $t_0 \in [0, 1]$ is the unique solution of the equation $G_t(-\xi_1) = 0$ where the unknown is t.

We deduce from all previous arguments that

- if $t < t_0$ then G_t is decreasing on $[-\xi_1, -\xi_t]$ and increasing on $[-\xi_t, \xi_1]$, and thus $-\xi_t = \operatorname{argmin}\{J_t(\xi),\ \xi \in [-\xi_1, \xi_1]\}$;
- if $t \geqslant t_0$ then G_t is increasing on $[-\xi_1, \xi_1]$ and thus $-\xi_1 = \operatorname{argmin}\{J_t(\xi),\ \xi \in [-\xi_1, \xi_1]\}$.

Let us study in particular the case $t < t_0$. It follows from an implicit function argument that the map $t \in [0, t_0] \mapsto -\xi_t$ is differentiable, since $-\xi_t$ is solution of $G_t(\xi) = 0$. Setting $g(t, \xi) = G_t(\xi)$, there holds moreover, for $t \in (0, t_0)$,

$$\frac{d(-\xi_t)}{dt}(t) = -\frac{\frac{\partial g}{\partial t}(t, -\xi_t)}{\frac{\partial g}{\partial \xi}(t, -\xi_t)} = -\frac{8 \sin(\pi L)(\pi L + \sin(\frac{\pi L}{2})\xi_t)^2}{t \frac{\partial G_t}{\partial \xi}(-\xi_t)} \xi_t,$$

and hence,

$$\operatorname{sign}\left(\frac{d(-\xi_t)}{dt}(t)\right) = \operatorname{sign}(-\xi_t) < 0,$$

for $t > 0$, since $G_t'(\xi) \geqslant 0$ on $[-\xi_1, 0]$. Observe also that $\xi_0 = \cos(2\alpha_0 + \frac{\pi L}{2}) = 0$, since $\alpha_0 = \frac{\pi}{4}(1 - L)$. Defining α_t as the inverse image of $-\xi_t$ by the initial change of variable yields the statement.

References

[1] N.I. Akhiezer, Elements of the Theory of Elliptic Functions, Translations of Mathematical Monographs, vol. 79, American Mathematical Society, 1990.

[2] B. Bonnard, J.-B. Caillau, E. Trélat, Second order optimality conditions in the smooth case and applications in optimal control, ESAIM Control Optim. Calc. Var. 13 (2) (2007) 207–236.

[3] D. Bucur, G. Buttazzo, Variational Methods in Shape Optimization Problems, Progress in Nonlinear Differential Equations, vol. 65, Birkhäuser Verlag, Basel, 2005.

[4] M. Cerf, T. Haberkorn, E. Trélat, Continuation from a flat to a round Earth model in the coplanar orbit transfer problem, Optimal Control Appl. Methods 33 (6) (2012) 654–675, http://dx.doi.org/10.1002/oca.1016.

[5] S. Cox, E. Zuazua, The rate at which energy decays in a damped string, Comm. Partial Differential Equations 19 (1994) 213–243.

[6] R. Fourer, D.M. Gay, B.W. Kernighan, AMPL: A Modeling Language for Mathematical Programming, second edition, Duxbury Press, 2002, 540 pp.

[7] P. Hébrard, A. Henrot, Optimal shape and position of the actuators for the stabilization of a string, Systems Control Lett. 48 (2003) 199–209.

[8] P. Hébrard, A. Henrot, A spillover phenomenon in the optimal location of actuators, SIAM J. Control Optim. 44 (2005) 349–366.

[9] A. Henrot, M. Pierre, Variation et optimisation de formes, Mathématiques Et Applications, Springer, 2005.

[10] A.E. Ingham, Some trigonometrical inequalities with applications to the theory of series, Math. Z. 41 (1936) 367–379.

[11] X.J. Li, J.M. Yong, Necessary conditions for optimal control of distributed parameter systems, SIAM J. Control Optim. 29 (4) (1991) 895–908.

[12] J.-L. Lions, Exact controllability, stabilizability and perturbations for distributed systems, SIAM Rev. 30 (1988) 1–68.

[13] J.-L. Lions, Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués, Tome 1, Recherches en Mathématiques Appliquées (Research in Applied Mathematics), Masson, 1988.

[14] S. Mandelbrojt, Quasi-analycité des séries de Fourier, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (2) 4 (3) (1935) 225–229.

[15] K. Morris, Linear-quadratic optimal actuator location, IEEE Trans. Automat. Control 56 (1) (2011) 113–124.

[16] A. Münch, Optimal location of the support of the control for the 1-D wave equation: numerical investigations, Comput. Optim. Appl. 42 (2009) 443–470.

[17] A. Münch, F. Periago, Optimal distribution of the internal null control for the 1D heat equation, J. Differential Equations 250 (2011) 95–111.

[18] S.L. Padula, R.K. Kincaid, Optimization strategies for sensor and actuator placement, Report NASA TM-1999-209126, 1999.

[19] F. Periago, Optimal shape and position of the support for the internal exact control of a string, Systems Control Lett. 58 (2) (2009) 136–140.

[20] Y. Privat, E. Trélat, E. Zuazua, Optimal observation of the one-dimensional wave equation, Preprint Hal, 2012.

[21] Y. Privat, E. Trélat, E. Zuazua, Optimal observability of the multi-dimensional wave and Schrödinger equations in quantum ergodic domains, Preprint Hal, 2012.

[22] L.S. Pontryagin, V.G. Boltyanskii, R.V. Gamkrelidze, E.F. Mishchenko, The Mathematical Theory of Optimal Processes, Interscience Publishers John Wiley & Sons, Inc., New York, London, 1962, 360 pp.

[23] D.L. Russell, Controllability and stabilizability theory for linear partial differential equations: recent progress and open questions, SIAM Rev. 20 (4) (1978) 639–739.
[24] O. Sigmund, J.S. Jensen, Systematic design of phononic band-gap materials and structures by topology optimization, Philos. Trans. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 361 (1806) (2003) 1001–1019.
[25] J. Simon, Compact sets in the space $L^p(0,T;B)$, Ann. Mat. Pura Appl. 146 (4) (1987) 65–96.
[26] E. Trélat, Contrôle optimal (Optimal control), Théorie Et Applications (Theory and Applications), Vuibert, Paris, 2005 (in French).
[27] E. Trélat, Optimal control and applications to aerospace: some results and challenges, J. Optim. Theory Appl. 154 (3) (2012) 713–758.
[28] M. van de Wal, B. Jager, A review of methods for input/output selection, Automatica 37 (4) (2001) 487–510.
[29] A. Wächter, L.T. Biegler, On the implementation of an interior-point filter line-search algorithm for large-scale nonlinear programming, Math. Program. 106 (2006) 25–57.