Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 32 (2015) 687–713

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Global solutions for the critical Burgers equation in the Besov spaces and the large time behavior

Tsukasa Iwabuchi

Department of Mathematics, Chuo University, Kasuga, Bunkyoku, Tokyo, 115-8551, Japan

Received 2 October 2013; received in revised form 20 March 2014; accepted 22 March 2014

Available online 2 April 2014

Abstract

We consider the Cauchy problem for the critical Burgers equation. The existence and the uniqueness of global solutions for small initial data are studied in the Besov space $\dot{B}^0_{\infty,1}(\mathbb{R}^n)$ and it is shown that the global solutions are bounded in time. We also study the large time behavior of the solutions with the initial data $u_0 \in L^1(\mathbb{R}^n) \cap \dot{B}^0_{\infty,1}(\mathbb{R}^n)$ to show that the solution behaves like the Poisson kernel.

Keywords: Burgers equation; Besov spaces; Large time behavior; Poisson kernel

1. Introduction

We consider the n-dimensional Burgers equation

$$\begin{cases} \partial_t u + \sum_{j=1}^{n} u \partial_{x_j} u + \Lambda^\alpha u = 0 & \text{for } t > 0,\ x \in \mathbb{R}^n, \\ u(0,x) = u_0(x) & \text{for } x \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where $\Lambda = (-\Delta)^{\frac{1}{2}}$. The Burgers equation with $\alpha = 0$ and $\alpha = 2$ has received an extensive amount of attention since the studies by Burgers in the 1940s. If $\alpha = 0$, the equation is the basic example of a PDE evolution leading to shocks. If $\alpha = 2$, it provides an accessible model for studying the interaction between nonlinear and dissipative phenomena. The value $\alpha = 1$ is a threshold for the occurrence of singularity in finite time or the global regularity (see [3,14,15,20]). The aim of this paper is to study the existence and the uniqueness of global solutions to (1.1) with $\alpha = 1$, which is bounded in time, and to show the solutions behaving like the Poisson kernel in the large time. There is also another purpose, namely, establishing a method to deal with such problems for a wider class of equations including the quasi-geostrophic equation which derivative orders are balanced on the linear part and the nonlinear part.

For the Cauchy problem (1.1), Kiselev, Nazarov and Shterenberg [20] considered in the periodic setting $\mathbb{S}^1$ to show the finite time blow up for the supercritical case $0 < \alpha < 1$ and the global well-posedness in $H^{\frac{1}{2}}(\mathbb{S}^1)$ for the

http://dx.doi.org/10.1016/j.anihpc.2014.03.002

critical case $\alpha = 1$ and in $H^s(\mathbb{S}^1)$ ($s > 3/2 - \alpha$) for the subcritical case $1 < \alpha < 2$. Dong, Du and Li [14] considered both of spaces $\mathbb{S}^1$ and $\mathbb{R}$ to show the finite time blow up for the supercritical case and the global well-posedness in $H^{\frac{1}{2}}(\mathbb{S}^1)$, $H^{\frac{1}{2}}(\mathbb{R})$ for the critical case and in $L^{1/(\alpha-1)}(\mathbb{S}^1)$, $L^{1/(\alpha-1)}(\mathbb{R})$ ($1 < \alpha < 2$) for the subcritical case. Finite time blow up for the supercritical case is also shown by Alibaud, Droniou and Vovelle [3]. Miao and Wu [23] showed the global well-posedness in the critical case $\alpha = 1$ for the initial data in the Besov spaces $\dot{B}^{\frac{1}{p}}_{p,1}(\mathbb{R})$ with $1 \leqslant p < \infty$. The notion of entropy solution has been introduced by Alibaud [1] to show the global well-posedness in $L^\infty(\mathbb{R}^n)$. On the study of weak solutions, Alibaud and Andreianov [2] showed that the uniqueness of a weak solution fails for $0 < \alpha < 1$. Chan and Czubak [13] establish global regularizing effects for the n-dimensional Burgers equation with sufficiently integrable initial data u_0 in the critical case $\alpha = 1$. We also refer to the results on the global regularizing effects by Droniou, Gallouet and Vovelle [15] in the subcritical case $\alpha > 1$ and Silvestre [27] for the Hamilton–Jacobi equations. Our goal on global solutions is considering the critical case $\alpha = 1$ to show the existence of global solutions, which are bounded in time, for small initial data $u_0 \in \dot{B}^0_{\infty,1}(\mathbb{R}^n)$ in all space dimensions. We note the following embeddings

$$\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n) \hookrightarrow \dot{B}^0_{\infty,1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n) \quad \text{for } 1 \leqslant p \leqslant \infty,$$

and that the boundedness in time of solutions are needed to deal with the large time behavior. We also mention the scaling invariance to (1.1) in the critical case $\alpha = 1$. For the solution u to (1.1) with $\alpha = 1$, let u_λ be defined by $u_\lambda(t,x) := u(\lambda t, \lambda x)$ for $\lambda > 0$. Then u_λ maintains Eq. (1.1) and we have the following norm invariance:

$$\sup_{t>0} \|u_\lambda(t)\|_{L^\infty} = \sup_{t>0} \|u(t)\|_{L^\infty} \quad \text{for any } \lambda > 0.$$

Then $L^\infty(\mathbb{R}^n)$ satisfies the above invariant property, and the spaces $\dot{H}^{\frac{1}{2}}(\mathbb{R})$, $L^\infty(\mathbb{R}^n)$ and $\dot{B}^{\frac{n}{p}}_{p,1}(\mathbb{R}^n)$ ($1 \leqslant p \leqslant \infty$) also satisfy such norm invariance.

On the large time behavior, Biler, Karch and Woyczynski [6] considered the equation with the semigroup generated by $(-\Delta)^{\frac{\alpha}{2}} - \Delta$ ($0 < \alpha < 2$) to study the asymptotic expansion of solutions. Biler, Karch and Woyczynski [7–9] also studied with a Lévy semigroup, which includes the semigroups generated by $(-\Delta)^{\frac{\alpha}{2}} - \varepsilon\Delta$ ($1 < \alpha < 2$, $\varepsilon > 0$), to show the large time behavior like the self-similar solutions. For Eq. (1.1), Karch, Miao and Xu [19] considered the subcritical case $1 < \alpha < 2$ in one space dimension to show that the large time asymptotic is described by the rarefaction waves. Alibaud, Imbert and Karch [4] considered the critical case $\alpha = 1$ and the supercritical case $0 < \alpha < 1$ to consider the initial data satisfying

$$u_0(x) = c + \int_{-\infty}^{x} m(dy), \qquad u_0(\cdot) - c, u_0(\cdot) - \left(c + \int_{-\infty}^{\infty} m(dy)\right) \in L^1(\mathbb{R}),$$

where $c \in \mathbb{R}$ and m is a finite signed measure on $\mathbb{R}$ with $\int_{-\infty}^{\infty} m > 0$. They [4] showed that for $\alpha = 1$ the solutions converge to the self-similar solution, and for $\alpha < 1$ the nonlinearity is negligible in the asymptotic expansion of solutions. Our goal on the large time behavior is considering the case $\alpha = 1$ to show that the solutions for initial data $u_0 \in L^1(\mathbb{R}^n) \cap \dot{B}^0_{\infty,1}(\mathbb{R}^n)$ behave like the Poisson kernel as $t \to \infty$. We also show a higher order asymptotic expansion, imposing the additional decay in the distance on the initial data.

Throughout this paper, P is defined by

$$P(x) := \mathcal{F}^{-1}\big[e^{-|\xi|}\big](x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}(1+|x|^2)^{\frac{n+1}{2}}} \quad \text{for } x \in \mathbb{R}^n,$$

where $\Gamma(\cdot)$ is the Gamma function, and let P_t be the Poisson kernel:

$$P_t(x) := t^{-n} P\big(t^{-1}x\big) \quad \text{for } t > 0,\ x \in \mathbb{R}^n.$$

To study Eq. (1.1) with $\alpha = 1$, we consider the following integral equation

$$u(t) = P_t * u_0 - \frac{1}{2}\int_0^t P_{t-\tau} * \left(\sum_{j=1}^n \partial_{x_j} u(\tau)^2\right) d\tau \quad \text{for } t \geqslant 0. \tag{1.2}$$

The following are our results on the global solutions and the large time behavior.

Theorem 1.1. *Let $n \geqslant 1$. There exist positive constants δ and C such that for any $u_0 \in \dot{B}^0_{\infty,1}(\mathbb{R}^n)$ with $\|u_0\|_{\dot{B}^0_{\infty,1}} \leqslant \delta$, a unique global solution u to (1.2) exists in the space $C([0,\infty); \dot{B}^0_{\infty,1}(\mathbb{R}^n)) \cap L^1(0,\infty; \dot{B}^1_{\infty,1}(\mathbb{R}^n))$. Furthermore, it holds that for $t \geqslant 0$*

$$\|u(t)\|_{\dot{B}^0_{\infty,1}} \leqslant C\|u_0\|_{\dot{B}^0_{\infty,1}} \exp\left\{C\int_0^t \|\nabla u(\tau)\|_{L^\infty}\,d\tau\right\}. \tag{1.3}$$

Theorem 1.2. *Let $n \geqslant 1$. There exists $\delta > 0$ such that for any initial data $u_0 \in L^1(\mathbb{R}^n) \cap \dot{B}^0_{\infty,1}(\mathbb{R}^n)$ with $\|u_0\|_{\dot{B}^0_{\infty,1}} \leqslant \delta$, a unique global solution u to (1.2) exists in the space $C([0,\infty); L^1(\mathbb{R}^n)) \cap C([0,\infty); \dot{B}^0_{\infty,1}(\mathbb{R}^n)) \cap L^1(0,\infty; \dot{B}^1_{\infty,1}(\mathbb{R}^n))$. Furthermore, the solution u satisfies the following.*

(i) *For $1 \leqslant p \leqslant \infty$, it holds that*

$$\lim_{t\to\infty} t^{n(1-\frac{1}{p})}\|u(t) - MP_t\|_{L^p} = 0, \tag{1.4}$$

where $M := \int_{\mathbb{R}^n} u_0(y)\,dy$.

(ii) *Let u_0 satisfy $|\cdot|u_0(\cdot) \in L^1(\mathbb{R}^n)$ and $1 \leqslant p \leqslant \infty$. Then it holds that for $t > 0$*

$$\|u(t) - MP_t\|_{L^p} \leqslant \begin{cases} Ct^{-(1-\frac{1}{p})-1}\log(e+t) & \text{if } n = 1, \\ Ct^{-n(1-\frac{1}{p})-1} & \text{if } n \geqslant 2, \end{cases} \tag{1.5}$$

$$\lim_{t\to\infty} t^{1+n(1-\frac{1}{p})}\Bigg\| u(t) - MP_t + \nabla P_t \cdot \int_{\mathbb{R}^n} y u_0(y)\,dy + \frac{1}{2}\sum_{j=1}^n (\partial_{x_j} P_t)\int_0^t\int_{\mathbb{R}^n} u(\tau,y)^2\,dy\,d\tau$$
$$+ \frac{1}{2}\sum_{j=1}^n \int_0^t (\partial_{x_j} P_{t-\tau}) * (MP_{\tau+1})^2\,d\tau - \frac{1}{2}\sum_{j=1}^n (\partial_{x_j} P_t)\int_0^t\int_{\mathbb{R}^n} \big(MP_{\tau+1}(y)\big)^2\,dy\,d\tau\Bigg\|_{L^p} = 0. \tag{1.6}$$

Remark 1.3. On the boundedness (1.5) in the case of one space dimension, the order is optimal and we cannot remove $\log(e+t)$. Indeed, it is possible to show that

$$\|u(t) - MP_t\|_{L^2} \geqslant cM^2 t^{-\frac{3}{2}}\log(e+t) \tag{1.7}$$

for large t. The above estimate will be proved in Section 7.

The methods of the proof of Theorem 1.1 are applying contraction argument and making use of a priori estimate. To see a key idea, we explain importance on introducing the space $L^1(0,\infty; \dot{B}^1_{\infty,1}(\mathbb{R}^n))$, and how to prove the uniqueness of the solution briefly. The space $L^1(0,\infty; \dot{B}^1_{\infty,1}(\mathbb{R}^n))$ plays an essential role in the proof of the above theorems. If we try to show the estimate in $L^\infty(\mathbb{R}^n)$, we hope to show the following estimate

$$\int_0^\infty \|\nabla P_t * f\|_{L^\infty}\,dt \leqslant C\|f\|_{L^\infty}.$$

However the above estimate never holds since for the function f defined by $f(x) = e^{ik\cdot x}$ ($k \in \mathbb{R}^n$) it holds that

$$\|\nabla P_t * f(x)\|_{L^\infty} = \|e^{ik\cdot x} x e^{-t|x|}\|_{L^\infty} = Ct^{-1}$$

and t^{-1} is not in $L^1(0,\infty)$. On the other hand, it follows from the embedding $\dot{B}^1_{\infty,1}(\mathbb{R}^n) \hookrightarrow \dot{W}^{1,\infty}(\mathbb{R}^n)$ that for Littlewood–Paley's dyadic decomposition $\{\phi_k\}_{k\in\mathbb{Z}}$ with $\operatorname{supp}\widehat{\phi_k} \subset \{\xi \in \mathbb{R}^n \mid 2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}\}$

$$\int_0^\infty \|\nabla P_t * f\|_{L^\infty}\,dt \leqslant C\int_0^\infty \|P_t * f\|_{\dot B^1_{\infty,1}}\,dt \leqslant C\sum_{k\in\mathbb{Z}} 2^k \int_0^\infty \|P_t*\phi_k * f\|_{L^\infty}\,dt$$
$$\leqslant C\sum_{k\in\mathbb{Z}} 2^k\int_0^\infty e^{-ct2^k}\,dt\,\|\phi_k * f\|_{L^\infty}$$
$$= C\|f\|_{\dot B^0_{\infty,1}}.$$

Then $\|\nabla P_t * f\|_{L^\infty}$ is integrable and it is one of the sufficient methods to study the Cauchy problem in the space $L^1(0,\infty;\dot B^1_{\infty,1}(\mathbb{R}^n))$ for the initial data $u_0\in \dot B^0_{\infty,1}(\mathbb{R}^n)$. On the proof of the uniqueness of the solution, we consider the following equation of the divergence form

$$\begin{cases} \partial_t u + \sum_{j=1}^n \partial_{x_j}(vu) + \Lambda u = 0,\\ u(0,x) = u_0(x), \end{cases} \tag{1.8}$$

for any given function $v\in L^1(0,\infty;\dot B^1_{\infty,1}(\mathbb{R}^n))$ and to show a priori estimate

$$\|u(t)\|_{\dot B^0_{\infty,1}} \leqslant C\|u_0\|_{\dot B^0_{\infty,1}} \exp\left(C\int_0^t \|v(\tau)\|_{\dot B^1_{\infty,1}}\,d\tau\right), \tag{1.9}$$

modifying the result by Miao and Wu [23], where the term $v\partial_{x_j}u$ is considered instead of $\partial_{x_j}(vu)$ of (1.8). We will consider a modified version of Eq. (1.8) to show a priori estimate. We should note that (1.9) is not enough for (1.3) but we will show that it is possible to replace $\|v\|_{\dot B^1_{\infty,1}}$ with $\|\nabla v\|_{L^\infty}$ for $v=u$ with certain adjustment.

