Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 32 (2015) 623–650

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

The generalized principal eigenvalue for Hamilton–Jacobi–Bellman equations of ergodic type

Naoyuki Ichihara

Department of Physics and Mathematics, Aoyama Gakuin University, 5-10-1 Fuchinobe, Chuo-ku, Sagamihara-shi, Kanagawa 252-5258, Japan

Received 2 August 2013; received in revised form 10 February 2014; accepted 28 February 2014

Available online 12 March 2014

Abstract

This paper is concerned with the generalized principal eigenvalue for Hamilton–Jacobi–Bellman (HJB) equations arising in a class of stochastic ergodic control. We give a necessary and sufficient condition so that the generalized principal eigenvalue of an HJB equation coincides with the optimal value of the corresponding ergodic control problem. We also investigate some qualitative properties of the generalized principal eigenvalue with respect to a perturbation of the potential function.

MSC: 35Q93; 60J60; 93E20

Keywords: Principal eigenvalue; Hamilton–Jacobi–Bellman equation; Ergodic control; Recurrence and transience

1. Introduction

This paper is concerned with the following Hamilton–Jacobi–Bellman (HJB) equation of ergodic type:

$$\lambda - A\phi + H(x, D\phi) + \beta V = 0 \quad \text{in } \mathbb{R}^N, \qquad \phi(0) = 0, \tag{EP}$$

where β is a real parameter, $D\phi = (\partial\phi/\partial x_1, \dots, \partial\phi/\partial x_N)$, A is a second order elliptic operator of the form

$$A := \frac{1}{2}\sum_{i,j=1}^{N} \left(\sigma\sigma^T\right)_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i(x)\frac{\partial}{\partial x_i}, \tag{1.1}$$

and σ^T denotes the transpose matrix of σ. The unknown of (EP) is the pair of a real constant λ and a function $\phi = \phi(x)$ on $\mathbb{R}^N$. Finding such a pair is called ergodic problem or nonlinear additive eigenvalue problem. The constraint $\phi(0) = 0$ in (EP) is always imposed to avoid the ambiguity of additive constants with respect to ϕ. Throughout the paper, we assume the following:

E-mail address: ichihara@gem.aoyama.ac.jp.

http://dx.doi.org/10.1016/j.anihpc.2014.02.003

(A1) $\sigma_{ij}, b_i \in W^{1,\infty}(\mathbb{R}^N)$ for all $1 \leqslant i, j \leqslant N$, and there exists a $\nu_1 > 0$ such that $\nu_1|\eta|^2 \leqslant |\sigma^T(x)\eta|^2 \leqslant \nu_1^{-1}|\eta|^2$ for all $x, \eta \in \mathbb{R}^N$.

(A2) There exist $m > 1$, $\nu_2 > 0$, and $\Sigma = (\Sigma_{ij}(x))_{1\leqslant i,j\leqslant N}$ with $\Sigma_{ij} \in W^{1,\infty}(\mathbb{R}^N)$ for all $1 \leqslant i, j \leqslant N$ such that

$$H(x,p) = \frac{1}{m}\left|\Sigma^T(x)p\right|^m, \qquad \nu_2|\eta|^2 \leqslant \left|\Sigma^T(x)\eta\right|^2 \leqslant \nu_2^{-1}|\eta|^2, \quad x, \eta \in \mathbb{R}^N.$$

(A3) $V \in W^{1,\infty}(\mathbb{R}^N)$, $V \not\equiv 0$, and $V(x) \to 0$ as $|x| \to \infty$, i.e., $\lim_{r\to\infty} \sup_{|x|\geqslant r} |V(x)| = 0$.

Here and in what follows, we identify the Sobolev space $W^{1,\infty}(\mathbb{R}^N)$ with the totality of bounded and Lipschitz continuous functions on $\mathbb{R}^N$. Note that assumption (A2) can be relaxed slightly (see Remark 3.8).

Ergodic problem (EP) is closely related to the following stochastic control problem:

$$\text{Minimize} \quad J_\beta(\xi) = \liminf_{T\to\infty} \frac{1}{T} E_x\left[\int_0^T \left\{L\left(X_t^\xi, \xi_t\right) - \beta V\left(X_t^\xi\right)\right\} dt\right], \tag{1.2}$$

$$\text{subject to} \quad dX_t^\xi = -\xi_t\, dt + b\left(X_t^\xi\right) dt + \sigma\left(X_t^\xi\right) dW_t, \tag{1.3}$$

where $L(x,\xi) := (1/m^*)|\Sigma^{-1}(x)\xi|^{m^*}$ for $m^* := m/(m-1) > 1$, and Σ^{-1} denotes the inverse matrix of Σ in (A2). Recall that (1.3) is interpreted as Ito's stochastic differential equation, where $W = (W_t)$ is an N-dimensional standard Brownian motion defined on some probability space, and $\xi = (\xi_t)$ stands for an $\mathbb{R}^N$-valued control process belonging to a suitable admissible class $\mathcal{A}$. Stochastic control of this type is called (stochastic) ergodic control. We refer to [8] for general information on the stochastic control theory. See also [2,5] for stochastic ergodic control in $\mathbb{R}^N$. We remark here that $L = L(x,\xi)$ is the Fenchel–Legendre transform of $H = H(x,p)$ with respect to the second variable, namely,

$$L(x,\xi) := \sup\left\{\xi \cdot p - H(x,p) \;\middle|\; p \in \mathbb{R}^N\right\}, \quad x, \xi \in \mathbb{R}^N.$$

The objective of this paper is to investigate several qualitative properties of the generalized principal eigenvalue for (EP) defined by

$$\lambda^*(\beta) := \sup\left\{\lambda \in \mathbb{R} \;\middle|\; \text{(EP) has a solution } \phi \in C^2\left(\mathbb{R}^N\right)\right\}. \tag{1.4}$$

Our main results consist of two parts. In the first half, we explore the relationship between the generalized principal eigenvalue $\lambda^* = \lambda^*(\beta)$ of (EP) and the optimal value $\Lambda = \Lambda(\beta) := \inf_{\xi\in\mathcal{A}} J_\beta(\xi)$ of the minimizing problem (1.2)–(1.3). More specifically, we discuss a necessary and sufficient condition so that these two values coincide. Such identity is valid in various contexts (e.g., [1,9–11,13,19]). The key lies in the ergodicity of the feedback diffusion $X = (X_t)$ governed by the stochastic differential equation

$$dX_t = -D_pH\left(X_t, D\phi(X_t)\right) dt + b(X_t)\, dt + \sigma(X_t)\, dW_t, \tag{1.5}$$

where $D_pH(x,p)$ stands for the gradient of H with respect to p, and ϕ is a solution of (EP) with $\lambda = \lambda^*(\beta)$. Note that (1.5) is a natural candidate for the optimal controlled process associated with (1.2)–(1.3). In this paper, we give a characterization of the identity $\lambda^*(\beta) = \Lambda(\beta)$, as a function of β, by regarding (EP) as a perturbation of the equation

$$\lambda - A\phi + H(x, D\phi) = 0 \quad \text{in } \mathbb{R}^N. \tag{1.6}$$

In contrast to the earlier results mentioned above, (A1)–(A3) do not deduce the desired identity even if (1.5) enjoys the ergodicity, in general. It will turn out in the following sections that two functions $\lambda^* = \lambda^*(\beta)$ and $\Lambda = \Lambda(\beta)$ coincide if and only if $\lambda^*(0) = 0$, where $\lambda^*(0)$ denotes the generalized principal eigenvalue of (1.6). The optimality of the controlled diffusion (1.5) can also be characterized in terms of λ^*. See Theorems 2.1 and 2.2 for the precise statement.

The value $\lambda^*(\beta)$ has a close connection with the generalized principal eigenvalue of the linear elliptic operator $A + \beta V$ provided $m = 2$ and $\Sigma \equiv \sigma$ in (A2), i.e., $H(x,p) = (1/2)|\sigma^T(x)p|^2$. Indeed, by the so-called Cole–Hopf transform $h := e^{-\phi}$, one can identify the totality of solutions ϕ of (EP) with that of positive solutions h of the linear eigenvalue problem

$$-(A + \beta V)h = \lambda h \quad \text{in } \mathbb{R}^N. \tag{1.7}$$

In particular, $\lambda^*(\beta)$ can be represented as

$$\lambda^*(\beta) = \sup\{\lambda \in \mathbb{R} \mid \text{(1.7) has a positive solution } h \in C^2(\mathbb{R}^N)\},$$

which agrees with one of the equivalent definitions of the generalized principal eigenvalue of $A + \beta V$ (cf. [7,20,23]). In this sense, (1.4) gives an extended notion of the principal eigenvalue which is also meaningful for nonlinear eigenvalue problem (EP). Note that, if $m \neq 2$ or $\Sigma = \Sigma(x)$ is not a constant multiple of the diffusion coefficient $\sigma = \sigma(x)$, then there is no such correspondence between linear and nonlinear problems.

In the second part of this paper, we discuss more specific properties of the function $\beta \mapsto \lambda^*(\beta)$ under some restrictive assumptions on the coefficients. In order to present our results briefly, we temporally assume that $\sigma \equiv I$ (the identity matrix) and $b \equiv 0$ in (A1), and that V has compact support. Then our ergodic control problem can be described as follows:

$$\begin{aligned} &\text{Minimize} \quad J_\beta(\xi) = \liminf_{T\to\infty} \frac{1}{T} E_x\left[\int_0^T \{L(X_t^\xi, \xi_t) - \beta V(X_t^\xi)\}\,dt\right], \\ &\text{subject to} \quad dX_t^\xi = -\xi_t\,dt + dW_t. \end{aligned} \tag{1.8}$$

Our interest is to investigate a "phase transition" which takes place in (1.8). In our previous work [12], which dealt with the case where $m = 2$ and $V \geqslant 0$, we proved that there exists a critical value β_c such that the following (a) and (b) hold:

(a) $\lambda^*(\beta) = \Lambda(\beta) = 0$ for any $\beta \leqslant \beta_c$ and $\lambda^*(\beta) = \Lambda(\beta) < 0$ for any $\beta > \beta_c$.
(b) X governed by (1.5) becomes transient for any $\beta < \beta_c$ and recurrent for any $\beta \geqslant \beta_c$.

Hence, qualitative properties of $\lambda^*(\beta)$ and X change in a vicinity of β_c. From the stochastic control point of view, this phase transition may be explained as follows. In the minimizing problem (1.8), the controller falls into a trade-off situation between the cost $L(x, \xi)$ and the "reward" $\beta V(x)$. If β is small, then the best strategy for the controller is to choose $\xi \equiv 0$ and get $J_\beta(0) = 0$ since taking a nonzero control $\xi \not\equiv 0$ is costly compared with the obtained reward. On the other hand, if β is sufficiently large, then his/her optimal strategy is to take a suitable control $\xi \not\equiv 0$ which forces the controlled process X^ξ to visit frequently the favorable position (i.e., around the bottom of $-\beta V(x)$). The value β_c represents, thus, the threshold at which the controller switches his/her optimal strategy from one to the other. Remark that, in (1.8), the feedback control $\xi_t := D_p H(X_t, D\phi(X_t))$ with X in (1.5) is optimal for any $\beta > \beta_c$, whereas $\xi \equiv 0$ is optimal for $\beta \leqslant \beta_c$.

As far as the value of β_c is concerned, it is known that $\beta_c = 0$ for $N = 1, 2$ and $\beta_c > 0$ for $N \geqslant 3$ provided $m = 2$ and $V \geqslant 0$ (see [12, Theorem 2.5]). In particular, if $N \geqslant 3$, the function $\beta \mapsto \lambda^*(\beta)$ possesses a "flat region" which contains the origin in its interior. This result exhibits a striking contrast between deterministic and stochastic ergodic control. To highlight the difference, let us consider the deterministic counterpart of our ergodic control problem defined by

$$\begin{aligned} &\text{Minimize} \quad \tilde{J}_\beta(\xi) = \liminf_{T\to\infty} \frac{1}{T} \int_0^T \{L(x_t^\xi, \xi_t) - \beta V(x_t^\xi)\}\,dt, \\ &\text{subject to} \quad \frac{d}{dt}x_t^\xi = -\xi_t, \qquad x_0^\xi = x. \end{aligned} \tag{1.9}$$

Let $\tilde{\mathcal{A}}$ denote the set of locally bounded functions ξ on $[0, \infty)$ with values in $\mathbb{R}^N$, and set $\tilde{\Lambda}(\beta) := \inf_{\xi\in\tilde{\mathcal{A}}} \tilde{J}_\beta(\xi)$. Then it is easy to see that $\tilde{\Lambda}(\beta) = -\max_{\mathbb{R}^N}(\beta V)$ for all β, so that no "flattening" occurs in any open interval containing the origin.

In this paper, we extend our previous results described above in two directions. Firstly, we allow V to be sign-changing. Secondly, we deal with arbitrary $m > 1$. Recall that both $V \geqslant 0$ and $m = 2$ are assumed in [12]. If V is sign-changing, then there exist two critical values $\overline{\beta} \geqslant 0$ and $\underline{\beta} \leqslant 0$ such that $\lambda^*(\beta) = 0$ if and only if $\underline{\beta} \leqslant \beta \leqslant \overline{\beta}$. Moreover, suppose that $N = 2$. Then it may happen that $\underline{\beta} < 0 < \overline{\beta}$ for $m > 2$, while $\underline{\beta} = \overline{\beta} = 0$ for any $1 < m \leqslant 2$.

The last fact shows that the shape of $\lambda^*(\beta)$ around the origin relies sensitively on m. In fact, several qualitative properties of solutions of (EP) change as to whether $1 < m \leqslant 2$ or $m > 2$. The novelty of this second part, compared to [12], is to clarify such dependence with respect to m. For instance, the diffusion X governed by (1.5) turns out to be transient for every $\beta \in (-\infty, \beta_c)$ if $1 < m \leqslant 2$ and $V \geqslant 0$, whereas this is not true, in general, for $m > 2$. See Section 6 for details.

Another remark to be pointed out is that the recurrence/transience of diffusion (1.5) gives an extended notion of the criticality/subcriticality of linear elliptic operator $A + \beta V + \lambda^*(\beta)$. To explain this, we recall that a linear second order elliptic operator, say P, is called subcritical if it admits a (minimal) Green function in $\mathbb{R}^N$, and called critical if there is no Green function in $\mathbb{R}^N$ but there exists a positive solution $h \in C^2(\mathbb{R}^N)$ of $Ph = 0$ in $\mathbb{R}^N$. For each positive solution h of $Ph = 0$ in $\mathbb{R}^N$, let $P^h := h^{-1}P(h\,\cdot)$ denote the h-transform of P. Then it is well known (e.g. [23, Section 4.3]) that P is critical (resp. subcritical) if and only if the P^h-diffusion is recurrent (resp. transient). Let us now consider the special case where $H(x,p) = (1/2)|\sigma^T(x)p|^2$, and let ϕ be a solution of (EP) with $\lambda = \lambda^*(\beta)$. Then $h := e^{-\phi}$ is a solution of $Ph = 0$ in $\mathbb{R}^N$ with $P := A + \beta V + \lambda^*(\beta)$, and the h-transform of P is written as $P^h = A - (\sigma\sigma^T)D\phi \cdot D$. Hence, P is critical (resp. subcritical) if and only if the P^h-diffusion

$$dX_t = -\big(\sigma\sigma^T\big)(X_t)D\phi(X_t)\,dt + b(X_t)\,dt + \sigma(X_t)\,dW_t \tag{1.10}$$

is recurrent (resp. transient). Since (1.10) is a particular case of (1.5) with $H(x,p) = (1/2)|\sigma^T(x)p|^2$, the notion of criticality/subcriticality of ergodic problem (EP) may be defined in terms of the recurrence/transience of diffusion (1.5). There is an extensive literature on the criticality of linear elliptic operators from both analytical and probabilistic viewpoint. See [18,20–24] and the references therein. Contrary to the linear case, little is known about the criticality of (EP), namely, the recurrence and transience of (1.5), except for some special cases discussed in [12]. The second part of this paper aims, as well, at developing a criticality theory for (EP) from our stochastic control point of view.

Before closing this introductory section, we mention that ergodic problems of type (EP) also appear in other mathematical problems such as singular perturbations, homogenizations, etc. In those problems, (EP) is called the cell problem and plays a crucial role to determine the so-called effective Hamiltonian. If the model has periodic structure, i.e., if (EP) is reduced to the equation in the N-dimensional unit torus $\mathbb{T}^N := \mathbb{R}^N/\mathbb{Z}^N$, then there is only one pair (λ, ϕ), up to an additive constant with respect to ϕ, which solves (EP), so that the situation becomes considerably simple. In recent years, singular perturbation problems without periodicity have also been a subject of interest in the context of mathematical finance. In those models, the analysis of ergodic problem (EP) is one of the main issues. We refer to [3,4] for more details in this direction.

This paper is organized as follows. In the next section we state our main results precisely. Section 3 is devoted to the preliminaries. Section 4 concerns fundamental properties of solutions to (EP). In Section 5, we discuss a necessary and sufficient condition so that $\Lambda = \lambda^*$. In Section 6, we investigate qualitative properties of $\lambda^*(\beta)$, especially, the "flattening" of the function $\lambda^*(\beta)$ around the origin $\beta = 0$.

