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On the limit $p \to \infty$ of global minimizers for a p-Ginzburg–Landau-type energy

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Abstract

We study the limit $p \to \infty$ of global minimizers for a p-Ginzburg–Landau-type energy

$$E_p(u) = \int\limits_{\mathbb{R}^2} |\nabla u|^p + \frac{1}{2}\left(1 - |u|^2\right)^2.$$

The minimization is carried over maps on $\mathbb{R}^2$ that vanish at the origin and are of degree one at infinity. We prove locally uniform convergence of the minimizers on $\mathbb{R}^2$ and obtain an explicit formula for the limit on $B(0, \sqrt{2})$. Some generalizations to dimension $N \geqslant 3$ are presented as well.

1. Introduction

For any $d \in \mathbb{Z}$, $N \geqslant 2$ and $p > N$ consider the class of maps

$$\mathcal{E}_p^d = \left\{u \in W^{1,p}_{loc}\left(\mathbb{R}^N, \mathbb{R}^N\right):\ E_p(u) < \infty,\ \deg(u) = d\right\},$$

where

$$E_p(u) = \int\limits_{\mathbb{R}^N} |\nabla u|^p + \frac{1}{2}\left(1 - |u|^2\right)^2.$$

By $\deg(u)$ we mean the degree of u "at infinity", which is properly defined since by Morrey's inequality (cf. [4, Theorem 9.12]), for any map $u \in W^{1,p}_{loc}(\mathbb{R}^N, \mathbb{R}^N)$ with $\int_{\mathbb{R}^N} |\nabla u|^p < \infty$ we have

$$u \in C^\alpha_{loc}\left(\mathbb{R}^2, \mathbb{R}^2\right), \quad \text{where } \alpha = 1 - N/p$$

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(except, perhaps, for a set of measure zero in $\mathbb{R}^2$) and

$$\left|u(x)-u(y)\right| \leqslant C_{p,N}\|\nabla u\|_{L^p(\mathbb{R}^N)}|x-y|^\alpha, \quad \forall x, y \in \mathbb{R}^N. \tag{1}$$

In fact, according to the proof given in [4], one can select

$$C_{p,N} = \frac{2^{2-N/p}}{1-N/p}. \tag{2}$$

It then easily follows (see [1] for the case $N=2$; the proof for any integer value of $N>2$ is identical) that

$$\lim_{|x|\to\infty}\left|u(x)\right| = 1. \tag{3}$$

Consequently, u has a well-defined degree, $\deg(u)$, equal to the degree of the S^{N-1}-valued map $\frac{u}{|u|}$ on any large circle $\{|z|=R\}$, $R \gg 1$.

In what follows, we assume that $N=2$ and, whenever appropriate, interpret $\mathbb{R}^2$-valued maps as complex-valued functions of the variable $z=x+iy$. We will return to the case $N \geqslant 3$ at the end of the Introduction and present some partial results for this case (Section 4).

For any $d \in \mathbb{Z}$, let

$$I_p(d) = \inf\left\{E_p(u)\colon\ u \in \mathcal{E}_p^d\right\}. \tag{4}$$

It has been established in [1] that $I_p(1)$ is attained for each $p>2$ and $N=2$. Denote by u_p a global minimizer of E_p in $\mathcal{E}_p^1$. It is clear that E_p is invariant with respect to translations and rotations. However, it is still unknown whether *uniqueness* of the minimizer u_p, modulo the above symmetries, is guaranteed. Such a uniqueness result would imply that, up to a translation and a rotation, u_p must take the form $f(r)e^{i\theta}$ (with $r=|x|$). Note that radial symmetry of a nontrivial *local minimizer* in the case $p=2$ was established by Mironescu in [7] (with a contribution from Sandier [8]). One way of inquiring whether the global minimizer u_p is radially symmetric or not for $p>2$, is by looking at the limiting behavior of $\{u_p\}_{p>2}$ as $p\to\infty$, which is the focus of the present contribution. We have already studied in [2] the behavior of minimizers in the class of radially symmetric functions when p is large and, in addition, showed their local stability for $2<p\leqslant 4$. The results presented in this work seem to support the radial symmetry conjecture (as in the case $p=2$ [7]); indeed, in the limit $p\to\infty$, we obtain the same asymptotic behavior for u_p as in the case of radially symmetric minimizers [2].

In view of the translational and rotational invariance properties of E_p, we may assume for each $p>2$ that

$$u_p(0)=0 \quad \text{and} \quad u_p(1)\in[0,\infty). \tag{5}$$

Our first main result is the following

Theorem 1. *For each $p>2$, let u_p denote a minimizer of E_p in $\mathcal{E}_p^1$ satisfying (5). Then, for a sequence $p_n\to\infty$, we have $u_{p_n}\to u_\infty$ in $C_{loc}(\mathbb{R}^2)$ and weakly in $\bigcap_{p>1} W^{1,p}_{loc}(\mathbb{R}^2,\mathbb{R}^2)$, where u_∞ satisfies*

$$\begin{cases} u_\infty(z) = \frac{z}{\sqrt{2}} & \text{on } B(0,\sqrt{2}) = \{|z|<\sqrt{2}\}, \\ \left|u_\infty(z)\right| = 1 & \text{on } \mathbb{R}^2\setminus B(0,\sqrt{2}). \end{cases} \tag{6}$$

Furthermore, the convergence $|u_{p_n}|\to|u_\infty|$ is uniform on $\mathbb{R}^2$.

Theorem 1 fails to identify the values in S^1 that the map u_∞ assumes on $\mathbb{R}^2\setminus B(0,\sqrt{2})$. A natural conjecture appears to be that $u_\infty(z)=\frac{z}{|z|}$ on $\mathbb{R}^2\setminus B(0,\sqrt{2})$, i.e., that $u_\infty = F$ where

$$F(z) = \begin{cases} \frac{z}{\sqrt{2}} & \text{on } B(0,\sqrt{2}), \\ \frac{z}{|z|} & \text{on } \mathbb{R}^2\setminus B(0,\sqrt{2}). \end{cases} \tag{7}$$

For simplicity, whenever appropriate, we will use the abbreviated notation u_p for u_{p_n}. Our second main result establishes explicit estimates for the rate of convergence of u_p to u_∞ inside the disc $B(0,\sqrt{2})$.

Theorem 2. *Under the assumptions of Theorem 1, for every $\beta < 1$ and $a < \sqrt{2}$, there exists $C_{\beta,a} > 0$ such that for all $p > 2$,*

$$\|u_p - u_\infty\|_{L^\infty(B(0,a))} \leqslant \frac{C_{\beta,a}}{p^{\beta/2}}. \tag{8}$$

Finally we consider the minimization of E_p in dimensions higher than 2. Although it is presently unknown whether $I_p(1)$ is attained for every $p > N \geqslant 3$, by using the same technique as in the proof of Theorem 1 we can show that the minimizer of E_p exists for sufficiently large values of p:

Theorem 3. *For every $N \geqslant 3$ there exists p_N such that for every $p > p_N$ the minimum value $I_p(1)$ of E_p is attained in $\mathcal{E}_p^1$ by some $u_p \in W^{1,p}_{loc}(\mathbb{R}^N, \mathbb{R}^N)$.*

In view of Theorem 3 it makes sense to investigate the asymptotic behavior of the set of minimizers $\{u_p\}_{p>2}$ as p tends to infinity for every $N \geqslant 3$. This is presented in the following

Theorem 4. *For each $p > p_N$, let u_p denote a minimizer of E_p in $\mathcal{E}_p^1$ satisfying $u_p(0) = 0$. Then, for a sequence $p_n \to \infty$, we have*

$$u_{p_n} \to u_\infty \text{ in } C_{loc}\big(\mathbb{R}^N\big) \text{ and weakly in } \bigcap_{p>1} W^{1,p}_{loc}\big(\mathbb{R}^N, \mathbb{R}^N\big), \tag{9}$$

where u_∞ satisfies

$$\begin{cases} u_\infty(x) = \frac{\mathcal{U}x}{\sqrt{N}} & \text{on } B(0, \sqrt{N}\,), \\ |u_\infty(x)| = 1 & \text{on } \mathbb{R}^N \setminus B(0, \sqrt{N}\,), \end{cases} \tag{10}$$

for some orthogonal $N \times N$ matrix $\mathcal{U}$ with $\det(\mathcal{U}) = 1$. We also have

$$\|\nabla u_\infty\|_{L^\infty(\mathbb{R}^N)} = 1 \tag{11}$$

and the convergence $|u_p| \to |u_\infty|$ is uniform on $\mathbb{R}^N$.