The method of the proof of Theorem 1.2 is to show the decay estimates of the solution u as $t\to\infty$ and to estimate the integral equation (1.2) with the decay estimates. It is easy to deal with the linear part $P_t * u_0$ by the analogous methods for the heat kernel. On the nonlinear part, it is difficult to show the decay estimates as $t\to\infty$ by the analogous argument for heat equations. Indeed, when we consider $\partial_t u - \Delta u = \partial_x u^2$ to treat the derivative, we use the estimate

$$\|\nabla e^{t\Delta} f\|_{L^p} \leqslant Ct^{-\frac12}\|f\|_{L^p}$$

and $t^{-\frac12}$ is integrable locally. One can deal with the solution in Lebesgue spaces $L^p(\mathbb{R}^n)$ as a solution satisfying a integral equation by the integrability in time. On the other hand, it holds for the Poisson kernel that

$$\|\nabla P_t * f\|_{L^p} \leqslant Ct^{-1}\|f\|_{L^p},$$

and t^{-1} is not integrable. Then, we need to impose the regularity of one order derivative on the solution, and $L^1(0,\infty;\dot B^1_{\infty,1}(\mathbb{R}^n))$ is important to obtain the decay estimate. Considering the solution in $L^1(0,\infty;\dot B^1_{\infty,1}(\mathbb{R}^n))$, we will apply a modified version of Gronwall's inequality to show decay estimates of the solutions. Once we obtain the decay estimates, we can show the large time behavior of the solution in the analogous way to that for the heat equations (see [16–18,24,25]). We also refer on the asymptotic expansion of solutions to the dissipative equation with the fractional Laplacian $(-\Delta)^{\frac{\alpha}{2}}$ $(1<\alpha\leqslant 2)$ to the result [29].

This paper is organized as follows. In Section 2, we introduce the definition of Besov spaces, its properties and some propositions on the Poisson kernel. We show a priori estimate related to (1.3) and (1.9) in Section 3, and Theorem 1.1 is proved in Section 4. We show the decay estimate of the solution in Section 5, and Theorem 1.2 is proved in Section 6. The estimate (1.7) is proved in Section 7.

Throughout this paper, we use the following notations. $\{\phi_k\}_{k\in\mathbb{Z}}$ denotes Littlewood–Paley's dyadic decomposition, i.e., let $\phi\in\mathcal{S}(\mathbb{R}^n)$ satisfy $\operatorname{supp}\widehat{\phi}\subset\{\xi\in\mathbb{R}^n \mid 2^{-1}\leqslant|\xi|\leqslant 2\}$, $\sum_{k\in\mathbb{Z}}\widehat{\phi}(2^{-k}\xi)\equiv 1$ for any $\xi\in\mathbb{R}^n\setminus\{0\}$, where $\widehat{\phi}$ is the Fourier transform of ϕ, and let $\{\phi_k\}_{k\in\mathbb{Z}}$ be defined by $\widehat{\phi_k}(\xi) := \widehat{\phi}(2^{-k}\xi)$. Let S_k be defined by $S_k f := \sum_{k'\leqslant k}\phi_{k'} * f$.

2. Preliminary

We introduce the definition of Besov spaces and propositions on Besov spaces, the linear estimates and the Poisson kernel. The asymptotic behavior of the solution to $\partial_t u + \Lambda u = 0$ is also shown. The idea of proof in this section is known for the heat kernel but we give the proof for the paper being self-contained.

Definition. Let $\{\phi_k\}_{k\in\mathbb{Z}}$ be Littlewood Paley's dyadic decomposition. For $s \in \mathbb{R}$ and $1 \leqslant p, q \leqslant \infty$, the homogeneous Besov space $\dot{B}^s_{p,q}(\mathbb{R}^n)$ is defined by

$$\dot{B}^s_{p,q}(\mathbb{R}^n) := \left\{ u \in \mathcal{S}'(\mathbb{R}^n) \;\middle|\; \|u\|_{\dot{B}^s_{p,q}} < \infty,\ \sum_{k\in\mathbb{Z}} \phi_k * u = u \text{ in } \mathcal{S}'(\mathbb{R}^n) \right\},$$
$$\|u\|_{\dot{B}^s_{p,q}} := \left\| \left\{ 2^{sk} \|\phi_k * u\|_{L^p} \right\}_{k\in\mathbb{Z}} \right\|_{\ell^q(\mathbb{Z})}.$$

Remark 2.1. By the argument in Kozono and Yamazaki [22], the Besov spaces of the above definition are complete if $s < n/p$ and $1 \leqslant q \leqslant \infty$, or $s = n/p$ and $q = 1$. The Besov space $\dot{B}^0_{\infty,1}(\mathbb{R}^n)$, in which we consider the Cauchy problem, is complete.

Proposition 2.2. *(See [12,28].) Let $s \in \mathbb{R}$ and $1 \leqslant p, q, r, p_j \leqslant \infty$ $(j = 1, 2, 3, 4)$.*

(i) $\dot{B}^s_{p,r}(\mathbb{R}^n) \hookrightarrow \dot{B}^s_{p,q}(\mathbb{R}^n)$ *if* $r \leqslant q$.
(ii) $\dot{B}^{s+n(\frac{1}{r}-\frac{1}{p})}_{r,q}(\mathbb{R}^n) \hookrightarrow \dot{B}^s_{p,q}(\mathbb{R}^n)$ *if* $r \leqslant p$.
(iii) $\dot{B}^0_{p,1}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n) \hookrightarrow \dot{B}^0_{p,\infty}(\mathbb{R}^n)$.
(iv) *Let $s > 0$ and $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$. Then it holds that*

$$\|fg\|_{\dot{B}^s_{p,1}} \leqslant C\left(\|f\|_{\dot{B}^s_{p_1,1}} \|g\|_{L^{p_2}} + \|f\|_{L^{p_3}} \|g\|_{\dot{B}^s_{p_4,1}}\right). \tag{2.1}$$

Proposition 2.3. *Let $s > 0$ and let p, p_j $(j = 1, 2, 3, 4)$ satisfy $1 \leqslant p, p_j \leqslant \infty$ $(j = 1, 2, 3, 4)$ and $1/p = 1/p_1 + 1/p_2 = 1/p_3 + 1/p_4$. Then it holds that*

$$\|fg\|_{\dot{B}^s_{p,1}} \leqslant C\left(\|f\|_{\dot{B}^s_{p_1,1}} \|g\|_{\dot{B}^0_{p_2,1}} + \|f\|_{\dot{B}^0_{p_3,1}} \|g\|_{\dot{B}^s_{p_4,1}}\right). \tag{2.2}$$

Proof. (2.2) is obtained by (2.1) and the embedding $\dot{B}^0_{p,1}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$. □

Proposition 2.4. *Let $s, \tilde{s} \in \mathbb{R}$, $1 \leqslant p, q \leqslant \infty$ and $\alpha \in (\mathbb{N} \cup \{0\})^n$.*

(i) *There exist $C, c > 0$ such that*

$$\|\phi_k * P_t * f\|_{L^p} \leqslant C e^{-ct2^k} \|\phi_k * f\|_{L^p}, \tag{2.3}$$

for any $k \in \mathbb{Z}$ and $f \in L^p(\mathbb{R}^n)$.
(ii) *If $s > \tilde{s}$, it holds that*

$$\|P_t * f\|_{\dot{B}^s_{p,1}} \leqslant C t^{-(s-\tilde{s})} \|f\|_{\dot{B}^{\tilde{s}}_{p,\infty}}. \tag{2.4}$$

(iii) *If $q \leqslant p$, it holds that*

$$\left\|\nabla^\alpha P_t * f\right\|_{L^p} \leqslant C t^{-|\alpha|-n(\frac{1}{q}-\frac{1}{p})} \|f\|_{L^q}, \tag{2.5}$$

where $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$.

Remark 2.5. We note on the estimate (2.4) that this is a smoothing effect on not only the derivative indices $s, \tilde{s}$ but also the interpolation indices $1, \infty$ and the power of t depends only on s, $\tilde{s}$. Such estimate for $e^{t\Delta}$ is known in the result by Kozono, Ogawa and Taniuchi [21].

Proof of Proposition 2.4. Let Ψ_k be defined by $\Psi_k = \phi_{k-1} + \phi_k + \phi_{k+1}$. We have from $\widehat{\phi_k} = \widehat{\Psi_k}\widehat{\phi_k}$ and Hausdorff Young's inequality

$$\left\|\phi_k * P_t * f(x)\right\|_{L^p} = \left\|\mathcal{F}^{-1}\left[e^{-t|\xi|}\widehat{\Psi_k}\widehat{\phi_k}\widehat{f}\right]\right\|_{L^p} \leqslant C\left\|\mathcal{F}^{-1}\left[e^{-t|\xi|}\widehat{\Psi_k}\right]\right\|_{L^1}\|\phi_k * f\|_{L^p}.$$

On the estimate of $\|\mathcal{F}^{-1}[e^{-t|\xi|}\widehat{\Psi_k}]\|_{L^1}$, let $s > n/2$ and we apply the change of variable $\xi \to 2^k\xi$, $x \to 2^{-k}x$ and Hölder's inequality, and estimate the norm of $H^s(\mathbb{R}^n)$ directly to obtain

$$\begin{aligned}\left\|\mathcal{F}^{-1}\left[e^{-t|\xi|}\widehat{\Psi_k}(\xi)\right]\right\|_{L^1} &= \left\|\mathcal{F}^{-1}\left[e^{-t2^k|\xi|}\widehat{\Psi_0}(\xi)\right](x)\right\|_{L^1_x} \\ &\leqslant C\left\|\left(1+|x|\right)^s\mathcal{F}^{-1}\left[e^{-t|\xi|}\widehat{\Psi_0}(\xi)\right](x)\right\|_{L^2_x} \\ &\leqslant C\left\|e^{-t2^k|\xi|}\widehat{\Psi_0}(\xi)\right\|_{H^s_\xi} \\ &\leqslant Ce^{-ct2^k}.\end{aligned}$$

Then (2.3) is obtained.

To prove (2.4), we have from (2.3)

$$\begin{aligned}\|P_t * f\|_{\dot{B}^s_{p,1}} &\leqslant C\sum_{k\in\mathbb{Z}} e^{-ct2^k}2^{sk}\|\phi_k * f\|_{L^p} \\ &= Ct^{-(s-\tilde{s})}\sum_{k\in\mathbb{Z}} e^{-ct2^k}\left(t2^k\right)^{s-\tilde{s}}2^{\tilde{s}k}\|\phi_k * f\|_{L^p} \\ &\leqslant Ct^{-(s-\tilde{s})}\|f\|_{\dot{B}^{\tilde{s}}_{p,\infty}}\sum_{k\in\mathbb{Z}} e^{-ct2^k}\left(t2^k\right)^{s-\tilde{s}}.\end{aligned}$$

It is sufficient to show that the supremum of the last sum with respect to $t > 0$ is finite. Let t be in $[2^{k_0}, 2^{k_0+1}]$ for some $k_0 \in \mathbb{Z}$, then it holds that $2^{k+k_0} \leqslant t2^k \leqslant 2^{k+k_0+1}$ and

$$\sum_{k\in\mathbb{Z}} e^{-ct2^k}\left(t2^k\right)^{s-\tilde{s}} \leqslant \sum_{k\in\mathbb{Z}} e^{-c2^{k+k_0}}\left(2^{k+k_0+1}\right)^{s-\tilde{s}} = \sum_{k\in\mathbb{Z}} e^{-c2^k}\left(2^{k+1}\right)^{s-\tilde{s}} < \infty.$$

This completes the proof of (2.4).

For the proof of (2.5) in the case $q = p$ and $\alpha = 0$, it is sufficient to apply Hausdorff–Young's inequality and the fact $\|P_t\|_{L^1} = 1$. In the case $q < p$ or $\alpha > 0$, it follows from Hausdorff–Young's inequality with $1/p = 1/r + 1/q - 1$ and $\|\nabla P_t\|_{L^r} = Ct^{-n(1-\frac{1}{r})} = Ct^{-n(\frac{1}{q}-\frac{1}{p})}$ that

$$\left\|\nabla^\alpha P_t * f\right\|_{L^p} \leqslant \left\|\nabla^\alpha P_t\right\|_{L^r}\|f\|_{L^q} = Ct^{-n(\frac{1}{q}-\frac{1}{p})}\|f\|_{L^q}.$$

Therefore, we complete the proof of (2.5). □

Proposition 2.6. *Let $s \in \mathbb{R}$ and $1 \leqslant p \leqslant \infty$.*

(i) *It holds that*

$$\|P_t * f\|_{L^1_t(0,\infty;\dot{B}^{s+1}_{p,1})} \leqslant C\|f\|_{\dot{B}^s_{p,1}} \tag{2.6}$$

for all $f \in \dot{B}^s_{p,1}(\mathbb{R}^n)$.

(ii) *It holds that*

$$\left\|\int_0^t P_{t-\tau} * f(\tau)\,d\tau\right\|_{L^1_t(0,\infty;\dot{B}^{s+1}_{p,1})} \leqslant C\|f\|_{L^1(0,\infty;\dot{B}^s_{p,1})} \tag{2.7}$$

for all $f \in L^1(0,\infty;\dot{B}^s_{p,1}(\mathbb{R}^n))$.

Proof. By (2.3), it holds that

$$\|P_t * \phi_k * f\|_{L^1_t(0,\infty;L^p)} \leqslant C\left\|e^{-ct2^k}\|\phi_k * f\|_{L^p}\right\|_{L^1_t(0,\infty)} \leqslant C2^{-k}\|\phi_k * f\|_{L^p}.$$

Therefore, we have from the above estimate

$$\|P_t * f\|_{L^1_t(0,\infty;\dot{B}^{s+1}_{p,1})} = \sum_{k\in\mathbb{Z}} 2^{(s+1)k}\|P_t * \phi_k * f\|_{L^1_t(0,\infty;L^p)} \leqslant C\sum_{k\in\mathbb{Z}} 2^{sk}\|\phi_k * f\|_{L^p} = \|f\|_{\dot{B}^s_{p,1}}.$$

This completes the proof of (2.6). In order to prove (2.7), we apply Minkowski's inequality and (2.6) to obtain

$$\begin{aligned}\left\|\int_0^t P_{t-\tau} * f(\tau)\,d\tau\right\|_{L^1_t(0,\infty;\dot{B}^{s+1}_{p,1})} &\leqslant \int_0^\infty \left\|P_{t-\tau} * f(\tau)\right\|_{L^1_t(\tau,\infty;\dot{B}^{s+1}_{p,1})}\,d\tau\\ &\leqslant C\int_0^\infty \left\|f(\tau)\right\|_{\dot{B}^s_{p,1}}\,d\tau\\ &= C\|f\|_{L^1(0,\infty;\dot{B}^s_{p,1})}.\end{aligned}$$

This completes the proof of (2.7). □

Proposition 2.7. *Let* $1 \leqslant p \leqslant \infty$.