2. Main results

Let $C^k(\mathbb{R}^N)$, $k \in \mathbb{N} \cup \{0\}$, be the set of C^k-functions on $\mathbb{R}^N$ equipped with the topology of locally uniform convergence. We say that a family $\{f_n\}$ converges in $C^k(\mathbb{R}^N)$ to a function f as $n \to \infty$ if f_n together with its partial derivatives of order up to k converge, locally uniformly in $\mathbb{R}^N$ as $n \to \infty$, to f and the corresponding partial derivatives, respectively. Given a $\gamma \in (0,1)$, we denote by $C^{k+\gamma}(\mathbb{R}^N)$ the Hölder space consisting of all $f \in C^k(\mathbb{R}^N)$ such that, for any compact $K \subset \mathbb{R}^N$,

$$\|f\|_{k+\gamma,K} := \sum_{|\alpha| \leqslant k} \sup_{x\in K} \big|D^\alpha f(x)\big| + \sum_{|\alpha|=k} \sup_{x,y\in K, x\neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^\gamma} < \infty,$$

where α denotes the multi-index of differential operator $D = (\partial/\partial x_1, \ldots, \partial/\partial x_N)$. Let $C_c^\infty(\mathbb{R}^N)$ be the set of infinitely differentiable functions on $\mathbb{R}^N$ with compact support. For $k \in \mathbb{N}$ and $q \in [1,\infty]$, we denote by $W^{k,q}(\mathbb{R}^N)$ the Sobolev space, i.e., the totality of $f \in L^q(\mathbb{R}^N)$ such that $\|f\|_{k,q} := (\int_{\mathbb{R}^N} \sum_{|\alpha|\leqslant k} |D^\alpha f|^q\,dx)^{1/q} < \infty$ for $1 < q < \infty$ and $\|f\|_{k,\infty} := \sum_{|\alpha|\leqslant k} \text{ess-sup}_{\mathbb{R}^N} |D^\alpha f| < \infty$ for $q = \infty$. We denote by $W^{k,q}_{\mathrm{loc}}(\mathbb{R}^N)$ the collection of Borel measurable functions f on $\mathbb{R}^N$ such that $f\zeta \in W^{k,q}(\mathbb{R}^N)$ for all $\zeta \in C_c^\infty(\mathbb{R}^N)$.

Let us consider ergodic problem (EP) and define its generalized principal eigenvalue by

$$\lambda^*(\beta) := \sup\{\lambda \in \mathbb{R} \mid \text{(EP) has a subsolution } \phi_0 \in \Phi\}, \tag{2.1}$$

where Φ is given by

$$\Phi := \left\{\phi \in \bigcap_{q>N} W^{3,q}_{\mathrm{loc}}(\mathbb{R}^N) \;\Big|\; \frac{\partial \phi}{\partial x_i}, A\phi \in L^\infty(\mathbb{R}^N),\ i = 1, \dots, N\right\}.$$

Note that $\Phi \subset C^{2+\gamma}(\mathbb{R}^N)$ for any $\gamma \in (0, 1)$ in view of the Sobolev embedding theorem. Throughout the paper, the notion of solution, subsolution, and supersolution will be understood in the classical sense. It will turn out in the next section that any classical solution of (EP) belongs to Φ, that $\lambda^*(\beta)$ is well-defined and finite for any β, and that its value coincides with the one defined by (1.4).

Now, fix any filtered probability space $(\Omega, \mathcal{F}, P; (\mathcal{F}_t))$ on which is defined an N-dimensional standard $(\mathcal{F}_t)$-Brownian motion $W = (W_t)$ with $W_0 = 0$, P-a.s. We denote by $\mathcal{A}$ the set of $(\mathcal{F}_t)$-progressively measurable control processes $\xi = (\xi_t)$ with values in $\mathbb{R}^N$ such that $\text{ess-sup}_{[0,T]\times\Omega} |\xi_t| < \infty$ for all $T > 0$. It is well known that, for any $x \in \mathbb{R}^N$ and $\xi \in \mathcal{A}$, there exists an $(\mathcal{F}_t)$-progressively measurable process $X^\xi = (X^\xi_t)$ with values in $\mathbb{R}^N$ such that

$$X^\xi_t = x - \int_0^t \xi_s\, ds + \int_0^t b(X^\xi_s)\, ds + \int_0^t \sigma(X^\xi_s)\, dW_s, \quad t \geqslant 0,\ P\text{-a.s.}$$

Moreover, X^ξ is uniquely determined up to a P-null set.

For each $\xi \in \mathcal{A}$, let $J_\beta(\xi)$ be the cost functional defined by (1.2). We denote the optimal value of the minimizing problem (1.2)–(1.3) by

$$\Lambda = \Lambda(\beta) := \inf_{\xi \in \mathcal{A}} J_\beta(\xi).$$

Our first main result is concerned with the relationship between $\Lambda(\beta)$ and the generalized principal eigenvalue $\lambda^*(\beta)$ of (EP) defined by (2.1). To describe the results, set $\bar{\lambda} := \sup\{\lambda^*(\beta) \mid \beta \in \mathbb{R}\}$. Since $\phi \equiv 0$ is a solution of (EP) with $\beta = 0$ and $\lambda = 0$, we observe that $\lambda^*(0) \geqslant 0$. In particular, $\bar{\lambda} \geqslant 0$.

Theorem 2.1. *Assume* (A1)–(A3). *Then the following* (i)–(iii) *hold.*

(i) *$\beta \mapsto \lambda^*(\beta)$ is a non-constant function which is concave in $\mathbb{R}$. Moreover, it is differentiable on the set $\{\beta \in \mathbb{R} \mid \lambda^*(\beta) < \bar{\lambda}\}$.*
(ii) *Suppose that $\lambda^*(0) = 0$. Then $\Lambda(\beta) = \lambda^*(\beta)$ for all β.*
(iii) *Suppose that $\lambda^*(0) > 0$. Then $\Lambda(\beta) = \min\{0, \lambda^*(\beta)\}$ for all β.*

In particular, two functions λ^* and Λ coincide if and only if $\lambda^*(0) = 0$.

We mention that, if $H(x, p) = (1/2)|\sigma^T(x)p|^2$, then (EP) can be transformed into the linear equation (1.7) by the Cole–Hopf transform. In this linear context, a similar result has been observed by [14]. See also Remark 5.14.

We next discuss a characterization of the optimal control for (1.2)–(1.3). To this end, let $\mathcal{S}(\beta)$ denote the set of solutions ϕ of (EP) with $\lambda = \lambda^*(\beta)$. For a given $\phi \in \mathcal{S}(\beta)$, we set $A^\phi := A - D_pH(x, D\phi(x)) \cdot D$. Note that A^ϕ is the infinitesimal generator of the diffusion $X = (X_t)$ governed by (1.5). Throughout the paper, X in (1.5) is called A^ϕ-diffusion. The following theorem gives an optimality criterion of the feedback control $\xi^*_t := D_pH(X_t, D\phi(X_t))$.

Theorem 2.2. *Assume* (A1)–(A3). *Let $\beta \in \mathbb{R}$ be such that $\lambda^*(\beta) < \bar{\lambda}$. Then there exists a unique solution $\phi \in C^2(\mathbb{R}^N)$ of* (EP) *with $\lambda = \lambda^*(\beta)$. Moreover, let X be the associated A^ϕ-diffusion and set $\xi^*_t := D_pH(X_t, D\phi(X_t))$. Then ξ^* is optimal if and only if $\lambda^*(\beta) = \Lambda(\beta)$. If $\lambda^*(\beta) \neq \Lambda(\beta)$, then $\xi_t \equiv 0$ is optimal.*

Now, we investigate the phase transition appearing in (1.2)–(1.3) under the assumption that $\bar{\lambda} = 0$. Note that this condition always holds when $b \equiv 0$. We give in Section 6 other sufficient conditions so that $\bar{\lambda} = 0$. In what follows, we assume $\bar{\lambda} = 0$ and set $J := \{\beta \in \mathbb{R} \mid \lambda^*(\beta) = 0\}$. By the concavity of $\lambda^*(\beta)$, J is a connected closed subset in $\mathbb{R}$. The following result concerns the recurrence and transience of A^ϕ-diffusions which characterizes the phase transition described in the introduction.

Theorem 2.3. *Let* (A1)–(A3) *hold, and assume that* $\bar{\lambda} = 0$. *Set* $J := \{\beta \in \mathbb{R} \mid \lambda^*(\beta) = 0\}$. *Then the following* (i)–(ii) *hold.*

(i) *For any* $\phi \in \mathcal{S}(\beta)$ *with* $\beta \notin J$, *the* A^ϕ*-diffusion is recurrent. Moreover, it is ergodic.*
(ii) *Assume that* $1 < m \leqslant 2$ *in* (A2). *Then the* A^ϕ*-diffusion is transient for any* $\phi \in \mathcal{S}(\beta)$ *with* $\beta \in \operatorname{Int} J$, *where* $\operatorname{Int} J$ *denotes the interior of* J.

The definitions of recurrence, transience, and ergodicity of diffusion processes will be reviewed in the next section. Notice here that the second claim of Theorem 2.3 may fail when $m > 2$ in (A2). A counterexample will be given in Section 6. We also remark that, under the hypothesis of Theorem 2.3, $\xi(x) \equiv 0$ becomes an optimal control policy of (1.2)–(1.3) for all $\beta \in J$, while the function $\xi(x) := D_p H(x, D\phi(x))$ is optimal for any $\beta \notin J$.

Let us now discuss more detailed properties of $\beta \mapsto \lambda^*(\beta)$ around the origin. In order to state our result precisely, we define $G_\alpha = G_\alpha(x)$ by

$$G_\alpha(x) := \frac{1}{2}\operatorname{tr}\big((\sigma\sigma^T)(x)\big) - \alpha\frac{|\sigma^T(x)x|^2}{2|x|^2} + b(x)\cdot x, \quad x \in \mathbb{R}^N, \tag{2.2}$$

where $\alpha \geqslant 0$ is a given constant. The following result gives a criterion which determines whether $\beta = 0$ belongs to $\operatorname{Int} J$ or ∂J, where ∂J stands for the boundary of J. In the former case, the "flattening" of $\lambda^*(\beta)$ occurs in a neighborhood of the origin.

Theorem 2.4. *Let* (A1)–(A3) *hold, and assume that* $\bar{\lambda} = 0$. *Set* $J := \{\beta \in \mathbb{R} \mid \lambda^*(\beta) = 0\}$. *Then the following* (i)–(ii) *hold.*

(i) *If* $1 < m \leqslant 2$ *in* (A2) *and* $G_\alpha(x) \leqslant 0$ *in* $\mathbb{R}^N \setminus B_R$ *for some* $\alpha \leqslant 2$ *and* $R > 0$, *then* $\partial J = \{0\}$, *namely,* $J = (-\infty, 0]$ *or* $J = [0, \infty)$ *or* $J = \{0\}$.
(ii) *If* $m \geqslant 2$ *in* (A2), $|x|^{m^*} V(x) \to 0$ *as* $|x| \to \infty$, *and* $\liminf_{|x|\to\infty} G_{m^*}(x) > 0$, *where* $m^* := m/(m-1)$, *then* $0 \in \operatorname{Int} J$.

As a direct consequence of Theorems 2.3 and 2.4, we are able to obtain a complete characterization of the criticality of (EP), namely, the recurrence/transience of (1.5) in the special case where $\sigma \equiv I$, $b \equiv 0$, and $m = 2$. More precisely, let us consider the following ergodic problem:

$$\lambda - \frac{1}{2}\Delta\phi + H(x, D\phi) + \beta V = 0 \quad \text{in } \mathbb{R}^N, \qquad \phi(0) = 0. \tag{2.3}$$

Let $\lambda^*(\beta)$ be the generalized principal eigenvalue of (2.3), and set $\bar{\lambda} := \sup\{\lambda^*(\beta) \mid \beta \in \mathbb{R}\}$. Then one has the following result which can be regarded as an extension of [12].

Theorem 2.5. *Let* (A2) *and* (A3) *hold. Suppose that* $m = 2$ *in* (A2) *and* $|x|^2 V(x) \to 0$ *as* $|x| \to \infty$. *Then the following* (i)–(iii) *hold.*

(i) $\bar{\lambda} = 0$.
(ii) *Set* $J := \{\beta \in \mathbb{R} \mid \lambda^*(\beta) = 0\}$. *Let* ϕ *be a solution of* (2.3) *with* $\lambda = \lambda^*(\beta)$. *Then the* A^ϕ*-diffusion is transient for* $\beta \in \operatorname{Int} J$ *and recurrent for* $\beta \notin \operatorname{Int} J$. *Moreover, it is ergodic for* $\beta \notin J$.
(iii) $\partial J = \{0\}$ *for* $N = 1, 2$, *and* $0 \in \operatorname{Int} J$ *for* $N \geqslant 3$.

Remark 2.6. Under the hypothesis of Theorem 2.5, it is known that the A^ϕ-diffusion with $\beta \in \partial J$ is null recurrent for $N \leqslant 4$ and positive recurrent for $N \geqslant 5$ provided $\Sigma \equiv I$ in (A2) (see [12, Theorem 6.8]). We do not know if the same result holds under the assumption of Theorem 2.3.

3. Preliminaries

In the rest of this paper, unless otherwise specified, we always assume (A1)–(A3). In this section, we collect several auxiliary results, most of which are fundamental and well known.

Let A be a second order elliptic operator of form (1.1), and let $X = (X_t)_{t \geqslant 0}$ be the diffusion process associated with A. Note that such process can be constructed by solving the stochastic differential equation

$$dX_t = b(X_t)\,dt + \sigma(X_t)\,dW_t, \qquad X_0 = x \in \mathbb{R}^N. \tag{3.1}$$

Recall also that the solution of (3.1) is unique (in any sense), does not explode in finite time, and has the strong Markov property (see, for instance, [23, Chapter 1]). Hereafter, we simply call the solution X of (3.1) A-diffusion. As is mentioned in the previous section, we use the wording "A^ϕ-diffusion" if $b(x)$ in (3.1) is replaced by $b(x) - D_pH(x, D\phi(x))$ for some $\phi \in C^2(\mathbb{R}^N)$.

For $y \in \mathbb{R}^N$ and $r > 0$, we set $B_r(y) := \{z \in \mathbb{R}^N \mid |z - y| < r\}$ and $\tau_{r,y} := \inf\{t > 0 \mid X_t \in B_r(y)\}$, where we use the convention $\inf \emptyset = \infty$. Set $B_r := B_r(0)$. An A-diffusion X is called recurrent if $P_x(\tau_{\varepsilon,y} < \infty) = 1$ for every $x, y \in \mathbb{R}^N$ and $\varepsilon > 0$, and called transient otherwise. Note that X is transient if and only if $P_x(\lim_{t\to\infty} |X_t| = \infty) = 1$ for all $x \in \mathbb{R}^N$. For any recurrent A-diffusion, there exists a σ-finite measure $\mu = \mu(dx)$ on $\mathbb{R}^N$, called the invariant measure, such that $\int_{\mathbb{R}^N} (A\varphi)(x)\mu(dx) = 0$ for all $\varphi \in C_c^\infty(\mathbb{R}^N)$. It is well known under our assumptions that such μ is unique up to a positive multiplicative constant. Furthermore, μ is absolutely continuous with respect to the Lebesgue measure, and the density $\mu = \mu(x) > 0$ belongs to $W^{1,q}_{\mathrm{loc}}(\mathbb{R}^N)$ for any $q > 1$ (see [6]). The function $\mu = \mu(x)$ is called the invariant density for the A-diffusion. A recurrent A-diffusion is called ergodic (or positive recurrent) if $\mu(\mathbb{R}^N) < \infty$, and called null-recurrent otherwise. In the former case, we always choose μ so that $\mu(\mathbb{R}^N) = 1$ and call it the invariant probability measure. We refer to [23, Chapter 4] for more details of these facts. In this paper, we abuse the notation of μ to denote both the invariant measure and its density.

The following theorem is a version of the famous criterion, known as Lyapunov's method, which gives sufficient conditions for the recurrence and transience of diffusion processes.

Theorem 3.1. *Let X be the A-diffusion. Then the following* (i)–(iii) *hold.*

(i) *X is transient if there exist $R > 0$ and $u \in C^2(\mathbb{R}^N \setminus B_R)$ such that*

$$\inf_{\partial B_R} u > \inf_{\mathbb{R}^N \setminus B_R} u > -\infty, \qquad \sup_{\mathbb{R}^N \setminus B_R} Au \leqslant 0.$$

(ii) *X is recurrent if there exist $R > 0$ and $u \in C^2(\mathbb{R}^N \setminus B_R)$ such that*

$$\lim_{|x|\to\infty} u(x) = \infty, \qquad \sup_{\mathbb{R}^N \setminus B_R} Au \leqslant 0.$$

(iii) *X is ergodic if there exist $R > 0$, $u \in C^2(\mathbb{R}^N \setminus B_R)$, and $\varepsilon > 0$ such that*

$$\inf_{\mathbb{R}^N \setminus B_R} u(x) > -\infty, \qquad \sup_{\mathbb{R}^N \setminus B_R} Au \leqslant -\varepsilon.$$

Proof. See, for instance, [23, Section 6.1] or [12, Theorem 4.1] for a complete proof. □

By using Theorem 3.1, we rediscover the fundamental result that the Brownian motions are null recurrent for $N = 1, 2$ and transient for $N \geqslant 3$.

Example 3.2. Fix any function $\psi \in C^3(\mathbb{R}^N)$ such that $\psi(x) = \log|x|$ in $\mathbb{R}^N \setminus B_1$. For a given $\alpha \in \mathbb{R}$, let $X = (X_t)$ denote the diffusion process governed by

$$dX_t = -\alpha D\psi(X_t)\,dt + dW_t.$$

Then X is transient if and only if $\alpha < (N/2) - 1$, null recurrent if and only if $(N/2) - 1 \leqslant \alpha \leqslant N/2$, and ergodic if and only if $\alpha > N/2$. In particular, by choosing $\alpha = 0$, we observe that Brownian motions are null recurrent for $N = 1, 2$ and transient for $N \geqslant 3$.

To justify this claim, suppose first that $\alpha < (N/2) - 1$ and fix any $\delta > 0$ such that $\delta \leqslant N - 2 - 2\alpha$. Let $u \in C^2(\mathbb{R}^N)$ be any function satisfying $u(x) = |x|^{-\delta}$ in $\mathbb{R}^N \setminus B_1$. Then we see that

$$Au = \frac{1}{2}\Delta u - \alpha D\psi(x) \cdot Du(x) = -\frac{\delta(N-2-\delta-2\alpha)}{2|x|^{2+\delta}} \leqslant 0 \quad \text{in } \mathbb{R}^N \setminus B_1.$$

Hence, X is transient in view of Theorem 3.1(i).

We next assume that $\alpha \geqslant (N/2) - 1$ and choose $u = \psi$. Then

$$Au = \frac{1}{2}\Delta \psi - \alpha \left|D\psi(x)\right|^2 = \frac{(N-2-2\alpha)}{2|x|^2} \leqslant 0 \quad \text{in } \mathbb{R}^N \setminus B_1.$$

In particular, X is recurrent in view of Theorem 3.1(ii). Furthermore, since the invariant density of X is given by $\mu = e^{-2\alpha\psi}$, we have

$$\mu(x) = e^{-2\alpha\psi(x)} = |x|^{-2\alpha}, \quad x \in \mathbb{R}^N \setminus B_1.$$

This implies that $\mu \in L^1(\mathbb{R}^N)$ if and only if $2\alpha > N$. Hence, X is ergodic if and only if $\alpha > N/2$.