Remark 1.1. We may alternatively state that (subsequences of) minimizers of E_p over $\mathcal{E}_p^1$ satisfying $u(0) = 0$ converge to a minimizer for the following problem:

$$\inf\bigg\{ \int_{\mathbb{R}^N} \big(1 - |u|^2\big)^2 \colon u \in W^{1,\infty}\big(\mathbb{R}^N, \mathbb{R}^N\big),\ u(0) = 0,\ \|\nabla u\|_\infty \leqslant 1 \bigg\}. \tag{12}$$

The latter result can, most probably, be appropriately formulated in terms of Γ-convergence. Theorem 4 shows that the minimizers of (12) are given by the set of maps in $W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ satisfying (10)–(11). The infinite size of this set is the source of our difficulty in identifying the limit map u_∞ outside the ball $B(0, \sqrt{N}\,)$. To confirm the natural conjecture that $u_\infty(x) = \frac{\mathcal{U}x}{|x|}$ for $|x| > \sqrt{N}$, a more delicate analysis of the energies $E_p(u_p)$ or of the Euler–Lagrange equation satisfied by u_p is required. In fact, our present arguments can be used to prove the same convergence result as in Theorem 4 not only for the minimizers $\{u_p\}$, but also for a sequence of "almost minimizers" $\{v_p\}$, satisfying $E_p(v_p) \leqslant I_p(1) + o(1)$ as $p \to \infty$.

2. Proof of Theorem 1

We first recall the upper-bound for the energy that was proved in [2] using the test function $U_p(re^{i\theta}) = f_p(r)e^{i\theta}$ with

$$f_p(r) = \begin{cases} \frac{1}{\sqrt{2}}\big(1 - \frac{\ln p}{p}\big)r, & r < \frac{\sqrt{2}}{1-\frac{\ln p}{p}}, \\ 1, & r \geqslant \frac{\sqrt{2}}{1-\frac{\ln p}{p}}. \end{cases}$$

Lemma 2.1. *We have*

$$I_p(1) \leqslant \frac{\pi}{3} + C\frac{\ln p}{p}, \quad \forall p > 3. \tag{13}$$

Remark 2.1. From (13) we clearly obtain that

$$\int_{\mathbb{R}^2} |\nabla u_p|^p \leqslant C, \quad \forall p > 3, \tag{14}$$

where C is independent of p. While this estimate is sufficient for our purpose, it should be noted that one can derive a more precise estimate

$$\int_{\mathbb{R}^2} |\nabla u_p|^p = \frac{2}{p} I_p(1) \leqslant \frac{C}{p},$$

via a Pohozaev-type identity (see [1, Lemma 4.1]).

Our next lemma provides a key estimate that will lead to a lower-bound for $I_p(1)$.

Lemma 2.2. *Let $\rho \in (0,1)$ be a regular value of u_p (which by Sard's lemma holds for almost every ρ) and set*

$$A_\rho = \{z \in \mathbb{R}^2 \colon \ |u_p(z)| < \rho\}. \tag{15}$$

Then, for any component V_ρ of A_ρ with $\deg(u, \partial V_\rho) = d$, we have for large p

$$\int_{V_\rho} \left(1 - |u_p|^2\right)^2 \geqslant |d| \left\{ 4\pi \left(\frac{\rho^4}{2} - \frac{\rho^6}{3} \right) + o(1) \right\}, \tag{16}$$

where $o(1)$ denotes a quantity that tends to zero as p goes to infinity, uniformly for $\rho \in (0,1)$.

Proof. Since ρ is a regular value of u_p, we can conclude from (3) that ∂V_ρ is a finite union of closed and simple C^1-curves, and hence $\deg(u, \partial V_\rho)$ is well-defined. Since the image of V_ρ by u_ρ covers the disc $B(0,\rho)$ (algebraically) d times, it follows by Hölder's inequality that

$$\pi |d| \rho^2 = \left| \int_{V_\rho} (u_p)_x \times (u_p)_y \right| \leqslant \frac{1}{2} \int_{V_\rho} |\nabla u_p|^2 \leqslant \frac{1}{2} \mu(V_\rho)^{\frac{p-2}{p}} \left(\int_{V_\rho} |\nabla u_p|^p \right)^{\frac{2}{p}}, \tag{17}$$

where μ denotes the Lebesgue measure in $\mathbb{R}^2$, which, in turn, yields

$$\mu(V_\rho) \geqslant \frac{(2\pi |d| \rho^2)^{\frac{p}{p-2}}}{(\int_{V_\rho} |\nabla u_p|^p)^{\frac{2}{p-2}}}. \tag{18}$$

From (18) and (14), we get

$$\begin{aligned}
\int_{V_\rho} \left(1 - |u_p|^2\right)^2 &= \int_{(1-\rho^2)^2}^{1} \mu\left(\left\{\left(1 - |u_p|^2\right)^2 > t\right\} \cap V_\rho\right) dt \\
&= \int_0^\rho 4r\left(1 - r^2\right) \mu(A_r \cap V_\rho)\, dr \geqslant \int_0^\rho 4r\left(1 - r^2\right) \frac{(2\pi |d| r^2)^{\frac{p}{p-2}}}{(\int_{V_r} |\nabla u_p|^p)^{\frac{2}{p-2}}}\, dr \\
&\geqslant |d| \left\{ \int_0^\rho 4r\left(1 - r^2\right)\left(2\pi r^2\right) dr + o(1) \right\} = |d| \left\{ 4\pi \left(\frac{\rho^4}{2} - \frac{\rho^6}{3} \right) + o(1) \right\}. \qquad \square
\end{aligned} \tag{19}$$

Corollary 2.1. *There exist $\rho_0 \in (\frac{3}{4}, 1)$, p_0 and R_0 such that for all $p > p_0$ the set A_{ρ_0} has a component $V_{\rho_0} \subset B(0, R_0)$ for which* $\deg(u, \partial V_{\rho_0}) = 1$ *and* $|u_p| \geqslant \frac{1}{2}$ *on* $\mathbb{R}^2 \setminus V_{\rho_0}$.