(i) *It holds that*

$$\left\|P_{t+1}(\cdot) - P_t(\cdot)\right\|_{L^p} \leqslant Ct^{-n(1-\frac{1}{p})-1} \tag{2.8}$$

for all $t > 0$.

(ii) *For* $f \in L^1(\mathbb{R}^n)$, *it holds that*

$$\lim_{t\to\infty} t^{n(1-\frac{1}{p})}\left\|P_t * f - P_t\int_{\mathbb{R}^n} f(y)\,dy\right\|_{L^p} = 0. \tag{2.9}$$

(iii) *For* $f \in L^1(\mathbb{R}^n)$ *with* $|\cdot|f(\cdot)L^1(\mathbb{R}^n)$, *it holds that*

$$\sup_{t>0} t^{n(1-\frac{1}{p})+1}\left\|P_t * f - P_t\int_{\mathbb{R}^n} f(y)\,dy\right\|_{L^p} < \infty, \tag{2.10}$$

$$\lim_{t\to\infty} t^{n(1-\frac{1}{p})+1}\left\|P_t * f - P_t\int_{\mathbb{R}^n} f(x)\,dx + \nabla P_t\cdot\int_{\mathbb{R}^n} yf(y)\,dy\right\|_{L^p} = 0. \tag{2.11}$$

Proof. On the proof of (2.8), since

$$\begin{aligned}P_{t+1}(x) - P_t(x) &= \int_0^1 \partial_\theta P_{t+\theta}(x)\,d\theta\\ &= \int_0^1 \left((-n)(t+\theta)^{-n-1}P\big((t+\theta)^{-1}x\big) + (t+\theta)^{-n}(\nabla P)\big((t+\theta)^{-1}x\big)\cdot\frac{-x}{(t+\theta)^2}\right)d\theta,\end{aligned}$$

we have from Minkowski's inequality and the change of variable

$$\begin{aligned}\|P_{t+1}-P_t\|_{L^p} &\leqslant C\int_0^1 \big\{(t+\theta)^{-n-1}\big\|P\big((t+\theta)^{-1}\cdot\big)\big\|_{L^p}+(t+\theta)^{-n-2}\big\||\cdot|(\nabla P)\big((t+\theta)^{-1}\cdot\big)\big\|_{L^p}\big\}\,d\theta \\ &\leqslant C\int_0^1 (t+\theta)^{-n(1-\frac{1}{p})-1}\,d\theta\,\big(\|P\|_{L^p}+\big\||\cdot|\nabla P\big\|_{L^p}\big) \\ &\leqslant Ct^{-n(1-\frac{1}{p})-1}.\end{aligned}$$

Therefore (2.8) is obtained.

For the proof of (2.9), we consider $f\in C_0^\infty(\mathbb{R}^n)$ firstly. If $f\in C_0^\infty(\mathbb{R}^n)$, there exists $R>0$ such that $\operatorname{supp} f\subset \{x\in\mathbb{R}^n \mid |x|\leqslant R\}$. It holds that

$$\begin{aligned}(P_t*f)(x)-P_t(x)\int_{\mathbb{R}^n} f(y)\,dy &= \int_{\mathbb{R}^n}\big\{P_t(x-y)-P_t(x)\big\}f(y)\,dy \\ &= \int_{\mathbb{R}^n}\int_0^1 \partial_\theta P_t(x-\theta y)\,d\theta\, f(y)\,dy \\ &= \int_{|y|\leqslant R}\int_0^1 t^{-n}(\nabla P)\big(t^{-1}(x-\theta y)\big)\cdot\frac{-y}{t}\,d\theta\, f(y)\,dy. \end{aligned} \tag{2.12}$$

Then, we take $L^p(\mathbb{R}^n)$ norm and apply Minkowski's inequality to obtain

$$\begin{aligned}t^{n(1-\frac{1}{p})}\bigg\|(P_t*f)(x)-P_t(x)\int_{\mathbb{R}^n} f(y)\,dy\bigg\|_{L^p} &\leqslant Ct^{n(1-\frac{1}{p})}\big\|t^{-n}\nabla P\big(t^{-1}\cdot\big)\big\|_{L^p}\int_{|y|\leqslant R}\frac{|y|}{t}f(y)\,dy \\ &\leqslant Ct^{-1}R\|f\|_{L^1} \\ &\to 0 \quad \text{as } t\to\infty.\end{aligned}$$

For $f\in L^1(\mathbb{R}^n)$, since $C_0^\infty(\mathbb{R}^n)$ is dense in $L^1(\mathbb{R}^n)$ and

$$t^{n(1-\frac{1}{p})}\bigg\|(P_t*f)-P_t\int_{\mathbb{R}^n} f(y)\,dy\bigg\|_{L^p}\leqslant C\|f\|_{L^1},$$

we also obtain (2.9) by density argument.

On the proof of (2.10), we have from (2.12) and Minkowski's inequality

$$\begin{aligned}t^{n(1-\frac{1}{p})+1}\bigg\|(P_t*f)-P_t\int_{\mathbb{R}^n} f(y)\,dy\bigg\|_{L^p} &\leqslant Ct^{n(1-\frac{1}{p})+1}\big\|t^{-n}\nabla P\big(t^{-1}\cdot\big)\big\|_{L^p}\int_{\mathbb{R}^n}\frac{|y|}{t}\big|f(y)\big|\,dy \\ &\leqslant C\big\||\cdot|f\big\|_{L^1}<\infty.\end{aligned} \tag{2.13}$$

Therefore (2.10) is obtained.

To prove (2.11), we assume $f\in C_0^\infty(\mathbb{R}^n)$ before dealing with the case $f\in L^1(\mathbb{R}^n)$ with $|\cdot|f(\cdot)\in L^1(\mathbb{R}^n)$. Let $R>0$ satisfy $\operatorname{supp} f\subset\{\xi\in\mathbb{R}^n \mid |x|\leqslant R\}$. It follows from (2.12) that

$$\begin{aligned}&(P_t*f)(x)-P_t(x)\int_{\mathbb{R}^n} f(y)\,dy+\nabla P_t\cdot\int_{\mathbb{R}^n} yf(y)\,dy \\ &\quad= -t^{-n}\sum_{j=1}^n\int_{\mathbb{R}^n}\int_0^1\big((\partial_{x_j}P)\big(t^{-1}(x-\theta y)\big)-(\partial_{x_j}P)\big(t^{-1}x\big)\big)\,d\theta\cdot\frac{y_j}{t}f(y)\,dy\end{aligned}$$

$$= -t^{-n} \sum_{j=1}^{n} \int_{\mathbb{R}^n} \int_0^1 \int_0^1 \partial_\mu \big((\partial_{x_j} P)\big(t^{-1}(x - \mu\theta y)\big)\big)\, d\theta\, \frac{y_j}{t} f(y)\, dy$$

$$= t^{-n} \sum_{j=1}^{n} \int_{\mathbb{R}^n} \int_0^1 \int_0^1 (\nabla \partial_{x_j} P)\big(t^{-1}(x - \mu\theta y)\big)\, d\mu \cdot \frac{\theta y}{t}\, d\theta\, \frac{y_j}{t} f(y)\, dy.$$

Then, we have from the above identity

$$t^{n(1-\frac{n}{p})+1} \Big\| (P_t * f)(\cdot) - P_t(\cdot) \int_{\mathbb{R}^n} f(y)\, dy + \nabla P_t(\cdot) \cdot \int_{\mathbb{R}^n} y f(y)\, dy \Big\|_{L^p}$$

$$\leqslant C t^{n(1-\frac{1}{p})+1} t^{-n} \sum_{j=1}^{n} \int_{|y| \leqslant R} \int_0^1 \int_0^1 \big\| \nabla \partial_{x_j} P\big(t^{-1}\cdot\big)\big\|_{L^p} \theta\, d\theta\, d\mu\, t^{-2} |y|^2 \big|f(y)\big|\, dy$$

$$\leqslant C t^{-1} R^2 \int_{|y| \leqslant R} \big|f(y)\big|\, dy$$

$$\to 0 \quad \text{as } t \to \infty.$$

For the proof in the case $f \in L^1(\mathbb{R}^n)$ with $|\cdot| f(\cdot) \in L^1(\mathbb{R}^n)$, we use the estimate (2.13) and the density argument. Then the proof of (2.11) is completed. □

3. A priori estimate

We consider a priori estimate of the following equation:

$$\begin{cases} \partial_t u + \sum_{j=1}^{n} \partial_{x_j} F(v_j, u) + (-\Delta)^{\frac{1}{2}} u = G(u) & \text{for } t > 0,\ x \in \mathbb{R}^n, \\ u(0, x) = u_0(x) & \text{for } x \in \mathbb{R}^n, \end{cases} \tag{3.1}$$

where $v = (v_1, v_2, \ldots, v_n)$ is the given vector field, and $F(\cdot,\cdot)$ be defined by

$$F(v, u) := \sum_{k_2 \in \mathbb{Z}} (S_{k_2-3} v)(\phi_{k_2} * u) = \sum_{k_2 \in \mathbb{Z}} \sum_{k_1 \leqslant k_2 - 3} (\phi_{k_1} * v)(\phi_{k_2} * u), \tag{3.2}$$

and G is a smooth function. $F(v, u)$ is one factor of the decomposition due to J.-M. Bony [10] on the interaction of low and high frequency of vu, and $G(u)$ can be regarded as the remainder term. We show a lemma related to the commutator estimates to show a priori estimate.

Lemma 3.1. *Let $u_k := \phi_k * u$ for simplicity and let $s \geqslant 0$ and $1 \leqslant p \leqslant \infty$. Then it holds that*

$$\sum_{k \in \mathbb{Z}} 2^{sk} \big\| \phi_k * \operatorname{div} F(v, u) - (S_{k-8} v) \cdot \nabla u_k \big\|_{L^p} \leqslant C \|\nabla v\|_{L^\infty} \|u\|_{\dot{B}^s_{p,1}}. \tag{3.3}$$

Remark 3.2. For the Euler equations, such estimate is known and we refer to [11,26] on the commutator estimates for divergence free vector field v.

Proof of Lemma 3.1. To prove (3.3), we show the following

$$\big\| \phi_k * \operatorname{div} F(S_{k-8} v, u) - (S_{k-8} v) \cdot \nabla u_k \big\|_{L^p}$$

$$\leqslant C \|\nabla v\|_{L^\infty} \sum_{\mu=-3}^{3} \|u_{k+\mu}\|_{L^p} + C \|\operatorname{div} v\|_{L^\infty} \sum_{\mu=-3}^{3} \|u_{k+\mu}\|_{L^p}, \tag{3.4}$$

$$\Big\| \phi_k * \operatorname{div} F\Big(\sum_{k_1 \geqslant k-7} \phi_{k_1} * v, u \Big) \Big\|_{L^p} \leqslant C \|v\|_{\dot{B}^1_{\infty,\infty}} \sum_{\mu=-3}^{3} \|u_{k+\mu}\|_{L^p}. \tag{3.5}$$

For the proof of (3.4), let Ψ_k be defined by $\Psi_k := \sum_{\mu=-3}^{3} \phi_{k+\mu}$. Since

$$
\begin{aligned}
& S_{k_2-3} S_{k-8} v = S_{k-8} v \quad \text{for } k_2 \geqslant k-3, \qquad \operatorname{supp} \widehat{\phi}_k \subset \{2^{k-1} \leqslant |\xi| \leqslant 2^{k+1}\}, \\
& \operatorname{supp} \mathcal{F}\big[(S_{k_2-3} S_{k-8} v)(\phi_{k_2} * u)\big] \subset \{\xi \in \mathbb{R}^n \mid 2^{k_2-2} \leqslant |\xi| \leqslant 2^{k_2+2}\}, \\
& \phi_k * \operatorname{div} F(v, \phi_{k_2} * u) = 0 \quad \text{for } k, k_2 \text{ with } |k_2 - k| \geqslant 4,
\end{aligned} \tag{3.6}
$$

we have

$$
\begin{aligned}
& \phi_k * \operatorname{div} F(S_{k-8} v, u) - (S_{k-8} v) \cdot \nabla u_k \\
& \quad = \phi_k * \operatorname{div}\big((S_{k-8} v)(\Psi_k * u)\big) - (S_{k-8} v) \cdot \nabla u_k \\
& \quad = \phi_k * \big((\operatorname{div} S_{k-8} v)(\Psi_k * u)\big) + \big(\phi_k * \big((S_{k-8} v) \cdot \nabla \Psi_k * u\big) - (S_{k-8} v) \cdot \nabla u_k\big) \\
& \quad =: I + II.
\end{aligned} \tag{3.7}
$$

On the estimate of I, it follows from Hausdorff–Young's inequality, $\|\phi_k\|_{L^1} = \|\phi_0\|_{L^1}$, Hölder's inequality and the definition of Ψ_k

$$
\|I\|_{L^p} \leqslant C \|\phi_k\|_{L^1} \big\|(\operatorname{div} S_{k-8} v)(\Psi_k * u)\big\|_{L^p} \leqslant C \|\operatorname{div} v\|_{L^\infty} \sum_{\mu=-3}^{3} \|u_{k+\mu}\|_{L^p}. \tag{3.8}
$$