The next ergodic theorem will be used in later discussions.

Theorem 3.3. *Let $X = (X_t)$ be the A-diffusion. Then the following* (i)–(ii) *hold.*

(i) *Suppose that X is ergodic with an invariant probability measure μ. Then, for any $f \in L^\infty_{\mathrm{loc}}(\mathbb{R}^N)$ with $\int_{\mathbb{R}^N} |f(y)|\mu(dy) < \infty$ and for any $x \in \mathbb{R}^N$,*

$$E_x\left[f(X_t)\right] \to \int_{\mathbb{R}^N} f(y)\mu(dy) \quad \text{as } t \to \infty.$$

In particular,

$$\frac{1}{T}E_x\left[\int_0^T f(X_t)\,dt\right] \to \int_{\mathbb{R}^N} f(y)\mu(dy) \quad \text{as } T \to \infty.$$

(ii) *Suppose that X is not ergodic. Then, for any $f \in C(\mathbb{R}^N)$ such that $f(y) \to 0$ as $|y| \to \infty$ and for any $x \in \mathbb{R}^N$,*

$$\frac{1}{T}E_x\left[\int_0^T f(X_t)\,dt\right] \to 0 \quad \text{as } T \to \infty.$$

Proof. We refer to [13, Proposition 2.6] for the first convergence result in (i). The second convergence can be easily deduced from the first one. The proof of (ii) can be found in [15, Theorem 1.3.10]. We emphasize here that claim (i) holds true not only for $f \in L^\infty(\mathbb{R}^N)$ but also for unbounded f provided it is integrable with respect to the invariant probability measure μ. □

We next recall a local gradient estimate of solutions ϕ of (EP). Hereafter, we use the notation $r_\pm := \max\{\pm r, 0\}$ for $r \in \mathbb{R}$.

Theorem 3.4. *For any $R > 0$, there exists a $K > 0$ depending only on N, m in* (A2)*, and the $W^{1,\infty}$-norm of σ, b, and Σ in B_{R+1} such that for any solution (λ, ϕ) of* (EP)*,*

$$\sup_{B_R} |D\phi| \leqslant K\left\{1 + \sup_{B_{R+1}} (\lambda + \beta V)_- + |\beta| \sup_{B_{R+1}} |DV|\right\}. \tag{3.2}$$

In particular, ϕ is Lipschitz continuous on $\mathbb{R}^N$.

Proof. If σ, b and Σ are sufficiently smooth, then (3.2) can be obtained by a version of the Bernstein method taking into account that $H(x,p)$ is superlinear in p. Namely, we first derive the equation for $w := (1/2)|D\phi|^2$ and then use the maximum principle after some localization arguments. See, for instance, [16,17] or [12, Appendix A] for details. For general $\sigma, b, \Sigma \in W^{1,\infty}(\mathbb{R}^N)$, we can take the standard approximation procedure. The Lipschitz continuity of ϕ is obvious from (A3) and (3.2). □

In view of Theorem 3.4, together with the classical regularity theory for elliptic equations, one has the following solvability result for (EP).

Theorem 3.5. *Let $\lambda^*(\beta)$ be defined by (2.1). Then $\lambda^*(\beta)$ is well-defined, finite, and (EP) has a solution $\phi \in C^2(\mathbb{R}^N)$ if and only if $\lambda \leqslant \lambda^*(\beta)$. In particular,*

$$\lambda^*(\beta) = \max\{\lambda \in \mathbb{R} \mid \text{(EP) has a subsolution } \phi_0 \in \Phi\} = \max\{\lambda \in \mathbb{R} \mid \text{(EP) has a solution } \phi \in C^2(\mathbb{R}^N)\}.$$

Proof. The proof is similar to that of [12, Theorem 2.1], so that we omit to reproduce it. □

Remark 3.6. Theorem 3.5 remains valid without (A3) provided $V \in W^{1,\infty}_{\mathrm{loc}}(\mathbb{R}^N)$ and $\sup_{\mathbb{R}^N} V < \infty$.

The next proposition collects some key properties of H that are deduced from (A2).

Proposition 3.7. *Let $H = H(x,p)$ be a function satisfying* (A2). *Then the following* (i)–(v) *hold.*

(i) *$H \in W^{1,\infty}(\mathbb{R}^N \times B_R)$ for all $R > 0$, and $p \mapsto H(x,p)$ is strictly convex for all $x \in \mathbb{R}^N$.*
(ii) *$p \mapsto H(x,p)$ is superlinear growing in the sense that, for suitable $\gamma > 1$ and $C > 0$, $C^{-1}|p|^\gamma - C \leqslant H(x,p)$ and $|D_pH(x,p)| \leqslant C(1+|p|^{\gamma-1})$ for all $x, p \in \mathbb{R}^N$, and $|D_xH(x,p)| \leqslant C(1+|p|^\gamma)$, a.e. in $x \in \mathbb{R}^N$ for all $p \in \mathbb{R}^N$.*
(iii) *For any $K > 0$, there exist some $\kappa_1, \kappa_2 > 0$ such that*
$$\kappa_1|p|^m \leqslant H(x,p) \leqslant \kappa_2|p|^m, \quad x \in \mathbb{R}^N,\ |p| \leqslant K,$$
where m is the constant in (A2).
(iv) *Suppose that $1 < m \leqslant 2$. Then for any $K > 0$, there exists a $\kappa_0 > 0$ such that*
$$H(x,p+q) - H(x,p) - D_pH(x,p)\cdot q \geqslant \frac{\kappa_0}{2}|q|^2, \quad x \in \mathbb{R}^N,\ |p|,|q| \leqslant K. \tag{3.3}$$
(v) *Suppose that $m > 2$. Then for any $K > 0$ and $\varepsilon > 0$, there exists a $\kappa_\varepsilon > 0$ such that*
$$H(x,p+q) - H(x,p) - D_pH(x,p)\cdot q \geqslant \frac{\kappa_\varepsilon}{2}|q|^2 - \varepsilon, \quad x \in \mathbb{R}^N,\ |p|,|q| \leqslant K. \tag{3.4}$$

Proof. Claims (i)–(iii) are obvious from (A2). Note that (ii) is valid with $\gamma = m$. It thus remains to prove (iv) and (v). To this end, choose an arbitrary $K > 0$ and fix any $x \in \mathbb{R}^N$ and $p, q \in \mathbb{R}^N$ with $|p|, |q| \leqslant K$. We assume that $p_t := p + (1-t)q \neq 0$ for all $t \in [0,1]$. Then by Taylor's theorem, we see that

$$\begin{aligned}
&H(x,p+q) - H(x,p) - D_pH(x,p)\cdot q \\
&= \int_0^1 tD_p^2H(x,p_t)q\cdot q\,dt \\
&= \int_0^1 t\left|\Sigma^T(x)p_t\right|^{m-2}\left\{\left|\Sigma^T(x)q\right|^2 + (m-2)\frac{(\Sigma^T(x)p_t\cdot\Sigma^T(x)q)^2}{|\Sigma^T(x)p_t|^2}\right\}dt \\
&\geqslant \min\{1, m-1\}\left|\Sigma^T(x)q\right|^2 \int_0^1 t\left|\Sigma^T(x)p_t\right|^{m-2}dt,
\end{aligned}$$

where $D_p^2H(x,p)$ denotes the Hessian matrix of $H(x,p)$ with respect to p.

We first assume $1 < m \leqslant 2$ and prove (iv). Since $|\Sigma^T(x)p_t|^{m-2} \geqslant \nu_2^{(2-m)/2}|p_t|^{m-2} \geqslant \nu_2^{(2-m)/2}(2K)^{m-2}$ for all $t \in [0,1]$, where ν_2 is the constant in (A2), we have

$$H(x,p+q) - H(x,p) - D_pH(x,p)\cdot q \geqslant \frac{1}{2}(m-1)\nu_2^{2-(m/2)}(2K)^{m-2}|q|^2.$$

This inequality is still valid if $p_t = 0$ for some $t \in [0,1]$. Hence, (iv) holds with $\kappa_0 := (m-1)\nu_2^{2-(m/2)}(2K)^{m-2}$.

We next assume $m > 2$ and prove (v). We first consider the case where $|q| \geqslant 4$. Then the Lebesgue measure of the set $E := \{t \in [0,1] \mid p_t \notin B_1\}$ is greater than $1/4$. In particular,

$$\int_0^1 t\left|\Sigma^T(x)p_t\right|^{m-2}dt \geqslant \nu_2^{(m/2)-1}\int_0^1 t|p_t|^{m-2}dt \geqslant \nu_2^{(m/2)-1}\int_E t\,dt \geqslant \frac{\nu_2^{(m/2)-1}}{32}.$$

Hence, we have

$$H(x,p+q) - H(x,p) - D_pH(x,p)\cdot q \geqslant \frac{\nu_2^{m/2}}{32}|q|^2.$$

We next consider the case where $|q| \leqslant 4$. Then, for any $\varepsilon > 0$, we have

$$H(x,p+q) - H(x,p) - D_pH(x,p)\cdot q \geqslant 0 \geqslant \frac{\varepsilon}{16}|q|^2 - \varepsilon.$$

Hence, (v) holds with $\kappa_\varepsilon := \min\{\nu_2^{m/2}/16, \varepsilon/8\}$. □

Remark 3.8. Theorems 2.1 to 2.5 remain valid if $H = H(x,p)$ in (EP) satisfies (i)–(v) of Proposition 3.7 instead of (A2).

The following verification theorem connects (EP) with the ergodic control problem (1.2)–(1.3).

Proposition 3.9. *Let* (λ,ϕ) *satisfy* (EP). *Then the following* (i)–(ii) *hold.*

(i) *For any* $T > 0$, $x \in \mathbb{R}^N$, *and* $\xi \in \mathcal{A}$,

$$\lambda T + \phi(x) \leqslant E_x\left[\int_0^T \left\{L\left(X_t^\xi,\xi_t\right) - \beta V\left(X_t^\xi\right)\right\}dt + \phi\left(X_T^\xi\right)\right]. \tag{3.5}$$

(ii) *Let* $X = (X_t)$ *be the* A^ϕ*-diffusion and set* $\xi_t^* := D_pH(X_t, D\phi(X_t))$. *Then for any* $T > 0$ *and* $x \in \mathbb{R}^N$,

$$\lambda T + \phi(x) = E_x\left[\int_0^T \left\{L\left(X_t,\xi_t^*\right) - \beta V(X_t)\right\}dt + \phi(X_T)\right]. \tag{3.6}$$

Proof. We first show (i). Fix any $\xi \in \mathcal{A}$ and apply Ito's formula to $\phi(X_t^\xi)$. Then, noting that $H(x,p) - \xi\cdot p \geqslant -L(x,\xi)$ for any $x,\xi,p \in \mathbb{R}^N$, and that $|D\phi|$ is bounded in $\mathbb{R}^N$, we have

$$E_x\left[\phi\left(X_T^\xi\right)\right] - \phi(x) = E_x\left[\int_0^T \left\{(A\phi)\left(X_t^\xi\right) - \xi_t\cdot D\phi\left(X_t^\xi\right)\right\}dt\right] \geqslant \lambda T - E_x\left[\int_0^T \left\{L\left(X_t^\xi,\xi_t\right) - \beta V\left(X_t^\xi\right)\right\}dt\right],$$

from which we obtain (3.5). We next prove (ii). Since $\xi^* \in \mathcal{A}$ and

$$H\left(X_t, D\phi(X_t)\right) - \xi_t^*\cdot D\phi(X_t) = -L\left(X_t,\xi_t^*\right)$$

for all $0 < t < T$, we obtain (3.6) by the same argument as in the proof of (i). □

In what follows, we use the notation

$$F[\phi](x) := -A\phi(x) + H\big(x, D\phi(x)\big), \quad \phi \in C^2\big(\mathbb{R}^N\big).$$

The following proposition is useful in later discussions.

Proposition 3.10. *Let $\phi, \psi \in C^2(\mathbb{R}^N)$, and set $A^\phi := A - D_pH(x, D\phi(x)) \cdot D$. Then the following* (i)–(ii) *hold.*

(i) *The function $u := \phi - \psi$ satisfies $A^\phi u \leqslant F[\psi] - F[\phi]$ in $\mathbb{R}^N$.*
(ii) *Suppose that $\sup_{\mathbb{R}^N}(|D\phi| + |D\psi|) < \infty$. Then, for any $\varepsilon > 0$, there exists a $\kappa > 0$ such that $u := e^{\kappa(\phi-\psi)}$ satisfies*

$$A^\phi u \leqslant \kappa u\big(F[\psi] - F[\phi] + \varepsilon\big) \quad \text{in } \mathbb{R}^N. \tag{3.7}$$

Moreover, if $1 < m \leqslant 2$ in (A2)*, then* (3.7) *is valid with $\varepsilon = 0$.*

Proof. This proposition has been essentially proved in [10]. We reproduce the proof for the convenience of the reader. We first prove (i). Since $H(x, p)$ is convex in p, we see that $H(x, q) - H(x, p) \geqslant D_pH(x, p)(q - p)$ for all $x, p, q \in \mathbb{R}^N$. In particular,

$$\begin{aligned} A^\phi(\phi - \psi) &= A(\phi - \psi) - D_pH(x, D\phi)(D\phi - D\psi) \\ &\leqslant A\phi - A\psi + H(x, D\psi) - H(x, D\phi) = F[\psi] - F[\phi]. \end{aligned}$$

We next prove (ii). Set $K := \sup_{\mathbb{R}^N}(|D\phi| + |D\psi|)$ and fix any $\varepsilon > 0$. We first consider the case where $m > 2$. In view of Proposition 3.7(iv), there exists some $\kappa_\varepsilon > 0$ such that (3.4) holds. In particular, $w := \phi - \psi$ satisfies

$$A^\phi w \leqslant F[\psi] - F[\phi] - \frac{\kappa_\varepsilon}{2}|Dw|^2 + \varepsilon \quad \text{in } \mathbb{R}^N.$$

Set $\kappa := \kappa_\varepsilon \nu_1$ and $u := e^{\kappa w}$, where ν_1 is the constant in (A1). Then, in view of the last inequality and (A1), we have

$$\begin{aligned} A^\phi u = \kappa u\left(A^\phi w + \frac{\kappa}{2}\left|\sigma^T Dw\right|^2\right) &\leqslant \kappa u\left(F[\psi] - F[\phi] - \frac{\kappa_\varepsilon}{2}|Dw|^2 + \varepsilon + \frac{\kappa}{2\nu_1}|Dw|^2\right) \\ &= \kappa u\big(F[\psi] - F[\phi] + \varepsilon\big). \end{aligned}$$

Thus, (3.7) is valid. Suppose next that $1 < m \leqslant 2$ in (A2). Then the stronger estimate (3.3) holds in place of (3.4). By a similar argument as above, we easily see that (3.7) is valid with $\varepsilon = 0$ and $\kappa = \kappa_0\nu_1$. Hence, we have completed the proof. □

4. General results on (EP)

This section is devoted to some fundamental results on the solvability of (EP). In this section, parameter β does not play any role, so that we always assume that $\beta = 1$. The generalized principal eigenvalue of (EP) is denoted by λ^* in place of $\lambda^*(1)$.

We begin with the following theorem which will play a substantial role in succeeding sections.

Theorem 4.1. *Let $\phi_0 \in \Phi$ satisfy*

$$\lambda^* + F[\phi_0] + V \leqslant -\rho \quad \text{in } \mathbb{R}^N \setminus B_R \tag{4.1}$$

for some $\rho > 0$ and $R > 0$. Then (EP) *with $\lambda = \lambda^*$ has a unique solution $\phi \in \Phi$. Moreover, the following* (i)–(ii) *hold:*

(i) *$\phi - \phi_0 \geqslant \delta|x| - M$ in $\mathbb{R}^N$ for some $\delta > 0$ and $M > 0$.*
(ii) *The associated A^ϕ-diffusion is ergodic with an invariant probability measure μ such that $\int_{\mathbb{R}^N} e^{\gamma|x|}\mu(dx) < \infty$ for some $\gamma > 0$.*

We divide the proof of Theorem 4.1 into several steps.

Proposition 4.2. *Under the hypothesis of Theorem 4.1, there exists a solution (λ, ϕ) of (EP) such that $\phi - \phi_0 \geqslant \delta|x| - M$ in $\mathbb{R}^N$ for some $\delta > 0$ and $M > 0$.*

Proof. Set $\psi(x) := \sqrt{1 + |x|^2}$ and $f := \phi_0 + \delta_0\psi$, where $\delta_0 \in (0, 1)$ will be specified later. We consider the following elliptic equation with small parameter $\varepsilon > 0$:

$$\varepsilon v + F[v] + V = \varepsilon f \quad \text{in } \mathbb{R}^N. \tag{4.2}$$

Set $C := \sup_{\mathbb{R}^N}(|F[f]| + |V|)$. Then $f - C/\varepsilon$ and $f + C/\varepsilon$ are, respectively, sub- and supersolutions of (4.2). Hence, we can construct a solution $v = v_\varepsilon \in C^2(\mathbb{R}^N)$ of (4.2) such that

$$f - \frac{C}{\varepsilon} \leqslant v_\varepsilon \leqslant f + \frac{C}{\varepsilon} \quad \text{in } \mathbb{R}^N. \tag{4.3}$$

See, for instance, [12, Proposition 5.2] for the construction of such solution. It is also not difficult to see that $\{\varepsilon v_\varepsilon(0) \mid \varepsilon > 0\}$ is bounded and $\{v_\varepsilon - v_\varepsilon(0) \mid \varepsilon > 0\}$ is precompact in $C^2(\mathbb{R}^N)$. Indeed, the former is obvious from (4.3), and the latter can be obtained in view of the local estimate of $|Dv_\varepsilon|$ uniformly in ε, which is deduced from a minor modification of Theorem 3.4 (see, e.g. [11, Appendix A]), as well as the classical regularity estimate for elliptic equations. In particular, there exists a sequence $\{\varepsilon_n\}$ with $\varepsilon_n \to 0$ as $n \to \infty$ such that $\lambda_n := \varepsilon_n v_{\varepsilon_n}(0)$ converges to some λ and $w_n := v_{\varepsilon_n} - v_{\varepsilon_n}(0)$ converges in $C^2(\mathbb{R}^N)$ to some $\phi \in C^2(\mathbb{R}^N)$ as $n \to \infty$. Since (λ_n, w_n) solves the equation

$$\lambda_n + \varepsilon_n w_n + F[w_n] + V = \varepsilon_n f \quad \text{in } \mathbb{R}^N, \qquad w_n(0) = 0, \tag{4.4}$$

we see that (λ, ϕ) is a solution of (EP) by letting $n \to \infty$ in (4.4). Furthermore, since $\lambda \leqslant \lambda^*$, we may assume, by renumbering $\{\varepsilon_n\}$ if necessary, that $\lambda_n \leqslant \lambda^* + \rho/2$ for all $n \geqslant 1$.