Proof. Note that by (2) one can select uniformly bounded $C_{p,2}$ in (1) for $p \geqslant 3$. This fact, together with (14) implies equicontinuity of the maps $\{u_p\}_{p\geqslant 3}$ on $\mathbb{R}^2$. Therefore, there exists $\lambda > 0$ such that

$$\bigl|u_p(z_0)\bigr| \leqslant \frac{1}{2} \quad \Rightarrow \quad \bigl|u_p(z)\bigr| \leqslant \frac{3}{4} \quad \text{on } B(z_0, \lambda) \quad \Rightarrow \quad \int\limits_{B(z_0,\lambda)} \bigl(1 - |u_p|^2\bigr)^2 \geqslant \nu := \pi\lambda^2\left(\frac{7}{16}\right)^2. \tag{20}$$

Fix $\rho_0 \in (\frac{3}{4}, 1)$ such that

$$4\pi\left(\frac{\rho_0^4}{2} - \frac{\rho_0^6}{3}\right) > \max\left(\frac{\pi}{3}, \frac{2\pi}{3} - \nu\right). \tag{21}$$

Let V_{ρ_0} be a component of A_{ρ_0} with $\deg(u_p, \partial V_{\rho_0}) \neq 0$ (we may assume w.l.o.g. that ρ_0 is a regular value of u_p). By (13), (16) and (21), it follows that there can be only one such component when p is sufficiently large (and thus $\deg(u_p, \partial V_{\rho_0}) = 1$). Moreover, by (20) and (21), on any other component of A_{ρ_0} (if there is one) we must have $|u_p| > \frac{1}{2}$.

It remains necessary to show that V_{ρ_0} is embedded in a sufficiently large disc. Similarly to (20), there exists $\lambda_0 > 0$ such that

$$\begin{aligned} &\bigl|u_p(z_0)\bigr| \leqslant \rho_0 \quad \Rightarrow \quad \bigl|u_p(z)\bigr| \leqslant \frac{1+\rho_0}{2} \quad \text{on } B(z_0, \lambda_0) \\ &\quad \Rightarrow \int\limits_{B(z_0,\lambda_0)} \bigl(1 - |u_p|^2\bigr)^2 \geqslant \nu_0 := \pi\lambda_0^2\left(1 - \left(\frac{1+\rho_0}{2}\right)^2\right)^2. \end{aligned} \tag{22}$$

Since V_{ρ_0} is connected and $0 \in V_{\rho_0}$, the set $\{|z|\colon z \in V_{\rho_0}\}$ is the interval $[0, R)$ for some positive R. For any integer k for which $2k\lambda_0 \leqslant R$ there exists a set of points $\{z_j\}_{j=0}^{k-1} \subset V_{\rho_0}$ with $|z_j| = 2j\lambda_0$. By (22) and (13) we have for sufficiently large p that

$$k\nu_0 \leqslant \sum_{j=0}^{k-1} \int\limits_{B(z_j,\lambda_0)} \bigl(1 - |u_p|^2\bigr)^2 < c_0 := \frac{2\pi}{3} + 1.$$

It follows that R is bounded from above by $R_0 := 2\lambda_0(\frac{c_0}{\nu_0} + 1)$. $\square$

In order to complete the proof of Theorem 1 we need to establish the convergence of $\{u_{p_n}\}_{n=1}^{\infty}$ to u_∞ and to identify the limit. We begin with the following lemma

Lemma 2.3. *For a sequence $p_n \to \infty$ we have*

$$\lim_{n\to\infty} u_{p_n} = u_\infty \text{ in } C_{loc}\bigl(\mathbb{R}^2\bigr) \text{ and weakly in } \bigcap_{p>1} W^{1,p}_{loc}\bigl(\mathbb{R}^2, \mathbb{R}^2\bigr). \tag{23}$$

Furthermore, the limit map u_∞ is a degree-one map in $W^{1,\infty}(\mathbb{R}^2, \mathbb{R}^2)$ satisfying also (5) *and*

$$\|\nabla u_\infty\|_\infty \leqslant 1. \tag{24}$$

Proof. Fix any $q > 3$. Since $\|u_p\|_{L^\infty} \leqslant 1$ (see [1]), we have by (13) on each disc $B(0, m)$, $m \geqslant 1$, that

$$\|u_p\|_{W^{1,q}(B(0,m))} \leqslant C_m, \quad p > q.$$

It follows that for all $m \geqslant 1$, there exists a sequence $p_n \uparrow \infty$, such that $\{u_{p_n}\}$ converges weakly in $W^{1,q}(B(0, m))$ to a limit u_∞. By Morrey's theorem, the convergence holds in $C(B(0, m))$ as well. Since the latter is true for every $m \geqslant 1$

and every $q > 3$, we may apply a diagonal subsequence argument to find a subsequence satisfying (23). The fact that u_∞ has degree one too follows from (23) and Corollary 2.1.

Finally, in order to prove (24), it suffices to note that for any disc $B \subset \mathbb{R}^2$, $\lambda > 1$ and $q > 1$, we have by (14) and the weak lower semicontinuity of the L^q-norm,

$$\lambda^q \mu\bigl(\bigl\{|\nabla u_\infty| > \lambda\bigr\} \cap B\bigr) \leqslant \int_B |\nabla u_\infty|^q \leqslant \liminf_{p\to\infty} \int_B |\nabla u_p|^q \leqslant \liminf_{p\to\infty} \mu(B)^{1-q/p}\biggl(\int_B |\nabla u_p|^p\biggr)^{q/p} \leqslant \mu(B). \tag{25}$$

Letting q tend to ∞ in (25) yields $\mu(\{|\nabla u_\infty| > \lambda\} \cap B) = 0$. The conclusion (24) follows since the disc B and $\lambda > 1$ are arbitrary. $\square$

A similar argument to the one used in the proof of Lemma 2.2 yields

Proposition 1.

$$\lim_{p\to\infty} \frac{1}{2}\int_{\mathbb{R}^2} \bigl(1 - |u_p|^2\bigr)^2 = \lim_{p\to\infty} I_p(1) = \frac{\pi}{3} = \frac{1}{2}\int_{\mathbb{R}^2} \bigl(1 - |F|^2\bigr)^2,$$

where F is as defined in (7).

Proof. As in (18) we have

$$\mu(A_\rho) \geqslant \frac{(2\pi\rho^2)^{\frac{p}{p-2}}}{(\int_{A_\rho} |\nabla u_p|^p)^{\frac{2}{p-2}}}. \tag{26}$$

Therefore,

$$\begin{aligned}\int_{\mathbb{R}^2} \bigl(1 - |u_p|^2\bigr)^2 &= \int_0^1 \mu\bigl(\bigl(1 - |u_p|^2\bigr)^2 > t\bigr)\,dt \\ &= \int_0^1 4\rho\bigl(1-\rho^2\bigr)\mu(A_\rho)\,d\rho \geqslant \int_0^1 4\rho\bigl(1-\rho^2\bigr)\frac{(2\pi\rho^2)^{\frac{p}{p-2}}}{(\int_{A_\rho} |\nabla u_p|^p)^{\frac{2}{p-2}}}\,d\rho.\end{aligned} \tag{27}$$

Since $\int_{A_\rho} |\nabla u_p|^p \leqslant I_p(1) \leqslant C$, taking the limit inferior of both sides of (27) yields, with the aid of (7)

$$\begin{aligned}\liminf_{p\to\infty} \int_{\mathbb{R}^2} \bigl(1 - |u_p|^2\bigr)^2 &\geqslant \int_0^1 4\rho\bigl(1-\rho^2\bigr)\Bigl(\liminf_{p\to\infty}\bigl(2\pi\rho^2\bigr)^{\frac{p}{p-2}}\Bigr)\,d\rho \\ &= \int_0^1 4\rho\bigl(1-\rho^2\bigr)\mu\bigl(|F| < \rho\bigr)\,d\rho = \int_{\mathbb{R}^2} \bigl(1 - |F|^2\bigr)^2 = \frac{2\pi}{3},\end{aligned} \tag{28}$$

and the proposition follows by combining (28) with (13). $\square$

Remark 2.2. In fact, for any d we have $\lim_{p\to\infty} I_p(d) = \frac{|d|\pi}{3}$ (see Proposition 2 in Section 4).

