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Multiple solutions of supercritical elliptic problems in perturbed domains ☆

Multiplicité de solutions de problèmes elliptiques surcritiques en domaines perturbés

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Abstract

We are concerned with existence and multiplicity of nontrivial solutions for the Dirichlet problem $\Delta u + |u|^{p-2}u = 0$ in Ω, $u = 0$ on $\partial\Omega$, where Ω is a bounded domain of $\mathbb{R}^n$, $n \geqslant 3$, and $p > \frac{2n}{n-2}$. We show that suitable perturbations of the domain, which modify its topological properties, give rise to a number of solutions which tends to infinity as the size of the perturbation tends to zero (some examples show that the perturbed domains may be even contractible). More precisely, we prove that for all $k \in \mathbb{N}$, if the size of the perturbation is small enough (depending on k), there exist at least k pairs of nontrivial solutions, which concentrate near the perturbation as the size of the perturbation tends to zero. The method we use, which is completely variational, gives also further informations on the qualitative properties of the solutions; in particular, these solutions (which may change sign) do not have more than k nodal regions and at least two solutions (which minimize the corresponding energy functional) have constant sign.

Résumé

Nous étudions l'existence et la multiplicité de solutions pour le problème de Dirichlet $\Delta u + |u|^{p-2}u = 0$, $u \not\equiv 0$ en Ω, $u = 0$ sur $\partial\Omega$, où Ω est un ouvert borné de $\mathbb{R}^n$, $n \geqslant 3$, et $p > \frac{2n}{n-2}$. Nous démontrons que l'existence et le nombre de solutions sont liés à certaines perturbations du domaine, qui modifient ses propriétés topologiques. Chaque perturbation dépend d'un paramètre ε (l'épaisseur de la perturbation) ; quand $\varepsilon \to 0$, le nombre de solutions tend à l'infini et les solutions se concentrent près de la perturbation (des examples montrent que les domaines perturbés peuvent même être contractiles). Plus précisément, nous prouvons que pour tout $k \in \mathbb{N}$ il existe $\varepsilon_k > 0$ tel que, pour tout $\varepsilon \in]0, \varepsilon_k[$, le problème a au moins k paires de solutions. La méthode que nous suivons, qui est complètement variationnelle, donne aussi des informations sur les propriétés qualitatives des solutions. En particulier, ces solutions (qui peuvent changer de signe) ne peuvent pas avoir plus que k régions nodales ; de plus, au moins deux solutions (qui minimisent la fonctionnelle de l'énergie) ont signe constant.

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1. Introduction

Let us consider the problem

$$P(\Omega)\qquad \begin{cases} \Delta u+|u|^{p-2}u=0, \quad u\not\equiv 0 & \text{in } \Omega,\\ u=0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$, $n\geqslant 3$, and $p>\frac{2n}{n-2}$ is a fixed exponent ($\frac{2n}{n-2}$ is the critical Sobolev exponent).

In this paper we show that suitable perturbations of a given domain give rise to solutions of the problem in the perturbed domains; when the size of the perturbation tends to zero, the number of the solutions tends to infinity while the solutions concentrate as Dirac masses near the perturbation; this perturbation is realized by removing a thin subset of the domain in such a way to modify its topological properties (see Theorem 2.1 and Remark 2.2).

It is well known that, when $2<p<\frac{2n}{n-2}$, $P(\Omega)$ has solutions in any domain Ω. On the contrary, when $p\geqslant\frac{2n}{n-2}$, the problem has no solution if Ω is starshaped (see [21]) while it has infinitely many solutions if Ω is, for example, an annulus (see [8]).

For $p=\frac{2n}{n-2}$, a sufficient condition for the existence of a positive solution is that Ω has nontrivial topology, in a suitable sense (see [1]); this condition is only sufficient but not necessary, as shown by some examples of contractible domains where the problem has positive solutions for $p=\frac{2n}{n-2}$ (see [5,7,16,18]). For $p>\frac{2n}{n-2}$, this nontriviality condition of the domain is neither a sufficient nor a necessary condition; in fact, nonexistence results hold for some $p>\frac{2n}{n-2}$ in some nontrivial domains (see [17,19]) while an arbitrarily large number of solutions can be obtained in some contractible domains for all $p>\frac{2n}{n-2}$ (see [20]).

It is also worth pointing out that existence of solutions in the supercritical case has been proved even in some "nearly starshaped" domains (see the definition introduced in [11]) for p sufficiently close to $\frac{2n}{n-2}$ (see [10,12,13]) or for p large enough (see [14]); on the other hand, a different definition of "nearly starshaped" domain is used in [6] in order to extend Pohožaev's nonexistence result to nonstarshaped domains when p is large enough.

Let us observe that in [20] several perturbations have been used in order to obtain several solutions in contractible domains. On the contrary, the result proved in the present paper shows that a unique perturbation can produce an arbitrarily large number of solutions.

In order to overcome the difficulties related to the presence of the supercritical exponent, we proceed as follows: we modify the nonlinear term $|u|^{p-2}u$ in such a way that the modified nonlinearity $g(x,u)$ has subcritical growth outside a neighbourhood of the perturbation; then (exploiting also the symmetry of the domain with respect to an axis) we use topological methods of Calculus of Variations to find multiple solutions of the modified problem in the perturbed domain; we also show that, as the size of the perturbation tends to zero, these solutions concentrate as Dirac masses near the perturbation (where the nonlinear term has not been modified); finally, we prove that these solutions solve also the unmodified problem when the size of the perturbation is small enough.

2. Notations and statement of the main result

Let Ω be a bounded domain of $\mathbb{R}^n$, having radial symmetry with respect to the x_n-axis, that is

$$x=(x_1,\dots,x_n)\in\Omega \quad\Longleftrightarrow\quad \left(\left(\sum_{i=1}^{n-1}x_i^2\right)^{1/2},0,\dots,0,x_n\right)\in\Omega. \tag{2.1}$$

Let $x^1=(0,\dots,0,x_n^1)$, $x^2=(0,\dots,0,x_n^2)$, $x^3=(0,\dots,0,x_n^3)$ be three points of the x_n-axis which satisfy

$$x_n^1<x_n^2<x_n^3,\quad x^1\notin\overline{\Omega},\ x^2\in\Omega,\ x^3\notin\overline{\Omega}. \tag{2.2}$$

For all $\varepsilon > 0$, set

$$\chi_\varepsilon = \left\{ x \in \mathbb{R}^n \colon \sum_{i=1}^{n-1} x_i^2 < \varepsilon^2,\ x_n^1 < x_n < x_n^3 \right\} \tag{2.3}$$

and

$$\Omega_\varepsilon = \Omega \setminus \bar{\chi}_\varepsilon. \tag{2.4}$$

Since the domain Ω_ε has radial symmetry with respect to the x_n-axis for all $\varepsilon > 0$, it is natural to look for solutions of problem $P(\Omega_\varepsilon)$ among the functions which have the same symmetry. Thus, we consider the subsets of $H_0^{1,2}(\Omega_\varepsilon)$, $H_0^{1,2}(\Omega)$ which consist of the functions radially symmetric with respect to the x_n-axis and denote by $H_S(\Omega_\varepsilon)$, $H_S(\Omega)$ these subspaces. Moreover, we intend that every function of $H_0^{1,2}(\Omega_\varepsilon)$ is extended in all of Ω by the value zero outside Ω_ε. In these spaces we shall use the norms

$$\|Du\|_2 = \left(\int_\Omega |Du|^2 \,\mathrm{d}x \right)^{1/2}$$

and

$$\|u\|_q = \left(\int_\Omega |u|^q \,\mathrm{d}x \right)^{1/q} \quad \forall u \in L^q(\Omega), \quad q \geqslant 1.$$

Theorem 2.1. *Let $p > \frac{2n}{n-2}$, Ω be a bounded domain radially symmetric with respect to the x_n-axis, x^1, x^2, x^3 be three points of the x_n-axis satisfying condition* (2.2) *and, for all $\varepsilon > 0$, χ_ε and Ω_ε be the domains defined in* (2.3) *and* (2.4)*. Then, for all positive integer k, there exists $\bar{\varepsilon}_k > 0$ such that, for all $\varepsilon \in]0, \bar{\varepsilon}_k[$, problem $P(\Omega_\varepsilon)$ has at least k pairs of solutions $\pm u_{1,\varepsilon}, \dots, \pm u_{k,\varepsilon}$ in $H_S(\Omega_\varepsilon) \cap L^p(\Omega_\varepsilon)$, which, for all $i = 1, \dots, k$, satisfy*:

(a) $\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |Du_{i,\varepsilon}|^2 \,\mathrm{d}x = 0$,
(b) $\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon} |u_{i,\varepsilon}|^p \,\mathrm{d}x = 0$,
(c) $\lim_{\varepsilon \to 0} \sup\{|u_{i,\varepsilon}(x)| \colon x \in \Omega_\varepsilon \setminus \chi_\rho\} = 0\ \forall \rho > 0$,
(d) $\lim_{\varepsilon \to 0} \sup\{|u_{i,\varepsilon}(x)| \colon x \in \Omega_\varepsilon \cap \chi_\rho\} = +\infty\ \forall \rho > 0$,
(e) $\lim_{\varepsilon \to 0} \int_{\Omega_\varepsilon \cap \chi_\rho} |u_{i,\varepsilon}|^q \,\mathrm{d}x = +\infty\ \forall q > \frac{n}{2}(p-1)$, $\forall \rho > 0$.