To estimate the lower bound of ϕ, we set $M := \sup_{n\geqslant 1} \max_{\overline{B}_R}(|w_n| + |\phi_0| + |\psi|) < \infty$, where $R > 0$ is the constant in (4.1). We claim that, if δ_0 is sufficiently small, then for any $n \geqslant 1$ and $\delta \in (0, \delta_0)$, we have

$$\phi_0 + \delta\psi - M \leqslant w_n \quad \text{in } \mathbb{R}^N. \tag{4.5}$$

Note that (4.5) yields the estimate $\phi - \phi_0 \geqslant \delta|x| - M$ in $\mathbb{R}^N$ by sending $n \to \infty$, so it remains to prove (4.5). To this end, we observe that $z := \phi_0 + \delta\psi - M$, with $\delta \in (0, \delta_0)$, satisfies

$$F[z] = F[\phi_0] - \delta A\psi + H(x, D\phi_0 + \delta D\psi) - H(x, D\phi_0).$$

Since $A\psi$, $D\psi$ and $D\phi_0$ are bounded in $\mathbb{R}^N$, we see that $F[z] \leqslant F[\phi_0] + \delta K$ in $\mathbb{R}^N$ for some $K > 0$ not depending on δ. Using (4.1) and noting that $z < f$ in $\mathbb{R}^N$, we have

$$\lambda_n + \varepsilon_n z + F[z] + V - \varepsilon_n f \leqslant \lambda^* + \frac{\rho}{2} + F[\phi_0] + V + \delta K \leqslant -\frac{\rho}{2} + \delta K \quad \text{in } \mathbb{R}^N \setminus B_R.$$

Fix $\delta_0 > 0$ so small that $\delta_0 K < \rho/4$. Then we obtain

$$\lambda_n + \varepsilon_n z + F[z] + V \leqslant \varepsilon_n f - \frac{\rho}{4} \quad \text{in } \mathbb{R}^N \setminus B_R. \tag{4.6}$$

This inequality, together with (4.4), yields that $z \leqslant w_n$ in $\mathbb{R}^N$. Indeed, $z \leqslant w_n$ in $\overline{B}_R$ by the definition of M. Furthermore, in view of (4.3), we see that $\inf_{\mathbb{R}^N}(w_n - f) > -\infty$. This implies that $(w_n - z)(x) = (w_n - f)(x) + (\delta_0 - \delta)\psi(x) + M \to \infty$ as $|x| \to \infty$. In particular, for each $n \geqslant 1$, there exists a bounded domain D_n, with $\overline{B}_R \subset D_n$, such that $z \leqslant w_n$ in $\mathbb{R}^N \setminus D_n$. Since w_n and z satisfy (4.4) and (4.6), respectively, we can apply the comparison theorem in the bounded domain $D_n \setminus \overline{B}_R$ to conclude that $z \leqslant w_n$ in $D_n \setminus \overline{B}_R$. Thus, we obtain $z \leqslant w_n$ in $\mathbb{R}^N$, and the proof is complete. □

Proposition 4.3. *Let (λ, ϕ) be the solution of (EP) constructed in Proposition 4.2. Then the A^ϕ-diffusion is ergodic with an invariant probability measure μ such that $\int_{\mathbb{R}^N} e^{\gamma|x|}\mu(dx) < \infty$ for some $\gamma > 0$.*

Proof. Since $\lambda \leqslant \lambda^*$ by the definition of λ^*, we see in view of Proposition 3.10(i) and (4.1) that

$$A^\phi(\phi - \phi_0) \leqslant F[\phi_0] - F[\phi] \leqslant -\lambda^* - \rho + \lambda \leqslant -\rho \quad \text{in } \mathbb{R}^N \setminus B_R.$$

Furthermore, since $(\phi - \phi_0)(x) \to \infty$ as $|x| \to \infty$, we can apply Theorem 3.1(iii) to conclude that the A^ϕ-diffusion is ergodic. To prove the integrability property, let $\mu = \mu(dx)$ be the invariant probability measure for the A^ϕ-diffusion X. Fix any $\varepsilon > 0$ such that $\varepsilon < \rho$. Then, by Ito's formula and Proposition 3.10(ii), there exists a $\kappa > 0$ such that $u := e^{\kappa(\phi-\phi_0)}$ satisfies

$$\begin{aligned} E_x\big[u(X_{T\wedge\tau_n})\big] - u(x) &= E_x\Bigg[\int_0^{T\wedge\tau_n} A^\phi u(X_t)\,dt\Bigg] \\ &\leqslant E_x\Bigg[\int_0^{T\wedge\tau_n} \kappa u(X_t)\big(F[\phi_0](X_t) - F[\phi](X_t) + \varepsilon\big)\,dt\Bigg] \\ &\leqslant -\kappa(\rho-\varepsilon)E_x\Bigg[\int_0^{T\wedge\tau_n} u(X_t)1_{\mathbb{R}^N\setminus B_R}(X_t)\,dt\Bigg] + \kappa K_1 E_x[T\wedge\tau_n], \end{aligned}$$

where $\tau_n := \inf\{t > 0 \mid X_t \notin B_n\}$ for $n \in \mathbb{N}$, and $K_1 := \sup_{B_R}\{u(F[\phi_0] + V + \lambda^* + \varepsilon)\}$. Sending $n \to \infty$ and noting that $u = e^{\kappa(\phi-\phi_0)} > 0$, we obtain

$$\kappa(\rho-\varepsilon)E_x\Bigg[\int_0^T u(X_t)\,dt\Bigg] \leqslant u(x) + \kappa K_1 T + \kappa(\rho-\varepsilon)K_2 T,$$

where $K_2 := \sup_{B_R} u$. We divide both sides by T and let $T \to \infty$. Then, in view of Theorem 3.3(i), the left-hand side can be estimated as

$$\int_{\mathbb{R}^N} (u1_{B_n})(y)\mu(dy) = \lim_{T\to\infty}\frac{1}{T}E_x\Bigg[\int_0^T (u1_{B_n})(X_t)\,dt\Bigg] \leqslant \limsup_{T\to\infty}\frac{1}{T}E_x\Bigg[\int_0^T u(X_t)\,dt\Bigg]$$

for any $n \in \mathbb{N}$. Noting that $\phi - \phi_0 \geqslant \delta|x| - M$ in $\mathbb{R}^N$ for some $\delta > 0$ and $M > 0$, we have

$$e^{-\kappa M}\int_{\mathbb{R}^N} e^{\kappa\delta|y|}\mu(dy) \leqslant \lim_{n\to\infty}\int_{\mathbb{R}^N} e^{\kappa(\phi-\phi_0)(y)}1_{B_n}(y)\mu(dy) \leqslant \frac{K_1}{\rho-\varepsilon} + K_2.$$

Hence, $\int_{\mathbb{R}^N} e^{\gamma|y|}\mu(dy) < \infty$ with $\gamma := \kappa\delta$, and we have completed the proof. □

Proposition 4.4. *Let (λ, ϕ) be the solution of (EP) constructed in Proposition 4.2. Then $\lambda = \lambda^*$.*

Proof. We argue by contradiction. Suppose that $\lambda < \lambda^*$. Then the A^ϕ-diffusion should be transient. Indeed, let ϕ^* be a solution of (EP) with $\lambda = \lambda^*$. Fix an $\varepsilon > 0$ such that $\lambda + \varepsilon < \lambda^*$. Then, in view of Proposition 3.10(ii), there exists a $\kappa > 0$ such that $u := e^{\kappa(\phi-\phi^*)}$ satisfies

$$A^\phi u \leqslant \kappa u\big(F\big[\phi^*\big] - F[\phi] + \varepsilon\big) = \kappa u\big(\lambda + \varepsilon - \lambda^*\big) < 0 \quad \text{in } \mathbb{R}^N. \tag{4.7}$$

This implies that u does not attain a minimum in $\mathbb{R}^N$. Otherwise, by the strong maximum principle, u is constant in $\mathbb{R}^N$, and so is $\phi - \phi^*$. But this is a contradiction. Hence, there exists an $x_0 \in \mathbb{R}^N \setminus B_1$ such that $u(x_0) < \inf_{|x|=1} u(x)$. Applying Theorem 3.1(i), we conclude that the A^ϕ-diffusion is transient. But, this is inconsistent with Proposition 4.3 claiming that the A^ϕ-diffusion is ergodic. Hence, $\lambda = \lambda^*$. □

We are now in a position to prove Theorem 4.1.

Proof of Theorem 4.1. It remains to prove that (EP) with $\lambda = \lambda^*$ has at most one solution. Let (λ, ϕ) be the solution of (EP) constructed in Proposition 4.2. In view of Proposition 4.4, we have $\lambda = \lambda^*$. Furthermore, Proposition 4.3

implies that the A^ϕ-diffusion X is ergodic with an invariant probability measure μ such that $\int_{\mathbb{R}^N} e^{\gamma|x|}\mu(dx) < \infty$ for some $\gamma > 0$.

We now set $\xi_t^* := D_pH(X_t, D\phi(X_t))$, and fix any solution ψ of (EP) with $\lambda = \lambda^*$. Then, in view of Proposition 3.9, we see that

$$\psi(x) \leqslant E_x\left[\int_0^T \{L(X_t, \xi_t^*) - V(X_t)\}\, dt + \psi(X_T)\right] - \lambda^* T = \phi(x) + E_x\big[(\psi - \phi)(X_T)\big].$$

Letting $T \to \infty$ and applying Theorem 3.3(i), we have

$$(\psi - \phi)(x) \leqslant \int_{\mathbb{R}^N} (\psi - \phi)(y)\mu(dy), \quad x \in \mathbb{R}^N.$$

Since $\operatorname{supp}\mu = \mathbb{R}^N$ and $x \in \mathbb{R}^N$ is arbitrary, we conclude that $\psi - \phi$ is constant in $\mathbb{R}^N$. Recalling that $\phi(0) = \psi(0) = 0$, we obtain $\psi = \phi$ in $\mathbb{R}^N$. Hence, we have completed the proof. □

Remark 4.5. By a careful reading of the arguments above, we see that Theorem 4.1 holds true without assuming that $V(x) \to 0$ as $|x| \to \infty$. Moreover, the positivity of γ in Theorem 4.1(ii) is locally uniform with respect to the $W^{1,\infty}$-norm of V. More precisely, suppose that there exists a $C > 0$ such that $\sup_{\mathbb{R}^N}(|V| + |DV|) \leqslant C$. Then there exist some $\gamma > 0$ and $K > 0$ depending on ρ and R in (4.1) and C, but independent of the specific choice of V, such that $\int_{\mathbb{R}^N} e^{\gamma|y|}\mu(dy) \leqslant K$.

As an easy corollary of Theorem 4.1, we obtain the following.

Theorem 4.6. *Let V_1, V_2 be two functions satisfying* (A3)*, and let λ_i^* denote the generalized principal eigenvalue of* (EP) *with $V = V_i$ for $i = 1, 2$. Assume that $\lambda_1^* < \lambda_2^*$. Then* (EP) *with $\lambda = \lambda_1^*$ and $V = V_1$ has a unique solution $\phi_1 \in \Phi$, and the A^{ϕ_1}-diffusion is ergodic with an invariant probability measure μ such that $\int_{\mathbb{R}^N} e^{\gamma|x|}\mu(dx) < \infty$ for some $\gamma > 0$. Moreover, for any solution ϕ_2 of* (EP) *with $\lambda = \lambda_2^*$ and $V = V_2$, there exist some $\delta > 0$ and $M > 0$ such that $\phi_1 - \phi_2 \geqslant \delta|x| - M$ in $\mathbb{R}^N$.*

Proof. Set $\rho := (\lambda_2^* - \lambda_1^*)/2 > 0$, and let ϕ_2 be a solution of (EP) with $\lambda = \lambda_2^*$ and $V = V_2$. Choose an $R > 0$ such that $V_1 - V_2 \leqslant \rho$ in $\mathbb{R}^N \setminus B_R$. Then we have

$$\lambda_1^* + F[\phi_2] + V_1 = \lambda_2^* - 2\rho + F[\phi_2] + V_2 + (V_1 - V_2) \leqslant -\rho \quad \text{in } \mathbb{R}^N \setminus B_R.$$

In particular, condition (4.1) in Theorem 4.1 is satisfied with ϕ_2 in place of ϕ_0. Hence, we have completed the proof. □

The next proposition will be used in Section 6.

Proposition 4.7. *Let $1 < m \leqslant 2$ in* (A2)*. Assume that there exists a subsolution $\psi \in C^2(\mathbb{R}^N)$ of* (EP) *with $\lambda = \lambda^*$ such that $\sup_{\mathbb{R}^N}|D\psi| < \infty$ and $\lambda^* + F[\psi](y) + V(y) < 0$ for some $y \in \mathbb{R}^N$. Then, for any solution ϕ of* (EP) *with $\lambda = \lambda^*$, the associated A^ϕ-diffusion is transient.*

Proof. Fix any solution ϕ of (EP) with $\lambda = \lambda^*$, and set $K := \sup_{\mathbb{R}^N}(|D\phi| + |D\psi|)$. Since $1 < m \leqslant 2$ in (A2), we see by Proposition 3.10(ii) that there exists a $\kappa > 0$ such that $u := e^{\kappa(\phi - \psi)}$ satisfies

$$A^\phi u \leqslant \kappa u\big(F[\psi] - F[\phi]\big) \leqslant 0 \quad \text{in } \mathbb{R}^N. \tag{4.8}$$

We claim here that u does not have a minimum in $\mathbb{R}^N$. Otherwise, in view of (4.8) and the strong maximum principle, we see that u, and therefore, $\phi - \psi$ is constant in $\mathbb{R}^N$. But, this does not agree with the assumption of ψ. Hence, there exists an $x_0 \in \mathbb{R}^N \setminus \overline{B}_1$ such that $u(x_0) < \min_{\bar{B}_1} u$. Applying Theorem 3.1(i), we conclude that the A^ϕ-diffusion is transient. □

By considering the contraposition of Proposition 4.7, we obtain a uniqueness result for (EP).

Corollary 4.8. *Let $1 < m \leqslant 2$ in* (A2). *Assume that there exists a solution ϕ of* (EP) *with $\lambda = \lambda^*$ such that the associated A^ϕ-diffusion is recurrent. Then* (EP) *with $\lambda = \lambda^*$ has no subsolution in Φ which is strict at some point in $\mathbb{R}^N$. In particular, ϕ is the unique solution of* (EP) *with $\lambda = \lambda^*$.*

Remark 4.9. A probabilistic interpretation of Proposition 4.7 can be stated as follows. Let $\phi \in C^2(\mathbb{R}^N)$ be a solution of (EP) with $\lambda = \lambda^*$, and let $X = (X_t)$ be the associated A^ϕ-diffusion. Fix any subsolution $\psi \in \Phi$ of (EP) with $\lambda = \lambda^*$ and set $f := -(\lambda^* + F[\psi] + V) \geqslant 0$. Then, in view of Proposition 3.10(ii) and Ito's formula, there exists a $\kappa > 0$ such that, for any $x \in \mathbb{R}^N$ and $T > 0$, the function $u := e^{\kappa(\phi-\psi)}$ satisfies

$$E_x\big[u(X_T)\big] - u(x) = E_x\left[\int_0^T A^\phi u(X_t)\,dt\right] \leqslant -\kappa E_x\left[\int_0^T (uf)(X_t)\,dt\right]. \tag{4.9}$$

We now choose any bounded domain $D \subset \mathbb{R}^N$ such that $D \subset \operatorname{supp}(f)$. Then, sending $T \to \infty$ in (4.9), we easily see that

$$E_x\left[\int_0^\infty 1_D(X_t)\,dt\right] < \infty \tag{4.10}$$

for all $x \in \mathbb{R}^N$, which implies the transience of X. This makes a striking contrast to the conclusion of Theorem 4.1(ii), where the A^ϕ-diffusion is ergodic, and therefore, the left-hand side of (4.10) is infinite for all $x \in \mathbb{R}^N$.

Remark 4.10. In the proof of Proposition 4.7, the assumption $1 < m \leqslant 2$ is crucial since we required (3.7) with $\varepsilon = 0$. We do not know, in general, whether Proposition 4.7 as well as Corollary 4.8 is still valid for $m > 2$.

5. Characterization of $\lambda^* = \Lambda$

In this section we prove Theorems 2.1 and 2.2. Let $\lambda^*(\beta)$ be the generalized principal eigenvalue of (EP). Recall that $\mathcal{S}(\beta)$ denotes the set of solutions of (EP) with $\lambda = \lambda^*(\beta)$.

Proposition 5.1. *The mapping $\beta \to \lambda^*(\beta)$ is concave.*

Proof. Let $\beta_1 < \beta_2$ and $\phi_i \in \mathcal{S}(\beta_i)$ for $i = 1, 2$. Then, by the convexity of $H(x, p)$ in p, the function $\phi := (1-\delta)\phi_1 + \delta\phi_2$ for $\delta \in (0, 1)$ satisfies

$$F[\phi] + \big\{(1-\delta)\beta_1 + \delta\beta_2\big\}V \leqslant (1-\delta)\big(F[\phi_1] + \beta_1 V\big) + \delta\big(F[\phi_2] + \beta_2 V\big) = -\big\{(1-\delta)\lambda^*(\beta_1) + \delta\lambda^*(\beta_2)\big\}.$$

In particular, $\lambda^*((1-\delta)\beta_1 + \delta\beta_2) \geqslant (1-\delta)\lambda^*(\beta_1) + \delta\lambda^*(\beta_2)$ by the definition of $\lambda^*(\beta)$. Hence, $\beta \mapsto \lambda^*(\beta)$ is concave. □

We next investigate the asymptotic behavior of $\lambda^*(\beta)$ as $|\beta| \to \infty$.