We can now complete the proof of Theorem 1.

Proof of Theorem 1. For each $\rho \in (0, 1]$, let $D_\rho = \{z \in \mathbb{R}^2\colon\ |u_\infty(z)| < \rho\}$. Using arguments similar to those used to establish Proposition 1, we obtain

$$\int_{\mathbb{R}^2} \left(1-|u_\infty|^2\right)^2 = \int_0^1 \mu\left(\left(1-|u_\infty|^2\right)^2 > t\right) dt = \int_0^1 4\rho\left(1-\rho^2\right)\mu(D_\rho)\, d\rho. \tag{29}$$

Since $\deg(u_\infty) = 1$ by Lemma 2.3, using (24) yields

$$\pi\rho^2 \leqslant \left| \int_{D_\rho} (u_\infty)_x \times (u_\infty)_y \right| \leqslant \int_{D_\rho} \left|(u_\infty)_x \times (u_\infty)_y\right| \leqslant \frac{1}{2}\int_{D_\rho} |\nabla u_\infty|^2 \leqslant \frac{1}{2}\mu(D_\rho). \tag{30}$$

From (29)–(30) it follows that

$$\int_{\mathbb{R}^2} \left(1-|u_\infty|^2\right)^2 \geqslant \int_0^1 8\pi\rho\left(1-\rho^2\right)\rho^2\, d\rho = \frac{2\pi}{3}. \tag{31}$$

On the other hand, by Lemma 2.3 and Proposition 1, for every $R > 0$

$$\int_{B(0,R)} \left(1-|u_\infty|^2\right)^2 = \lim_{n\to\infty} \int_{B(0,R)} \left(1-|u_{p_n}|^2\right)^2 \leqslant \frac{2\pi}{3},$$

which together with (31) implies that

$$\int_{\mathbb{R}^2} \left(1-|u_\infty|^2\right)^2 = \frac{2\pi}{3}. \tag{32}$$

Therefore, for any $\rho \in (0,1)$, pointwise equalities between the integrands in (30) must hold almost everywhere in D_ρ. It follows that

$$\begin{cases} (u_\infty)_x \perp (u_\infty)_y, \qquad \left|(u_\infty)_x\right| = \left|(u_\infty)_y\right| \quad \text{and} \quad \left|(u_\infty)_x\right|^2 + \left|(u_\infty)_y\right|^2 = 1, \\ \operatorname{sign}\left\{(u_\infty)_x \times (u_\infty)_y\right\} \equiv \sigma \in \{-1, 1\}, \end{cases} \tag{33}$$

a.e. in D_1. From (33) we conclude that u_∞ is a conformal map a.e. in D_1 (it cannot be anti-conformal because $\deg(u_\infty) = 1$) with $|\nabla u_\infty| \equiv 1$. Hence, u_∞ must be of the form $u_\infty(z) = az + b$ with $|a| = \frac{1}{\sqrt{2}}$. Since u_∞ satisfies (5), we finally conclude that (6) holds.

Finally, to prove that $|u_p| \to |u_\infty|$ uniformly on $\mathbb{R}^2$ assume, on the contrary, that for some $\rho_0 < 1$ there exists a sequence $\{z_n\}_{n=1}^\infty$ with $|z_n| \to \infty$ such that $|u_{p_n}(z_n)| \leqslant \rho_0$ for all n. But then using (22) we are led immediately to a contradiction with Proposition 1 since we have already established that

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \left(1-|u_{p_n}|^2\right)^2 = \lim_{n\to\infty} \int_{B(0,\sqrt{2})} \left(1-|u_{p_n}|^2\right)^2 = \frac{2\pi}{3}. \qquad \square$$

3. Proof of Theorem 2

Let

$$\frac{\partial}{\partial z} \overset{def}{=} \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right); \qquad \frac{\partial}{\partial \bar z} \overset{def}{=} \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right).$$

We begin with a simple lemma that establishes the existence of an approximate holomorphic map for a given map u such that the L^2-norm of $\frac{\partial u}{\partial \bar z}$ is "small". To this end we introduce some additional notation. For a function $f \in L^1(\Omega)$ we denote by f_Ω its average value over Ω, i.e.,

$$f_\Omega = \frac{1}{\mu(\Omega)} \int_\Omega f.$$

We further set $\nabla_\perp u = (u_y, -u_x)$.

Lemma 3.1. *Let Ω be a bounded, simply connected domain in $\mathbb{R}^2$ with $\partial\Omega \in C^1$. Let $u = u_r + iu_i \in H^1(\Omega, \mathbb{C})$ satisfy*

$$\int_\Omega |\nabla u + i\nabla_\perp u|^2 \leqslant \epsilon^2, \tag{34}$$

for some $\epsilon > 0$. Then, there exists v which is holomorphic in Ω and such that $v_\Omega = u_\Omega$,

$$\int_\Omega \big|\nabla(u - v)\big|^2 \leqslant 4\epsilon^2 \tag{35}$$

and

$$\int_\Omega |\nabla u|^2 = \int_\Omega |\nabla v|^2 + \int_\Omega \big|\nabla(u - v)\big|^2. \tag{36}$$

Proof. Consider the Hilbert space $\mathcal{H} = \{U \in H^1(\Omega, \mathbb{C}): U_\Omega = 0\}$ with the norm $\|U\|^2_{\mathcal{H}} = \int_\Omega |\nabla U|^2$ and its closed subspace $\mathcal{K} = \{V \in \mathcal{H}: V \text{ is holomorphic in } \Omega\}$. Let $v = V + u_\Omega$ where $V \in \mathcal{K}$ is the nearest point projection of $u - u_\Omega \in \mathcal{H}$ on $\mathcal{K}$. Clearly v satisfies (36). To prove (35), it is sufficient, in view of the definition of v, to construct a single function $\tilde{v} \in H^1(\Omega, \mathbb{C})$, which is holomorphic in Ω, and satisfies

$$\int_\Omega \big|\nabla(u - \tilde{v})\big|^2 \leqslant 4\epsilon^2. \tag{37}$$

Set $\tilde{v} = \tilde{v}_r + i\tilde{v}_i$ where $\tilde{v}_r \in H^1_0(\Omega, \mathbb{C}) + u_r$ is harmonic and $\tilde{v}_i$ is the conjugate harmonic function to $\tilde{v}_r$ satisfying $(\tilde{v}_i)_\Omega = (u_i)_\Omega$. Let $\phi \in C_0^\infty(\Omega, \mathbb{C})$. Clearly,

$$\int_\Omega \nabla\bar{\phi} \cdot \nabla_\perp w = 0, \quad \forall w \in H^1(\Omega, \mathbb{C}), \tag{38}$$

and since $\tilde{v}$ is harmonic, we have

$$\int_\Omega \nabla\bar{\phi} \cdot \nabla\tilde{v} = 0. \tag{39}$$

By density of $C_0^\infty(\Omega, \mathbb{C})$ in $H^1_0(\Omega, \mathbb{C})$, (38)–(39) hold for every $\phi \in H^1_0(\Omega, \mathbb{C})$. In particular, employing the identity

$$\nabla\tilde{v} + i\nabla_\perp\tilde{v} = 0, \tag{40}$$

and using (38) we obtain for $\phi = u_r - \tilde{v}_r$ that

$$\begin{aligned}\big\|\nabla(u_r - \tilde{v}_r)\big\|_2^2 &= \Re\int_\Omega \nabla(u_r - \tilde{v}_r) \cdot \nabla(u - \tilde{v}) = \Re\int_\Omega \nabla(u_r - \tilde{v}_r) \cdot \big\{\nabla(u - \tilde{v}) + i\nabla_\perp(u - \tilde{v})\big\} \\ &= \Re\int_\Omega \nabla(u_r - \tilde{v}_r) \cdot (\nabla u + i\nabla_\perp u) \leqslant \big\|\nabla(u_r - \tilde{v}_r)\big\|_2 \big\|\nabla u + i\nabla_\perp u\big\|_2.\end{aligned} \tag{41}$$