On the estimate of II, let $\widetilde{u}_k := \Psi_k * u$ and it follows from the change of variables, $u_k = \phi_k * \widetilde{u}_k$ and integration by parts that

$$
\begin{aligned}
II(x) &= \int_{\mathbb{R}^n} 2^{nk} \phi_0(2^k y)\big(S_{k-8} v(x-y) - S_{k-8} v(x)\big) \cdot \nabla \widetilde{u}_k(x-y)\, dy \\
&= \int_{\mathbb{R}^n} 2^{nk} \phi_0(2^k y)\left(\int_0^1 (\nabla S_{k-8} v)(x - \theta y) \cdot (-y)\, d\theta\right) \cdot \nabla \widetilde{u}_k(x-y)\, dy \\
&= -2^{(n+1)k} \int_{\mathbb{R}^n} (\nabla \phi_0)(2^k y) \cdot \left(\int_0^1 (\nabla S_{k-8} v)(x - \theta y) \cdot (-y)\, d\theta\right) \widetilde{u}_k(x-y)\, dy \\
&\quad + 2^{nk} \int_{\mathbb{R}^n} \phi_0(2^k y)\left(\int_0^1 (\nabla \operatorname{div} S_{k-8} v)(x - \theta y) \cdot \theta y\, d\theta\right) \widetilde{u}_k(x-y)\, dy \\
&\quad + 2^{nk} \int_{\mathbb{R}^n} \phi_0(2^k y)\left(\int_0^1 \sum_{j=1}^{n} (\nabla S_{k-8} v_j)(x - \theta y) \cdot (-1)\, d\theta\right) \widetilde{u}_k(x-y)\, dy \\
&= -\int_{\mathbb{R}^n} \nabla \phi_0(y) \cdot \left(\int_0^1 (\nabla S_{k-8} v)\big(x - \theta 2^{-k} y\big) \cdot (-y)\, d\theta\right) \widetilde{u}_k\big(x - 2^{-k} y\big)\, dy \\
&\quad + 2^{-k} \int_{\mathbb{R}^n} \phi_0(y)\left(\int_0^1 (\nabla \operatorname{div} S_{k-8} v)\big(x - \theta 2^{-k} y\big) \cdot (-y)\, d\theta\right) \widetilde{u}_k\big(x - 2^{-k} y\big)\, dy \\
&\quad + \int_{\mathbb{R}^n} \phi_0(y)\left(\int_0^1 \sum_{j=1}^{n} (\nabla S_{k-8} v_j)\big(x - \theta 2^{-k} y\big) \cdot (-1)\, d\theta\right) \widetilde{u}_k\big(x - 2^{-k} y\big)\, dy.
\end{aligned}
$$

Then we taking the $L^p(\mathbb{R}^n)$ norm on the above inequality, it follows from $(1+|y|)(|\phi_0(y)| + |\nabla \phi_0(y)|) \in L^1_y(\mathbb{R}^n)$, the support of $\mathcal{F}[S_{k-8} v]$ being restricted to the set $\{|\xi| \leqslant 2^{k-7}\}$ and Hölder's inequality that

$$\begin{aligned}\|II\|_{L^p} &\leqslant C\|\nabla S_{k-8}v\|_{L^\infty}\|\widetilde{u}_k\|_{L^p} + 2^{-k}\big\|\nabla \operatorname{div} S_{k-8}v\big\|_{L^\infty}\|\widetilde{u}_k\|_{L^p} \\ &\leqslant C\|\nabla v\|_{L^\infty}\|\widetilde{u}_k\|_{L^p}.\end{aligned} \tag{3.9}$$

Then we obtain (3.4) by (3.7), (3.8), (3.9) and the definition of $\widetilde{u}_k$, Ψ_k.

For the proof of (3.5), it follows from (3.6), Hausdorff–Young inequality, Hölder's inequality and the number of k_1 with $k-3 \leqslant k_2 \leqslant k+3$ and $k-7 \leqslant k_1 \leqslant k_2 - 3$ being 8

$$\begin{aligned}\bigg\|\phi_k * \operatorname{div} F\bigg(\sum_{k_1 \geqslant k-7} \phi_{k_1} * v, u\bigg)\bigg\|_{L^p} &\leqslant C2^k \bigg\|\phi_k * \sum_{k_2=k-3}^{k+3}\sum_{k_1=k-7}^{k_2-3} (\phi_{k_1} * v)(\phi_{k_2} * u)\bigg\|_{L^p} \\ &\leqslant C2^k \sum_{k_2=k-3}^{k+3}\sum_{k_1=k-7}^{k_2-3} \|\phi_k\|_{L^1}\big\|(\phi_{k_1} * v)(\phi_{k_2} * u)\big\|_{L^p} \\ &\leqslant C2^k \sum_{k_2=k-3}^{k+3}\sum_{k_1=k-7}^{k_2-3} \|\phi_{k_1} * v\|_{L^\infty}\|\phi_{k_2} * u\|_{L^p} \\ &\leqslant C\|v\|_{\dot{B}^1_{\infty,\infty}} \sum_{k_2=k-3}^{k+3} \|u_{k_2}\|_{L^p}.\end{aligned}$$

Then (3.5) is obtained.

We show (3.3) by the use of (3.4) and (3.5). It follows from (3.4), (3.5) and the embedding $\dot{W}^{1,\infty}(\mathbb{R}^n) \hookrightarrow \dot{B}^1_{\infty,\infty}(\mathbb{R}^n)$

$$\begin{aligned}\sum_{k\in\mathbb{Z}} 2^{sk}\big\|\phi_k * \operatorname{div} F(v,u) - (S_{k-8}v)\cdot\nabla u_k\big\|_{L^p} &\leqslant C\big(\|\nabla v\|_{L^\infty} + \|v\|_{\dot{B}^1_{\infty,\infty}}\big)\sum_{k\in\mathbb{Z}} 2^{sk}\sum_{\mu=-3}^{3}\|u_{k+\mu}\|_{L^p} \\ &\leqslant C\|\nabla v\|_{L^\infty}\sum_{\mu=-3}^{3} 2^{-s\mu}\sum_{k\in\mathbb{Z}} 2^{s(k+\mu)}\|u_{k+\mu}\|_{L^p} \\ &\leqslant C\|\nabla v\|_{L^\infty}\|u\|_{\dot{B}^s_{p,1}}.\end{aligned}$$

Therefore we complete the proof of (3.3). □

Proposition 3.3. *Let G satisfy*

$$\big\|G(u)\big\|_{\dot{B}^s_{\infty,1}} \leqslant Cw(t)\|u\|_{\dot{B}^s_{\infty,1}}, \tag{3.10}$$

for some nonnegative function w. Then it holds that for the solution u to (3.1)

$$\big\|u(t)\big\|_{\dot{B}^s_{\infty,1}} \leqslant \|u_0\|_{\dot{B}^s_{\infty,1}} \exp\bigg(C\int_0^t \big(\big\|\nabla v(\tau)\big\|_{L^\infty} + w(\tau)\big)\,d\tau\bigg). \tag{3.11}$$

Proof. For simplicity, let $u_k = \phi_k * u$ and $G_k = \phi_k * G$. It follows from (3.1)

$$\partial_t u_k + (S_{k-8}v)\cdot\nabla u_k + (-\Delta)^{\frac{1}{2}}u_k = -\phi_k * \big(\operatorname{div} F(v,u)\big) + (S_{k-8}v)\cdot\nabla u_k + G_k.$$

Let ψ_k be the flow of the regularized vector field $S_{k-8}v$ and let $\widetilde{u}_k$ be defined by $\widetilde{u}_k := u_k \circ \psi_k$. It holds that

$$\begin{aligned}\partial_t\widetilde{u}_k + (-\Delta)^{\frac{1}{2}}\widetilde{u}_k = &-\big(\phi_k * \big(\operatorname{div} F(v,u)\big)\big)\circ\psi_k + \big((S_{k-8}v)\cdot\nabla u_k\big)\circ\psi_k \\ &+ (-\Delta)^{\frac{1}{2}}\widetilde{u}_k - \big((-\Delta)^{\frac{1}{2}}u_k\big)\circ\psi_k + G_k\circ\psi_k.\end{aligned} \tag{3.12}$$

Then, it follows in the analogous way to the proof of Theorem 1.2 in the result by Miao and Wu [23] that

$$\sum_{k\in\mathbb{Z}} 2^{sk}\|u_k\|_{L^\infty(0,t;L^\infty)} \leqslant \|u_0\|_{\dot B^s_{\infty,1}} + C\int_0^t \sum_{k\in\mathbb{Z}} \bigl(2^{sk}\bigl\|\phi_k * \bigl(\operatorname{div} F(v,u)\bigr) - (S_{k-8}v)\cdot\nabla u_k\bigr\|_{L^\infty} + 2^{sk}\|G_k\|_{L^\infty}\bigr)\,d\tau.$$

It follows from the above estimate, (3.3) and (3.10) that

$$\begin{aligned}\sum_{k\in\mathbb{Z}} 2^{sk}\|u_k\|_{L^\infty(0,t;L^\infty)} &\leqslant \|u_0\|_{\dot B^s_{\infty,1}} + C\int_0^t \bigl(\|\nabla v\|_{L^\infty} + w(\tau)\bigr)\|u\|_{\dot B^s_{\infty,1}}\,d\tau \\ &\leqslant \|u_0\|_{\dot B^s_{\infty,1}} + C\int_0^t \bigl(\|\nabla v\|_{L^\infty} + w(\tau)\bigr)\Bigl(\sum_{k\in\mathbb{Z}} 2^{sk}\|u_k\|_{L^\infty(0,\tau;L^\infty)}\Bigr)\,d\tau.\end{aligned}$$

By Gronwall's inequality, we obtain

$$\bigl\|u(t)\bigr\|_{\dot B^s_{\infty,1}} \leqslant \sum_{k\in\mathbb{Z}} 2^{sk}\|u_k\|_{L^\infty(0,t;L^\infty)} \leqslant \|u_0\|_{\dot B^s_{\infty,1}} \exp\Bigl(C\int_0^t \bigl(\|\nabla v\|_{L^\infty} + w(\tau)\bigr)\,d\tau\Bigr).$$

Then the proof of (3.11) is completed. □

4. Proof of Theorem 1.1

We prove Theorem 1.1 by contraction argument. Let Ψ and X be defined by

$$\begin{aligned}&\Psi(u)(t) := P_t * u_0 - \frac{1}{2}\int_0^t P_{t-\tau} * \sum_{j=1}^n \partial_{x_j} u^2\,d\tau,\\ &X := \bigl\{u \in C\bigl([0,\infty); \dot B^0_{\infty,1}(\mathbb{R}^n)\bigr) \bigm| \|u\|_X \leqslant \rho\bigr\},\\ &\|u\|_X := \|u\|_{L^\infty(0,\infty;\dot B^0_{\infty,1})} + \|u\|_{L^1(0,\infty;\dot B^1_{\infty,1})},\end{aligned}$$

where ρ will be taken later. We see that X is a complete metric space since the Besov space $\dot B^0_{\infty,1}(\mathbb{R}^n)$ is complete by the argument in [22]. For the estimate of the nonlinear part, we introduce the following proposition.

Proposition 4.1. *It holds that*

$$\Bigl\|\int_0^t P_{t-\tau} * \sum_{j=1}^n \partial_{x_j} u^2\,d\tau\Bigr\|_X \leqslant C\|u\|_{L^\infty(0,\infty;\dot B^0_{\infty,1})}\|u\|_{L^1(0\infty;\dot B^1_{\infty,1})}, \tag{4.1}$$

$$\Bigl\|\int_0^t P_{t-\tau} * \sum_{j=1}^n \partial_{x_j} u^2\,d\tau - \int_0^t P_{t-\tau} * \sum_{j=1}^n \partial_{x_j} v^2\,d\tau\Bigr\|_X \leqslant C\bigl(\|u\|_X + \|v\|_X\bigr)\|u-v\|_X. \tag{4.2}$$

Remark 4.2. The constant C in Proposition 4.1 is independent of ρ, which is in the definition of X, and depends only on the dimension n.

Proof of Proposition 4.1. On the proof of (4.1), we have from the boundedness of P_t* in $\dot B^0_{\infty,1}(\mathbb{R}^n)$, (2.7), (2.2) and Hölder's inequality

$$\Bigl\|\int_0^t P_{t-\tau} * \sum_{j=1}^n \partial_{x_j} u^2\,d\tau\Bigr\|_X \leqslant C\int_0^\infty \|u^2\|_{\dot B^1_{\infty,1}}\,d\tau$$

$$\leqslant C\int_0^\infty \|u\|_{\dot B^0_{\infty,1}}\|u\|_{\dot B^1_{\infty,1}}\,d\tau$$
$$\leqslant C\|u\|_{L^\infty(0,\infty;\dot B^0_{\infty,1})}\|u\|_{L^1(0,\infty;\dot B^1_{\infty,1})}.$$

On the proof of (4.2), we also have from the boundedness of P_t* in $\dot B^0_{\infty,1}(\mathbb{R}^n)$, (2.7), (2.2) and Hölder's inequality

$$\left\|\int_0^t P_{t-\tau}*\sum_{j=1}^n \partial_{x_j}u^2\,d\tau-\int_0^t P_{t-\tau}*\sum_{j=1}^n \partial_{x_j}v^2\,d\tau\right\|_X$$
$$\leqslant C\int_0^\infty \big\|(u+v)(u-v)\big\|_{\dot B^1_{\infty,1}}\,d\tau$$
$$\leqslant C\int_0^\infty \big(\|u+v\|_{\dot B^1_{\infty,1}}\|u-v\|_{\dot B^0_{\infty,1}}+\|u+v\|_{\dot B^0_{\infty,1}}\|u-v\|_{\dot B^1_{\infty,1}}\big)\,d\tau$$
$$\leqslant C\big(\|u\|_X+\|v\|_X\big)\|u-v\|_X.$$

Then, we complete the proof. □

By the use of the above proposition, we prove Theorem 1.1. On the proof of uniqueness, the uniqueness with smallness condition is shown easily by the contraction argument, while we give the proof without the smallness. We also note that the uniqueness in the space $L^\infty((0,\infty)\times\mathbb{R}^n)$ is already known for the entropy solutions by [1] and $L^\infty(0;\infty;\dot B^0_{\infty,1}(\mathbb{R}^n))\subset L^\infty((0,\infty)\times\mathbb{R}^n)$.

Proof of Theorem 1.1. We prove that Ψ is contraction map from X to itself. Let $C_0>0$ be the constant which satisfies (2.6), (4.1) and (4.2), and let $\rho>0$ satisfy $\rho\leqslant(4C_0)^{-1}$. We consider the initial data $u_0\in\dot B^0_{\infty,1}(\mathbb{R}^n)$ with

$$\|u_0\|\leqslant\frac{\rho}{2C_0}.$$

By (2.6), (4.1) and (4.2), there exists $C_0>0$ such that we have for $u,v\in X$

$$\big\|\Psi(u)\big\|_X\leqslant C_0\|u_0\|_{\dot B^0_{\infty,1}}+C_0\|u\|_X^2\leqslant C_0\cdot\frac{\rho}{2C_0}+C_0\rho^2\leqslant\rho,$$
$$\big\|\Psi(u)-\Psi(v)\big\|_X\leqslant C_0\big(\|u\|_X+\|v\|_X\big)\|u-v\|_X\leqslant C_0\cdot 2\rho\|u-v\|_X\leqslant\frac12\|u-v\|.$$

Therefore, we obtain a unique solution $u\in X$ by the contraction mapping principle.