Proposition 5.2. *The mapping $\beta \mapsto \lambda^*(\beta)$ is not constant in $\mathbb{R}$. More precisely, the following* (i)–(ii) *hold.*

(i) *Suppose that* $\sup_{\mathbb{R}^N} V > 0$. *Then $\lambda^*(\beta) \to -\infty$ as $\beta \to \infty$.*
(ii) *Suppose that* $\inf_{\mathbb{R}^N} V < 0$. *Then $\lambda^*(\beta) \to -\infty$ as $\beta \to -\infty$.*

Proof. We rewrite (EP) with $\lambda = \lambda^*(\beta)$ in the divergence form, namely,

$$\lambda^*(\beta) - \operatorname{div}\big(a(x)D\phi\big) - g(x)\cdot D\phi + H(x, D\phi) + \beta V = 0 \quad \text{in } \mathbb{R}^N,$$

where $a_{ij} := (1/2)(\sigma\sigma^T)_{ij}$ and $g_i := b_i - \sum_{k=1}^N \partial a_{ik}/\partial x_k$ for $i, j = 1, \dots, N$. Let $\eta \in C_c^\infty(\mathbb{R}^N)$ be any nonnegative function satisfying $\int_{\mathbb{R}^N} \eta^{m^*}(x)\,dx = 1$, where $m^* := m/(m-1) > 1$. Then, for any $\phi \in \mathcal{S}(\beta)$, we see that

$$\lambda^*(\beta) + \int_{\mathbb{R}^N} H\big(x, D\phi(x)\big)\eta(x)^{m^*}\,dx + \beta \int_{\mathbb{R}^N} V(x)\eta(x)^{m^*}\,dx$$
$$= -\int_{\mathbb{R}^N} a(x)D\phi(x)\cdot D\big(\eta(x)^{m^*}\big)\,dx + \int_{\mathbb{R}^N} g(x)D\phi(x)\eta(x)^{m^*}\,dx.$$

Using (A2), Young's inequality, and the relation $1/m + 1/m^* = 1$, we have

$$\lambda^*(\beta) + \kappa_1 \int_{\mathbb{R}^N} \big|D\phi(x)\big|^m \eta(x)^{m^*}\,dx + \beta \int_{\mathbb{R}^N} V(x)\eta(x)^{m^*}\,dx$$
$$\leqslant m^* \int_{\mathbb{R}^N} \big|D\phi(x)\big|\big|a(x)D\eta(x)\big|\eta(x)^{m^*-1}\,dx + \int_{\mathbb{R}^N} \big|D\phi(x)\big|\big|g(x)\big|\eta(x)^{m^*}\,dx$$
$$\leqslant \frac{\kappa_1}{2} \int_{\mathbb{R}^N} \big|D\phi(x)\big|^m \eta(x)^{m^*}\,dx + C_0 \int_{\mathbb{R}^N} \Big(\big|a(x)D\eta(x)\big|^{m^*} + \big|g(x)\big|^{m^*}\eta(x)^{m^*}\Big)\,dx$$

for some $\kappa_1 > 0$ and $C_0 > 0$ not depending on β. In particular, there exists a $C > 0$ independent of β such that

$$\lambda^*(\beta) + \beta \int_{\mathbb{R}^N} V(x)\eta(x)^{m^*}\,dx \leqslant C.$$

We now assume $\sup_{\mathbb{R}^N} V > 0$ and choose η so that $\int_{\mathbb{R}^N} V(x)\eta(x)^{m^*}\,dx > 0$. Then, by sending $\beta \to \infty$, we see that $\lambda^*(\beta) \to -\infty$ as $\beta \to \infty$. Hence, (i) is valid. Similarly, if $\inf_{\mathbb{R}^N} V < 0$, then we have $\lambda^*(\beta) \to -\infty$ as $\beta \to -\infty$ by choosing η so that $\int_{\mathbb{R}^N} V(x)\eta(x)^{m^*}\,dx < 0$. Hence, we have completed the proof. □

Let us now study the differentiability of $\lambda^*(\beta)$ with respect to β. Recall that $\bar{\lambda} := \sup\{\lambda^*(\beta) \mid \beta \in \mathbb{R}\}$.

Proposition 5.3. *Let β_i be such that $\lambda^*(\beta_i) < \bar{\lambda}$ for $i = 1, 2$, and let ϕ_i and $\mu_i = \mu_i(dx)$ be, respectively, the unique solution of (EP) with $\lambda = \lambda^*(\beta_i)$ and the invariant probability measure associated with the A^{ϕ_i}-diffusion. Then*

$$-(\beta_2 - \beta_1) \int_{\mathbb{R}^N} V(x)\mu_2(dx) \leqslant \lambda^*(\beta_2) - \lambda^*(\beta_1) \leqslant -(\beta_2 - \beta_1) \int_{\mathbb{R}^N} V(x)\mu_1(dx).$$

Proof. Set $v := \phi_2 - \phi_1$. Then, we see that

$$\lambda^*(\beta_2) - \lambda^*(\beta_1) - Av + H(x, D\phi_2) - H(x, D\phi_1) + (\beta_2 - \beta_1)V = 0.$$

In view of the convexity $H(x,p) - H(x,q) \geqslant D_pH(x,q)\cdot(p-q)$, we have

$$\lambda^*(\beta_2) - \lambda^*(\beta_1) - A^{\phi_1}v + (\beta_2 - \beta_1)V \leqslant 0. \tag{5.1}$$

Let $X = (X_t)$ denote the A^{ϕ_1}-diffusion, and apply Ito's formula to $v(X_t)$. Then, noting (5.1) and the fact that $|Dv|$ is bounded in $\mathbb{R}^N$, we have

$$E_x\big[v(X_T)\big] - v(x) = E_x\Bigg[\int_0^T A^{\phi_1}v(X_t)\,dt\Bigg] \geqslant \big(\lambda^*(\beta_2) - \lambda^*(\beta_1)\big)T + (\beta_2 - \beta_1)E_x\Bigg[\int_0^T V(X_t)\,dt\Bigg].$$

Since $\int_{\mathbb{R}^N} |v(y)|\mu_1(dy) < \infty$ by Theorem 4.6, we conclude in view of Theorem 3.3(i) that

$$\lambda^*(\beta_2) - \lambda^*(\beta_1) \leqslant \lim_{T\to\infty} \frac{E_x[v(X_T)]}{T} - \lim_{T\to\infty} \frac{\beta_2 - \beta_1}{T} E_x\Bigg[\int_0^T V(X_t)\,dt\Bigg] = -(\beta_2 - \beta_1) \int_{\mathbb{R}^N} V(x)\mu_1(dx).$$

Changing the role of β_1 and β_2, we also have

$$\lambda^*(\beta_1) - \lambda^*(\beta_2) \leqslant -(\beta_1 - \beta_2) \int_{\mathbb{R}^N} V(x)\mu_2(dx).$$

Hence, we have completed the proof. □

Proposition 5.4. *$\lambda^*(\beta)$ is differentiable at any $\beta \in I := \{\beta \mid \lambda^*(\beta) < \bar{\lambda}\}$. Moreover, the following representation formula holds:*

$$\frac{d\lambda^*}{d\beta}(\beta) = -\int_{\mathbb{R}^N} V(y)\mu_\beta(dy), \quad \beta \in I, \tag{5.2}$$

where $\mu_\beta = \mu_\beta(dx)$ denotes the invariant probability measure associated with the A^ϕ-diffusion for $\phi \in \mathcal{S}(\beta)$.

Proof. Fix any $\beta \in I$. Let $\{\beta_n\}$ be any sequence such that $\beta_n \to \beta$ as $n \to \infty$. We may assume without loss of generality that $\beta_n \in I$ for all n. Let $\phi_n \in \mathcal{S}(\beta_n)$, and let $\mu_n = \mu_n(dx)$ be the invariant probability measure for the A^{ϕ_n}-diffusion. By virtue of Proposition 5.3, it suffices to prove that μ_n converges weakly to $\mu = \mu_\beta$ as $n \to \infty$.

Observe first that the family $\{\mu_n\}$ is tight since $\sup_n \int_{\mathbb{R}^N} e^{\gamma|x|}\mu_n(dx) < \infty$ for some $\gamma > 0$ (see Remark 4.5). By choosing a suitable subsequence of $\{\beta_n\}$ if necessary, we may assume that μ_n converges weakly to a probability measure μ_∞ as $n \to \infty$. Since μ_n satisfies

$$\int_{\mathbb{R}^N} \big(A\eta(x) - D_pH\big(x, D\phi_n(x)\big) \cdot D\eta(x)\big)\mu_n(dx) = 0, \quad \eta \in C_c^\infty\big(\mathbb{R}^N\big), \tag{5.3}$$

and $\{\phi_n\}$ is precompact in $C^2(\mathbb{R}^N)$, which can be verified from Theorem 3.4 and the classical regularity theory for elliptic equations, we see by taking $n \to \infty$ in (5.3) that

$$\int_{\mathbb{R}^N} \big(A\eta(x) - D_pH\big(x, D\phi(x)\big) \cdot D\eta(x)\big)\mu_\infty(dx) = 0, \quad \eta \in C_c^\infty\big(\mathbb{R}^N\big),$$

where ϕ is the unique solution of (EP) with $\lambda = \lambda^*(\beta)$. The last equality, together with the uniqueness of the invariant probability measure for the A^ϕ-diffusion, yields that $\mu_\infty = \mu$. In particular, μ_n converges weakly to μ as $n \to \infty$. Hence, we obtain (5.2) by virtue of Proposition 5.3. □

In view of the propositions above, we obtain claim (i) of Theorem 2.1.

Remark 5.5. Proposition 5.4 shows that $\lambda^*(\beta)$ is differentiable on $\mathbb{R} \setminus \partial I$. We conjecture that $\lambda^*(\beta)$ is differentiable at $\beta \in \partial I$ if and only if there is no $\phi \in \mathcal{S}(\beta)$ such that the associated A^ϕ-diffusion is ergodic, although we do not have any rigorous proof at this stage. Note that this conjecture is true in the "linear" case, more precisely, in the case where $\sigma = \Sigma = I$ and $b = 0$ (e.g., [25]). The method in [25] is based on the analysis of the linear Schrödinger operator $-(1/2)\Delta - \beta V$ which is completely different from the one developed in this paper.

Now, we proceed to the proof of (ii) and (iii) in Theorem 2.1. Let $\Lambda(\beta) = \inf_{\xi \in \mathcal{A}} J_\beta(\xi)$ be the optimal value of the ergodic control problem (1.2)–(1.3). We first claim that one side inequality $\lambda^*(\beta) \geqslant \Lambda(\beta)$ is always valid.

Proposition 5.6. *For any $\beta \in \mathbb{R}$, one has $\lambda^*(\beta) \geqslant \Lambda(\beta)$.*

Proof. Fix any β. Let $\{f_n\}$ be a family of C^2-functions such that $\sup_n |Df_n| < \infty$, $0 \leqslant f_n \leqslant |\lambda^*(\beta)| + \sup_{\mathbb{R}^N} |\beta V| + 2$ in $\mathbb{R}^N$, $f_n = 0$ in B_n, and $f_n = |\lambda^*(\beta)| + \sup_{\mathbb{R}^N} |\beta V| + 2$ in $\mathbb{R}^N \setminus B_{n+1}$ for each n. Clearly, $\{f_n\}$ is decreasing and converges to zero in $C(\mathbb{R}^N)$ as $n \to \infty$.

Let $\lambda_n = \lambda_n(\beta)$ be the generalized principal eigenvalue of

$$\lambda + F[\phi] + \beta V - f_n = 0 \quad \text{in } \mathbb{R}^N, \qquad \phi(0) = 0. \tag{5.4}$$

Notice here that, for each n, λ_n is well defined and finite in view of Remark 3.6. Furthermore, by the choice of $\{f_n\}$, we easily see that $\lambda_n \geqslant \lambda_{n+1} \geqslant \lambda^*(\beta)$ for all n. In particular, there exists a $\lambda_\infty \geqslant \lambda^*(\beta)$ such that $\lambda_n \to \lambda_\infty$ as $n \to \infty$.

We now claim that $\lambda_\infty = \lambda^*(\beta)$. Let ϕ_n be the solution of (5.4) with $\lambda = \lambda_n$ such that $\phi_n(0) = 0$. Since $\{\phi_n\}$ is precompact in $C^2(\mathbb{R}^N)$, we can extract a subsequence of $\{\phi_n\}$, still denoted by $\{\phi_n\}$, such that ϕ_n converges in $C^2(\mathbb{R}^N)$ to a $\phi \in C^2(\mathbb{R}^N)$ as $n \to \infty$. Noting $f_n \to 0$ in $C(\mathbb{R}^N)$ and $\lambda_n \to \lambda_\infty$ as $n \to \infty$, we see that (λ_∞, ϕ) is a solution of (EP). By the definition of $\lambda^*(\beta)$, we have $\lambda_\infty \leqslant \lambda^*(\beta)$. Hence, $\lambda_\infty = \lambda^*(\beta)$. In what follows, we may assume without loss of generality that $\lambda_n \leqslant \lambda^*(\beta) + 1$ for all n.

Let us investigate the ergodicity of the A^{ϕ_n}-diffusion. To this end, observe first that $\phi_0 \equiv 0$ satisfies

$$\lambda_n + F[\phi_0] + \beta V - f_n \leqslant \lambda^*(\beta) + 1 + \beta V - \Bigl(\bigl|\lambda^*(\beta)\bigr| + \max_{\mathbb{R}^N} |\beta V| + 2\Bigr) \leqslant -1 \quad \text{in } \mathbb{R}^N \setminus B_{n+1}.$$

Thus, we can apply Theorem 4.1 (see also Remark 4.5) to conclude that ϕ_n is the unique solution of (5.4) with $\lambda = \lambda_n$ such that $\phi_n(0) = 0$, and that the associated A^{ϕ_n}-diffusion is ergodic with an invariant probability measure μ_n satisfying $\int_{\mathbb{R}^N} e^{\gamma|x|}\mu_n(dx) < \infty$ for some $\gamma > 0$.

Let $X = (X_t)$ denote the A^{ϕ_n}-diffusion. Set $\xi_t^* := D_p H(X_t, D\phi_n(X_t))$. Then, in view of Theorem 3.3(i), Proposition 3.9(ii) with $\beta V - f_n$ in place of βV, and the nonnegativity of f_n, we have

$$\begin{aligned}\lambda_n &= \lim_{T\to\infty} \frac{1}{T} E_x\Biggl[\int_0^T \bigl\{L\bigl(X_t, \xi_t^*\bigr) - \beta V(X_t) + f_n(X_t)\bigr\}\,dt + \phi_n(X_T)\Biggr] \\ &\geqslant \liminf_{T\to\infty} \frac{1}{T} E_x\Biggl[\int_0^T \bigl\{L\bigl(X_t, \xi_t^*\bigr) - \beta V(X_t)\bigr\}\,dt\Biggr] \geqslant \Lambda(\beta).\end{aligned}$$

Sending $n \to \infty$, we obtain $\lambda^*(\beta) \geqslant \Lambda(\beta)$. Hence, we have completed the proof. □

We next discuss the validity of the opposite inequality $\lambda^*(\beta) \leqslant \Lambda(\beta)$. For this purpose, let us consider the following stochastic control problem of finite time horizon:

$$\begin{aligned}&\text{Minimize} \quad J_\beta(\xi; T, x) := E_x\Biggl[\int_0^T \bigl\{L\bigl(X_t^\xi, \xi_t\bigr) - \beta V\bigl(X_t^\xi\bigr)\bigr\}\,dt\Biggr] \\ &\text{subject to} \quad dX_t^\xi = -\xi_t\,dt + b(X_t)\,dt + \sigma\bigl(X_t^\xi\bigr)\,dW_t.\end{aligned}$$

It is well known that the value function

$$u_\beta(T, x) := \inf_{\xi \in \mathcal{A}} J_\beta(\xi; T, x) \tag{5.5}$$

belongs to $C([0,\infty) \times \mathbb{R}^N) \cap C^{1,2}((0,\infty) \times \mathbb{R}^N)$ and u_β is a solution of the HJB equation

$$\begin{cases} \dfrac{\partial u}{\partial t} - Au + H(x, Du) + \beta V = 0 & \text{in } (0,\infty) \times \mathbb{R}^N, \\ u(0, \cdot) = 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{5.6}$$

Moreover, for each $T > 0$, there exists a $C_T > 0$ such that $|u_\beta(t, x)| \leqslant C_T(1 + |x|)$ in $[0, T] \times \mathbb{R}^N$. We also mention here that the comparison theorem holds for classical sub- and supersolutions of (5.6). That is, if u_1 and u_2 are, respectively, sub- and supersolutions of (5.6) such that $u_1(0, \cdot) \leqslant u_2(0, \cdot)$ in $\mathbb{R}^N$ and $|u_i(t, x)| \leqslant C_T(1 + |x|)$ in $[0, T] \times \mathbb{R}^N$ for $i = 1, 2$, then $u_1 \leqslant u_2$ in $[0, T] \times \mathbb{R}^N$. See, for instance, [8, Chapter IV] for details.

We begin with establishing a few propositions.

Proposition 5.7. *The function $\beta \mapsto \Lambda(\beta)$ is concave. Moreover, $\Lambda(0) = 0$.*

Proof. Fix any $\beta_0 < \beta_1$, and set $\beta_\delta := (1-\delta)\beta_0 + \delta\beta_1$ for $\delta \in (0,1)$. Then, for any $S > 0$,

$$\inf_{T \geqslant S} \frac{J_{\beta_\delta}(\xi; T, x)}{T} \geqslant (1-\delta) \inf_{T \geqslant S} \frac{J_{\beta_0}(\xi; T, x)}{T} + \delta \inf_{T \geqslant S} \frac{J_{\beta_1}(\xi; T, x)}{T}.$$

Sending $S \to \infty$ and then taking the infimum over all $\xi \in \mathcal{A}$, we conclude that $\Lambda(\beta_\delta) \geqslant (1-\delta)\Lambda(\beta_0) + \delta\Lambda(\beta_1)$ for all $\delta \in (0,1)$. Hence, $\Lambda(\beta)$ is concave.