Hence, by (34) and (41),

$$\big\|\nabla(u_r - \tilde{v}_r)\big\|_2 \leqslant \epsilon. \tag{42}$$

Set $w = u - \tilde{v}$. By (34) and (40)

$$\|\nabla w + i\nabla_\perp w\|_2 \leqslant \epsilon. \tag{43}$$

However, as w_r is real we have by (42)

$$\|\nabla w_r + i\nabla_\perp w_r\|_2 = \sqrt{2}\|\nabla w_r\|_2 \leqslant \sqrt{2}\epsilon. \tag{44}$$

Since

$$\nabla w + i\nabla_\perp w = \nabla w_r + i\nabla_\perp w_r + i(\nabla w_i + i\nabla_\perp w_i),$$

we get from (43)–(44) that

$$\|\nabla w_i\|_2 = \frac{1}{\sqrt{2}}\|\nabla w_i + i\nabla_\perp w_i\|_2 \leqslant \frac{1}{\sqrt{2}}\big(\|\nabla w + i\nabla_\perp w\|_2 + \|\nabla w_r + i\nabla_\perp w_r\|_2\big) \leqslant \left(1 + \frac{1}{\sqrt{2}}\right)\epsilon,$$

which together with (42) clearly implies (37) □

By Poincaré inequality and (35) we immediately deduce:

Corollary 3.1. *Let v be given by Lemma 3.1. Then,*

$$\|u - v\|_{H^1(\Omega)} \leqslant C\epsilon, \tag{45}$$

where C depends only on Ω.

Lemma 3.2. *Let f be holomorphic in $\Omega \subset \mathbb{R}^2$. Suppose that for every disc $B(x_0, s) \subset \Omega$ we have*

$$\int\limits_{B(x_0,s)} \big(|f|^2 - 1\big) \leqslant \epsilon, \tag{46}$$

for some $\epsilon > 0$. Then,

$$\|f\|^2_{L^\infty(\Omega_s)} \leqslant 1 + \frac{\epsilon}{\mu(B(x_0, s))},$$

where

$$\Omega_s = \big\{x \in \Omega \;\big|\; d(x, \partial\Omega) > s\big\}.$$

Proof. As f is holomorphic, $|f|^2$ is subharmonic. By the mean value principle we obtain for any $x_0 \in \Omega_s$

$$\big|f(x_0)\big|^2 \leqslant \frac{1}{\mu(B(x_0, s))} \int\limits_{B(x_0,s)} |f|^2 = 1 + \frac{1}{\mu(B(x_0, s))} \int\limits_{B(x_0,s)} \big(|f|^2 - 1\big), \tag{47}$$

from which the lemma easily follows. □

Lemma 3.3. *Let f be holomorphic in $B_R = B(0, R) \subset \mathbb{R}^2$. Suppose that*

$$\int\limits_{B_R} \big(1 - |f|^2\big) \leqslant \epsilon, \tag{48}$$

for some $\epsilon > 0$. Suppose further that

$$\|f\|^2_{L^\infty(B_R)} \leqslant 1 + \epsilon. \tag{49}$$

Then, there exist $\alpha \in [-\pi, \pi)$ and $C > 0$, depending only on R, such that

$$\big|f(x) - e^{i\alpha}\big| \leqslant C\frac{\epsilon}{d_x^2}, \quad x \in B_R, \tag{50}$$

where $d_x = R - |x|$.

Proof. By (48)–(49),

$$\int\limits_{B_R} \big||f|^2 - 1 - \epsilon\big| = \int\limits_{B_R} \big(1 - |f|^2\big) + \pi R^2\epsilon \leqslant C\epsilon,$$

hence,

$$\int_{B_R} \left||f|^2 - 1\right| \leqslant C\epsilon \tag{51}$$

(we denote by C and c different constants, depending on R only). Since the function $||f|^2 - 1|$ is subharmonic, we deduce from (51) that for every $x \in B_R$,

$$\left|\left|f(x)\right|^2 - 1\right| \leqslant \frac{1}{\pi d_x^2} \int_{B(x,d_x)} \left||f|^2 - 1\right| \leqslant \frac{c\epsilon}{d_x^2}. \tag{52}$$

It follows in particular that

$$\left|f(x)\right|^2 \geqslant \frac{1}{2}, \qquad |x| \leqslant R - \sqrt{2c\epsilon}. \tag{53}$$

In $B(0, R - \sqrt{2c\epsilon}\,)$ we may write then $f = e^{U+iV}$, where V is the conjugate harmonic function of U that satisfies $V(0) \in [-\pi, \pi)$. By (52) we have

$$\left|U(x)\right| \leqslant \frac{C\epsilon}{d_x^2}, \qquad |x| \leqslant R - \sqrt{2c\epsilon}. \tag{54}$$

From (54) we get an interior estimate for the derivatives of U (see (2.31) in [5]):

$$\left|\nabla U(x)\right| \leqslant C\frac{\epsilon}{d_x^3}, \qquad |x| \leqslant R - \sqrt{4c\epsilon}. \tag{55}$$

Note that by the Cauchy–Riemann equations, (55) holds for V as well, i.e.,

$$\left|\nabla V(x)\right| \leqslant C\frac{\epsilon}{d_x^3}, \qquad |x| \leqslant R - \sqrt{4c\epsilon}. \tag{56}$$

For any $x \in B(0, R - \sqrt{4c\epsilon}\,) \setminus \{0\}$ we obtain, using (56), the estimate

$$\left|V(x) - V(0)\right| \leqslant \int_{d_x}^{R} \left|\nabla V\left((R-s)\frac{x}{|x|}\right)\right| ds \leqslant C\epsilon \int_{d_x}^{R} \frac{ds}{s^3} \leqslant \frac{C\epsilon}{d_x^2}. \tag{57}$$

Therefore, setting $\alpha = V(0)$ and using (54) and (57), we obtain for every $x \in B(0, R - \sqrt{4c\epsilon}\,)$ that

$$\left|f(x) - e^{i\alpha}\right| \leqslant \left|f(x) - e^{iV(x)}\right| + \left|e^{iV(x)} - e^{iV(0)}\right| \leqslant \left|e^{U(x)} - 1\right| + \left|V(x) - V(0)\right| \leqslant \frac{C\epsilon}{d_x^2}.$$

For $x \in B_R \setminus B(0, R - \sqrt{4c\epsilon}\,)$, i.e., when $d_x \leqslant \sqrt{4c\epsilon}$, we have clearly $|f(x) - e^{i\alpha}| \leqslant 2 + \epsilon$, so choosing C big enough yields (50) for all $x \in B_R$. $\square$

Let A_ρ be defined in (15). The following lemma lists some of its properties.