To show the uniqueness of the solution, let u,v be solutions to (1.2) in the space $C([0,\infty);\dot B^0_{\infty,1}(\mathbb{R}^n))\cap L^1(0,\infty;\dot B^1_{\infty,1}(\mathbb{R}^n))$. Then u and v satisfy

$$\partial_t(u-v)+\frac12\sum_{j=1}^n\partial_{x_j}\big\{(u+v)(u-v)\big\}+\Lambda(u-v)=0.$$

By the use of the notation F defined by (3.2), it holds that

$$\partial_t(u-v)+\frac12\sum_{j=1}^n\partial_{x_j}F(u+v,u-v)+\Lambda(u-v)$$
$$=-\frac12\sum_{j=1}^n\partial_{x_j}\big\{(u+v)(u-v)\big\}+\frac12\sum_{j=1}^n\partial_{x_j}F(u+v,u-v). \tag{4.3}$$

In order to apply Proposition 3.3, we show the following

$$\Big\| \sum_{j=1}^{n} \big(\partial_{x_j}(fg) - \partial_{x_j} F(f,g) \big) \Big\|_{\dot B^0_{\infty,1}} \leqslant C \|f\|_{\dot B^1_{\infty,1}} \|g\|_{\dot B^0_{\infty,1}}. \tag{4.4}$$

Once (4.4) is proved, we obtain $u(t) = v(t)$ for all $t \geqslant 0$ by (4.3), Proposition 3.3, the embedding $\dot B^1_{\infty,1}(\mathbb{R}^n) \hookrightarrow \dot W^{1,\infty}(\mathbb{R}^n)$ and $u, v \in L^1(0,\infty; \dot B^1_{\infty,1}(\mathbb{R}^n))$. Then it remains to prove (4.4). By the definition of F, we have

$$\begin{aligned} \sum_{j=1}^{n} \big(\partial_{x_j}(fg) - \partial_{x_j} F(f,g) \big) &= \sum_{j=1}^{n} \sum_{k_2 \in \mathbb{Z}} \sum_{k_1 \geqslant k_2+3} \partial_{x_j}(f_{k_1} g_{k_2}) + \sum_{j=1}^{n} \sum_{k_2 \in \mathbb{Z}} \sum_{|k_1-k_2| \leqslant 2} \partial_{x_j}(f_{k_1} g_{k_2}) \\ &=: I + II, \end{aligned} \tag{4.5}$$

where $f_{k_1} := \phi_{k_1} * f$ and $g_{k_2} := \phi_{k_2} * g$. On the estimate of I, since $\phi_k * (f_{k_1} g_{k_2}) = 0$ for $|k-k_1| \geqslant 3$ and $k_1 \geqslant k_2 + 3$, it is sufficient to consider the sum with $|k-k_1| \leqslant 2$ and it follows from Hausdorff–Young's inequality and $\|\phi_k\|_{L^1} = \|\phi_0\|_{L^1}$

$$\begin{aligned} \|I\|_{\dot B^0_{\infty,1}} &= \sum_{k \in \mathbb{Z}} \sum_{|k-k_1| \leqslant 2} \sum_{k_1 \geqslant k_2+3} 2^k \big\| \phi_k * (f_{k_1} g_{k_2}) \big\|_{L^\infty} \\ &\leqslant \sum_{k_1 \in \mathbb{Z}} \sum_{|k-k_1| \leqslant 2} \sum_{k_2 \in \mathbb{Z}} 2^k \|\phi_k\|_{L^1} \|f_{k_1}\|_{L^\infty} \|g_{k_2}\|_{L^\infty} \\ &\leqslant C \|g\|_{\dot B^0_{\infty,1}} \sum_{k_1 \in \mathbb{Z}} 2^{k_1} \|f_{k_1}\|_{L^\infty} \\ &= C \|f\|_{\dot B^1_{\infty,1}} \|g\|_{\dot B^0_{\infty,1}}. \end{aligned} \tag{4.6}$$

On the estimate of II, since $\phi_k * (f_{k_1} g_{k_2}) = 0$ for $|k_1 - k_2| \leqslant 2$ and $k - 4 > k_1$, it follows from Hausdorff–Young's inequality and $\|\phi_k\|_{L^1} = \|\phi_0\|_{L^1}$

$$\begin{aligned} \|II\|_{\dot B^0_{\infty,1}} &\leqslant \sum_{k \in \mathbb{Z}} \sum_{|k_1-k_2| \leqslant 2,\ k_1 \geqslant k-4} 2^k \|f_{k_1}\|_{L^\infty} \|g_{k_2}\|_{L^\infty} \\ &\leqslant C \sum_{|k_1-k_2| \leqslant 2} 2^{k_1} \|f_{k_1}\|_{L^\infty} \|g_{k_2}\|_{L^\infty} \\ &\leqslant C \|f\|_{\dot B^1_{\infty,\infty}} \|g_{k_2}\|_{\dot B^0_{\infty,1}}. \end{aligned} \tag{4.7}$$

We obtain (4.4) by (4.5), (4.6), (4.7) and the embedding $\dot B^1_{\infty,1}(\mathbb{R}^n) \hookrightarrow \dot B^1_{\infty,\infty}(\mathbb{R}^n)$. Therefore we complete the proof of the uniqueness of the solution.

It remains to show a priori estimate (1.3). Since it follows from (1.1) and (4.7) that the solution u satisfies

$$\partial_t u + \sum_{j=1}^{n} \partial_{x_j} F(u,u) + \Lambda u = -\frac{1}{2} \sum_{j=1}^{n} \partial_{x_j} \sum_{|k_1-k_2| \leqslant 2} (\phi_{k_1} * u)(\phi_{k_2} * u),$$

$$\Big\| \sum_{j=1}^{n} \partial_{x_j} \sum_{|k_1-k_2| \leqslant 2} (\phi_{k_1} * u)(\phi_{k_2} * u) \Big\|_{\dot B^0_{\infty,1}} \leqslant C \|u\|_{\dot B^1_{\infty,\infty}} \|u\|_{\dot B^0_{\infty,1}} \leqslant C \|\nabla u\|_{L^\infty} \|u\|_{\dot B^0_{\infty,1}},$$

where F is defined by (3.2), we obtain (1.3) by Proposition 3.3. Then we complete the proof of Theorem 1.1. □

5. Decay estimates of solutions

We introduce a modified version of Gronwall's inequality to show the proposition on the decay estimate of the solution.

Lemma 5.1. *Let a, $f, g \in C([0,\infty))$ be nonnegative functions, and they satisfy*

$$f(t) \leqslant a(t) + \int_{\frac{t}{2}}^{t} g(\tau) f(\tau)\, d\tau.$$

Then, it holds that for $t > 0$

$$f(t) \leqslant a(t) + \int_{\frac{t}{2}}^{t} a(\tau) g(\tau) \exp\left(\int_{\tau}^{t} g(\widetilde{\tau})\, d\widetilde{\tau}\right) d\tau. \tag{5.1}$$

Proof. We may assume $a \in C^1([0,\infty))$ since we can deal with the case $a \in C([0,\infty))$ by the density argument. Let $t_0 > 0$ and let $\Phi(t)$ be defined by

$$\Phi(t) := a(t) + \int_{\frac{t_0}{2}}^{t} g(\tau) f(\tau)\, d\tau \quad \text{for } t \geqslant \frac{t_0}{2}.$$

Then, it follows from $f(t) \leqslant \Phi(t)$ that

$$\Phi'(t) = a'(t) + g(t) f(t) \leqslant a'(t) + g(t)\Phi(t).$$

We multiply $\exp(-\int_0^t g(\tau)\, d\tau)$ and integrate both sides to obtain

$$\left\{\Phi(t) \exp\left(-\int_0^t g(\tau)\, d\tau\right)\right\}' \leqslant a'(t) \exp\left(-\int_0^t g(\tau)\, d\tau\right),$$

$$\Phi(t) \exp\left(-\int_0^t g(\tau)\, d\tau\right) - \Phi(t_0/2) \exp\left(-\int_0^{\frac{t_0}{2}} g(\tau)\, d\tau\right) \leqslant \int_{\frac{t_0}{2}}^{t} a'(\tau) \exp\left(-\int_0^{\tau} g(\widetilde{\tau})\, d\widetilde{\tau}\right) d\tau. \tag{5.2}$$

On the right hand side of the above inequality, we have from the integration by parts

$$\begin{aligned} \int_{\frac{t_0}{2}}^{t} a'(\tau) \exp\left(-\int_0^{\tau} g(\widetilde{\tau})\, d\widetilde{\tau}\right) d\tau &= a(t) \exp\left(-\int_0^{t} g(\widetilde{\tau})\, d\widetilde{\tau}\right) - a(t_0/2) \exp\left(-\int_0^{\frac{t_0}{2}} g(\widetilde{\tau})\, d\widetilde{\tau}\right) \\ &\quad + \int_{\frac{t_0}{2}}^{t} a(\tau) g(\tau) \exp\left(-\int_0^{\tau} g(\widetilde{\tau})\, d\widetilde{\tau}\right) d\tau. \end{aligned} \tag{5.3}$$

It follows from (5.2), (5.3) and $\Phi(t_0/2) = a(t_0/2)$

$$\Phi(t) \exp\left(-\int_0^t g(\tau)\, d\tau\right) \leqslant a(t) \exp\left(-\int_0^t g(\widetilde{\tau})\, d\widetilde{\tau}\right) + \int_{\frac{t_0}{2}}^{t} a(\tau) g(\tau) \exp\left(-\int_0^{\tau} g(\widetilde{\tau})\, d\widetilde{\tau}\right) d\tau,$$

$$\Phi(t) \leqslant a(t) + \int_{\frac{t_0}{2}}^{t} a(\tau) g(\tau) \exp\left(\int_{\tau}^{t} g(\widetilde{\tau})\, d\widetilde{\tau}\right) d\tau.$$

Then, it holds that for $t = t_0$

$$f(t_0) \leqslant \Phi(t_0) \leqslant a(t_0) + \int_{\frac{t_0}{2}}^{t_0} a(\tau)g(\tau)\exp\left(\int_{\tau}^{t_0} g(\tilde{\tau})\,d\tilde{\tau}\right)d\tau.$$

The proof of (5.1) is completed since t_0 is an arbitrary positive real number. □

Proposition 5.2. *Let $u_0 \in L^1(\mathbb{R}^n)$ and let u be a solution to the integral equation (1.2) in $L^\infty(0,\infty; L^1(\mathbb{R}^n)) \cap L^1(0,\infty; \dot{B}^1_{\infty,1}(\mathbb{R}^n))$. Then, it holds that for p with $1 < p \leqslant \infty$*

$$\|u(t)\|_{L^p} \leqslant Ct^{-n(1-\frac{1}{p})}, \qquad \|u(t)\|_{\dot{B}^0_{p,1}} \leqslant Ct^{-n(1-\frac{1}{p})} \tag{5.4}$$

for all $t > 0$. Furthermore, if $t^\alpha u(t) \in L^1_t(0,\infty; \dot{B}^{\alpha+1}_{\infty,1}(\mathbb{R}^n))$ for some $\alpha > 0$, it holds that for $1 \leqslant p \leqslant \infty$

$$\||\nabla|^\alpha u(t)\|_{L^p} \leqslant Ct^{-n(1-\frac{1}{p})-\alpha}, \qquad \|u(t)\|_{\dot{B}^\alpha_{p,1}} \leqslant Ct^{-n(1-\frac{1}{p})-\alpha} \tag{5.5}$$

for all $t > 0$.

Remark 5.3. The problem in the proof is how to use the decay estimate on the Poisson kernel for the nonlinear term, since $\|\nabla P_{t-\tau}\|_{L^\infty} \leqslant C(t-\tau)^{-1}$ and $(t-\tau)^{-1}$ is not integrable. The idea is to divide the interval $[0,t]$ into $[0,t/2]$ and $[t/2,t]$ and apply Lemma 5.1

Proof of Proposition 5.2. For the proof of (5.4), it is sufficient to consider the estimate of the norm in Besov space $\dot{B}^0_{p,1}(\mathbb{R}^n)$ since the embedding $\dot{B}^0_{p,1}(\mathbb{R}^n) \hookrightarrow L^p(\mathbb{R}^n)$ holds. It follows from Proposition 2.2 (ii), Proposition 2.4, the embedding $L^1(\mathbb{R}^n) \hookrightarrow \dot{B}^0_{1,\infty}(\mathbb{R}^n)$, Hölder's inequality, (2.2), and the embedding $\dot{B}^0_{\infty,1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$ that

$$\begin{aligned}
\|u(t)\|_{\dot{B}^0_{p,1}} &\leqslant Ct^{-n(1-\frac{1}{p})}\|u_0\|_{\dot{B}^0_{1,\infty}} + C\int_0^{\frac{t}{2}} (t-\tau)^{-n(1-\frac{1}{p})}\|\nabla(u^2)\|_{\dot{B}^0_{1,\infty}}\,d\tau + \int_{\frac{t}{2}}^{t} \|u^2\|_{\dot{B}^1_{p,1}}\,d\tau \\
&\leqslant Ct^{-n(1-\frac{1}{p})}\|u_0\|_{L^1} + Ct^{-n(1-\frac{1}{p})}\int_0^\infty \|u\|_{L^1}\|\nabla u\|_{L^\infty}\,d\tau + C\int_{\frac{t}{2}}^{t} \|u\|_{\dot{B}^1_{\infty,1}}\|u\|_{\dot{B}^0_{p,1}}\,d\tau \\
&\leqslant C\left(\|u_0\|_{L^1} + \|u\|_{L^\infty(0,\infty;L^1)}\|u\|_{L^1(0,\infty;\dot{B}^1_{\infty,1})}\right)t^{-n(1-\frac{1}{p})} + C\int_{\frac{t}{2}}^{t} \|u\|_{\dot{B}^1_{\infty,1}}\|u\|_{\dot{B}^0_{p,1}}\,d\tau.
\end{aligned} \tag{5.6}$$

By Lemma 5.1, we have

$$\begin{aligned}
\|u(t)\|_{\dot{B}^0_{p,1}} &\leqslant Ct^{-n(1-\frac{1}{p})} + \int_{\frac{t}{2}}^{t} C\tau^{-n(1-\frac{1}{p})}\|u\|_{\dot{B}^1_{\infty,1}}\exp\left(C\int_{\tau}^{t}\|u\|_{\dot{B}^1_{\infty,1}}\,d\tilde{\tau}\right)d\tau \\
&\leqslant Ct^{-n(1-\frac{1}{p})} + Ct^{-n(1-\frac{1}{p})}\|u\|_{L^1(0,\infty;\dot{B}^1_{\infty,1})}\exp\left(\|u\|_{L^1(0,\infty;\dot{B}^1_{\infty,1})}\right) \\
&\leqslant Ct^{-n(1-\frac{1}{p})},
\end{aligned}$$

where the above constant C depends on the norms of u_0 and u. Then, we complete the proof of (5.4).