The proof is reported in Section 3, where also other properties of the solutions are specified (see Propositions 3.8 and 3.9).

Remark 2.2. A result analogous to Theorem 2.1 holds in some more general perturbed domains. In fact, let Ω, x^1, x^2, x^3 be as in Theorem 2.1; consider a sequence of subdomains Ω_j of Ω, satisfying the radial symmetry condition (2.1) for all $j \in \mathbb{N}$; now, set

$$\operatorname{cap}(\Omega \setminus \Omega_j) = \inf\left\{ \int_{\mathbb{R}^n} |Du|^2 \,\mathrm{d}x \colon u \in C_0^\infty(\mathbb{R}^n),\ u(x) \geqslant 1\ \forall x \in \Omega \setminus \Omega_j \right\} \tag{2.5}$$

and assume that

(a) $\lim_{j \to \infty} \operatorname{cap}(\Omega \setminus \Omega_j) = 0$,
(b) for all $j \in \mathbb{N}$ there exists $\rho_j > 0$ such that $\chi_{\rho_j} \cap \Omega_j = \emptyset$.

Then, arguing as in the proof of Theorem 2.1, one can prove that, for all $p > \frac{2n}{n-2}$, problem $P(\Omega_j)$, for j large enough, has at least k pairs of solutions $\pm u_{1,j}, \dots, \pm u_{k,j}$ in $H_S(\Omega_j) \cap L^p(\Omega_j)$, which, as $j \to \infty$, present the same asymptotic behaviour that the solutions given by Theorem 2.1 present as $\varepsilon \to 0$ (see (a)–(e) in Theorem 2.1 as well as the properties described in Propositions 3.8 and 3.9).

Finally, let us remark that the methods we use in this paper can be easily adapted to deal with more general nonlinear terms having supercritical growth.

Examples 2.3. Theorem 2.1 allows us to obtain an arbitrarily large number of solutions even in some contractible domains. Consider, for example, the domain $A \setminus \bar{\chi}_\varepsilon$ (a pierced annulus) where A is the annulus

$$A = \{x \in \mathbb{R}^n\colon\ 0 < r_1 < |x| < r_2\}$$

and χ_ε is the cylinder defined as in (2.3) with $x_n^1 = 0$ and $x_n^3 > r_2$. It is clear that, because of this choice of x^1 and x^3, $A \setminus \bar{\chi}_\varepsilon$ is a contractible domain for all $\varepsilon > 0$; moreover, Theorem 2.1 applies when $\varepsilon > 0$ is small enough and guarantees the existence of an arbitrarily large number of solutions. As $\varepsilon \to 0$, the number of solutions tends to infinity and the solutions concentrate near points of the x_n-axis. Notice that these solutions do not converge as $\varepsilon \to 0$ to solutions of the problem in the limit domain (i.e. the annulus A where, on the other hand, it is easy to find infinitely many radial solutions).

Let us remark that, in order to have an arbitrarily large number of solutions in contractible domains, in [20] some domains Ω_ε^h of the following form have been considered: $\Omega_\varepsilon^h = \Omega^h \setminus \bigcup_{j=1}^h \bar{\chi}_\varepsilon^j$ where χ_ε^j, for $j = 1, \dots, h$, is the cylinder of the form (2.3) with $x_n^1 = j$ and $x_n^3 = j + 1$ and $\Omega^h = T^h \setminus \bigcup_{j=1}^h \overline{B(c_j, r_j)}$, where

$$T^h = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n\colon\ \sum_{i=1}^{n-1} x_i^2 < 1,\ 0 < x_n < h + 1 \right\} \tag{2.6}$$

and $B(c_j, r_j)$ is the ball of radius $r_j \in]0, 1/2[$ and centre $c_j = (0, \dots, 0, j) \in \mathbb{R}^n$. In [20] it is proved that for all $j = 1, \dots, h$ there exists a solution of $P(\Omega_\varepsilon^h)$ whose positive or negative part concentrates as $\varepsilon \to 0$ near the cylinder χ_ε^j. Now, it is clear that Theorem 2.1 can be applied in the domain Ω^h with h separate terns of points x^1, x^2, x^3 satisfying condition (2.2); so, for each $j = 1, \dots, h$, for ε small enough, we obtain many solutions localized near χ_ε^j, whose number tends to infinity as $\varepsilon \to 0$.

Notice that, taking into account Remark 2.2, similar multiplicity results can be obtained also in contractible domains Ω_r^s of the form

$$\Omega_r^s = \left\{ x = (x_1, \dots, x_n) \in \mathbb{R}^n\colon\ 1 < |x| < r,\ \left(\sum_{i=1}^{n-1} x_i^2 \right)^{1/2} > s x_n \right\} \tag{2.7}$$

for all $r > 1$, when $s > 0$ is small enough.

Domains of this type have been also considered in [10,12–14]. When $p > \frac{2n}{n-2}$ is a fixed exponent, the concentration phenomena, which allow us to obtain the multiplicity result proved in the present paper, occur as $s \to 0$; on the contrary, other different concentration phenomena, which arise as $p \to \frac{2n}{n-2}$ (see [10,12,13]) or $p \to +\infty$ (see [14]), have been used to find solutions for all $r > 1$ and $s > 0$, not necessarily small, when p is close to $\frac{2n}{n-2}$ or it is large enough.

In the case $p \leqslant \frac{2n}{n-2}$, the solutions of $P(\Omega)$ are obtained as critical points of the functional

$$\tilde{f}(u) = \frac{1}{2} \int_\Omega |Du|^2 \,\mathrm{d}x - \frac{1}{p} \int_\Omega |u|^p \,\mathrm{d}x.$$

In the case $p > \frac{2n}{n-2}$, there are some difficulties in dealing with this functional. Therefore, we introduce the following modified functional $f_\varepsilon : H_S(\Omega_\varepsilon) \to \mathbb{R}$, defined by

$$f_\varepsilon(u) = \frac{1}{2} \int_{\Omega_\varepsilon} |Du|^2 \,\mathrm{d}x - \int_{\Omega_\varepsilon} G(x, u) \,\mathrm{d}x, \tag{2.8}$$

where $G(x, u) = \int_0^u g(x, t) \,\mathrm{d}t$ with $g(x, t)$ defined as follows. For all $x \in \Omega \cap \bar{\chi}_1$ we set $g(x, t) = |t|^{p-2} t$. If $\Omega \setminus \bar{\chi}_1 \neq \emptyset$, in order to define $g(x, t)$ for $x \in \Omega \setminus \bar{\chi}_1$, we first remark that

$$\inf \left\{ \int_{\Omega \setminus \bar{\chi}_1} |Du|^2 \,\mathrm{d}x\colon\ u \in H_0^{1,2}(\Omega),\ \int_{\Omega \setminus \bar{\chi}_1} u^2 \,\mathrm{d}x = 1 \right\} > 0; \tag{2.9}$$

then we choose $t_0 > 0$ small enough such that

$$(p-1)t_0^{p-2} < \inf\left\{ \int_{\Omega\setminus\bar\chi_1} |Du|^2\,\mathrm{d}x:\ u \in H_0^{1,2}(\Omega),\ \int_{\Omega\setminus\bar\chi_1} u^2\,\mathrm{d}x = 1 \right\}; \tag{2.10}$$

hence, for all $x \in \Omega \setminus \bar\chi_1$, we set

$$g(x,t) = \begin{cases} |t|^{p-2}t & \text{if } |t| \leqslant t_0,\\ t_0^{p-1} + (p-1)t_0^{p-2}(t-t_0) & \text{if } t \geqslant t_0,\\ -t_0^{p-1} + (p-1)t_0^{p-2}(t+t_0) & \text{if } t \leqslant -t_0. \end{cases} \tag{2.11}$$

Let us remark that this choice of t_0 implies the existence of a constant $\tilde c > 0$ such that

$$\int_{\Omega\setminus\bar\chi_1} \left[|Du|^2 - g(x,u)u\right]\mathrm{d}x \geqslant \tilde c \int_{\Omega\setminus\bar\chi_1} |Du|^2\,\mathrm{d}x \quad \forall u \in H_0^{1,2}(\Omega), \tag{2.12}$$

as one can easily verify.