Lemma 3.4. *There exist $p_0 > 2$ and $C > 0$ such that for all $p > p_0$ and $\rho > \frac{1}{2}$ we have*

$$\mu(A_\rho) \geqslant 2\pi\rho^2\left(1 - \frac{C}{p}\right), \tag{58a}$$

$$\int_0^1 \rho\left(1 - \rho^2\right)\left|\mu(A_\rho) - 2\pi\rho^2\right| d\rho \leqslant C\frac{\ln p}{p}. \tag{58b}$$

Proof. The estimate (58a) follows directly from (26) and (14). Since by (58a)

$$\mu(A_\rho) \geqslant \left|\mu(A_\rho) - 2\pi\rho^2\right| + 2\pi\rho^2 - \frac{C}{p},$$

we obtain using (27) that

$$I_p(1) \geqslant \int_0^1 2\rho\left(1-\rho^2\right)\mu(A_\rho)\,d\rho \geqslant \int_0^1 2\rho\left(1-\rho^2\right)\left|\mu(A_\rho) - 2\pi\rho^2\right|d\rho + \frac{\pi}{3} - \frac{C}{p}.$$

Combining the above with (13) yields (58b). □

Lemma 3.5. *Let* $\ln p/p \ll \delta_p < 1/4$. *There exists* $1 - 2\delta_p < \rho < 1 - \delta_p$, *such that for all* $p > p_0$

$$\int_{A_\rho} |\nabla u_p + i\nabla_\perp u_p|^2 \leqslant C\delta_p^{-2}\frac{\ln p}{p}. \tag{59}$$

Proof. By (58b) there exists $1 - 2\delta_p < \rho < 1 - \delta_p$ such that

$$\left|\mu(A_\rho) - 2\pi\rho^2\right| \leqslant C\delta_p^{-2}\frac{\ln p}{p}. \tag{60}$$

Applying (60) yields

$$\begin{aligned}
\frac{1}{4}\int_{A_\rho} |\nabla u_p + i\nabla_\perp u_p|^2 &= \int_{A_\rho}\left[\frac{1}{2}|\nabla u_p|^2 - (u_p)_x \times (u_p)_y\right] \\
&= \int_{A_\rho}\frac{1}{2}|\nabla u_p|^2 - \pi\rho^2 \leqslant \frac{1}{2}\left(\int_{A_\rho}|\nabla u_p|^p\right)^{2/p}\mu(A_\rho)^{1-2/p} - \pi\rho^2 \\
&\leqslant \frac{1}{2}\left(1+\frac{C}{p}\right)\left(2\pi\rho^2 + C\delta_p^{-2}\frac{\ln p}{p}\right)^{1-2/p} - \pi\rho^2 \leqslant \frac{C}{\delta_p^2}\frac{\ln p}{p}. \qquad \Box
\end{aligned} \tag{61}$$

Proof of Theorem 2. Set $\eta = \frac{\sqrt{2}-a}{10}$ and then

$$b_j = a + j\eta, \quad j = 1,\dots,9.$$

Let ρ be given by Lemma 3.5 for $\delta_p = \eta/\sqrt{2}$, so that $\rho \in (b_8/\sqrt{2}, b_9/\sqrt{2})$. We can also assume without loss of generality that ρ is a regular value for $|u_p|$. By Theorem 1 we have for sufficiently large p,

$$B(0,b_8) \subset A_\rho \subset B(0,b_9). \tag{62}$$

By (62) and Lemma 3.5 we have

$$\int_{B(0,b_8)} |\nabla u_p + i\nabla_\perp u_p|^2 \leqslant \int_{A_\rho} |\nabla u_p + i\nabla_\perp u_p|^2 \leqslant \frac{C}{(a-\sqrt{2})^2}\frac{\ln p}{p} = C_a\frac{\ln p}{p}.$$

Applying Corollary 3.1 yields the existence of a holomorphic function v_p in $B(0,b_8)$ such that $(v_p)_{B(0,b_8)} = (u_p)_{B(0,b_8)}$ and such that (36) holds with $u = u_p$, $v = v_p$ and

$$\|u_p - v_p\|^2_{H^1(B(0,b_8))} \leqslant C_a\frac{\ln p}{p}. \tag{63}$$

We denote $w_p(z) = \sqrt{2}v_p'(z)$ (where $v_p' = \frac{\partial v_p}{\partial z}$ is the derivative of the holomorphic map v_p) and note that $|w_p(z)| = |\nabla v_p(z)|$. As a is kept fixed, we suppress in the sequel the dependence of the constants on a.

For any ball $B \subset B(0,b_8)$ we apply the same estimates as in (17),

$$\int_B |\nabla u_p|^2 - 1 \leqslant \left(\int_{B(0,b_8)} |\nabla u_p|^p \right)^{2/p} \mu(B)^{1-2/p} - \mu(B) \leqslant (1 + C/p)\big(\mu(B)\big)^{1-2/p} - \mu(B) \leqslant \frac{C}{p}.$$

Combining the above with (36) yields

$$\int_B \left(|w_p|^2 - 1\right) = \int_B \left(|\nabla v_p|^2 - 1\right) \leqslant \int_B |\nabla u_p|^2 - 1 \leqslant \frac{C}{p}, \quad \forall B \subset B(0, b_8).$$

By Lemma 3.2 it then follows that

$$\|w_p\|^2_{L^\infty(B(0,b_7))} \leqslant 1 + \frac{C_1}{p}. \tag{64}$$

Next, we apply Lemma 3.5 again, this time with $\delta_p = 3\eta/\sqrt{2}$, to find a corresponding $\tilde{\rho} \in (b_4/\sqrt{2}, b_7/\sqrt{2})$. For p large we have $B(0, b_4) \subset A_{\tilde{\rho}} \subset B(0, b_7)$. Arguing as in (17) we obtain, using (60),

$$\int_{A_{\tilde{\rho}}} |\nabla u_p|^2 - 1 \geqslant 2 \int_{A_{\tilde{\rho}}} (u_p)_x \times (u_p)_y - \mu(A_{\tilde{\rho}}) \geqslant 2\pi \tilde{\rho}^2 - \mu(A_{\tilde{\rho}}) \geqslant -C \frac{\ln p}{p}.$$

By (36), once again, we have that

$$\int_{A_{\tilde{\rho}}} \left(|w_p|^2 - 1\right) \geqslant -C \frac{\ln p}{p}. \tag{65}$$

Next, we apply the same argument as the one used in the beginning of the proof of Lemma 3.3 to obtain, using (64) and (65),

$$\int_{A_{\tilde{\rho}}} \left| |w_p|^2 - 1 - \frac{C_1}{p} \right| = \int_{A_{\tilde{\rho}}} \left(1 - \frac{C_1}{p} - |w_p|^2\right) \leqslant C \frac{\ln p}{p}.$$

Hence, also

$$\int_{B(0,b_4)} \big| |w_p|^2 - 1 \big| \leqslant \int_{A_{\tilde{\rho}}} \big| |w_p|^2 - 1 \big| \leqslant C \frac{\ln p}{p}. \tag{66}$$

We can now use (64) and (66) and apply Lemma 3.3 to obtain the existence of $\alpha_p \in [-\pi, \pi)$ such that

$$\left| w_p(z) - e^{i\alpha_p} \right| \leqslant C \frac{\ln p}{p}, \quad z \in B(0, a). \tag{67}$$

Consequently, there exists a constant γ_p such that

$$\left| \sqrt{2} v_p(z) - e^{i\alpha_p} z - \gamma_p \right| \leqslant C \frac{\ln p}{p}, \quad z \in B(0, a). \tag{68}$$

Set

$$U = u_p - v_p.$$

For every $q > 2$ we have for $p > q$, by (63), (68), and the fact that $|u_p| \leqslant 1$,

$$\|U\|^q_{L^q(B(0,a))} \leqslant \|U\|^{q-2}_{L^\infty(B(0,a))} \|U\|^2_{L^2(B(0,a))} \leqslant C \left(\frac{\ln p}{p} \right).$$