We next show that $\Lambda(0) = 0$. Since $L \geqslant 0$ in $\mathbb{R}^{2N}$, we see that $\Lambda(0) \geqslant 0$. On the other hand, by choosing $\xi \equiv 0$, we have

$$\Lambda(0) \leqslant \lim_{T\to\infty} \frac{1}{T} E_x\left[\int_0^T L\big(X_t^0, 0\big)\,dt\right] = 0.$$

Hence, $\Lambda(0) = 0$, and we have completed the proof. □

Proposition 5.8. *Let β be such that $\lambda^*(\beta) < 0$, and let $u_\beta(T,x)$ be the value function defined by (5.5). Then*

$$\lambda^*(\beta) \leqslant \liminf_{T\to\infty} \frac{u_\beta(T,x)}{T}.$$

Proof. Set $\phi_0(x) := \log(1+|x|^2)$. Note that ϕ_0 satisfies (4.1) with $\rho := (1/2)|\lambda^*(\beta)|$ for some $R > 0$ since $\lambda^*(\beta) < 0$ and $|F[\phi_0](x)| \to 0$ and $V(x) \to 0$ as $|x| \to \infty$. Then, applying the same argument as in the proof of Theorem 4.1, we can find a decreasing sequence $\{\varepsilon_n\}$ converging to zero as $n \to \infty$ and a family of solutions (λ_n, w_n) of

$$\lambda + \varepsilon_n w + F[w] + \beta V = \varepsilon_n \phi_0 \quad \text{in } \mathbb{R}^N, \qquad w(0) = 0,$$

such that $\lambda_n \to \lambda^*(\beta)$ as $n \to \infty$ and $\phi_0 - M \leqslant w_n \leqslant \phi_0 + C_n$ in $\mathbb{R}^N$ for some $M > 0$ not depending on n, and for some $C_n > 0$ which may depend on n. Hereafter, we may assume without loss of generality that $\lambda_n < 0$ for all n.

Now, fix any $\gamma > 0$ and set

$$v(t,x) := \big(1 - e^{-t}\big)\Big\{(1-\delta)w_n - \delta\sqrt{1+|x|^2} - K\Big\} + \big(\lambda^*(\beta) - \gamma\big)t,$$

where $n \geqslant 1$, $\delta \in (0,1)$, and $K > 0$ will be specified later. Since $|F[-\sqrt{1+|x|^2}]| \leqslant C$ in $\mathbb{R}^N$ for some $C > 0$, we have

$$\frac{\partial v}{\partial t} + F[v] + \beta V \leqslant \lambda^*(\beta) - \lambda_n - \gamma + \varepsilon_n M + \delta|\beta V| + \delta C + e^{-t}\Big(\phi_0 - \delta\sqrt{1+|x|^2} + |\beta V| + C_n - K\Big).$$

By choosing n, δ, and then K so that the right-hand side becomes less than zero, we see that v is a subsolution of (5.6) with $v(0,\cdot) \equiv 0$ in $\mathbb{R}^N$. It is also obvious from the definition of v that, for any $T > 0$, there exists a $C_T > 0$ such that $|v(t,x)| \leqslant C_T(1+|x|)$ in $[0,T] \times \mathbb{R}^N$. We can thus apply the comparison theorem for solutions of (5.6) to conclude that $v \leqslant u_\beta$ in $[0,\infty) \times \mathbb{R}^N$. In particular,

$$\lambda^*(\beta) - \gamma = \lim_{T\to\infty} \frac{v(T,x)}{T} \leqslant \liminf_{T\to\infty} \frac{u_\beta(T,x)}{T}.$$

Since $\gamma > 0$ is arbitrary, we obtain the desired estimate. □

Set $J_- := \{\beta \in \mathbb{R} \mid \lambda^*(\beta) < 0\}$, $J := \{\beta \in \mathbb{R} \mid \lambda^*(\beta) = 0\}$, and $J_+ := \{\beta \in \mathbb{R} \mid \lambda^*(\beta) > 0\}$. We investigate the validity of $\lambda^* = \Lambda$ on each set.

Proposition 5.9. $\lambda^*(\beta) = \Lambda(\beta)$ *for all* $\beta \in J_-$.

Proof. Fix any $\beta \in J_-$. In view of Propositions 5.6, it suffices to show that $\lambda^*(\beta) \leqslant \Lambda(\beta)$. Let $u_\beta(T,x)$ be the value function defined by (5.5). Then, by the definition of u_β, we see that

$$\inf_{T\geqslant S}\frac{u_\beta(T,x)}{T}\leqslant \inf_{T\geqslant S}\frac{1}{T}E_x\left[\int_0^T\left\{L\left(X_t^\xi,\xi_t\right)-\beta V(X_t)\right\}dt\right],\quad S>0,\ \xi\in\mathcal{A}.$$

Sending $S\to\infty$, and then taking the infimum over all $\xi\in\mathcal{A}$, we obtain

$$\liminf_{T\to\infty}\frac{u_\beta(T,x)}{T}\leqslant \Lambda(\beta).$$

This inequality, together with Proposition 5.8, implies that $\lambda^*(\beta)\leqslant\Lambda(\beta)$. Hence, we have completed the proof. □

Proposition 5.10. *$\lambda^*(\beta)=\Lambda(\beta)$ for all $\beta\in J$.*

Proof. By the concavity of $\lambda^*(\beta)$, the form of J turns out to be one of the following: (a) $J=\{\beta_0\}$ for some $\beta_0\in\mathbb{R}$; (b) $J=\{\beta_-,\beta_+\}$ for some $\beta_\pm\in\mathbb{R}$ with $\beta_-<\beta_+$; (c) $J=[\beta_-,\beta_+]$ for some $\beta_\pm\in\mathbb{R}$ with $\beta_-<\beta_+$; (d) $J=(-\infty,\beta_0]$ for some $\beta_0\in\mathbb{R}$; (e) $J=[\beta_0,\infty)$ for some $\beta_0\in\mathbb{R}$.

Suppose first that (a) holds. Then, by Proposition 5.2, there exists a sequence $\{\beta_n\}$ such that $\lambda^*(\beta_n)<0$ for all $n\geqslant 1$ and $\beta_n\to\beta_0$ as $n\to\infty$. Since $\lambda^*(\beta_n)=\Lambda(\beta_n)$ in view of Proposition 5.9, we obtain $\lambda^*(\beta_0)=\Lambda(\beta_0)$ by sending $n\to\infty$. Hence, the claim is valid in case (a). We can also prove the identity $\lambda^*(\beta)=\Lambda(\beta)$ in case (b) since the same argument can be applied to β_- and β_+ in place of β_0. We now assume (c). Since two functions $\lambda^*(\beta)$ and $\Lambda(\beta)$ are both concave and their values coincide in $J_-=\mathbb{R}\setminus[\beta_-,\beta_+]$, we observe in combination with Proposition 5.6 that $\lambda^*(\beta)=\Lambda(\beta)=0$ for all $\beta\in[\beta_-,\beta_+]$. Hence, the claim is true in case (c). We next suppose that (d) holds. Note by virtue of Proposition 5.2 that this situation happens only when $V\geqslant 0$ in $\mathbb{R}^N$. In particular, by the definition of Λ and the nonnegativity of V, we have $\Lambda(\beta)\geqslant 0$ for all $\beta<0$. Since λ^* and Λ are both concave and they coincide in (β_0,∞), we conclude, in view of Proposition 5.6, that $\lambda^*(\beta)=\Lambda(\beta)=0$ in $(-\infty,\beta_0]$. Thus, the claim is valid in case (d). We can also apply the same argument in case (e). Hence, we have completed the proof. □

We now compare the values of λ^* and Λ in J_+. For this purpose, we first remark the following result.

Proposition 5.11. *Suppose that the A-diffusion is recurrent. Then $\lambda^*(0)=0$.*

Proof. Since $\lambda^*(0)\geqslant 0$, it suffices to prove that $\lambda^*(0)\leqslant 0$. We argue by contradiction assuming that $\lambda^*(0)>0$. Let $\phi\in\mathcal{S}(0)$. Note that ϕ cannot be constant. Fix any $\varepsilon\in(0,\lambda^*(0))$ and set $K:=\sup_{\mathbb{R}^N}|D\phi|$. Then, from Proposition 3.10(ii), there exists a $\kappa>0$ such that $u:=e^{-\kappa\phi}$ satisfies

$$Au=A^0u\leqslant\kappa u\left(F[\phi]-F[0]+\varepsilon\right)=\kappa u\left(\varepsilon-\lambda^*(0)\right)<0\quad\text{in }\mathbb{R}^N.$$

Since the A-diffusion is recurrent, we can see, as in the proof of Proposition 4.7, that u attains a minimum in $\mathbb{R}^N$. Applying the strong maximum principle, we conclude that u, and therefore, ϕ is constant in $\mathbb{R}^N$. This is a contradiction. Hence, $\lambda^*(0)\leqslant 0$. □

Proposition 5.12. *The following* (i) *and* (ii) *hold.*

(i) *Suppose that $\lambda^*(0)>0$. Then $\Lambda(\beta)=0$ for all $\beta\in J_+$.*
(ii) *Suppose that $\lambda^*(0)=0$. Then $\lambda^*(\beta)=\Lambda(\beta)$ for all $\beta\in J_+$.*

Proof. We first assume that $\lambda^*(0)>0$. Then $\Lambda(\beta)\leqslant 0$ for all $\beta\in\mathbb{R}$. Indeed, since the A-diffusion is transient by Proposition 5.11, we see in view of Theorem 3.3(ii) that

$$\Lambda(\beta)\leqslant\liminf_{T\to\infty}\frac{1}{T}E_x\left[\int_0^T\left\{L\left(X_t^0,0\right)-\beta V\left(X_t^0\right)\right\}dt\right]=-\limsup_{T\to\infty}\frac{1}{T}E_x\left[\int_0^T\beta V\left(X_t^0\right)dt\right]=0.$$

Hence, $\Lambda(\beta)\leqslant 0$ for all $\beta\in\mathbb{R}$. On the other hand, by the concavity of $\beta\mapsto\lambda^*(\beta)$, the form of J_+ becomes one of the following (a)–(c): (a) $J_+=(-\infty,\beta_+)$ for some $\beta_+>0$; (b) $J_+=(\beta_-,\infty)$ for some $\beta_-<0$; (c) $J_+=(\beta_-,\beta_+)$ for some $\beta_\pm\in\mathbb{R}$ with $\beta_-<0<\beta_+$.

Let us consider (a). Note that this situation happens only when $V \geqslant 0$ in $\mathbb{R}^N$ by Proposition 5.2. We can also see that $\Lambda(\beta) \geqslant 0$ for all $\beta < 0$ by the definition of Λ and the nonnegativity of V. In particular, $\Lambda(\beta) = 0$ in $(-\infty, 0]$. Since $\Lambda(\beta_+) = \lambda^*(\beta_+) = 0$ and $\Lambda(\beta) = \lambda^*(\beta) < 0$ in (β_+, ∞), we conclude that $\Lambda(\beta) = 0$ for all $\beta \in (-\infty, \beta_+)$. Hence, claim (i) is true in case (a). By a symmetric argument, we can also see that (i) is true in case (b). Let us assume (c). Then $\Lambda(\beta_\pm) = \lambda^*(\beta_\pm) = 0$. Since Λ is concave and $\Lambda(0) = 0$, we conclude that $\Lambda(\beta) = 0$ in (β_-, β_+). Hence, (i) is proved.

Suppose next that $\lambda^*(0) = 0$. Fix any $\beta \in J_+$ and let $\phi \in \mathcal{S}(\beta)$. Since $\lambda^*(0) = 0 < \lambda^*(\beta)$, we see by Theorem 4.6 with $(V_1, \phi_1) = (0, 0)$ and $(V_2, \phi_2) = (\beta V, \phi)$ that ϕ is bounded above in $\mathbb{R}^N$. We now fix any $\xi \in \mathcal{A}$. Then, in view of Proposition 3.9(i) and the fact that ϕ is bounded above, we have

$$\lambda^*(\beta) \leqslant \liminf_{T\to\infty} \frac{1}{T} E_x\left[\int_0^T \left\{L\left(X_t^\xi, \xi_t\right) - \beta V\left(X_t^\xi\right)\right\} dt + \phi\left(X_T^\xi\right)\right] \leqslant \liminf_{T\to\infty} \frac{1}{T} E_x\left[\int_0^T \left\{L\left(X_t^\xi, \xi_t\right) - \beta V\left(X_t^\xi\right)\right\}\right].$$

Taking the infimum over all $\xi \in \mathcal{A}$, we obtain $\lambda^*(\beta) \leqslant \Lambda(\beta)$, and therefore $\lambda^*(\beta) = \Lambda(\beta)$ by Proposition 5.6. Hence, we have completed the proof. □

It is now easy to prove Theorem 2.1 by combining Propositions 5.9, 5.10, and 5.12. We can also prove Theorem 2.2 as follows.

Proof of Theorem 2.2. Since $\lambda^*(\beta) < \bar{\lambda}$, we see in view of Theorem 4.6 that (EP) with $\lambda = \lambda^*(\beta)$ has a unique solution $\phi \in C^2(\mathbb{R}^N)$, and that the associated A^ϕ-diffusion X is ergodic with an invariant probability density μ such that $e^{\gamma|x|}\mu \in L^1(\mathbb{R}^N)$ for some $\gamma > 0$. Applying Proposition 3.9(ii), together with the ergodicity of X, we conclude that

$$\lambda^*(\beta) = \lim_{T\to\infty} \frac{1}{T} E_x\left[\int_0^T \left\{L\left(X_t, \xi_t^*\right) - \beta V(X_t)\right\}\right],$$

where $\xi_t^* := D_p H(X_t, D\phi(X_t))$. This implies that ξ^* is optimal if $\lambda^*(\beta) = \Lambda(\beta)$. It is also obvious that ξ^* is not optimal if $\lambda^*(\beta) > \Lambda(\beta)$. In the latter case, we have $\Lambda(\beta) = 0$ by virtue of Theorem 2.1, and similarly as in the proof of Proposition 5.12(i), we see that $\xi_t \equiv 0$ is optimal. Hence, we have completed the proof. □

Remark 5.13. It can happen that $\lambda^*(0) > 0$. For instance, let $\psi \in C^3(\mathbb{R}^N)$ satisfy $\psi(x) = |x|$ in $\mathbb{R}^N \setminus B_1$, and set $\sigma := I$ and $b := \alpha D\psi$, where $\alpha > 0$ will be specified later. Note that the A-diffusion is transient for any $\alpha > 0$. Indeed, let $u \in C^2(\mathbb{R}^N)$ be such that $u(x) = |x|^{-1}$ in $\mathbb{R}^N \setminus B_1$. Then

$$Au = \frac{1}{2}\Delta u + \alpha D\psi \cdot Du = \frac{3-N}{2|x|^3} - \frac{\alpha}{|x|^2} \leqslant 0 \quad \text{in } \mathbb{R}^N \setminus B_R$$

for some $R > 0$. Applying Theorem 3.1(i), we conclude that the A-diffusion is transient.

We now claim that $\lambda^*(0) > 0$ if α is sufficiently large. We argue by contradiction supposing that $\lambda^*(0) = 0$. Then, since $H(x, p) \leqslant \kappa_2|p|^m$ in $\mathbb{R}^{2N}$ for some $\kappa_2 > 0$, we have

$$F[\psi] \leqslant -\frac{N-1}{2|x|} - \alpha + \kappa_2 \quad \text{in } \mathbb{R}^N \setminus B_1.$$

In particular, if $\alpha > \kappa_2$, then there exist some $\rho > 0$ and $R > 0$ such that $\lambda^*(0) + F[\psi] \leqslant -\rho$ in $\mathbb{R}^N \setminus B_R$. This implies that condition (4.1) in Theorem 4.1 is satisfied with $\phi_0 = \psi$. Hence, we conclude that $\phi \equiv 0$ is the unique solution of (EP) with $\lambda = \lambda^*(0)$ and $V \equiv 0$, and that the $A^\phi = A$-diffusion is ergodic. But, this is a contradiction since the A-diffusion is transient for any $\alpha > 0$. Hence, $\lambda^*(0) > 0$ as far as $\alpha > \kappa_2$.

Remark 5.14. Let us consider the special case where $H(x, p) = (1/2)|\sigma^T(x)p|^2$, and let $u(T, x)$ be the solution of (5.6). Then $v := e^{-u}$ turns out to be a solution to the Cauchy problem for linear parabolic equation

$$\begin{cases} \dfrac{\partial v}{\partial t} - Av - \beta V v = 0 & \text{in } (0,\infty)\times\mathbb{R}^N, \\ v(0,\cdot)\equiv 1 & \text{in } \mathbb{R}^N. \end{cases}$$

In particular, by the Feynman–Kac formula for v, we obtain

$$\frac{u(T,x)}{T} = -\frac{1}{T}\log E_x\left[\exp\left(\beta\int_0^T V(X_t)\,dt\right)\right]. \tag{5.7}$$

The long time behavior of the right-hand side of (5.7) was studied in [14], while we investigated that of the left-hand side, which is valid for any H satisfying (A2). Hence, Theorem 2.1 covers [14] as a particular case.

6. Qualitative properties of $\lambda^*(\beta)$

In this section we prove Theorems 2.3, 2.4, and 2.5. Namely, we investigate the recurrence and transience of diffusion X governed by (1.5) as well as qualitative properties of $\lambda^*(\beta)$ around the origin. Throughout this section, we always assume that $\bar\lambda := \sup\{\lambda^*(\beta) \mid \beta\in\mathbb{R}\} = 0$. We begin with a few sufficient conditions so that $\bar\lambda = 0$.

Proposition 6.1. *Suppose one of the following* (a) *or* (b) *holds. Then* $\bar\lambda = 0$.

(a) *The A-diffusion is transient and* $\lambda^*(0) = 0$.
(b) *The A-diffusion is null-recurrent.*

Proof. We first assume (a). We argue by contradiction assuming that $\lambda^*(\beta) > 0 = \lambda^*(0)$ for some $\beta \neq 0$. Then, by Theorem 4.6, $\phi \equiv 0$ is the unique solution of (EP) with $\beta = 0$ and $\lambda = \lambda^*(0)$. Furthermore, the associated A^0-diffusion is ergodic. But this is inconsistent with (a). Hence, $\bar\lambda = 0$.