Notice that all nontrivial critical points of f_ε belong to the set

$$M_\varepsilon = \left\{u \in H_S(\Omega_\varepsilon):\ u \not\equiv 0,\ f'_\varepsilon(u)[u] = 0\right\}. \tag{2.13}$$

The solutions of problem $P(\Omega_\varepsilon)$ will be obtained, for $\varepsilon > 0$ small enough, as critical points for f_ε constrained on M_ε.

3. Proof of Theorem 2.1 and behaviour of the solutions

Lemma 3.1. *For all $\varepsilon > 0$, the functional f_ε (see (2.8)) is well defined and belongs to the class C^2. Moreover, the following properties hold.*

(a) *Let $u \in H_S(\Omega_\varepsilon)$, $u \not\equiv 0$; then either $f'_\varepsilon(tu)[u] > 0$ $\forall t > 0$ (what happens, for example, if $u \equiv 0$ in χ_1) or there exists a unique $\bar t > 0$ such that $\bar t u \in M_\varepsilon$; in the second case, $f'_\varepsilon(tu)[u] > 0$ $\forall t \in]0, \bar t[$ and $f'_\varepsilon(tu)[u] < 0$ $\forall t > \bar t$ (the second case occurs, for example, if $u \equiv 0$ in $\Omega_\varepsilon \setminus \chi_1$).*
(b) *For all $\varepsilon > 0$ such that $M_\varepsilon \neq \emptyset$, we have $\inf_{M_\varepsilon} f_\varepsilon > 0$.*
(c) *There exists $\bar\varepsilon > 0$ such that $M_\varepsilon \neq \emptyset$ $\forall \varepsilon \in]0, \bar\varepsilon[$.*
(d) $\lim_{\varepsilon\to 0} \inf_{M_\varepsilon} f_\varepsilon = 0$.
(e) *If $M_\varepsilon \neq \emptyset$, then M_ε is a C^1-manifold of codimension 1.*
(f) *Every critical point for f_ε constrained on M_ε is a critical point for f_ε.*

Proof. First, let us remark that the functionals

$$u \longrightarrow \int_{\Omega_\varepsilon\setminus\bar\chi_1} G(x,u)\,\mathrm{d}x \quad \text{and} \quad u \longrightarrow \int_{\Omega_\varepsilon\cap\chi_1} G(x,u)\,\mathrm{d}x, \tag{3.1}$$

defined respectively in $L^2(\Omega_\varepsilon \setminus \bar\chi_1)$ and in $L^p(\Omega_\varepsilon \cap \chi_1)$, are C^2-functionals. Moreover, notice that

$$\int_{\Omega_\varepsilon\setminus\bar\chi_1} u^2\,\mathrm{d}x \leqslant \bar c \int_{\Omega_\varepsilon} |Du|^2\,\mathrm{d}x \quad \forall u \in H_0^{1,2}(\Omega_\varepsilon) \tag{3.2}$$

for a suitable constant $\bar c > 0$ and, for all $\varepsilon > 0$, there exists $\bar c_\varepsilon > 0$ such that

$$\left(\int_{\Omega_\varepsilon\cap\chi_1} |u|^p\,\mathrm{d}x\right)^{2/p} \leqslant \bar c_\varepsilon \int_{\Omega_\varepsilon} |Du|^2\,\mathrm{d}x \quad \forall u \in H_S(\Omega_\varepsilon). \tag{3.3}$$

Hence, by standard arguments, it is easy to verify that f_ε is a well defined C^2-functional in $H_S(\Omega_\varepsilon)$.

(a) Notice that, for all $t > 0$,

$$f'_\varepsilon(tu)[u] = t\left[\int_{\Omega_\varepsilon} |Du|^2\,dx - \int_{\Omega_\varepsilon} \frac{g(x,tu)}{t} u\,dx\right],$$

where (by the definition of g) the function $t \mapsto \int_{\Omega_\varepsilon} \frac{g(x,tu)}{t} u\,dx$ is strictly increasing in $]0,+\infty[$ for all $u \not\equiv 0$. Then the assertion (a) follows, taking also into account that $g(x,tu)u = t^{p-1}|u|^p$ if $u \equiv 0$ in $\Omega_\varepsilon \setminus \chi_1$ while $g(x,tu)u \leqslant (p-1)t_0^{p-2}tu^2$ if $u \equiv 0$ in χ_1.

(b) From property (a) we infer that, for all $u \in M_\varepsilon$,

$$f_\varepsilon(u) \geqslant f_\varepsilon(tu) \quad \forall t \geqslant 0. \tag{3.4}$$

On the other hand, we have $f'_\varepsilon(0) = 0$ and $f''_\varepsilon(0)[u,u] = \int_{\Omega_\varepsilon} |Du|^2\,dx$. Therefore there exists $r > 0$ and $\alpha > 0$ such that

$$\inf\left\{ f_\varepsilon(u)\colon\ u \in H_S(\Omega_\varepsilon),\ \int_{\Omega_\varepsilon} |Du|^2\,dx = r^2 \right\} \geqslant \alpha.$$

It follows that

$$\inf_{M_\varepsilon} f_\varepsilon \geqslant \alpha > 0.$$

(c) Since $x^2 \in \Omega$, there exists $\rho_2 > 0$ such that $B(x^2,\rho_2) \subset \Omega$. Choose $\bar\varphi \in H_S(B(x^2,\rho_2) \setminus \bar\chi_{\rho_2/2})$ and, for all $\varepsilon > 0$, set

$$\bar\varphi_\varepsilon(x) = \bar\varphi\left(x^2 + \frac{\rho_2}{2\varepsilon}\left(x - x^2\right)\right) \tag{3.5}$$

(we intend that $\bar\varphi$ is extended by zero outside $B(x^2,\rho_2) \setminus \bar\chi_{\rho_2/2}$). Then, it is easy to verify that $\bar\varphi_\varepsilon \in H_S(\Omega_\varepsilon)$ for $\varepsilon \in]0,\rho_2/2[$ and $\bar\varphi_\varepsilon \equiv 0$ in $\Omega_\varepsilon \setminus \chi_1$ for $\varepsilon \in]0,1/2[$; thus, for $0 < \varepsilon < \min\{1/2,\rho_2/2\}$, there exists $\bar t_\varepsilon > 0$ such that $\bar t_\varepsilon \bar\varphi_\varepsilon \in M_\varepsilon$.

(d) For the proof of (d), it suffices to observe that, since $p > \frac{2n}{n-2}$,

$$\lim_{\varepsilon \to 0} f_\varepsilon(\bar t_\varepsilon \bar\varphi_\varepsilon) = 0 \tag{3.6}$$

as one can easily verify by a direct computation.

(e) Let us set

$$F_\varepsilon(u) = f'_\varepsilon(u)[u] \tag{3.7}$$

(notice that F_ε is a C^1-functional in $H_S(\Omega_\varepsilon)$). We shall prove that $F'_\varepsilon(u) \neq 0$ for all $u \in M_\varepsilon$, so the assertion will follow by the implicit function theorem. In fact, if $u \in M_\varepsilon$, then $u \not\equiv 0$ and $F_\varepsilon(u) = 0$, that is

$$\int_{\Omega_\varepsilon} |Du|^2\,dx - \int_{\Omega_\varepsilon} g(x,u)u\,dx = 0.$$

Therefore, we have

$$\begin{aligned} F'_\varepsilon(u)[u] &= 2\int_{\Omega_\varepsilon} |Du|^2\,dx - \int_{\Omega_\varepsilon} g'(x,u)u^2\,dx - \int_{\Omega_\varepsilon} g(x,u)u\,dx \\ &= \int_{\Omega_\varepsilon} \left[g(x,u)u - g'(x,u)u^2\right]dx \end{aligned}$$

(here $g'(x,t)$ denotes the derivative of $g(x,t)$ with respect to t).