For the proof of (5.5), it is sufficient to consider the estimate in Besov space $\dot{B}^\alpha_{p,1}(\mathbb{R}^n)$ since $\||\nabla|^\alpha u\|_{L^p} \leqslant C\|u\|_{\dot{B}^\alpha_{p,1}}$. It follows from the analogous argument to (5.6) and the decay estimates (5.4) that

$$\begin{aligned}\|u(t)\|_{\dot{B}^{\alpha}_{p,1}} &\leqslant Ct^{-n(1-\frac{1}{p})-\alpha}\|u_0\|_{\dot{B}^0_{1,\infty}} + C\int_0^{\frac{t}{2}}(t-\tau)^{-n(1-\frac{1}{p})-\alpha}\|\nabla(u^2)\|_{\dot{B}^0_{1,\infty}}\,d\tau + C\int_{\frac{t}{2}}^{t}\|u^2\|_{\dot{B}^{\alpha+1}_{p,1}}\,d\tau\\
&\leqslant Ct^{-n(1-\frac{1}{p})-\alpha} + C\int_{\frac{t}{2}}^{t}\|u\|_{\dot{B}^{\alpha+1}_{\infty,1}}\|u\|_{\dot{B}^0_{p,1}}\,d\tau\\
&\leqslant Ct^{-n(1-\frac{1}{p})-\alpha} + Ct^{-\alpha}\int_{\frac{t}{2}}^{t}\tau^{\alpha}\|u\|_{\dot{B}^{\alpha+1}_{\infty,1}}\tau^{-n(1-\frac{1}{p})}\,d\tau\\
&\leqslant Ct^{-n(1-\frac{1}{p})-\alpha} + Ct^{-n(1-\frac{1}{p})-\alpha}\big\||\tau|^{\alpha}u(\tau)\big\|_{L^1_\tau(0,\infty;\dot{B}^{\alpha+1}_{\infty,1})}.\end{aligned}$$

Then, we complete the proof of (5.5). □

6. Proof of Theorem 1.2

We give a proposition on the existence of global solutions in $L^1(\mathbb{R}^n)$ and $\dot{B}^0_{\infty,1}(\mathbb{R}^n)$ to prove Theorem 1.2.

Proposition 6.1. *Let $\alpha > 0$. Then there exists $\delta > 0$ such that for any initial data $u_0 \in L^1(\mathbb{R}^n) \cap \dot{B}^0_{\infty,1}(\mathbb{R}^n)$ with $\|u_0\|_{\dot{B}^0_{\infty,1}} \leqslant \delta$, there exists a unique global solution to the integral equation (1.2) in the space $C([0,\infty);\dot{B}^0_{\infty,1}(\mathbb{R}^n)) \cap L^1(0,\infty;\dot{B}^1_{\infty,1}(\mathbb{R}^n))$ with the following property*

$$\big\|u(t)\big\|_{L^\infty(0,\infty;L^1)} < \infty, \qquad \big\|t^{\alpha}u(t)\big\|_{L^1_t(0,\infty;\dot{B}^{\alpha+1}_{\infty,1})} < \infty. \tag{6.1}$$

Proof. By Theorem 1.1, there exists $\delta_1 > 0$ such that if $\|u_0\|_{\dot{B}^0_{\infty,1}} \leqslant \delta_1$, a unique solution exists in the space $C([0,\infty);\dot{B}^0_{\infty,1}(\mathbb{R}^n)) \cap L^1(0,\infty;\dot{B}^1_{\infty,1}(\mathbb{R}^n))$.

For the proof of (6.1), we utilize the following asymptotic expansion of the solution;

$$u(t) = \sum_{m=1}^{\infty} U_m(t), \tag{6.2}$$

where U_m is defined by

$$\begin{cases} U_1(t) := P_t * u_0 & \text{for } m = 1,\\ U_m(t) := -\dfrac{1}{2}\displaystyle\sum_{m_1+m_2=m,\ m_1,m_2\geqslant 1}\int_0^t P_{t-\tau} * \left(\sum_{j=1}^{n}\partial_{x_j}\big(U_{m_1}(\tau)U_{m_2}(\tau)\big)\right)d\tau & \text{for } m \geqslant 2,\end{cases}$$

and we refer to [5] for the derivation of the above expansion. It follows from Theorem 3 in [5] that there exists $\delta_2 > 0$ such that for any u_0 with $\|u_0\|_{\dot{B}^0_{\infty,1}} \leqslant \delta_2$ the solution to (1.2) exists globally in time, the right hand side of (6.2) converges absolutely in the space $L^\infty(0,\infty;\dot{B}^0_{\infty,1}(\mathbb{R}^n)) \cap L^1(0,\infty;\dot{B}^1_{\infty,1}(\mathbb{R}^n))$, and the solution is expanded by (6.2) with Proposition 4.1 and the estimate $\|P_t * u_0\|_{L^\infty(0,\infty;\dot{B}^0_{\infty,1})\cap L^1(0,\infty;\dot{B}^1_{\infty,1})} \leqslant C\|u_0\|_{\dot{B}^0_{\infty,1}}$.

We show by the induction that there exists $C_0 > 0$ such that

$$\sup_{t>0}\big\|U_m(t)\big\|_{L^1} \leqslant \frac{C_0^m}{(1+m)^2}\|u_0\|^{m-1}_{\dot{B}^0_{\infty,1}}\|u_0\|_{L^1} \quad \text{for } m \geqslant 1, \tag{6.3}$$

$$\big\|t^{\alpha}U_m(t)\big\|_{L^1(0,\infty;\dot{B}^{\alpha+1}_{\infty,1})} \leqslant \frac{C_0^m}{(1+m)^2}\|u_0\|^m_{\dot{B}^0_{\infty,1}} \quad \text{for } m \geqslant 1. \tag{6.4}$$

Once we obtain (6.3) and (6.4), we see that

$$\sum_{m=1}^{\infty} \sup_{t>0} \| U_m(t) \|_{L^1} \leqslant \|u_0\|_{L^1} \sum_{m=1}^{\infty} \frac{C_0^m \|u_0\|_{\dot B^0_{\infty,1}}^{m-1}}{(1+m)^2},$$

$$\sum_{m=1}^{\infty} \| t^\alpha U_m(t) \|_{L^1(0,\infty;\dot B^\alpha_{\infty,1})} \leqslant \sum_{m=1}^{\infty} \frac{C_0^m \|u_0\|_{\dot B^0_{\infty,1}}^{m}}{(1+m)^2},$$

and the two series converge if u_0 is sufficiently small in $\dot B^0_{\infty,1}(\mathbb{R}^n)$. Then (6.1) is obtained for initial data u_0 with $\|u_0\|_{\dot B^0_{\infty,1}} \leqslant \min\{\delta_1, \delta_2, C_0^{-1}\}$.

For the proof of (6.3) and (6.4), we use the following estimate (6.5) which is obtained by the proof of the absolute convergence of the expansion (6.2) for small initial data in $\dot B^0_{\infty,1}(\mathbb{R}^n)$ and the solution in $L^\infty(0,\infty;\dot B^0_{\infty,1}(\mathbb{R}^n)) \cap L^1(0,\infty;\dot B^1_{\infty,1}(\mathbb{R}^n))$ (see [5]). It holds that there exists C_1 such that for any $m \geqslant 1$

$$\|U_m\|_{L^\infty(0,\infty;\dot B^0_{\infty,1})} + \|U_m\|_{L^1(0,\infty;\dot B^1_{\infty,1})} \leqslant \frac{C_1^m}{(1+m)^2} \|u_0\|_{\dot B^0_{\infty,1}}^m. \tag{6.5}$$

On the estimate (6.3) for $m = 1$, it is easy to see that

$$\| U_1(t) \|_{L^1} = \|P_t * u_0\|_{L^1} \leqslant \|u_0\|_{L^1}.$$

Let $m \geqslant 2$ and it follows on U_m from the boundedness of P_t* in $L^1(\mathbb{R}^n)$, the embedding $\dot B^0_{1,1}(\mathbb{R}^n) \hookrightarrow L^1(\mathbb{R}^n)$ and (2.1)

$$\begin{aligned} \| U_m(t) \|_{L^1} &\leqslant C \sum_{m_1+m_2=m} \int_0^t \|U_{m_1} U_{m_2}\|_{\dot B^1_{1,1}}\, d\tau \\ &\leqslant C \sum_{m_1+m_2=m} \int_0^t \big(\|U_{m_1}\|_{\dot B^1_{\infty,1}} \|U_{m_2}\|_{L^1} + \|U_{m_1}\|_{L^1} \|U_{m_2}\|_{\dot B^1_{\infty,1}} \big)\, d\tau \\ &\leqslant C \sum_{m_1+m_2=m} \|U_{m_1}\|_{L^1(0,\infty;\dot B^1_{\infty,1})} \|U_{m_2}\|_{L^\infty(0,\infty;L^1)}. \end{aligned} \tag{6.6}$$

Let C_2 be a constant which satisfies (6.6), let C_3 be a constant with $C_3 > 8C_2$, and we assume

$$\sup_{t>0} \| U_i(t) \|_{L^1} \leqslant \frac{C_3^{i-1} C_1^i}{(1+i)^2} \|u_0\|_{\dot B^0_{\infty,1}}^{i-1} \|u_0\|_{L^1} \quad \text{for } i = 1, 2, \ldots, m-1. \tag{6.7}$$

Then we have from (6.6), (6.5) and (6.7)

$$\begin{aligned} \| U_m(t) \|_{L^1} &\leqslant C_2 \sum_{m_1+m_2=m} \frac{C_1^{m_1}}{(1+m_1)^2} \|u_0\|_{\dot B^0_{\infty,1}}^{m_1} \cdot \frac{C_3^{m_2-1} C_1^{m_2}}{(1+m_2)^2} \|u_0\|_{\dot B^0_{\infty,1}}^{m_2-1} \|u_0\|_{L^1} \\ &\leqslant C_2 C_3^{m-2} C_1^m \|u_0\|_{\dot B^0_{\infty,1}}^{m-1} \|u_0\|_{L^1} \sum_{m_1+m_2=m} \frac{1}{(1+m_1)^2 (1+m_2)^2}. \end{aligned} \tag{6.8}$$

It follows on the above sum that

$$\begin{aligned} \sum_{m_1=1}^{m-1} \frac{1}{(1+m_1)^2(1+m-m_1)^2} &\leqslant 2 \sum_{1 \leqslant m_1 \leqslant \frac{m}{2}} \frac{1}{(1+m_1)^2 (1+m-\frac{m}{2})^2} \\ &= \frac{8}{(2+m)^2} \sum_{1 \leqslant m_1 \leqslant \frac{m}{2}} \frac{1}{(1+m_1)^2} \\ &\leqslant \frac{8}{(1+m)^2}. \end{aligned}$$

Then, we have from (6.8), the above estimate and $C_3 > 8C_2$

$$\big\|U_m(t)\big\|_{L^1} \leqslant C_2 C_3^{m-2} C_1^m \|u_0\|_{\dot B^0_{\infty,1}}^{m-1} \|u_0\|_{L^1} \cdot \frac{8}{(1+m)^2} \leqslant \frac{C_3^{m-1} C_1^m}{(1+m)^2} \|u_0\|_{\dot B^0_{\infty,1}}^{m-1} \|u_0\|_{L^1}.$$

Then we obtain (6.3).

On the estimate of (6.4) for $m=1$, we have from (2.4) and (2.6)

$$\big\|t^\alpha U_1(t)\big\|_{L^1(0,\infty;\dot B^{\alpha+1}_{\infty,1})} \leqslant C\|P_{\frac{t}{2}} * u_0\|_{L^1(0,\infty;\dot B^1_{\infty,1})} \leqslant C\|u_0\|_{\dot B^0_{\infty,1}}.$$

Let $m \geqslant 2$ and it follows on U_m from (2.4) and the boundedness of P_t* in $\dot B^{\alpha+1}_{\infty,1}(\mathbb{R}^n)$

$$\begin{aligned}\big\|U_m(t)\big\|_{\dot B^{\alpha+1}_{\infty,1}} &\leqslant C\sum_{m_1+m_2=m}\int_0^{\frac{t}{2}}(t-\tau)^{-\alpha}\|P_{\frac{t-\tau}{2}} * U_{m_1}U_{m_2}\|_{\dot B^2_{\infty,1}}\,d\tau + C\sum_{m_1+m_2=m}\int_{\frac{t}{2}}^t \|P_{t-\tau} * U_{m_1}U_{m_2}\|_{\dot B^{\alpha+2}_{\infty,1}}\,d\tau\\ &\leqslant Ct^{-\alpha}\sum_{m_1+m_2=m}\int_0^{\frac{t}{2}}\|P_{\frac{t-\tau}{2}} * U_{m_1}U_{m_2}\|_{\dot B^2_{\infty,1}}\,d\tau + Ct^{-\alpha}\sum_{m_1+m_2=m}\int_{\frac{t}{2}}^t \|P_{t-\tau} * U_{m_1}U_{m_2}\|_{\dot B^{\alpha+2}_{\infty,1}}\tau^\alpha\,d\tau.\end{aligned}$$

Multiplying t^α and taking $L^1_t(0,\infty)$ norm, we have from Minkowski's inequality (2.6), (2.2) and Hölder's inequality

$$\begin{aligned}\big\|t^\alpha\big\|U_m(t)\big\|_{\dot B^{\alpha+1}_{\infty,1}}\big\|_{L^1_t(0,\infty)} &\leqslant C\sum_{m_1+m_2=m}\left(\int_0^\infty \|P_{\frac{t-\tau}{2}} * U_{m_1}U_{m_2}\|_{L^1_t(2\tau,\infty;\dot B^2_{\infty,1})}\,d\tau\right.\\ &\quad\left. + \int_0^\infty \|P_{t-\tau} * U_{m_1}U_{m_2}\|_{L^1(\tau,2\tau;\dot B^{\alpha+2}_{\infty,1})}\tau^\alpha\,d\tau\right)\\ &\leqslant C\sum_{m_1+m_2=m}\left(\int_0^\infty \|U_{m_1}U_{m_2}\|_{\dot B^1_{\infty,1}}\,d\tau + \int_0^\infty \|U_{m_1}U_{m_2}\|_{\dot B^{\alpha+1}_{\infty,1}}\tau^\alpha\,d\tau\right)\\ &\leqslant C\sum_{m_1+m_2=m}\left(\int_0^\infty \|U_{m_1}\|_{\dot B^1_{\infty,1}}\|U_{m_2}\|_{\dot B^0_{\infty,1}}\,d\tau + \int_0^\infty \|U_{m_1}\|_{\dot B^{\alpha+1}_{\infty,1}}\|U_{m_2}\|_{\dot B^0_{\infty,1}}\tau^\alpha\,d\tau\right)\\ &\leqslant C\sum_{m_1+m_2=m}\big(\|U_{m_1}\|_{L^1(0,\infty;\dot B^1_{\infty,1})} + \big\|t^\alpha U_{m_1}\big\|_{L^1(0,\infty;\dot B^{\alpha+1}_{\infty,1})}\big)\|U_{m_2}\|_{L^\infty(0,\infty;\dot B^0_{\infty,1})}.\end{aligned}$$