We next assume (b). Suppose that $\lambda^*(\beta) > 0$ for some $\beta \neq 0$. Since $\lambda^*(0) = 0$ by virtue of Proposition 5.11, we can apply the same argument as above to deduce a contradiction. Hence, $\bar\lambda = 0$. □

The following proposition provides another sufficient condition in terms of the asymptotic behavior of b as $|x|\to\infty$.

Proposition 6.2. *Suppose that* $|x|^{-1}(b(x)\cdot x) \to 0$ *as* $|x|\to\infty$. *Then* $\bar\lambda = 0$.

Proof. Let $\psi\in C^3(\mathbb{R}^N)$ be any function such that $|D\psi|\leqslant 1$ in $\mathbb{R}^N$ and $\psi(x) = |x|$ in $\mathbb{R}^N\setminus B_1$. Fix any $\varepsilon\in(0,1)$ and let $X^\varepsilon = (X_t^\varepsilon)$ be the solution of

$$dX_t^\varepsilon = \varepsilon D\psi\big(X_t^\varepsilon\big)\,dt + b\big(X_t^\varepsilon\big)\,dt + \sigma\big(X_t^\varepsilon\big)\,dW_t. \tag{6.1}$$

Note that X^ε is transient. Indeed, choose any $u\in C^2(\mathbb{R}^N)$ such that $u(x) = |x|^{-1}$ in $\mathbb{R}^N\setminus B_1$. Then

$$Au + \varepsilon D\psi\cdot Du = -\frac{\operatorname{tr}(\sigma\sigma^T)}{|x|^3} + 3\frac{|\sigma^T(x)x|^2}{|x|^5} - |x|^{-2}\left(\frac{b(x)\cdot x}{|x|} + \varepsilon\right) \quad \text{in } \mathbb{R}^N\setminus B_1.$$

Since $|x|^{-1}b(x)\cdot x\to 0$ as $|x|\to\infty$, we conclude that $Au + \varepsilon D\psi\cdot Du \leqslant 0$ in $\mathbb{R}^N\setminus B_R$ for some $R>0$. Hence, X^ε is transient in view of Theorem 3.1(i).

We next show that

$$\limsup_{T\to\infty}\frac{E_x[|X_T^\varepsilon|]}{T} \leqslant 4\varepsilon. \tag{6.2}$$

Since $b(x)\cdot x\leqslant \varepsilon|x|$ in $\mathbb{R}^N\setminus B_R$ for some $R>0$, we have $(\varepsilon D\psi(x) + b(x))\cdot x \leqslant 2\varepsilon|x|$ in $\mathbb{R}^N\setminus B_R$. In particular,

$$\big(\varepsilon D\psi(x) + b(x)\big)\cdot x \leqslant \sup_{B_R}\big(|D\psi| + |b|\big)R + 2\varepsilon|x| \quad \text{in } \mathbb{R}^N.$$

We now apply Ito's formula to $|X_t^\varepsilon|^2$. Then,

$$E_x\big[\big|X^\varepsilon_{T\wedge\tau_n}\big|^2\big]-|x|^2=E_x\Bigg[\int_0^{T\wedge\tau_n}\big\{2\big(\varepsilon D\psi\big(X_t^\varepsilon\big)+b\big(X_t^\varepsilon\big)\big)\cdot X_t^\varepsilon+\operatorname{tr}\big(\sigma\sigma^T\big(X_t^\varepsilon\big)\big)\big\}\,dt\Bigg]$$

$$\leqslant 2R\Big(1+\sup_{\mathbb{R}^N}|b|\Big)T+4\varepsilon E_x\Bigg[\int_0^T\big|X^\varepsilon_{t\wedge\tau_n}\big|\,dt\Bigg]+\sup_{\mathbb{R}^N}\operatorname{tr}\big(\sigma\sigma^T\big)T,$$

where $\tau_n:=\inf\{t>0\mid |X_t^\varepsilon|>n\}$. This implies that

$$\sup_{0\leqslant t\leqslant T}E_x\big[\big|X^\varepsilon_{t\wedge\tau_n}\big|^2\big]\leqslant CT+4\varepsilon T\sup_{0\leqslant t\leqslant T}E_x\big[\big|X^\varepsilon_{t\wedge\tau_n}\big|\big]\leqslant CT+8\varepsilon^2T^2+\frac12\sup_{0\leqslant t\leqslant T}E_x\big[\big|X^\varepsilon_{t\wedge\tau_n}\big|^2\big]$$

for some $C>0$ not depending on n and T. Thus, we conclude that

$$E_x\big[\big|X_T^\varepsilon\big|\big]\leqslant\liminf_{n\to\infty}E_x\big[\big|X^\varepsilon_{T\wedge\tau_n}\big|\big]\leqslant\liminf_{n\to\infty}\sqrt{E_x\big[\big|X^\varepsilon_{T\wedge\tau_n}\big|^2\big]}\leqslant\sqrt{2CT+16\varepsilon^2T^2}.$$

Dividing both sides by T and sending $T\to\infty$, we obtain (6.2).

We finally show that $\bar\lambda=0$. Since $\lambda^*(0)\geqslant 0$, it suffices to prove that $\lambda^*(\beta)\leqslant 0$ for all β. Fix any $\varepsilon\in(0,1)$, and set $\xi_t^\varepsilon:=\varepsilon D\psi(X_t^\varepsilon)$. Observe that $\xi^\varepsilon\in\mathcal{A}$. Let $\phi\in\mathcal{S}(\beta)$. Since $L(x,\xi)\leqslant C|\xi|^{m^*}$ and $|\phi(x)|\leqslant C(1+|x|)$ for some $C>0$, we see that

$$\phi(x)+\lambda^*(\beta)T\leqslant E_x\Bigg[\int_0^T\big\{L\big(X_t^\varepsilon,\xi_t^\varepsilon\big)-\beta V\big(X_t^\varepsilon\big)\big\}\,dt+\phi\big(X_T^\varepsilon\big)\Bigg]$$

$$\leqslant\varepsilon^{m^*}CT-\beta E_x\Bigg[\int_0^T V\big(X_t^\varepsilon\big)\,dt\Bigg]+C\big(1+E_x\big[\big|X_T^\varepsilon\big|\big]\big).$$

We now divide both sides by T and send $T\to\infty$. Then, noting (6.2) and Theorem 3.3(ii), we have $\lambda^*(\beta)\leqslant C\varepsilon^{m^*}+4C\varepsilon$. Letting $\varepsilon\to 0$, we obtain $\lambda^*(\beta)\leqslant 0$. □

We now set $J:=\{\beta\in\mathbb{R}\mid\lambda^*(\beta)=0\}$. By the concavity of $\lambda^*(\beta)$ and the assumption that $\bar\lambda=0$, we observe that J is a nonempty, connected, and closed subset of $\mathbb{R}$.

Proposition 6.3. *Assume that $\bar\lambda=0$. Then the following* (i)–(iii) *hold.*

(i) *$J=(-\infty,\overline{\beta}]$ for some $\overline{\beta}\geqslant 0$ if $V\geqslant 0$ in $\mathbb{R}^N$.*
(ii) *$J=[\underline{\beta},\infty)$ for some $\underline{\beta}\leqslant 0$ if $V\leqslant 0$ in $\mathbb{R}^N$.*
(iii) *$J=[\underline{\beta},\overline{\beta}]$ for some $\underline{\beta}\leqslant 0\leqslant\overline{\beta}$ if V is sign-changing.*

Proof. We first assume that $V\geqslant 0$ in $\mathbb{R}^N$. Then the mapping $\beta\mapsto\lambda^*(\beta)$ is non-increasing by the definition of $\lambda^*(\beta)$. Since $\lambda^*(0)\geqslant 0$ and $\bar\lambda=0$ by assumption, we have $\lambda^*(\beta)=0$ for all $\beta<0$. On the other hand, in view of Proposition 5.2, we have $\lambda^*(\beta)\to-\infty$ as $\beta\to\infty$. In particular, there exists a $\overline{\beta}\geqslant 0$ such that $J=(-\infty,\overline{\beta}]$. Suppose next that $V\leqslant 0$ in $\mathbb{R}^N$. Then, similarly as above, we see that $J=[\underline{\beta},\infty)$ for some $\underline{\beta}\leqslant 0$. We finally assume that V is sign-changing. Then, by Proposition 5.2 and the concavity of $\lambda^*(\beta)$, we conclude that $J=[\underline{\beta},\overline{\beta}]$ for some $\underline{\beta}\leqslant 0\leqslant\overline{\beta}$. Hence, we have completed the proof. □

We now prove our main theorems. We first show Theorem 2.3.

Proof of Theorem 2.3. It suffices to prove (ii) since (i) is obvious from Theorem 4.6. In what follows, we assume $1<m\leqslant 2$ in (A2) and $\operatorname{Int}J\neq\emptyset$.

Fix any $\beta \in \operatorname{Int} J$ and $\phi \in \mathcal{S}(\beta)$. Suppose first that $V \geqslant 0$ in $\mathbb{R}^N$. Then there exists a $\overline{\beta} \geqslant 0$ such that $J = (-\infty, \overline{\beta}]$. In particular, any $\psi \in \mathcal{S}(\overline{\beta})$ is a subsolution of (EP) with $\lambda = \lambda^*(\beta) = 0$ which is strict at some point in $\mathbb{R}^N$. Applying Proposition 4.7, we conclude that the A^ϕ-diffusion is transient. By the symmetric argument as above, we also see that (ii) is valid when $V \leqslant 0$ in $\mathbb{R}^N$.

It thus remains to consider the case where V is sign-changing. In this case, $J = [\underline{\beta}, \overline{\beta}]$ for some $\underline{\beta} < \overline{\beta}$. Fix any $\psi_1 \in \mathcal{S}(\underline{\beta})$ and $\psi_2 \in \mathcal{S}(\overline{\beta})$. Then there exists a $\delta \in (0,1)$ such that $\beta = (1-\delta)\underline{\beta} + \delta\overline{\beta}$. Set $\psi := (1-\delta)\psi_1 + \delta\psi_2$. In view of the convexity of $H(x,p)$ in p, we easily see that ψ is a subsolution of $F[\psi] + \beta V = 0$ in $\mathbb{R}^N$. Moreover, there exists a $y \in \mathbb{R}^N$ such that $F[\psi](y) + \beta V(y) < 0$. Indeed, if this is not true, then ψ is a solution of $F[\psi] + \beta V = 0$ in $\mathbb{R}^N$. In particular, $H(x, D\psi(x)) = (1-\delta)H(x, D\psi_1(x)) + \delta H(x, D\psi_2(x))$ for all $x \in \mathbb{R}^N$. Since $H(x,p)$ is strictly convex in p, we have $D\psi_1 = D\psi_2$ in $\mathbb{R}^N$. This implies that $\psi_1 - \psi_2$ is constant in $\mathbb{R}^N$, and therefore $\overline{\beta} V = \underline{\beta} V$ in $\mathbb{R}^N$. But this is a contradiction since $\underline{\beta} < \overline{\beta}$ and $V \not\equiv 0$. Hence, ψ is a subsolution of $F[\psi] + \beta V = 0$ in $\mathbb{R}^N$ which is strict at some $y \in \mathbb{R}^N$. Applying Proposition 4.7 again, we conclude that the A^ϕ-diffusion is transient. Hence, we have completed the proof of Theorem 2.3. $\square$

We turn to the proof of Theorem 2.4. To this end, we establish a few propositions.

Lemma 6.4. *Assume that $\overline{\lambda} = 0$. Then, for any $\beta \notin J$, there exists a $C > 0$ such that the unique solution ϕ of (EP) with $\lambda = \lambda^*(\beta)$ satisfies*

$$C^{-1}|x| - C \leqslant \phi(x) \leqslant C\bigl(1 + |x|\bigr), \quad x \in \mathbb{R}^N.$$

Proof. Since $\lambda^*(0) = 0$, we obtain the first inequality by choosing $(V_1, \phi_1) = (\beta V, \phi)$ and $(V_2, \phi_2) = (0, 0)$ in Theorem 4.6. The second inequality is obvious from the Lipschitz continuity of ϕ. $\square$

Proposition 6.5. *Let $\overline{\lambda} = 0$, and let G_α be the function defined by (2.2). Assume that*

$$\lim_{|x|\to\infty} |x|^{m^*} V(x) = 0, \qquad \liminf_{|x|\to\infty} G_{m^*}(x) > 0,$$

where $m^* := m/(m-1)$. Then, for any $\beta \in \partial J$, there exists a $\phi \in \mathcal{S}(\beta)$ such that $\inf_{\mathbb{R}^N} \phi > -\infty$. Moreover, if $m \geqslant 2$ in (A2)*, then $\phi(x) \to \infty$ as $|x| \to \infty$, and the associated A^ϕ-diffusion is recurrent.*

Proof. Let $\beta \in \partial J$. It suffices to prove the claim when $\beta = \max J < \infty$. The opposite case $\beta = \min J > -\infty$ can be proved similarly. Set $\gamma := (m-2)/(m-1) \in (-\infty, 1)$ and choose any $\phi_0 \in C^3(\mathbb{R}^N)$ such that

$$\phi_0(x) = \begin{cases} c \log|x| & (m=2), \\ \frac{c}{\gamma}|x|^\gamma & (m \neq 2) \end{cases} \quad \text{in } \mathbb{R}^N \setminus B_2,$$

where $c \in (0,1)$ is some constant which will be specified later. Note that ϕ_0 is bounded below on $\mathbb{R}^N$, and $\delta|x| - \phi_0(x) \to \infty$ as $|x| \to \infty$ for any $\delta > 0$. Furthermore, observing that $\gamma - 2 = m(\gamma - 1) = -m^*$, we have

$$F[\phi_0](x) \leqslant c|x|^{-m^*}\bigl(-G_{m^*}(x) + Kc^{m-1}\bigr) \quad \text{in } \mathbb{R}^N \setminus B_2$$

for some $K > 0$. Since $G_{m^*} \geqslant \varepsilon$ in $\mathbb{R}^N \setminus B_{R_0}$ for some $\varepsilon > 0$ and $R_0 > 0$, we can choose $c > 0$ and $\rho > 0$ so that

$$F[\phi_0] \leqslant -\rho|x|^{-m^*} \quad \text{in } \mathbb{R}^N \setminus B_{R_0}. \tag{6.3}$$

Hereafter, we fix such c and ρ.

Let $\{\beta_n\}$ be any decreasing sequence such that $\beta_n \to \beta$ as $n \to \infty$. We may assume without loss of generality that $\beta < \beta_n < \beta + 1$ for all $n \geqslant 1$. Then, in view of (6.3) and the assumption on V, there exists an $R > R_0$ such that for all $\mu \in (1/2, 1)$ and $n \geqslant 1$,

$$\mu F[\phi_0] + \beta_n V \leqslant 0 \quad \text{in } \mathbb{R}^N \setminus B_R. \tag{6.4}$$

Let $\phi_n \in \mathcal{S}(\beta_n)$ and define $M_1 > 0$ by

$$M_1 := \sup\{|\phi_0(x)| + |\phi_n(x)| \mid n \geqslant 1,\ x \in B_R\},$$

where $R > 0$ is the constant in (6.4). Notice that $M_1 < \infty$ since $|\phi_n(x)| = |\phi_n(x) - \phi_n(0)| \leqslant C|x|$ in $\mathbb{R}^N$ for some $C > 0$ not depending on n.

We now claim that $\phi_n \geqslant \phi_0 - M_1$ in $\mathbb{R}^N$ for all n. Indeed, if we set $w_0 := \phi_0 - M_1$, then $\phi_n \geqslant w_0$ in B_R by the definition of M_1. Since $\phi_n(x)$ is linearly growing as $|x| \to \infty$ by Lemma 6.4, there exists a bounded domain D_n in $\mathbb{R}^N$ such that $B_R \subset D_n$ and $\phi_n \geqslant w_0$ in $\mathbb{R}^N \setminus D_n$. Observing that $\lambda^*(\beta_n) < 0$ for any $n \geqslant 1$ and that ϕ_0 satisfies (6.4), we have

$$F[w_0] + \beta_n V = F[\phi_0] + \beta_n V \leqslant 0 < -\lambda^*(\beta_n) \quad \text{in } D_n \setminus \bar{B}_R.$$

This implies that w_0 is a strict subsolution of $\lambda^*(\beta_n) + F[\phi] + \beta_n V = 0$ in $D_n \setminus \bar{B}_R$. Since ϕ_n is a solution to the same equation, we obtain $\phi_n \geqslant w_0$ in $D_n \setminus \bar{B}_R$ by the standard comparison theorem. Hence, $\phi_n \geqslant \phi_0 - M_1$ in $\mathbb{R}^N$. On the other hand, in view of Theorem 3.4 and the classical regularity theory for elliptic equations, we can extract a subsequence of $\{\phi_n\}$, still denoted by $\{\phi_n\}$, such that ϕ_n converges in $C^2(\mathbb{R}^N)$ to a function ϕ as $n \to \infty$. Sending $n \to \infty$ and noting that $\lambda^*(\beta_n) \to \lambda^*(\beta)$ as $n \to \infty$, we conclude that $\phi \in \mathcal{S}(\beta)$ and $\inf_{\mathbb{R}^N}(\phi - \phi_0) > -\infty$. Since ϕ_0 is bounded below on $\mathbb{R}^N$, we obtain $\inf_{\mathbb{R}^N} \phi > -\infty$.

We now assume $m \geqslant 2$ in (A2) and prove the recurrence of the associated A^ϕ-diffusion. Note that $\phi_0(x) \to \infty$ as $|x| \to \infty$, and therefore $\phi(x) \to \infty$ as $|x| \to \infty$. Fix any $\mu \in (1/2, 1)$. Then, observing that $\mu F[\phi_0] + \beta V \leqslant 0$ in $\mathbb{R}^N \setminus B_R$ by virtue of (6.4) and that $\lambda^*(\beta) = 0$, we have

$$A^\phi(\phi - \mu\phi_0) \leqslant \mu F[\phi_0] - F[\phi] \leqslant \mu F[\phi_0] + \beta V + \lambda^*(\beta) \leqslant 0 \quad \text{in } \mathbb{R}^N \setminus B_R.$$

Since $(\phi - \mu\phi_0)(x) \to \infty$ as $|x| \to \infty$, we conclude in view of Theorem 3.1(ii) that the A^ϕ-diffusion is recurrent. Hence, we have completed the proof. □

We are now in a position to prove Theorem 2.4.