Since $g(x,t)t - g'(x,t)t^2 < 0$ $\forall t \neq 0$ and since $u \not\equiv 0$, we infer that $F'_\varepsilon(u)[u] < 0$ and so $F'_\varepsilon(u) \neq 0$.

(f) If $u \in M_\varepsilon$ is a critical point for f_ε constrained on M_ε, there exists a Lagrange multiplier μ such that

$$f'_\varepsilon(u) + \mu F'_\varepsilon(u) = 0. \tag{3.8}$$

In particular, we have

$$f'_\varepsilon(u)[u] + \mu F'_\varepsilon(u)[u] = 0, \tag{3.9}$$

which implies $\mu = 0$ since $f'_\varepsilon(u)[u] = 0$ (because $u \in M_\varepsilon$) and $F'_\varepsilon(u)[u] \neq 0$. Therefore, $f'_\varepsilon(u) = 0$. $\square$

Thus, finding nontrivial critical points for f_ε is equivalent to finding critical points for f_ε constrained on M_ε. The compactness property proved in the following lemma plays an important role to find critical points for f_ε constrained on M_ε.

Lemma 3.2. *Let $\varepsilon > 0$ be such that $M_\varepsilon \neq \emptyset$ and $(u_i)_i$ be a sequence in M_ε. The following properties hold.*

(a) *If $\sup_{i\in\mathbb{N}} f_\varepsilon(u_i) < +\infty$, then the sequence $(u_i)_i$ is bounded in $H_0^{1,2}(\Omega_\varepsilon)$.*

(b) *Let us set $F_\varepsilon(u) = f'_\varepsilon(u)[u]$; if $u_i \rightharpoonup u$ weakly in $H_0^{1,2}(\Omega_\varepsilon)$ and there exists a sequence $(\mu_i)_i$ in $\mathbb{R}$ such that*

$$f'_\varepsilon(u_i) + \mu_i F'_\varepsilon(u_i) \longrightarrow 0 \quad \text{in } H^{-1}(\Omega_\varepsilon), \tag{3.10}$$

then $\lim_{i\to\infty} \mu_i = 0$, $u \in M_\varepsilon$ and $u_i \to u$ in $H_0^{1,2}(\Omega_\varepsilon)$.

(c) *The functional f_ε constrained on M_ε satisfies the Palais–Smale condition at any level $c \in \mathbb{R}$, i.e. every sequence $(u_i)_i$ in M_ε, such that*

$$\lim_{i\to\infty} f_\varepsilon(u_i) = c \quad \text{and} \quad \operatorname{grad}_{M_\varepsilon} f_\varepsilon(u_i) \to 0 \quad \text{in } H^{-1}(\Omega_\varepsilon), \tag{3.11}$$

is relatively compact in $H_0^{1,2}(\Omega_\varepsilon)$.

Proof. (a) Since $u_i \in M_\varepsilon$, taking into account the definition of $G(x,t)$, we have

$$\begin{aligned} f_\varepsilon(u_i) &= \frac{1}{2}\int_{\Omega_\varepsilon} |Du_i|^2\,\mathrm{d}x - \int_{\Omega_\varepsilon} G(x,u_i)\,\mathrm{d}x \\ &\geqslant \frac{1}{2}\int_{\Omega_\varepsilon} |Du_i|^2\,\mathrm{d}x - \frac{1}{2}\int_{\Omega_\varepsilon\setminus\bar\chi_1} g(x,u_i)u_i\,\mathrm{d}x - \frac{1}{p}\int_{\Omega_\varepsilon\cap\chi_1} |u_i|^p\,\mathrm{d}x \\ &= \left(\frac{1}{2}-\frac{1}{p}\right)\int_{\Omega_\varepsilon} |Du_i|^2\,\mathrm{d}x - \left(\frac{1}{2}-\frac{1}{p}\right)\int_{\Omega_\varepsilon\setminus\bar\chi_1} g(x,u_i)u_i\,\mathrm{d}x. \end{aligned}$$

It follows that

$$\sup_{i\in\mathbb{N}}\left\{\int_{\Omega_\varepsilon} |Du_i|^2\,\mathrm{d}x - \int_{\Omega_\varepsilon\setminus\bar\chi_1} g(x,u_i)u_i\,\mathrm{d}x\right\} < +\infty. \tag{3.12}$$

Hence (3.12) and (2.12) imply $\sup_{i\in\mathbb{N}}\int_{\Omega_\varepsilon} |Du_i|^2\,\mathrm{d}x < +\infty$.

(b) First, let us prove that $u \not\equiv 0$. Notice that

$$\lim_{i\to\infty}\int_{\Omega_\varepsilon\setminus\bar\chi_1} |u_i - u|^2\,\mathrm{d}x = 0; \tag{3.13}$$

moreover, since $\varepsilon > 0$ and $u_i \in H_S(\Omega_\varepsilon)$,

$$\lim_{i\to\infty}\int_{\Omega_\varepsilon\cap\chi_1} |u_i - u|^p\,\mathrm{d}x = 0. \tag{3.14}$$

Notice that $u_i \in M_\varepsilon$ implies

$$\int_{\Omega_\varepsilon\setminus\bar\chi_1} \left[|Du_i|^2 - g(x,u_i)u_i\right]\mathrm{d}x + \int_{\Omega_\varepsilon\cap\chi_1} \left[|Du_i|^2 - |u_i|^p\right]\mathrm{d}x = 0. \tag{3.15}$$

Let us observe that we must have

$$\int_{\Omega_\varepsilon \cap \chi_1} |u_i|^p \,\mathrm{d}x > 0 \quad \forall i \in \mathbb{N}, \tag{3.16}$$

otherwise, since $u_i \not\equiv 0$ in Ω_ε, we should have

$$\int_{\Omega_\varepsilon \setminus \bar{\chi}_1} \left[|Du_i|^2 - g(x, u_i)u_i\right] \mathrm{d}x > 0,$$

because of the choice of t_0 (see (2.10)), which contradicts (3.15).

Since

$$\int_{\Omega_\varepsilon \setminus \bar{\chi}_1} \left[|Du|^2 - g(x, u)u\right] \mathrm{d}x \geqslant 0 \quad \forall u \in H_0^{1,2}(\Omega_\varepsilon), \tag{3.17}$$

we infer from (3.15)

$$\int_{\Omega_\varepsilon \cap \chi_1} |u_i|^p \,\mathrm{d}x \geqslant \int_{\Omega_\varepsilon \cap \chi_1} |Du_i|^2 \,\mathrm{d}x.$$

On the other hand, as $\varepsilon > 0$ and $u_i \in H_S(\Omega_\varepsilon)$, there exists a constant $\bar{c} > 0$ such that

$$\int_{\Omega_\varepsilon \cap \chi_1} |Du|^2 \,\mathrm{d}x \geqslant \bar{c}\Bigg(\int_{\Omega_\varepsilon \cap \chi_1} |u|^p \,\mathrm{d}x\Bigg)^{2/p} \quad \forall u \in H_S(\Omega_\varepsilon). \tag{3.18}$$

Hence we get

$$\int_{\Omega_\varepsilon \cap \chi_1} |u_i|^p \,\mathrm{d}x \geqslant \int_{\Omega_\varepsilon \cap \chi_1} |Du_i|^2 \,\mathrm{d}x \geqslant \bar{c}\Bigg(\int_{\Omega_\varepsilon \cap \chi_1} |u_i|^p \,\mathrm{d}x\Bigg)^{2/p} \quad \forall i \in \mathbb{N},$$

which, because of (3.16), implies

$$\inf_{i \in \mathbb{N}} \int_{\Omega_\varepsilon \cap \chi_1} |u_i|^p \,\mathrm{d}x > 0.$$

Taking into account (3.14), it follows that

$$\int_{\Omega_\varepsilon \cap \chi_1} |u|^p \,\mathrm{d}x > 0. \tag{3.19}$$

Notice that, as $u_i \in M_\varepsilon$, (3.10) implies

$$\lim_{i \to \infty} \mu_i F'_\varepsilon(u_i)[u_i] = 0. \tag{3.20}$$

Moreover, $u_i \in M_\varepsilon$ implies

$$F'_\varepsilon(u_i)[u_i] = 2\int_{\Omega_\varepsilon} |Du_i|^2 \,\mathrm{d}x - \int_{\Omega_\varepsilon} g'(x, u_i)u_i^2 \,\mathrm{d}x - \int_{\Omega_\varepsilon} g(x, u_i)u_i \,\mathrm{d}x = \int_{\Omega_\varepsilon} \left[g(x, u_i)u_i - g'(x, u_i)u_i^2\right] \mathrm{d}x. \tag{3.21}$$