Then it is possible to show the estimate (6.4) in the analogous way to (6.3) by induction since we can regard the last estimate as (6.6) and it is possible to show estimates corresponding to (6.7) and (6.8). Hence the proof is completed. □

By Proposition 6.1 with $\alpha = 2$, there exists $\delta > 0$ such that for any $u_0 \in L^1(\mathbb{R}^n) \cap \dot B^0_{\infty,1}(\mathbb{R})$ with $\|u_0\|_{\dot B^0_{\infty,1}} \leqslant \delta$, a unique solution u to the integral equation (1.2) exists in the space $L^\infty(0,\infty;L^1(\mathbb{R}^n)) \cap C([0,\infty);\dot B^0_{\infty,1}(\mathbb{R}^n)) \cap L^1(0,\infty;\dot B^1_{\infty,1}(\mathbb{R}^n))$ with $t^2u(t) \in L^1_t(0,\infty;\dot B^3_{\infty,1}(\mathbb{R}^n))$. We also see that $u \in C([0,\infty),L^1(\mathbb{R}^n))$ by $P_t * u_0 \in C([0,\infty),L^1(\mathbb{R}^n))$, the integrability on $\tau \in (0,t)$ of the function $\|P_{t-\tau} * \sum_{j=1}^n \partial_{x_j}(u^2)\|_{L^1}$ and applying the dominated convergence theorem with the estimate

$$\int_0^t \left\|P_{t-\tau} * \sum_{j=1}^n \partial_{x_j}\big(u^2\big)\right\|_{L^1} d\tau \leqslant C\int_0^t \big\|\nabla\big(u^2\big)\big\|_{L^1}\,d\tau \leqslant C\|u\|_{L^\infty(0,\infty;L^1)}\|u\|_{L^1(0,\infty;\dot B^1_{\infty,1})} < \infty.$$

Then it follows from Proposition 5.2 that

$$\|u(t)\|_{L^p} \leqslant C(1+t)^{-n(1-\frac{1}{p})}, \tag{6.9}$$

$$\||\nabla|^{\alpha} u(t)\|_{L^p} \leqslant Ct^{-n(1-\frac{1}{p})-\alpha}, \qquad \|u(t)\|_{\dot{B}^{\alpha}_{p,1}} \leqslant Ct^{-n(1-\frac{1}{p})-\alpha} \tag{6.10}$$

for all $t>0$, where $1 \leqslant p \leqslant \infty$, $0<\alpha \leqslant 2$. By the use of (6.9) and (6.10), we show the large time behavior (1.4), (1.5) and (1.6).

Proof of (1.4) in Theorem 1.2. By the following estimate:

$$\|u(t) - MP_t\|_{L^p} \leqslant \|P_t * u_0 - MP_t\|_{L^p} + \left\| \int_0^t P_{t-\tau} * \sum_{j=1}^n \partial_{x_j}(u^2)\, d\tau \right\|_{L^p}$$

and (2.9), it is sufficient to consider the nonlinear part. On the nonlinear part, we have on the integral for $\tau \in [0, t/2]$ from (2.5) and (6.9)

$$\begin{aligned}
\left\| \int_0^{\frac{t}{2}} P_{t-\tau} * \sum_{j=1}^n \partial_{x_j} u^2\, d\tau \right\|_{L^p} &\leqslant C \int_0^{\frac{t}{2}} (t-\tau)^{-n(1-\frac{1}{p})-1} \|u^2\|_{L^1}\, d\tau \\
&\leqslant Ct^{-n(1-\frac{1}{p})-1} \int_0^{\frac{t}{2}} \|u\|_{L^2}^2\, d\tau \\
&\leqslant Ct^{-n(1-\frac{1}{p})-1} \int_0^{\frac{t}{2}} (1+\tau)^{-n}\, d\tau \\
&\leqslant \begin{cases} Ct^{-(1-\frac{1}{p})-1} \log(e+t) & \text{if } n=1, \\ Ct^{-n(1-\frac{1}{p})-1} & \text{if } n \geqslant 2. \end{cases}
\end{aligned} \tag{6.11}$$

Then, it follows from the above estimate that

$$\lim_{t\to\infty} t^{n(1-\frac{1}{p})} \left\| \int_0^{\frac{t}{2}} P_{t-\tau} * \sum_{j=1}^n \partial_{x_j} u^2\, d\tau \right\|_{L^p} = 0.$$

On the integral over $\tau \in [t/2, t]$, we have from the boundedness of P_t* in $L^p(\mathbb{R}^n)$, Hölder's inequality, (6.9) and (6.10)

$$\begin{aligned}
\left\| \int_{\frac{t}{2}}^{t} P_{t-\tau} * \sum_{j=1}^n \partial_{x_j} u^2\, d\tau \right\|_{L^p} &\leqslant C \int_{\frac{t}{2}}^{t} \|u\|_{L^p} \|\nabla u\|_{L^\infty}\, d\tau \\
&\leqslant C \int_{\frac{t}{2}}^{t} \tau^{-n(1-\frac{1}{p})} \tau^{-n-1}\, d\tau \\
&\leqslant Ct^{-n(1-\frac{1}{p})-n}.
\end{aligned} \tag{6.12}$$

Then, it holds that

$$\lim_{t\to\infty} t^{n(1-\frac{1}{p})} \left\| \int_{\frac{t}{2}}^{t} P_{t-\tau} * \sum_{j=1}^n \partial_{x_j} u^2\, d\tau \right\|_{L^p} = 0.$$

Therefore, we complete the proof of (1.4). □

Proof of (1.5) in Theorem 1.2. (1.5) is obtained by (2.10), (6.11) and (6.12). □

Proof of (1.6) in Theorem 1.2. Since the convergence on the linear part is obtained by (2.11), it is sufficient to consider the nonlinear part. Then, it holds that

$$\begin{aligned}&\Bigg\| -\int_0^t P_{t-\tau} * \sum_{j=1}^n \partial_{x_j}\big(u(\tau)^2\big)\,d\tau + \sum_{j=1}^n (\partial_{x_j} P_t) \int_0^t \int_{\mathbb{R}^n} u(\tau, y)^2\,dy\,d\tau \\ &\quad + \int_0^t \sum_{j=1}^n (\partial_{x_j} P_{t-\tau}) * (M P_{\tau+1})^2\,d\tau - \sum_{j=1}^n (\partial_{x_j} P_t) \int_0^t \int_{\mathbb{R}^n} \big(M P_{\tau+1}(y)\big)^2\,dy\,d\tau \Bigg\|_{L^p} \\ &\leqslant \sum_{j=1}^n \Bigg\| \int_0^t \int_{\mathbb{R}^n} \big((\partial_{x_j} P_{t-\tau})(x-y) - (\partial_{x_j} P_t)(x)\big)\big(u(\tau, y)^2 - \big(M P_{\tau+1}(y)\big)^2\big)\,dy\,d\tau \Bigg\|_{L^p_x}. \end{aligned} \tag{6.13}$$

For the estimate of the right hand side of (6.13), we introduce the following proposition.

Proposition 6.2. *Let* $U(\tau, y) := u(\tau, y)^2 - (M P_{\tau+1}(y))^2$.

(i) *For* p *with* $1 \leqslant p \leqslant \infty$, *it holds that*

$$\big\|\nabla U(\tau)\big\|_{L^p} \leqslant \begin{cases} C\tau^{-1-(1-\frac{1}{p})-2} \log(e+\tau) & \text{if } n = 1, \\ C\tau^{-n-n(1-\frac{1}{p})-2} & \text{if } n \geqslant 2, \end{cases} \tag{6.14}$$

for all $\tau > 0$.

(ii) *It holds that*

$$\int_0^\infty \int_{\mathbb{R}^n} \big|U(\tau, y)\big|\,dy\,d\tau < \infty. \tag{6.15}$$

(iii) *For* ε *with* $0 < \varepsilon < 1/2$, *it holds that*

$$t^{n(1-\frac{1}{p})+1} \Bigg\| \int_0^{\varepsilon t} \int_{|y| \leqslant \varepsilon t} \big((\partial_{x_j} P_{t-\tau})(x-y) - (\partial_{x_j} P_t)(x)\big) U(\tau, y)\,dy\,d\tau \Bigg\|_{L^p} \leqslant C\varepsilon \tag{6.16}$$

for all $t > 0$.

Proof. For the proof of (6.14), it follows from Hölder's inequality, (2.5), (6.9), and (6.10)

$$\begin{aligned}\big\|\nabla U(\tau)\big\|_{L^p} &\leqslant \big\|u(\tau) + M P_{\tau+1}\big\|_{L^p} \big\|\nabla u(\tau) - \nabla M P_{\tau+1}\big\|_{L^\infty} + \big\|\nabla u(\tau) + \nabla M P_{\tau+1}\big\|_{L^p} \big\|u(\tau) - M P_{\tau+1}\big\|_{L^\infty} \\ &\leqslant C\tau^{-n(1-\frac{1}{p})} \big\|\nabla u(\tau) - \nabla M P_{\tau+1}\big\|_{L^\infty} + C\tau^{-n(1-\frac{1}{p})-1} \big\|u(\tau) - M P_{\tau+1}\big\|_{L^\infty}. \end{aligned} \tag{6.17}$$

On the estimate of $\|\nabla u(\tau) - \nabla M P_{\tau+1}\|_{L^\infty}$, we have from the triangle inequality

$$\begin{aligned}\big\|\nabla u(\tau) - \nabla M P_{\tau+1}\big\|_{L^\infty} &\leqslant \|\nabla P_\tau * u_0 - M\nabla P_\tau\|_{L^\infty} + \|M\nabla P_\tau - M\nabla P_{\tau+1}\|_{L^\infty} \\ &\quad + \Bigg\|\nabla \int_0^{\frac{\tau}{2}} P_{\tau-\tilde\tau} * \sum_{j=1}^n \partial_{x_j}\big(u^2\big)\,d\tilde\tau\Bigg\|_{L^\infty} + \Bigg\|\nabla \int_{\frac{\tau}{2}}^{\tau} P_{\tau-\tilde\tau} * \sum_{j=1}^n \partial_{x_j}\big(u^2\big)\,d\tilde\tau\Bigg\|_{L^\infty}. \end{aligned} \tag{6.18}$$

It follows on the first and the second terms in the right hand side of (6.18) from (2.5), (2.8) and (2.10) that

$$\begin{aligned}
&\|\nabla P_\tau * u_0 - M\nabla P_\tau\|_{L^\infty} + \|M\nabla P_\tau - M\nabla P_{\tau+1}\|_{L^\infty} \\
&\quad \leqslant C\tau^{-1}\big(\|P_{\frac{\tau}{2}} * u_0 - MP_{\frac{\tau}{2}}\|_{L^\infty} + \|MP_{\frac{\tau}{2}} - MP_{\frac{\tau}{2}+1}\|_{L^\infty}\big) \\
&\quad \leqslant C\tau^{-1}\tau^{-n-1}.
\end{aligned} \tag{6.19}$$

We have on the third term in the right hand side of (6.18) from (2.5) and (6.9)

$$\begin{aligned}
\left\|\nabla \int_0^{\frac{\tau}{2}} P_{\tau-\tilde\tau} * \sum_{j=1}^n \partial_{x_j}\big(u^2\big)\,d\tilde\tau\right\|_{L^\infty} &\leqslant C\int_0^{\frac{\tau}{2}} (\tau-\tilde\tau)^{-2-n}\|u^2\|_{L^1}\,d\tilde\tau \\
&\leqslant C\tau^{-2-n}\int_0^{\frac{\tau}{2}} \|u\|_{L^2}^2\,d\tilde\tau \\
&\leqslant C\tau^{-2-n}\int_0^{\frac{\tau}{2}} (1+\tilde\tau)^{-n}\,d\tilde\tau \\
&\leqslant \begin{cases} C\tau^{-2-1}\log(e+\tau) & \text{if } n=1, \\ C\tau^{-2-n} & \text{if } n\geqslant 2. \end{cases}
\end{aligned} \tag{6.20}$$

We also have on the fourth term in the right hand side of (6.18) from the boundedness of P_t* in $L^\infty(\mathbb{R}^n)$, the embedding $\dot B^0_{\infty,1}(\mathbb{R}^n) \hookrightarrow L^\infty(\mathbb{R}^n)$, (2.2), (6.9) and (6.10)

$$\begin{aligned}
\left\|\nabla \int_{\frac{\tau}{2}}^{\tau} P_{\tau-\tilde\tau} * \sum_{j=1}^n \partial_{x_j}\big(u^2\big)\,d\tilde\tau\right\|_{L^\infty} &\leqslant C\int_{\frac{\tau}{2}}^{\tau} \big\||\nabla|^2 u^2\big\|_{\dot B^0_{\infty,1}}\,d\tilde\tau \\
&\leqslant C\int_{\frac{\tau}{2}}^{\tau} \|u\|_{\dot B^2_{\infty,1}}\|u\|_{L^\infty}\,d\tilde\tau \\
&\leqslant C\int_{\frac{\tau}{2}}^{\tau} \tilde\tau^{-n-2}(1+\tilde\tau)^{-n}\,d\tilde\tau \\
&\leqslant C\tau^{-2-n}.
\end{aligned} \tag{6.21}$$

Then, it follows from (6.18), (6.19), (6.20) and (6.21)

$$\big\|\nabla u(\tau) - \nabla MP_{\tau+1}\big\|_{L^\infty} \leqslant \begin{cases} C\tau^{-2-1}\log(e+\tau) & \text{if } n=1, \\ C\tau^{-2-n} & \text{if } n\geqslant 2. \end{cases} \tag{6.22}$$

On the estimate of $\|u(\tau) - MP_{\tau+1}\|_{L^\infty}$, we have from (1.5) and (2.8)

$$\begin{aligned}
\big\|u(\tau) - MP_{\tau+1}\big\|_{L^\infty} &\leqslant \big\|u(\tau) - MP_\tau\big\|_{L^\infty} + \|MP_\tau - MP_{\tau+1}\|_{L^\infty} \\
&\leqslant \begin{cases} C\tau^{-1-1}\log(e+\tau) & \text{if } n=1, \\ C\tau^{-1-n} & \text{if } n\geqslant 2. \end{cases}
\end{aligned} \tag{6.23}$$

By (6.17), (6.22) and (6.23), we obtain (6.14).