Furthermore, by Hölder's inequality, (67), (63) and (13) we have that

$$\|\nabla U\|^q_{L^q(B(0,a))} \leqslant \|\nabla U\|^{2\frac{p-q}{p-2}}_{L^2(B(0,a))} \|\nabla U\|^{p\frac{q-2}{p-2}}_{L^p(B(0,a))} \leqslant C \left(\frac{\ln p}{p} \right)^{\frac{p-q}{p-2}}.$$

Consequently, for each fixed $q > 2$ we have

$$\|U\|_{W^{1,q}(B(0,a))} \leqslant C\left(\frac{\ln p}{p}\right)^{\frac{p-q}{q(p-2)}}. \tag{69}$$

By Sobolev embedding the bound in (69) holds also for $\|U\|_{L^\infty(B(0,a))}$ and, in particular, we get that for every $0 < \beta < 1$,

$$\|U\|_{L^\infty(B(0,a))} \leqslant C_\beta p^{-\beta/2}. \tag{70}$$

Combining (70) and (68) we obtain that

$$\left|\sqrt{2}u_p(z) - e^{i\alpha_p}z - \gamma_p\right| \leqslant C_\beta p^{-\beta/2}, \quad z \in B(0,a).$$

As $u_p(0) = 0$, it immediately follows that $|\gamma_p| \leqslant C_\beta p^{-\beta/2}$, and hence

$$\left|\sqrt{2}u_p(z) - e^{i\alpha_p}z\right| \leqslant C_\beta p^{-\beta/2}, \quad z \in B(0,a). \tag{71}$$

Substituting $z = 1$ into (71) we obtain using (5) that $|\alpha_p| \leqslant C_\beta p^{-\beta/2}$ and (8) follows. □

4. The problem in dimension $N \geqslant 3$

This section is mainly devoted to the proofs of Theorem 3 and Theorem 4. We begin with the computation of $\lim_{p\to\infty} I_p(d)$. Denote by ω_N the volume of the unit ball in $\mathbb{R}^N$. It turns out that the constant

$$\tau_N := \frac{4\omega_N}{(N+2)(N+4)}N^{N/2} \tag{72}$$

generalizes the constant $\frac{\pi}{3}$ in (13) for dimensions higher than $N = 2$.

Proposition 2. *We have*

$$\lim_{p\to\infty} I_p(d) = |d|\tau_N. \tag{73}$$

Proof. (i) First we establish an upper bound. When $d = 1$, following a construction similar to the one used in the proof of Lemma 2.1, we define a map U_p by

$$U_p(x) = \begin{cases} \frac{x}{r_p}, & |x| < r_p, \\ \frac{x}{|x|}, & |x| \geqslant r_p, \end{cases} \tag{74}$$

with $r_p := \frac{\sqrt{N}}{1-\frac{\ln p}{p}}$. A direct computation shows that for $p \geqslant N+1$ we have

$$\begin{aligned} E_p(U_p) &\leqslant \frac{1}{2}\int\limits_{B(0,r_p)}\left(1-|U_p|^2\right)^2 + C\frac{\ln p}{p} = \frac{1}{2}\int\limits_0^{\sqrt{N}}\left(1-\frac{r^2}{N}\right)^2 N\omega_N r^{N-1}\,dr + C\frac{\ln p}{p} \\ &= \frac{4\omega_N}{(N+2)(N+4)}N^{N/2} + C\frac{\ln p}{p}. \end{aligned} \tag{75}$$

Next we turn to the case $d > 1$. Fix d distinct points $q_1, \dots, q_d$ in $\mathbb{R}^N$ with

$$\delta := \frac{1}{4}\min\left\{|q_i - q_j|\colon i \neq j\right\} > 4\sqrt{N}.$$

Fix K satisfying

$$K > \max_{1\leqslant j\leqslant d}|q_j| + 4\delta,$$

and set $\Omega = B(0,K) \setminus \bigcup_{j=1}^{d} \overline{B(q_j,\delta)}$. Fix a smooth map $V : \overline{\Omega} \to S^{N-1}$ satisfying

$$V(x) = \frac{x - q_j}{|x - q_j|} \text{ on } \partial B(q_j,\delta), \quad j = 1,\dots,d.$$

Let $M = \|\nabla V\|_{L^\infty(\Omega)}$ and fix $R > M\sqrt{N-1}$. We finally define

$$W_p(x) = \begin{cases} U_p(x - Rq_j), & x \in B(Rq_j, r_p),\ j = 1,\dots,d, \\ \frac{x-Rq_j}{|x-Rq_j|}, & x \in B(Rq_j, R\delta) \setminus B(Rq_j, r_p),\ j = 1,\dots,d, \\ V(x/R), & x \in R\Omega, \\ V(K\frac{x}{|x|}), & x \in \mathbb{R}^N \setminus B(0, RK). \end{cases}$$

By our construction $\|\nabla W_p\|_{L^\infty(\mathbb{R}^N \setminus \bigcup_{j=1}^{d} B(q_j,r_p))} \leqslant \gamma < 1$, and hence, it follows from (75) that

$$E_p(W_p) \leqslant d\tau_N + o(1), \tag{76}$$

which is the desired upper bound.

(ii) We next obtain a lower bound. Assume that $d \geqslant 1$ and let u denote a map in $\mathcal{E}_p^d$. We attempt to prove that

$$E_p(u) \geqslant d\tau_N + o(1) \quad \text{as } p \to \infty, \tag{77}$$

where $o(1)$ is a quantity that goes to zero when p goes to infinity (i.e., it is independent of u). We establish (77) for $u \in C^\infty(\mathbb{R}^N, \mathbb{R}^N)$. The proof for any $u \in \mathcal{E}_p^d$ then follows by density. Furthermore, in view of (76), we may suppose that

$$E_p(u) \leqslant d\tau_N + 1. \tag{78}$$

We continue to argue as in the proof of Lemma 2.2. Given a regular value $\rho \in (0,1)$ of u, let V_ρ denote a component or a finite union of components of $A_\rho = \{x \in \mathbb{R}^N\colon\ |u(x)| < \rho\}$ with $\deg(u, \partial V_\rho) = D$. We claim that

$$\int_{V_\rho} \left(1 - |u|^2\right)^2 \geqslant |D| \left\{ 4\omega_N N^{N/2} \left(\frac{\rho^{N+2}}{N+2} - \frac{\rho^{N+4}}{N+4} \right) + o(1) \right\}, \tag{79}$$

as $p \to \infty$, where the decay of the $o(1)$ term is uniform on $\rho \in (0,1)$. To obtain the generalization of (17) to any N, we use Hadamard's inequality and the inequality of arithmetic and geometric means (see [3] for both inequalities) as follows:

$$|D|\omega_N \rho^N = \left| \int_{V_\rho} \det(\nabla u) \right| \leqslant \int_{V_\rho} \prod_{j=1}^{N} \left| \frac{\partial u}{\partial x_j} \right| \leqslant \frac{1}{N^{N/2}} \int_{V_\rho} |\nabla u|^N \leqslant \frac{1}{N^{N/2}} \mu(V_\rho)^{\frac{p-N}{p}} \left(\int_{V_\rho} |\nabla u|^p \right)^{\frac{N}{p}}. \tag{80}$$

From (80) we get a lower bound for $\mu(V_\rho)$ which yields (79) by the same argument as in (19) (thanks to (78) we have a bound for $\int_{V_\rho} |\nabla u|^p$). Finally we apply (79) with $V_\rho = A_\rho$ (so that $D = d$) and let $\rho \uparrow 1^-$ to obtain (77). □

We next prove Theorem 3, or the existence of a minimizer in (4) for sufficiently large values of p (we emphasize that for $N = 2$ this existence has been established in [1] for *any* $p > 2$, hence we expect it to hold for any $p > N$ when $N \geqslant 3$).