Hence, from (3.13) and (3.14) we infer that

$$\lim_{i \to \infty} F'_\varepsilon(u_i)[u_i] = \int_{\Omega_\varepsilon} \left[g(x, u)u - g'(x, u)u^2\right] \mathrm{d}x. \tag{3.22}$$

Notice that

$$\int_{\Omega_\varepsilon} \left[g(x, u)u - g'(x, u)u^2\right] \mathrm{d}x < 0$$

because $g(x,t)t - g'(x,t)t^2 < 0\ \forall t \neq 0$ and $u \not\equiv 0$ in Ω_ε (as (3.19) holds). Taking into account (3.20), it follows that

$$\lim_{i\to\infty} \mu_i = 0. \tag{3.23}$$

Using again (3.10), we obtain

$$\lim_{i\to\infty} \left\{ f'_\varepsilon(u_i)[u-u_i] + \mu_i F'_\varepsilon(u_i)[u-u_i] \right\} = 0, \tag{3.24}$$

which, taking into account (3.13), (3.14), (3.20), (3.23) and the weak convergence of u_i, implies

$$\lim_{i\to\infty} \int_{\Omega_\varepsilon} |Du_i|^2\,\mathrm{d}x = \int_{\Omega_\varepsilon} |Du|^2\,\mathrm{d}x.$$

Thus, $u_i \to u$ strongly in $H_0^{1,2}(\Omega_\varepsilon)$ and $u \in M_\varepsilon$.

(c) Property (a) implies that every Palais–Smale sequence $(u_i)_i$ is bounded in $H_0^{1,2}(\Omega_\varepsilon)$; hence, up to a subsequence, it converges weakly in $H_0^{1,2}(\Omega_\varepsilon)$; then, property (b) guarantees the strong convergence in $H_0^{1,2}(\Omega_\varepsilon)$ to a function $u \in M_\varepsilon$. $\square$

Lemma 3.3. *Let $(\varepsilon_i)_i$ be a sequence of positive numbers and $(u_i)_i$ a sequence of functions in $H_S(\Omega_{\varepsilon_i})$ such that $f'_{\varepsilon_i}(u_i) = 0$ and $u_i \not\equiv 0$ in Ω_{ε_i} for all $i \in \mathbb{N}$.*

If $\lim_{i\to\infty} f_{\varepsilon_i}(u_i) = 0$, then

(a) $\lim_{i\to\infty} \int_{\Omega_{\varepsilon_i}} |Du_i|^2\,\mathrm{d}x = 0$,
(b) $\lim_{i\to\infty} \varepsilon_i = 0$,
(c) $\lim_{i\to\infty} \sup\{|u_i(x)|\colon x \in \Omega_{\varepsilon_i} \setminus \chi_\rho\} = 0\ \forall \rho > 0$.

Proof. (a) Clearly, it suffices to prove that there exists a constant $\bar{c} > 0$ for which

$$f_\varepsilon(u) \geqslant \bar{c} \int_{\Omega_\varepsilon} |Du|^2\,\mathrm{d}x \tag{3.25}$$

for all $\varepsilon > 0$ and for all $u \in H_S(\Omega_\varepsilon)$ such that $f'_\varepsilon(u) = 0$.

If $f'_\varepsilon(u) = 0$, arguing as in the proof of Lemma 3.2, we have

$$\begin{aligned} f_\varepsilon(u) &\geqslant \frac{1}{2}\int_{\Omega_\varepsilon} |Du|^2\,\mathrm{d}x - \frac{1}{2}\int_{\Omega_\varepsilon\setminus\bar{\chi}_1} g(x,u)u\,\mathrm{d}x - \frac{1}{p}\int_{\Omega_\varepsilon\cap\chi_1} |u|^p\,\mathrm{d}x \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\left\{ \int_{\Omega_\varepsilon\setminus\bar{\chi}_1} \left[|Du|^2 - g(x,u)u\right]\mathrm{d}x + \int_{\Omega_\varepsilon\cap\chi_1} |Du|^2\,\mathrm{d}x \right\}. \end{aligned} \tag{3.26}$$

Hence, the existence of a constant $\bar{c} > 0$, such that (3.25) holds, follows easily from (2.12).

(b) Arguing by contradiction, assume that, up to a subsequence, $\inf_{i\in\mathbb{N}} \varepsilon_i > 0$ and choose $\bar{\varepsilon} \in\ ]0, \inf_{i\in\mathbb{N}} \varepsilon_i[$. Thus, $u_i \in H_S(\Omega_{\bar{\varepsilon}})$ for all $i \in \mathbb{N}$. Since $\bar{\varepsilon} > 0$, there exists a constant $\bar{c}_{\bar{\varepsilon}} > 0$ such that

$$\int_{\Omega_{\bar{\varepsilon}}\cap\chi_1} |Du|^2\,\mathrm{d}x \geqslant \bar{c}_{\bar{\varepsilon}} \left(\int_{\Omega_{\bar{\varepsilon}}\cap\chi_1} |u|^p\,\mathrm{d}x \right)^{2/p} \quad \forall u \in H_S(\Omega_{\bar{\varepsilon}}). \tag{3.27}$$

Hence, property (a) implies

$$\lim_{i\to\infty} \int_{\Omega_{\bar{\varepsilon}}\cap\chi_1} |u_i|^p\,\mathrm{d}x = 0. \tag{3.28}$$

On the other hand, we have $f'_{\bar{\varepsilon}}(u_i)[u_i] = 0$, that is

$$\int_{\Omega_{\bar{\varepsilon}}\cap\chi_1} \left[|Du_i|^2 - |u_i|^p\right]\mathrm{d}x + \int_{\Omega_{\bar{\varepsilon}}\setminus\bar{\chi}_1} \left[|Du_i|^2 - g(x,u_i)u_i\right]\mathrm{d}x = 0. \tag{3.29}$$

First, let us remark that (3.29) implies

$$\int_{\Omega_{\bar\varepsilon}\cap\chi_1} |u_i|^p\,\mathrm{d}x > 0 \quad \forall i \in \mathbb{N} \tag{3.30}$$

otherwise, as $u_i \not\equiv 0$ in $\Omega_{\bar\varepsilon}$, we should have (because of (2.12))

$$\int_{\Omega_{\bar\varepsilon}\setminus\bar\chi_1} \left[|Du_i|^2 - g(x,u_i)u_i\right]\mathrm{d}x > 0$$

and so (3.29) cannot hold.

Moreover, from (3.29) and (2.12) we infer that

$$\int_{\Omega_{\bar\varepsilon}\cap\chi_1} |u_i|^p\,\mathrm{d}x \geqslant \int_{\Omega_{\bar\varepsilon}\cap\chi_1} |Du_i|^2\,\mathrm{d}x \quad \forall i \in \mathbb{N}. \tag{3.31}$$

Therefore, taking into account (3.27), (3.30) and (3.31), we get

$$\inf_{i\in\mathbb{N}} \int_{\Omega_{\bar\varepsilon}\cap\chi_1} |u_i|^p\,\mathrm{d}x > 0, \tag{3.32}$$

which contradicts (3.28).