Proof of Theorem 2.4. We first show (i). Since $G_\alpha(x) \leqslant 0$ in $\mathbb{R}^N \setminus B_R$ for some $\alpha \leqslant 2$ and $R > 0$, we see that the A-diffusion is recurrent. Indeed, let $u \in C^2(\mathbb{R}^N)$ be any function satisfying

$$u(x) = \begin{cases} \log|x| & (\alpha = 2), \\ (2-\alpha)^{-1}|x|^{2-\alpha} & (\alpha < 2) \end{cases} \quad \text{in } \mathbb{R}^N \setminus B_2.$$

Then, a straightforward calculation shows that $Au(x) = |x|^{-\alpha} G_\alpha(x) \leqslant 0$ in $\mathbb{R}^N \setminus B_R$ for some $R > 0$. Since $u(x) \to \infty$ as $|x| \to \infty$, we conclude in view of Theorem 3.1(ii) that the A-diffusion is recurrent.

We now assume $0 \in \operatorname{Int} J$ and deduce a contradiction. Suppose first that $V \geqslant 0$ in $\mathbb{R}^N$. Then, in view of Proposition 6.3(i), there exists a $\overline{\beta} \geqslant 0$ such that $J = (-\infty, \overline{\beta}]$. By assumption, we have $\overline{\beta} > 0$. Since any $\phi \in \mathcal{S}(\overline{\beta})$ is a subsolution of (EP) with $\beta = 0$ and $\lambda = \lambda^*(0) = 0$ which is strict at some point in $\mathbb{R}^N$, we see in view of Proposition 4.7 that the $A = A^0$-diffusion is transient. But this is a contradiction. Hence, $\partial J = \{0\}$. Suppose next that $V \leqslant 0$ in $\mathbb{R}^N$. Then, by a similar argument as above, we can also see that $\partial J = \{0\}$. We finally consider the case where V is sign-changing. In this case, as in the proof of Theorem 2.3, we can construct a subsolution ψ of (EP) with $\beta = 0$ and $\lambda = \lambda^*(0) = 0$ which is strict at some point in $\mathbb{R}^N$. In particular, the $A = A^0$-diffusion is transient. But this is again a contradiction. Hence, $\partial J = \{0\}$. The proof is complete.

We next prove (ii). Suppose contrarily that $0 \in \partial J$. Then, in view of Proposition 6.5, there exists a $\phi \in \mathcal{S}(0)$ such that $\phi(x) \to \infty$ as $|x| \to \infty$ and the associated A^ϕ-diffusion is recurrent. Since $A^\phi \phi \leqslant -F[\phi] = 0$ in $\mathbb{R}^N$ and ϕ has a minimum in $\mathbb{R}^N$, we can apply the strong maximum principle to conclude that ϕ is constant in $\mathbb{R}^N$. But this is a contradiction. Hence, $0 \notin \partial J$, namely, $0 \in \operatorname{Int} J$. □

Theorem 2.5 is now easily proved as a corollary of Theorems 2.3 and 2.4.

Proof of Theorem 2.5. Claim (i) follows directly from Proposition 6.2. Claim (ii) is also obvious from Theorem 2.3 and Proposition 6.5. To prove (iii), we observe that $G_\alpha(x) = (N - \alpha)/2$. If $N \leqslant 2$, then $G_2 \leqslant 0$ in $\mathbb{R}^N$. In particular, $\partial J = \{0\}$ by virtue of Theorem 2.4(i). On the other hand, if $N \geqslant 3$, then $G_2(x) \geqslant 1/2$ in $\mathbb{R}^N$. Thus, we can apply Theorem 2.4(ii) to conclude that $0 \notin \partial J$. Hence, we have completed the proof of Theorem 2.5. □

We close this section with an example which shows that Theorem 2.3(ii) as well as Theorem 2.5(ii)–(iii) are not true when $m > 2$. Let us consider the simple equation

$$\lambda - \frac{1}{2}\Delta\phi + \frac{1}{m}|D\phi|^m + \beta V = 0 \quad \text{in } \mathbb{R}^N, \qquad \phi(0) = 0. \tag{6.5}$$

In what follows, we assume the following:

(H) $m > 2$, $V \not\equiv 0$, $V \geqslant 0$ in $\mathbb{R}^N$, and $|x|^{N+\varepsilon}V(x) \to 0$ as $|x| \to \infty$ for some $\varepsilon > 0$.

Let $\lambda^*(\beta)$ denote the generalized principal eigenvalue of (6.5). Since (6.5) does not contain the first-order term, we see in view of Proposition 6.2 that $\overline{\lambda} = 0$. Set $J := \{\beta \in \mathbb{R} \mid \lambda^*(\beta) = 0\}$.

Proposition 6.6. *Let* (H) *hold. Assume that* $N \geqslant 2$. *Then* $J = (-\infty, \overline{\beta}]$ *for some* $\overline{\beta} > 0$. *Furthermore, for any* $\beta \in (0, \overline{\beta})$, *there exists a solution* ϕ *of* (6.5) *with* $\lambda = \lambda^*(\beta)$ *such that* $\inf_{\mathbb{R}^N} \phi > -\infty$.

Proof. The positivity of $\overline{\beta}$ is obvious from Proposition 6.5 since $G_{m^*}(x) \equiv (N - m^*)/2$ and $1 < m^* < 2$. To prove the latter claim, fix any $\alpha \in (0, 1)$ and $\psi \in C^3(\mathbb{R}^N)$ such that $\psi(x) = |x|$ in $\mathbb{R}^N \setminus B_1$. We define the differential operator A_α by $A_\alpha := (1/2)\Delta - \alpha D\psi \cdot D$. Note that the A_α-diffusion is ergodic. Indeed, if we set $u(x) = |x|^2$, then $A_\alpha u(x) = N - 2\alpha|x| \leqslant -1$ in $\mathbb{R}^N \setminus B_R$ for some large $R > 0$. Hence, the A_α-diffusion is ergodic in view of Theorem 3.1(iii). Furthermore, its invariant probability density is given by $\mu_\alpha(x) = Z^{-1}e^{-2\alpha\psi(x)}$, where $Z := \int_{\mathbb{R}^N} e^{-2\alpha\psi(x)}\,dx < \infty$. In particular, $e^{\kappa|x|}\mu_\alpha \in L^1(\mathbb{R}^N)$ for any $0 < \kappa < 2\alpha$.

Fix any $\beta \in (0, \overline{\beta})$ and consider the following ergodic problem with parameter α:

$$\lambda - A_\alpha\phi + \frac{1}{m}|D\phi|^m + \beta V = 0 \quad \text{in } \mathbb{R}^N, \qquad \phi(0) = 0. \tag{6.6}$$

Let λ^*_α be the generalized principal eigenvalue of (6.6). Then we see that $\lambda^*_\alpha < 0$ for all $\alpha \in (0, 1)$. To verify this, let $X = (X_t)$ be the A_α-diffusion, and let ϕ be any solution of (6.6) with $\lambda = \lambda^*_\alpha$. Then, in view of Theorem 3.9(i) with $L(x, \xi) := (1/m^*)|\xi|^{m^*}$ and the ergodicity of X, we have

$$\lambda^*_\alpha \leqslant \lim_{T\to\infty} \frac{1}{T} E_x\left[\int_0^T \{L(X_t, 0) - \beta V(X_t)\} + \phi(X_t)\right] = -\beta \int_{\mathbb{R}^N} V(y)\mu_\alpha(dy) < 0.$$

In particular, we observe from Theorem 4.1 with $\phi_0 \equiv 0$ that there exists a unique solution $\phi_\alpha \in C^2(\mathbb{R}^N)$ of (6.6) with $\lambda = \lambda^*_\alpha$, and that $\phi_\alpha(x) \to \infty$ as $|x| \to \infty$.

We now fix any $\gamma \in (N - 2, N - 2 + \varepsilon)$, where $\varepsilon > 0$ is the constant in (H). Let $\phi_0 \in C^3(\mathbb{R}^N)$ be any function such that $\phi_0(x) = (1/\gamma)|x|^{-\gamma}$ in $\mathbb{R}^N \setminus B_1$. Then, by a straightforward calculation, we see that

$$F_\alpha[\phi_0] := -A_\alpha\phi_0 + \frac{1}{m}|D\phi_0|^m \leqslant -\frac{\gamma + 2 - N}{2|x|^{\gamma+2}} - \frac{\alpha}{|x|^{\gamma+1}} + \frac{K}{|x|^{m(\gamma+1)}} \quad \text{in } \mathbb{R}^N \setminus B_1$$

for some $K > 0$ not depending on α. Since $\gamma + 2 - N > 0$ and $\gamma + 2 < m(\gamma + 1)$ by the assumption on m, there exist some $R > 0$ and $\rho > 0$ not depending on α such that

$$F_\alpha[\phi_0] + \beta V \leqslant -\rho \quad \text{in } \mathbb{R}^N \setminus B_R. \tag{6.7}$$

We claim here that there exists a solution (λ, ϕ) of (6.5) such that $\inf_{\mathbb{R}^N} \phi > -\infty$. Since the family $\{\lambda^*_\alpha\}_\alpha$ is bounded and $\{\phi_\alpha\}_\alpha$ is precompact in $C^2(\mathbb{R}^N)$, we can extract a subsequence $\{\alpha_n\}$ converging to zero as $n \to \infty$ such that $\lambda_n := \lambda^*_{\alpha_n} \to \lambda$ for some $\lambda \in \mathbb{R}$ and $\phi_n := \phi_{\alpha_n} \to \phi$ in $C^2(\mathbb{R}^N)$ for some ϕ as $n \to \infty$. In particular, (λ, ϕ) is a solution of (6.5). Observe that $\lambda \leqslant 0$ since $\lambda^*(\beta) = 0$. We now prove that $\inf_{\mathbb{R}^N} \phi > -\infty$. Set $M := \sup\{|\phi_n(x)| + |\phi_0(x)| \mid n \geqslant 1,\ x \in B_R\}$, where $R > 0$ is the constant given in (6.7). Note in view of Theorem 3.4 that $M < \infty$ since the $W^{1,\infty}$-norm of $\alpha_n D\psi$ is bounded uniformly in n. Then, by the definition of M, we see that $\phi_n \geqslant \phi_0 - M$ in B_R. On the other hand, since $\phi_n(x) \to \infty$ as $|x| \to \infty$ and ϕ_0 is bounded in $\mathbb{R}^N$, there exists a bounded domain D_n containing B_R such that $\phi_n \geqslant \phi_0 - M$ in $\mathbb{R}^N \setminus D_n$. Furthermore, we observe that

$$F_{\alpha_n}[\phi_n] + \beta V = -\lambda^*_{\alpha_n} > 0 \quad \text{in } \mathbb{R}^N.$$

In particular, ϕ_n is a supersolution of $F_{\alpha_n}[\phi] + \beta V = 0$ in $D_n \setminus B_R$. Since $\phi_0 - M$ is a strict subsolution of the same equation in $D_n \setminus B_R$, we can apply the standard comparison theorem to conclude that $\phi_n \geqslant \phi_0 - M$ in $D_n \setminus B_R$. Thus, $\phi_n \geqslant \phi_0 - M$ in $\mathbb{R}^N$. Letting $n \to \infty$, we obtain $\phi \geqslant \phi_0 - M$ in $\mathbb{R}^N$. Hence, $\inf_{\mathbb{R}^N} \phi > -\infty$.

We finally show that $\lambda = 0$ arguing by contradiction. Suppose that $\lambda < 0$. Then, by the same argument as in the proof of Proposition 4.4, we see that the A^ϕ-diffusion is transient. On the other hand, set $\psi(x) := -c\sqrt{1 + |x|^2}$, where $c > 0$ is chosen so that $|F[\psi]| \leqslant (1/2)|\lambda|$ in $\mathbb{R}^N$. Then, there exists an $R > 0$ such that

$$A^\phi(\phi - \psi) \leqslant F[\psi] - F[\phi] \leqslant \frac{1}{2}|\lambda| + \lambda + \beta V \leqslant -\frac{1}{4}|\lambda| \quad \text{in } \mathbb{R}^N \setminus B_R.$$

Since $(\phi - \psi)(x) \to \infty$ as $|x| \to \infty$, we conclude that the A^ϕ-diffusion is recurrent. But this is a contradiction. Hence, $\lambda = 0$, and we have completed the proof. □

In view of Proposition 6.6, we obtain the recurrence of A^ϕ-diffusions in the two dimensional case.

Proposition 6.7. *Let* (H) *hold and assume that $N = 2$. For a given $\beta \in (0, \overline{\beta})$, let ϕ be the solution of* (6.5) *constructed in Proposition* 6.6. *Then, the A^ϕ-diffusion is recurrent.*

Proof. Let $\phi_0 \in C^3(\mathbb{R}^2)$ be such that $\phi_0(x) = -\log\log|x|$ in $\mathbb{R}^2 \setminus B_2$. Then a direct computation shows that

$$F[\phi_0] = -\frac{1}{2}\Delta\phi_0 + \frac{1}{m}|D\phi_0|^m \leqslant -\frac{1}{2(|x|\log|x|)^2} + \frac{1}{m(|x|\log|x|)^m} \quad \text{in } \mathbb{R}^2 \setminus B_2.$$

Since $|x|^{2+\varepsilon}V(x) \to 0$ as $|x| \to \infty$ for some $\varepsilon > 0$, there exists an $R > 0$ such that

$$F[\phi_0] + \beta V \leqslant 0 \quad \text{in } \mathbb{R}^2 \setminus B_R.$$

In particular, we have

$$A^\phi(\phi - \phi_0) \leqslant \bigl(F[\phi_0] - F[\phi]\bigr) \leqslant 0 \quad \text{in } \mathbb{R}^2 \setminus B_R.$$

Since $(\phi - \phi_0)(x) \to \infty$ as $|x| \to \infty$, we can apply Theorem 3.1(ii) to conclude that the A^ϕ-diffusion is recurrent. Hence, we have completed the proof. □

The last proposition implies that Theorem 2.3(ii) is not true, in general, for $m > 2$.

Acknowledgements

The author would like to thank Professor Shuenn-Jyi Sheu for fruitful discussions on Proposition 5.3. This research was supported in part by JSPS KAKENHI Grant No. 24740089.

References

[1] A. Arapostathis, V.S. Borkar, A relative value iteration algorithm for nondegenerate controlled diffusions, SIAM J. Control Optim. 50 (2012) 1886–1902.

[2] A. Arapostathis, V.S. Borkar, M.K. Ghosh, Ergodic Control of Diffusion Processes, Encycl. Math. Appl., vol. 143, Cambridge University Press, Cambridge, UK, 2011.

[3] M. Bardi, A. Cesaroni, Optimal control with random parameters: a multiscale approach, Eur. J. Control 17 (2011) 30–45.

[4] M. Bardi, A. Cesaroni, L. Manca, Convergence by viscosity methods in multiscale financial models with stochastic volatility, SIAM J. Financ. Math. 1 (2010) 230–265.

[5] A. Bensoussan, J. Frehse, Bellman equations of ergodic control in $\mathbb{R}^N$, J. Reine Angew. Math. 429 (1992) 125–160.

[6] V.I. Bogachev, N.V. Krylov, M. Röckner, On regularity of transition probabilities and invariant measures of singular diffusions under minimal conditions, Commun. Partial Differ. Equ. 26 (2001) 2037–2080.

[7] H. Berestycki, L. Nirenberg, S.R.S. Varadhan, The principal eigenvalue and maximum principle for second-order elliptic operators in general domains, Commun. Pure Appl. Math. 47 (1994) 47–92.

[8] W.H. Fleming, M. Soner, Controlled Markov Processes and Viscosity Solutions, Appl. Math., vol. 25, Springer-Verlag, New York, 1993.

[9] H. Hata, H. Nagai, S.J. Sheu, Asymptotics of the probability minimizing a "down side" risk, Ann. Appl. Probab. 20 (2010) 52–89.
[10] N. Ichihara, Recurrence and transience of optimal feedback processes associated with Bellman equations of ergodic type, SIAM J. Control Optim. 49 (2011) 1938–1960.
[11] N. Ichihara, Large time asymptotic problems for optimal stochastic control with superlinear cost, Stoch. Process. Appl. 122 (2012) 1248–1275.
[12] N. Ichihara, Criticality of viscous Hamilton–Jacobi equations and stochastic ergodic control, J. Math. Pures Appl. 100 (2013) 368–390.
[13] N. Ichihara, S.J. Sheu, Large time behavior of solutions of Hamilton–Jacobi–Bellman equations with quadratic nonlinearity in gradients, SIAM J. Math. Anal. 45 (2013) 279–306.
[14] H. Kaise, S.J. Sheu, Evaluation of large time expectations for diffusion processes, unpublished preprint.
[15] H. Kunita, Stochastic Flows and Stochastic Differential Equations, Camb. Stud. Adv. Math., vol. 24, 1990.
[16] P.-L. Lions, Résolution de problèmes elliptiques quasilinéaires, Arch. Ration. Mech. Anal. 74 (1980) 335–353.
[17] P.-L. Lions, Quelques remarques sur les problèmes elliptiques quasilinéaires du second ordre, J. Anal. Math. 45 (1985) 234–254.
[18] M. Murata, Structure of positive solutions to $(-\Delta + V)u = 0$ in $\mathbb{R}^n$, Duke Math. J. 53 (1986) 869–943.
[19] H. Nagai, Down side risk minimization via a large deviations approach, Ann. Appl. Probab. 22 (2012) 608–669.
[20] Y. Pinchover, On positive solutions of second order elliptic equations, stability results and classification, Duke Math. J. 57 (1988) 955–980.
[21] Y. Pinchover, Criticality and ground states for second order elliptic equations, J. Differ. Equ. 80 (1989) 237–250.
[22] Y. Pinchover, On criticality and ground states for second order elliptic equations, II, J. Differ. Equ. 87 (1990) 353–364.
[23] R.G. Pinsky, Positive Harmonic Functions and Diffusion, Camb. Stud. Adv. Math., vol. 45, 1995.
[24] B. Simon, Large time behavior of the L^p norm of Schrödinger semigroups, J. Funct. Anal. 40 (1981) 66–83.
[25] M. Takeda, K. Tsuchida, Criticality of generalized Schrödinger operators and differentiability of spectral functions, in: Stochastic Analysis and Related Topics in Kyoto, in: Adv. Stud. Pure Math., vol. 41, Math. Soc. Japan, Tokyo, 2004, pp. 333–350.