Proof of Theorem 3. For any fixed $p \geqslant N + 1$ consider a minimizing sequence $\{v_n\} \subset \mathcal{E}_1$. We may assume that these maps are smooth, satisfy $v_n(0) = 0$ and thanks to (77) that

$$E_p(v_n) \leqslant I_p(1) + \frac{1}{n} \leqslant C, \quad \forall n. \tag{81}$$

Combined together, (81) and Morrey's inequality (1) imply equicontinuity of the sequence $\{v_n\}$. Hence we can repeat with slight modifications (e.g., using (79) instead of (16)) the arguments of Corollary 2.1 to arrive at an analogous conclusion: there exist $\rho_0 \in (\frac{3}{4}, 1)$ as well as $p_N > N+1$ and R_0 such that for all $p > p_N$ the set $A_{\rho_0}^{(n)} := \{|v_n(x)| < \rho\}$ has a component $V_{\rho_0}^{(n)} \subset B(0, R_0)$ for which

$$\deg\big(v_n, \partial V_{\rho_0}^{(n)}\big) = 1 \quad \text{and} \quad |v_n| \geqslant \frac{1}{2} \quad \text{on } \mathbb{R}^2 \setminus V_{\rho_0}^{(n)}. \tag{82}$$

Next, for $p > p_N$, let $\{v_{n_k}\}_{k=1}^{\infty}$ be a subsequence of the minimizing sequence $\{v_n\}$ that converges weakly in $W_{loc}^{1,p}$ and strongly in C_{loc} to a limit v. Since $|v(x)| \geqslant \frac{1}{2}$ for $|x| \geqslant R_0$ by (82) we conclude that $v \in \mathcal{E}_1$. By lower semicontinuity

$$E_p(v) \leqslant \liminf_{n\to\infty} E_p(v_n) = I_p(1)$$

and hence, v is the desired minimizer. □

We conclude this section with the proof of Theorem 4.

Proof of Theorem 4. The arguments we use here are similar in nature to those employed in the proofs of Lemma 2.3 and Theorem 1. We first extract a bounded subsequence $\{u_{p_n}\}$ in $W^{1,q}(B(0,m))$ for some $q > N+1$ and any fixed integer m. Passing to a subsequence, we may assume that the subsequence converges weakly in $W^{1,q}(B(0,m))$ and strongly in $C(B(0,m))$ to a limit u_∞. Repeating the process for each m and different values of q and passing then to a diagonal subsequence yields a subsequence satisfying (9). The estimates (77) and (1)–(2) imply equicontinuity of the maps $\{u_{p_n}\}$ on $\mathbb{R}^N$. This implies, in conjunction with (73), as in the proof of Corollary 2.1 and Theorem 3, that there exist ρ_0, R_0 and a component $V_{\rho_0}^{(n)}$ of $A_{\rho_0}^{(n)} = \{|u_{p_n}(x)| < \rho_0\}$, such that the analog of (82) holds for u_{p_n}, namely

$$\deg\big(u_{p_n}, \partial V_{\rho_0}^{(n)}\big) = 1 \quad \text{and} \quad |u_{p_n}| \geqslant \frac{1}{2} \quad \text{on } \mathbb{R}^2 \setminus V_{\rho_0}^{(n)}.$$

It follows that the degree of the limit u_∞ equals to one as claimed. In addition the inequality

$$\|\nabla u_\infty\|_{L^\infty(\mathbb{R}^N)} \leqslant 1 \tag{83}$$

follows by an argument identical to the one used in the proof of (24).

Next, we attempt to obtain the explicit formulae in (10). As in the proof of Theorem 1 we denote by D_ρ the domain

$$D_\rho = \big\{x \in \mathbb{R}^N\colon \big|u_\infty(x)\big| < \rho\big\} \quad \forall \rho \in (0,1].$$

As in (29), we have

$$\int_{\mathbb{R}^N} \big(1 - |u_\infty|^2\big)^2 = \int_0^1 \mu\big(\big(1 - |u_\infty|^2\big)^2 > t\big)\, dt = \int_0^1 4\rho\big(1-\rho^2\big)\mu(D_\rho)\, d\rho. \tag{84}$$

Since $\deg(u_\infty) = 1$, using (83), Hadamard's inequality, and the AM-GM inequality as in (80) yields

$$\omega_N \rho^N \leqslant \bigg|\int_{D_\rho} \det(\nabla u_\infty)\bigg| \leqslant \int_{D_\rho} \big|\det(\nabla u_\infty)\big| \leqslant \int_{D_\rho} \prod_{j=1}^N \bigg|\frac{\partial u_\infty}{\partial x_j}\bigg| \leqslant \frac{1}{N^{N/2}} \int_{D_\rho} |\nabla u_\infty|^N \leqslant \frac{1}{N^{N/2}} \mu(D_\rho), \tag{85}$$

and hence,

$$\mu(D_\rho) \geqslant N^{N/2} \omega_N \rho^N. \tag{86}$$

On the other hand, the same argument as in the proof of Theorem 1 gives

$$\int_{\mathbb{R}^N} \big(1 - |u_\infty|^2\big)^2 = \lim_{n\to\infty} \int_{\mathbb{R}^N} \big(1 - |u_{p_n}|^2\big)^2 = 2\tau_N = \int_0^1 4\rho\big(1-\rho^2\big) N^{N/2} \omega_N \rho^N\, d\rho. \tag{87}$$

Combining (84) with (86)–(87) implies that

$$\mu(D_\rho) = N^{N/2} \omega_N \rho^N, \quad \rho < 1.$$

Thus equalities must hold between all integrals in (85), and hence also, almost everywhere, between the integrands. Consequently, the rows of the Jacobian matrix ∇u_∞ are orthogonal to each other a.e. in D_1, and each row has norm equal to $\sqrt{N}$ and the sign of $\det(\nabla u_\infty)$ must be constant (and hence positive because the degree of u_∞ is equal to 1). In particular we deduce that u_∞ is conformal in the sense that it is a *weak solution* of the Cauchy–Riemann system in D_1 as defined in [6, Chapter 5]. Namely, $u_\infty \in W^{1,N}_{loc}(D_1, \mathbb{R}^N)$ (in our case it belongs even to $W^{1,\infty}$), $\det(\nabla u_\infty)$ has constant sign in D_1 and

$$(\nabla u_\infty)^T \nabla u_\infty = \big(\det(\nabla u_\infty)\big)^{2/N} \mathbf{1} \quad \text{a.e. in } D_1. \tag{88}$$

The generalization of Liouville's theorem for this case (see [6, Chapter 5]) implies that u_∞ must be a "Mobius map", i.e., of the form

$$u_\infty(x) = b + \frac{\alpha \mathcal{U}(x-a)}{|x-a|^\epsilon} \tag{89}$$

for some $b \in \mathbb{R}^N$, $\alpha \in \mathbb{R}$, $a \in \mathbb{R}^N \setminus D_1$, $\mathcal{U}$ an orthogonal matrix and ϵ is either 0 or 2. However, since in our case we already know that

$$\big|\nabla u_\infty(x)\big| = 1 \quad \text{a.e. in } D_1, \tag{90}$$

it follows that $\epsilon = 0$ in (89). Using the fact that $u_\infty(0) = 0$ and $\det(\nabla u_\infty) > 0$ in conjunction with (90), leads to (10). From (90) we conclude that the inequality in (83) is, in fact, an equality and (11) readily follows. Finally, the uniform convergence of $|u_p|$ follows as in the case $N = 2$. □

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