(c) For all $\rho > 0$, let us consider the cylinder

$$C_\rho = \left\{x = (x_1,\dots,x_n) \in \mathbb{R}^n:\ \sum_{i=1}^{n-1} x_i^2 \leqslant \rho^2\right\}.$$

Since the subspace of the radial functions in $H^1(\Omega\setminus C_\rho)$ is embedded in $L^q(\Omega\setminus C_\rho)$ for all $\rho > 0$ and $q \geqslant 1$, property (a) implies

$$\lim_{i\to\infty} \int_{\Omega_{\varepsilon_i}\setminus C_\rho} |u_i|^q\,\mathrm{d}x = 0 \quad \forall q \geqslant 1,\ \forall \rho > 0. \tag{3.33}$$

Now, let us introduce the function w_i defined by

$$w_i(x) = \frac{1}{n(n-2)\omega_n} \int_{\Omega_{\varepsilon_i}} \frac{|g(y,u_i(y))|}{|x-y|^{n-2}}\,\mathrm{d}y,$$

where ω_n is the measure of the unit sphere of $\mathbb{R}^n$. It is well known that w_i solves the equation

$$\Delta w_i(x) + \left|g\big(x,u_i(x)\big)\right| = 0;$$

moreover, we have $w_i \geqslant 0$ on $\partial\Omega_{\varepsilon_i}$, which implies

$$w_i(x) \geqslant \left|u_i(x)\right| \quad \forall x \in \Omega_{\varepsilon_i}. \tag{3.34}$$

Now, let us prove that

$$\lim_{i\to\infty} \sup\left\{w_i(x):\ x \in \Omega\setminus C_\rho\right\} = 0 \quad \forall \rho > 0. \tag{3.35}$$

For all $x \in \Omega\setminus C_\rho$, we write w_i as

$$w_i(x) = \frac{1}{n(n-2)\omega_n}\left[\int_{\Omega_{\varepsilon_i}\setminus B(x,\rho/2)} \frac{|g(y,u_i(y))|}{|x-y|^{n-2}}\,\mathrm{d}y + \int_{\Omega_{\varepsilon_i}\cap B(x,\rho/2)} \frac{|g(y,u_i(y))|}{|x-y|^{n-2}}\,\mathrm{d}y\right]. \tag{3.36}$$

The first integral in (3.36) can be estimated as follows:

$$\int_{\Omega_{\varepsilon_i}\setminus B(x,\rho/2)} \frac{|g(y,u_i(y))|}{|x-y|^{n-2}}\,\mathrm{d}y \leqslant \left(\frac{2}{\rho}\right)^{n-2} \int_{\Omega_{\varepsilon_i}} |g(y,u_i(y))|\,\mathrm{d}y. \tag{3.37}$$

As $f'_{\varepsilon_i}(u_i) = 0$, property (a) implies

$$\lim_{i\to\infty} \int_{\Omega_{\varepsilon_i}} g(x, u_i)u_i \, \mathrm{d}x = 0. \tag{3.38}$$

Since

$$\left|g(x,t)\right| \leqslant g(x,t)t + g(x,1) \quad \forall x \in \Omega, \ \forall t \in \mathbb{R}, \tag{3.39}$$

taking into account the definition of $g(x,t)$, we easily infer that

$$\lim_{i\to\infty} \int_{\Omega_{\varepsilon_i}} \left|g\big(y, u_i(y)\big)\right| \mathrm{d}y = 0,$$

which, by (3.37), implies

$$\lim_{i\to\infty} \int_{\Omega_{\varepsilon_i}\setminus B(x,\rho/2)} \frac{|g(y,u_i(y))|}{|x-y|^{n-2}} \, \mathrm{d}y = 0 \tag{3.40}$$

uniformly with respect to $x \in \Omega \setminus C_\rho$.

In order to deal with the second integral in (3.36), let us remark that, since $x \in \Omega \setminus C_\rho$,

$$\lim_{i\to\infty} \int_{\Omega_{\varepsilon_i}\cap B(x,\rho/2)} \left|g\big(y, u_i(y)\big)\right|^q \mathrm{d}y = 0 \quad \forall q \geqslant 1, \ \forall \rho > 0$$

because of (3.33). Since $\Omega_{\varepsilon_i} \cap B(x,\rho/2) \subset \Omega \setminus C_{\rho/2}$ for all $x \in \Omega \setminus C_\rho$, for $q > \frac{n}{2}$ we obtain

$$\int_{\Omega_{\varepsilon_i}\cap B(x,\rho/2)} \frac{|g(y,u_i(y))|}{|x-y|^{n-2}} \, \mathrm{d}y \leqslant c(p,q)\Bigg(\int_{\Omega_{\varepsilon_i}\setminus C_{\rho/2}} \left|g\big(y,u_i(y)\big)\right|^q \mathrm{d}y\Bigg)^{1/q} \quad \forall x \in \Omega \setminus C_\rho,$$

where $c(p,q)$ is a suitable constant depending only on p and q. Taking into account (3.33), it follows that

$$\lim_{i\to\infty} \int_{\Omega_{\varepsilon_i}\cap B(x,\rho/2)} \frac{|g(y,u_i(y))|}{|x-y|^{n-2}} \, \mathrm{d}y = 0 \tag{3.41}$$

uniformly with respect to $x \in \Omega \setminus C_\rho$. Hence, (3.35) follows from (3.36), (3.40) and (3.41). Notice that, since x^1 and x^2 do not belong to $\overline{\Omega}$, (3.34) and (3.35) imply, in particular, that $u_i \to 0$ uniformly on the boundary of $\Omega \setminus \bar{\chi}_1$. Therefore, taking into account the definition of $g(x,t)$ for $x \in \Omega \setminus \bar{\chi}_1$, it follows that $u_i \to 0$ uniformly in $\Omega \setminus \bar{\chi}_1$. Thus, property (c) is completely proved. □

Corollary 3.4. *There exists a positive constant $\bar{c}(\Omega, p)$ (depending only on Ω and p) such that, if $u \in H_S(\Omega_\varepsilon)$ is a critical point for f_ε and $f_\varepsilon(u) \leqslant \bar{c}(\Omega, p)$, then*

(a) $|u(x)| \leqslant t_0 \ \forall x \in \Omega_\varepsilon \setminus \bar{\chi}_1$,
(b) *u solves problem $P(\Omega_\varepsilon)$ (provided $u \not\equiv 0$ in Ω_ε).*

The proof follows easily from property (c) of Lemma 3.3.

Proposition 3.5. *For all $\varepsilon > 0$, the minimum of the functional f_ε constrained on M_ε is achieved. Moreover,*

(a) *the minimizing functions have constant sign*;
(b) *there exists $\varepsilon_1 > 0$ such that, for all $\varepsilon \in]0, \varepsilon_1[$, the minimizing functions solve problem $P(\Omega_\varepsilon)$.*

Proof. The existence of the minimum of f_ε on M_ε follows from Lemma 3.2 and (b) of Lemma 3.1. Property (a) is a consequence of Lemma 3.1. In fact, if $u \in M_\varepsilon$ is a minimizing function for f_ε on M_ε and assume, for example, that $u^+ \not\equiv 0$, then we must have $u^- \equiv 0$; otherwise, u^+ and u^- belong to M_ε and, by (b) of Lemma 3.1, $f_\varepsilon(u^\pm) > \inf_{M_\varepsilon} f_\varepsilon > 0$; it follows that

$$f_\varepsilon(u) = f_\varepsilon(u^+) + f_\varepsilon(u^-) > f_\varepsilon(u^+),$$

which is a contradiction, since $f_\varepsilon(u) = \inf_{M_\varepsilon} f_\varepsilon$.

Property (b) follows easily from Corollary 3.4, taking into account (d) of Lemma 3.1. □

Remark 3.6. When $2 < p < \frac{2n}{n-2}$, since $H_0^{1,2}(\Omega_\varepsilon)$ is compactly embedded in $L^p(\Omega_\varepsilon)$, a positive solution of problem $P(\Omega_\varepsilon)$ can be easily obtained for all $\varepsilon \geqslant 0$ by minimizing the functional

$$\tilde{f}(u) = \frac{1}{2}\int_\Omega |Du|^2\,\mathrm{d}x - \frac{1}{p}\int_\Omega |u|^p\,\mathrm{d}x$$

constrained on the manifold

$$\widetilde{M}_\varepsilon = \left\{u \in H_S(\Omega_\varepsilon):\ u \not\equiv 0,\ \int_{\Omega_\varepsilon} |Du|^2\,\mathrm{d}x = \int_{\Omega_\varepsilon} |u|^p\,\mathrm{d}x\right\}.$$

On the contrary, when $p \geqslant \frac{2n}{n-2}$, the existence of the minimum of $\tilde{f}$ on $\widetilde{M}_\varepsilon$ is not guaranteed. Indeed, if Ω_ε meets the x_n-axis, it is easy to verify that, for all $\varepsilon \geqslant 0$,

(a) if $p > \frac{2n}{n-2}$, then

$$\inf_{\widetilde{M}_\varepsilon} \tilde{f} = 0, \tag{3.42}$$

which, of course, implies that the minimum does not exist;

(b) if $p = \frac{2n}{n-2}$, then

$$\inf_{\widetilde{M}_\varepsilon} \tilde{f} = S > 0, \tag{3.43}$$

where S denotes the best Sobolev constant; as it is well known (see [4,21], etc.), the infimum S cannot be achieved for any bounded domain Ω_ε.

Hence, for $p \geqslant \frac{2n}{n-2}$ the problem cannot be solved by minimization. For $p = \frac{2n}{n-2}$ and $\varepsilon > 0$, positive solutions of $P(\Omega_\varepsilon)$ are obtained in [16] as local minimum points for $\tilde{f}$ constrained on $\widetilde{M}_\varepsilon$. On the contrary, this approach does not work any more for $p > \frac{2n}{n-2}$, since in this case, if Ω_ε meets the x_n-axis, $\tilde{f}$ constrained on $\widetilde{M}_\varepsilon$ cannot have any local minimum point. In fact, for all $u \in \widetilde{M}_\varepsilon$, we can find a sequence $(u_i)_i$ in $\widetilde{M}_\varepsilon$, which converges to u in $H_0^{1,2}(\Omega_\varepsilon)$ and in $L^p(\Omega_\varepsilon)$ and satisfies

$$\tilde{f}(u_i) < \tilde{f}(u) \quad \forall i \in \mathbb{N}.$$

Such a sequence $(u_i)_i$ can be obtained as follows. Let $\tilde{x}^2 = (0, \ldots, 0, \tilde{x}_n^2)$, with $\tilde{x}_n^2 \notin [x_n^1, x_n^3]$, be a point of the x_n-axis, which belongs to Ω; let $z \in C_0^\infty(B(0,1))$, $z \not\equiv 0$, be a fixed function having radial symmetry with respect to the origin; for all $\rho > 0$, let us set

$$z_\rho(x) = z\left(\frac{x - \tilde{x}^2}{\rho}\right).$$

Notice that

$$\tilde{u}_\rho = \left(\frac{\|Dz_\rho\|_2^2}{\|z_\rho\|_p^p}\right)^{1/(p-2)} z_\rho \in \widetilde{M}_\varepsilon \quad \forall \varepsilon \geqslant 0$$

for $\rho > 0$ small enough; moreover, $\lim_{\rho\to 0} \tilde f(\tilde u_\rho) = 0$ for all $p > \frac{2n}{n-2}$ (which, in particular, implies (3.42)). Now, for all $i \in \mathbb{N}$, set

$$u_{i,\rho} = \left(1 - \frac{1}{i}\right)^{1/p} u + \left(\frac{1}{i}\right)^{1/p} \frac{\|u\|_p}{\|z_\rho\|_p} z_\rho.$$

One can verify that

$$\lim_{\rho\to 0} \|u_{i,\rho}\|_p = \|u\|_p \quad \forall i \in \mathbb{N}$$

and

$$\lim_{\rho\to 0} \|Du_{i,\rho}\|_2^2 = \left(1 - \frac{1}{i}\right)^{2/p} \|Du\|_2^2 < \|Du\|_2^2 \quad \forall i \in \mathbb{N},$$

where the strict inequality holds because $u \not\equiv 0$ in Ω_ε. It follows that there exists a sequence of positive numbers $\rho_i \to 0$ such that, if we set

$$u_i = \left(\frac{\|Du_{i,\rho_i}\|_2^2}{\|u_{i,\rho_i}\|_p^p}\right)^{1/(p-2)} u_{i,\rho_i} \quad \forall i \in \mathbb{N},$$

then the sequence $(u_i)_i$ satisfies the desired properties.

Proposition 3.7. *Let $p > \frac{2n}{n-2}$ and Ω be a bounded domain satisfying the same assumptions as in Theorem* 2.1. *Then, for all positive integer k, there exists $\varepsilon_k > 0$ such that, for all $\varepsilon \in]0, \varepsilon_k[$, the functional f_ε has at least $2k$ nontrivial critical points $\pm u_{1,\varepsilon}, \ldots, \pm u_{k,\varepsilon}$. Moreover, we have*

$$\lim_{\varepsilon\to 0} f_\varepsilon(u_{i,\varepsilon}) = 0 \quad \forall i = 1, \ldots, k. \tag{3.44}$$

Proof. Since $x^2 \in \Omega$, there exist k distinct points $\bar x^1, \ldots, \bar x^k$ on the x_n-axis, which belong to Ω and satisfy

$$\bar x^i = \left(0, \ldots, 0, \bar x^i_n\right) \quad \text{with } x^1_n < \bar x^i_n < x^3_n \ \forall i = 1, \ldots, k.$$

Set

$$C_1 = \left\{x = (x_1, \ldots, x_n) \in \mathbb{R}^n \colon \sum_{i=1}^{n-1} x_i^2 \leqslant 1\right\}$$

and choose a function $\varphi \in H_0^{1,2}(B(0,2) \setminus C_1)$, $\varphi \not\equiv 0$, having radial symmetry with respect to the x_n-axis.

For all $i = 1, \ldots, k$ and $\varepsilon > 0$, let $\varphi_{i,\varepsilon}$ be the function defined by

$$\varphi_{i,\varepsilon}(x) = \varphi\left(\frac{x - \bar x_i}{\varepsilon}\right)$$

(here we intend that φ is extended by zero outside $B(0,2) \setminus C_1$).

Since $\bar x^1, \ldots, \bar x^k$ are distinct points in Ω, there exists $\varepsilon_k > 0$ such that, for all $\varepsilon \in]0, \varepsilon_k[$, the functions $\varphi_{1,\varepsilon}, \ldots, \varphi_{k,\varepsilon}$ belong to $H_S(\Omega_\varepsilon)$ and have disjoint supports. Notice that, for all $\sigma = (\sigma_1, \ldots, \sigma_k) \in \mathbb{R}^k$, such that $\sum_{i=1}^k \sigma_i^2 = 1$, there exists a unique positive number $\tau_\varepsilon(\sigma)$ such that

$$\tau_\varepsilon(\sigma) \sum_{i=1}^k \sigma_i \varphi_{i,\varepsilon} \in M_\varepsilon;$$

moreover, $\tau_\varepsilon(\sigma)$ depends continuously on σ. In fact, we have $\sum_{i=1}^k \sigma_i \varphi_{i,\varepsilon} \not\equiv 0$ because $\varphi \not\equiv 0$, $\varphi_{1,\varepsilon}, \ldots, \varphi_{k,\varepsilon}$ have disjoint supports and $\sum_{i=1}^k \sigma_i^2 = 1$; a direct computation shows that

$$\tau_\varepsilon(\sigma) = \left(\frac{\|\sum_{i=1}^k \sigma_i D\varphi_{i,\varepsilon}\|_2^2}{\|\sum_{i=1}^k \sigma_i \varphi_{i,\varepsilon}\|_p^p}\right)^{1/(p-2)}, \tag{3.45}$$

which satisfies all the properties above described.

Now, let us introduce the following subset of M_ε

$$S_k^\varepsilon = \left\{ \tau_\varepsilon(\sigma) \sum_{i=1}^{k} \sigma_i \varphi_{i,\varepsilon} \colon \sigma = (\sigma_1, \ldots, \sigma_k) \in \mathbb{R}^k,\ \sum_{i=1}^{k} \sigma_i^2 = 1 \right\}. \tag{3.46}$$

Notice that S_k^ε is radially diffeomorphic to the unit sphere of $\mathbb{R}^k$, as follows easily from (3.45). Hence, since f_ε constrained on M_ε satisfies the Palais–Smale condition (see Lemma 3.2), well known multiplicity results of the critical points theory for even functionals (see, for example, [9]) guarantee that there exist at least k pairs $\pm u_{1,\varepsilon}, \ldots, \pm u_{k,\varepsilon}$ of critical points for f_ε constrained on M_ε, which satisfy

$$f_\varepsilon(u_{i,\varepsilon}) \leqslant \sup_{S_k^\varepsilon} f_\varepsilon \quad \forall i = 1, \ldots, k. \tag{3.47}$$

Notice that the functions $\pm u_{1,\varepsilon}, \ldots, \pm u_{k,\varepsilon}$ are nontrivial critical points for f_ε because of Lemma 3.1. Finally, let us remark that

$$\lim_{\varepsilon \to 0} \sup_{S_k^\varepsilon} f_\varepsilon = 0, \tag{3.48}$$

as one can easily verify taking into account that

$$\lim_{\varepsilon \to 0} \frac{\|D\varphi_{i,\varepsilon}\|_2}{\|\varphi_{i,\varepsilon}\|_p} = 0 \quad \forall i = 1, \ldots, k \tag{3.49}$$

(indeed, $\|D\varphi_{i,\varepsilon}\|_2$ and $\|\varphi_{i,\varepsilon}\|_p$ do not depend on $i = 1, \ldots, k$).

Thus, (3.44) follows from (3.47) and (3.48). □

The following proposition states a general property of the solutions of problem $P(\Omega)$, that we shall use to describe the behaviour of the solutions of $P(\Omega_\varepsilon)$ as $\varepsilon \to 0$.

Proposition 3.8. *Let Ω be a bounded domain of $\mathbb{R}^n$ and u be a solution of problem $P(\Omega)$ with $p > 2$. Let λ_1 be the first eigenvalue of the Laplace operator $-\Delta$ in $H_0^{1,2}(\Omega)$. Then,*

$$u^+ \not\equiv 0 \quad \textit{implies} \quad \sup_\Omega u^+ \geqslant \lambda_1^{1/(p-2)} \tag{3.50}$$

and

$$u^- \not\equiv 0 \quad \textit{implies} \quad \sup_\Omega u^- \geqslant \lambda_1^{1/(p-2)}. \tag{3.51}$$

Proof. Since u solves problem $P(\Omega)$, we have in particular

$$\int_\Omega |Du^+|^2 \, dx = \int_\Omega (u^+)^p \, dx. \tag{3.52}$$

On the other hand, since $u^+ \in H_0^{1,2}(\Omega)$,

$$\int_\Omega |Du^+|^2 \, dx \geqslant \lambda_1 \int_\Omega (u^+)^2 \, dx. \tag{3.53}$$

Therefore,

$$\int_\Omega \left[(u^+)^p - \lambda_1 (u^+)^2 \right] dx \geqslant 0 \tag{3.54}$$

which is impossible if $u^+ \not\equiv 0$ and $\sup_\Omega u^+ < \lambda_1^{1/(p-2)}$ because $(u^+(x))^p - \lambda_1 (u^+(x))^2 < 0$ for all $x \in \Omega$ such that $0 < u(x) < \lambda_1^{1/(p-2)}$.

A similar argument holds for u^-. □

Proposition 3.9. *Let $(\varepsilon_i)_i$ be a sequence of positive numbers and $(u_i)_i$ a sequence of functions in $H_S(\Omega_{\varepsilon_i})$ such that $f'_{\varepsilon_i}(u_i) = 0$ $\forall i \in \mathbb{N}$ and $\lim_{i\to\infty} f_{\varepsilon_i}(u_i) = 0$.*

Then, if $u_i^+ \not\equiv 0$ $\forall i \in \mathbb{N}$,

(a) $\lim_{i\to\infty} \int_{\Omega_{\varepsilon_i}\cap\chi_\rho} (u_i^+)^q \,\mathrm{d}x = +\infty$ $\forall q > \frac{n}{2}(p-1)$, $\forall \rho > 0$,

(b) $\lim_{i\to\infty} \sup\{u_i^+(x)\colon x \in \Omega_{\varepsilon_i} \cap \chi_\rho\} = +\infty$ $\forall \rho > 0$.

Similar properties hold if $u_i^- \not\equiv 0$ $\forall i \in \mathbb{N}$.

Proof. Arguing by contradiction, assume that there exists $\bar q > \frac{n}{2}(p-1)$ and $\bar\rho > 0$ such that, up to a subsequence,

$$\lim_{i\to\infty} \int\limits_{\Omega_{\varepsilon_i}\cap\chi_{\bar\rho}} (u_i^+)^{\bar q} \,\mathrm{d}x < +\infty. \tag{3.55}$$

Let us prove that, as a consequence,

$$\lim_{i\to\infty} \int\limits_{\Omega_{\varepsilon_i}} (u_i^+)^q \,\mathrm{d}x = 0 \quad \forall q < \bar q. \tag{3.56}$$

In fact, for all $\rho > 0$, we have

$$\int\limits_{\Omega_{\varepsilon_i}} (u_i^+)^q \,\mathrm{d}x \leqslant \Bigg(\int\limits_{\Omega_{\varepsilon_i}\cap\chi_\rho} (u_i^+)^{\bar q} \,\mathrm{d}x \Bigg)^{q/\bar q} (\operatorname{meas}\chi_\rho)^{1-q/\bar q} + \int\limits_{\Omega_{\varepsilon_i}\setminus\chi_\rho} (u_i^+)^q \,\mathrm{d}x; \tag{3.57}$$

taking into account (c) of Lemma 3.3, it follows that

$$\limsup_{i\to\infty} \int\limits_{\Omega_{\varepsilon_i}} (u_i^+)^q \,\mathrm{d}x \leqslant \Bigg(\lim_{i\to\infty} \int\limits_{\Omega_{\varepsilon_i}\cap\chi_{\bar\rho}} (u_i^+)^{\bar q} \,\mathrm{d}x \Bigg)^{q/\bar q} (\operatorname{meas}\chi_\rho)^{1-q/\bar q} \quad \forall \rho \in]0, \bar\rho]. \tag{3.58}$$

So, letting $\rho \to 0$ and taking into account (3.55), we get (3.56).

Now observe that, arguing as in the proof of property (c) of Lemma 3.3, one can prove that

$$u_i(x) \leqslant \frac{1}{n(n-2)\omega_n} \int\limits_{\Omega_{\varepsilon_i}} \frac{[u_i^+(y)]^{p-1}}{|x-y|^{n-2}} \,\mathrm{d}y \quad \forall x \in \Omega_{\varepsilon_i}.$$

Then, for $q \in]\frac{n}{2}(p-1), \bar q[$, we obtain

$$u_i(x) \leqslant c(p,q) \Bigg(\int\limits_{\Omega_{\varepsilon_i}} \big(u_i^+(y)\big)^q \,\mathrm{d}y \Bigg)^{(p-1)/q} \qquad \forall x \in \Omega_{\varepsilon_i}$$

for a suitable constant $c(p,q)$ (depending only on p and q). Taking into account (3.56), it follows that $\lim_{i\to\infty} \sup_{\Omega_{\varepsilon_i}} u_i^+ = 0$, which contradicts Proposition 3.8 because we assumed $u_i^+ \not\equiv 0$ $\forall i \in \mathbb{N}$. So, property (a) is proved.

Property (b) follows easily from (a), taking into account (c) of Lemma 3.3. In fact, if (b) does not hold, (c) of Lemma 3.3 (by the dominated convergence theorem) implies that, up to a subsequence,

$$\lim_{i\to\infty} \int\limits_{\Omega_\varepsilon} (u_i^+)^q \,\mathrm{d}x = 0 \quad \forall q > \frac{n}{2}(p-1),$$

which contradicts (a).

Analogous arguments hold if we assume $u_i^- \not\equiv 0$ $\forall i \in \mathbb{N}$. $\square$

Proof of Theorem 2.1. For $\varepsilon > 0$ small enough, let $\pm u_{1,\varepsilon}, \dots, \pm u_{k,\varepsilon}$ be the critical points for f_ε given by Proposition 3.7. Taking into account Corollary 3.4, these functions solve problem $P(\Omega_\varepsilon)$ for $\varepsilon > 0$ small enough. Moreover, (a) follows from (a) of Lemma 3.3; (b) is equivalent to (a) because these functions belong to M_ε; (c) follows from (c) of Lemma 3.3; (d) and (e) are proved in Proposition 3.9. □

Remark 3.10. The methods of critical points theory used in the proof of Proposition 3.7 give also information about the Morse index of the critical points $\pm u_{1,\varepsilon}, \dots, \pm u_{k,\varepsilon}$ obtained for the functional f_ε (which is not greater than k). Moreover, it is possible to estimate, in terms of the Morse index, the number of nodal regions of these functions (see, for example, [2,3]). Thus, in particular, we obtain that the solutions given by Theorem 2.1 do not have more than k nodal regions.

Remark 3.11. The main step in the proof of Theorem 2.1 has been the fact that the critical points for the functional f_ε (obtained in Proposition 3.7) correspond to critical values which tend to zero as $\varepsilon \to 0$. It is clear that this fact occurs because $p > \frac{2n}{n-2}$, but it cannot hold for $p = \frac{2n}{n-2}$. Therefore, these arguments cannot work in the critical case; indeed, for $p = \frac{2n}{n-2}$, a solution having k nodal regions corresponds to a critical value at least equal to $\frac{k}{n} S^{n/2}$ (where S is the best Sobolev constant).

However, using a different approach, also in the critical case it is possible to state similar multiplicity results and find solutions having analogous qualitative properties; in particular, one can find solutions which concentrate near the x_n-axis as $\varepsilon \to 0$, even if the corresponding critical values do not tend to zero (see [15]).

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