For the proof of (6.15), it follows from Hölder's inequality, $\|u(\tau)\|_{L^1} + \|P_{\tau+1}\|_{L^1} \leqslant C$ for all τ, (1.5) and (2.8) that

$$\begin{aligned}
\int_{\mathbb{R}^n} \big|U(\tau,y)\big|\,dy &\leqslant \big\|u(\tau) + MP_{\tau+1}\big\|_{L^1}\big\|u(\tau) - MP_{\tau+1}\big\|_{L^\infty} \\
&\leqslant C\big(\big\|u(\tau) - MP_\tau\big\|_{L^\infty} + \|MP_\tau - MP_{\tau+1}\|_{L^\infty}\big)
\end{aligned}$$

$$\leqslant \begin{cases} C(\tau^{-1-1}\log(e+\tau)+\tau^{-n-1}) & \text{if } n=1, \\ C(\tau^{-n-1}+\tau^{-n-1}) & \text{if } n\geqslant 2. \end{cases}$$

Then we consider the integral on the variable τ, the integrability for τ with $\tau \geqslant 1$ is guaranteed by the above estimate and the integrability for τ with $0<\tau<1$ is guaranteed by the following estimate

$$\|U(\tau)\|_{L^1} \leqslant C\Big(\|u\|^2_{L^\infty(0,\infty;L^2)} + \sup_{\tau>0}\|P_{\tau+1}\|^2_{L^2}\Big) \leqslant C\Big(\|u\|^2_{L^\infty(0,\infty;L^2)} + \sup_{\tau>0}(\tau+1)^{-\frac{n}{2}\cdot 2}\Big) < \infty.$$

Then the proof of (6.15) is completed.

For the proof of (6.16), we show that

$$\big\|(\partial_{x_j}P_{t-\tau})(x-y) - (\partial_{x_j}P_t)(x)\big\|_{L^p_x} \leqslant C\varepsilon t^{-n(1-\frac{1}{p})-1} \tag{6.24}$$

for all τ, y with $\tau \leqslant \varepsilon t$ and $|y| \leqslant \varepsilon t$. Once we obtain (6.24), it follows from (6.24) and (6.15)

$$\begin{aligned}
&t^{n(1-\frac{1}{p})+1}\Bigg\|\int_0^{\varepsilon t}\int_{|y|\leqslant\varepsilon t}\big((\partial_{x_j}P_{t-\tau})(x-y)-(\partial_{x_j}P_t)(x)\big)U(\tau,y)\,dy\,d\tau\Bigg\|_{L^p} \\
&\quad\leqslant t^{n(1-\frac{1}{p})+1}\int_0^{\varepsilon t} C\varepsilon t^{-n(1-\frac{1}{p})-1}\int_{|y|\leqslant\varepsilon t}\big|U(\tau,y)\big|\,dy\,d\tau \\
&\quad\leqslant C\varepsilon\int_0^\infty\int_{\mathbb{R}^n}\big|U(\tau,y)\big|\,dy\,d\tau \\
&\quad\leqslant C\varepsilon.
\end{aligned}$$

Then, it remains to show (6.24). We have from $|y| \leqslant \varepsilon t$, $\tau \leqslant \varepsilon t$, $0<\varepsilon<1/2$ and $t^{-1} \leqslant (t-\theta\tau)^{-1} \leqslant 2t^{-1}$ for $\theta \in [0,1]$ and $\tau \in [0,\varepsilon t]$

$$\begin{aligned}
&\big|(\partial_{x_j}P_{t-\tau})(x-y)-(\partial_{x_j}P_t)(x)\big| \\
&\quad=\Bigg|\int_0^1 \partial_\theta\big((t-\theta\tau)^{-n-1}(\partial_{x_j}P)\big((t-\theta\tau)^{-1}(x-\theta y)\big)\big)\,d\theta\Bigg| \\
&\quad=\Bigg|\int_0^1 \big\{(-n-1)(t-\theta\tau)^{-n-2}(-\tau)(\partial_{x_j}P)\big((t-\theta\tau)^{-1}(x-\theta y)\big) \\
&\qquad+(t-\theta\tau)^{-n-1}(\nabla\partial_{x_j}P)\big((t-\theta\tau)^{-1}(x-\theta y)\big)\cdot(t-\theta\tau)^{-2}\tau(x-\theta y) \\
&\qquad+(t-\theta\tau)^{-n-1}(\nabla\partial_{x_j}P)\big((t-\theta\tau)^{-1}(x-\theta y)\big)\cdot(t-\theta\tau)^{-1}(-y)\big\}\,d\theta\Bigg| \\
&\quad\leqslant C\int_0^1\big\{t^{-n-1}\varepsilon\big|(\partial_{x_j}P)\big((t-\theta\tau)^{-1}(x-\theta y)\big)\big|+t^{-n-2}\varepsilon\big|(\nabla\partial_{x_j}P)\big((t-\theta\tau)^{-1}(x-\theta y)\big)\big||x-\theta y| \\
&\qquad+t^{-n-1}\varepsilon\big|(\nabla\partial_{x_j}P)\big((t-\theta\tau)^{-1}(x-\theta y)\big)\big|\big\}\,d\theta.
\end{aligned} \tag{6.25}$$

Since

$$\begin{aligned}
&\big\|(\partial_{x_j}P)\big((t-\theta\tau)^{-1}(x-\theta y)\big)\big\|_{L^p_x} = (t-\theta\tau)^{\frac{n}{p}}\|\partial_{x_j}P\|_{L^p} \leqslant Ct^{\frac{n}{p}}, \\
&\big\|\big|(\nabla\partial_{x_j}P)\big((t-\theta\tau)^{-1}(x-\theta y)\big)\big||x-\theta y|\big\|_{L^p_x} = (t-\theta\tau)^{\frac{n}{p}+1}\big\|\nabla\partial_{x_j}P\cdot|x|\big\|_{L^p} \leqslant Ct^{\frac{n}{p}+1}, \\
&\big\|(\nabla\partial_{x_j}P)\big((t-\theta\tau)^{-1}(x-\theta y)\big)\big\|_{L^p_x} = (t-\theta\tau)^{\frac{n}{p}}\|\nabla\partial_{x_j}P\|_{L^p} \leqslant Ct^{\frac{n}{p}},
\end{aligned}$$

it follows on the $L^p(\mathbb{R}^n)$ norm of (6.25) that

$$\big\|(\partial_{x_j}P_{t-\tau})(x-y)-(\partial_{x_j}P_t)(x)\big\|_{L^p_x}\leqslant C\int_0^1\big(t^{-n-1}\varepsilon t^{\frac{n}{p}}+t^{-n-2}\varepsilon t^{\frac{n}{p}+1}\big)\,d\theta$$
$$\leqslant C\varepsilon t^{-n(1-\frac{1}{p})-1}.$$

Therefore, we obtain (6.24). The proof of Proposition 6.2 is completed. □

To prove (1.6), we consider the estimate of the right hand side of (6.13) by the use of Proposition 6.2. For ε with $0<\varepsilon<1/2$, we decompose the interval $[0,t]$ into $[0,\varepsilon t]$ and $[\varepsilon t,t]$.

On the integral for $\tau\in[\varepsilon t,t]$, it follows from $(\partial_{x_j}P_{t-\tau})*U=P_{t-\tau}*(\partial_{x_j}U)$, the boundedness of $P_{t-\tau}$ in $L^p(\mathbb{R}^n)$ and (6.14)

$$\begin{aligned}
&t^{n(1-\frac{1}{p})+1}\sum_{j=1}^n\bigg\|\int_{\varepsilon t}^t\int_{\mathbb{R}^n}\big((\partial_{x_j}P_{t-\tau})(x-y)-(\partial_{x_j}P_t)(x)\big)U(\tau,y)\,dy\,d\tau\bigg\|_{L^p_x}\\
&\quad\leqslant Ct^{n(1-\frac{1}{p})+1}\bigg(\int_{\varepsilon t}^t\big\|P_{t-\tau}*\nabla U(\tau)\big\|_{L^p}\,d\tau+\int_{\varepsilon t}^t\int_{\mathbb{R}^n}\|\nabla P_t\|_{L^p}\big|U(\tau,y)\big|\,dy\,d\tau\bigg)\\
&\quad\leqslant Ct^{n(1-\frac{1}{p})+1}\bigg(\int_{\varepsilon t}^t\big\|\nabla U(\tau)\big\|_{L^p}\,d\tau+t^{-n(1-\frac{1}{p})-1}\int_{\varepsilon t}^t\int_{\mathbb{R}^n}\big|U(\tau,y)\big|\,dy\,d\tau\bigg)\\
&\quad\leqslant Ct^{n(1-\frac{1}{p})+1}\int_{\varepsilon t}^t\tau^{-n-n(1-\frac{1}{p})-2}\log(1+\tau)\,d\tau+C\int_{\varepsilon t}^t\int_{\mathbb{R}^n}\big|U(\tau,y)\big|\,dy\,d\tau.
\end{aligned}\tag{6.26}$$

On the first term of the right hand side of (6.26), we have

$$t^{n(1-\frac{1}{p})+1}\int_{\varepsilon t}^t\tau^{-n-n(1-\frac{1}{p})-2}\log(1+\tau)\,d\tau\leqslant C\varepsilon^{-n-n(1-\frac{1}{p})-1}t^{-n}\log(1+t)\to 0\quad\text{as } t\to\infty.$$

On the second term, we have from (6.15) and Lebesgue's dominated convergence theorem

$$\lim_{t\to\infty}\int_{\varepsilon t}^t\int_{\mathbb{R}^n}\big|U(\tau,y)\big|\,dy\,d\tau=0.$$

Then, we obtain the following

$$\lim_{t\to\infty}t^{n(1-\frac{1}{p})+1}\sum_{j=1}^n\bigg\|\int_{\varepsilon t}^t\int_{\mathbb{R}^n}\big((\partial_{x_j}P_{t-\tau})(x-y)-(\partial_{x_j}P_t)(x)\big)U(\tau,y)\,dy\,d\tau\bigg\|_{L^p_x}=0\tag{6.27}$$

for any ε with $0<\varepsilon<1/2$.

On the integral for $\tau\in[0,\varepsilon t]$, we first consider the case $|y|\geqslant\varepsilon t$ on the integral over $\mathbb{R}^n$. It follows from $|t-\tau|^{-1}\leqslant 2t^{-1}$ for $\tau\in[0,\varepsilon t]$, (6.15) and Lebesgue's dominated convergence theorem

$$\begin{aligned}
&t^{n(1-\frac{1}{p})+1}\sum_{j=1}^n\bigg\|\int_0^{\varepsilon t}\int_{|y|\geqslant\varepsilon t}\big((\partial_{x_j}P_{t-\tau})(x-y)-(\partial_{x_j}P_t)(x)\big)U(\tau,y)\,dy\,d\tau\bigg\|_{L^p_x}\\
&\quad\leqslant Ct^{n(1-\frac{1}{p})+1}\int_0^{\varepsilon t}\int_{|y|\geqslant\varepsilon t}\big(\|\partial_{x_j}P_{t-\tau}\|_{L^p}+\|\partial_{x_j}P_t\|_{L^p}\big)\big|U(\tau,y)\big|\,dy\,d\tau
\end{aligned}$$

$$
\begin{aligned}
&\leqslant C t^{n(1-\frac{1}{p})+1} \int_0^{\varepsilon t} \int_{|y|\geqslant \varepsilon t} \left((t-\tau)^{-n(1-\frac{1}{p})-1} + t^{-n(1-\frac{1}{p})-1}\right)\left|U(\tau,y)\right| dy\, d\tau \\
&\leqslant C \int_0^{\infty} \int_{|y|\geqslant \varepsilon t} \left|U(\tau,y)\right| dy\, d\tau \\
&\to 0 \quad \text{as } t\to\infty.
\end{aligned} \tag{6.28}
$$

On the integral for $\tau \in [0, \varepsilon t]$ and y with $|y| \leqslant \varepsilon t$, we apply the estimate (6.16). Then, we have from (6.13), (6.27), (6.28) and (6.16)

$$
\begin{aligned}
&\limsup_{t\to\infty} t^{n(1-\frac{1}{p})+1} \Bigg\| \int_0^t P_{t-\tau} * \sum_{j=1}^n \partial_{x_j}\left(u(\tau)^2\right) d\tau - \sum_{j=1}^n (\partial_{x_j} P_t) \int_0^t \int_{\mathbb{R}^n} u(\tau,y)^2\, dy\, d\tau \\
&\quad + \int_0^t \sum_{j=1}^n (\partial_{x_j} P_{t-\tau}) * (M P_{\tau+1})^2\, d\tau - \sum_{j=1}^n (\partial_{x_j} P_t) \int_0^t \int_{\mathbb{R}^n} \left(M P_{\tau+1}(y)\right)^2 dy\, d\tau \Bigg\|_{L^p} \leqslant C\varepsilon
\end{aligned}
$$

for any ε with $0 < \varepsilon < 1/2$. Since ε is an arbitrary real positive number, the proof of (1.6) is completed. □

7. Proof of the estimate (1.7)

By the use of (1.6) and $\|\nabla P_t\|_{L^2} \leqslant t^{-2}$, it is sufficient to show that

$$
\begin{aligned}
&\Bigg\| (\partial_x P_t) \int_0^t \int_{\mathbb{R}} u(\tau,y)^2\, dy\, d\tau + \int_0^t (\partial_x P_{t-\tau}) * (M P_{\tau+1})^2\, d\tau \\
&\quad - (\partial_x P_t) \int_0^t \int_{\mathbb{R}} \left(M P_{\tau+1}(y)\right)^2 dy\, d\tau \Bigg\|_{L^2} \geqslant c M^2 t^{-\frac{3}{2}} \log(e+t)
\end{aligned} \tag{7.1}
$$

for large t. The function in the $L^2(\mathbb{R})$ norm in the left hand side of (7.1) is equal to

$$
\begin{aligned}
&(\partial_x P_t) \int_0^t \int_{\mathbb{R}} \left\{u(\tau,y)^2 - \left(M P_{\tau+1}(y)\right)^2\right\} dy\, d\tau + \int_0^t (\partial_x P_{t-\tau}) * (M P_{\tau+1})^2\, d\tau \\
&=: \int_0^t (I + II)\, d\tau.
\end{aligned} \tag{7.2}
$$

On the estimate of I, it follows from $\|\partial_x P_t\|_{L^2} \leqslant C t^{-\frac{3}{2}}$ and (6.15) that

$$
\left\| \int_0^t I\, d\tau \right\|_{L^2} \leqslant C t^{-\frac{3}{2}}.
$$

On the estimate of II, we decompose $[0,t]$ into $[0,t/2]$ and $[t/2,t]$. For the interval $[t/2,t]$, it follows from $\partial_x P_{t-\tau} * (P_{\tau+1})^2 = P_{t-\tau} * \partial_x (P_{\tau+1})^2$ the boundedness of $P_t *$ in $L^2(\mathbb{R}^n)$ and Hölder's inequality that

$$
\left\| \int_{\frac{t}{2}}^t II\, d\tau \right\|_{L^2} \leqslant C \int_{\frac{t}{2}}^t \left\| P_{t-\tau} * \partial_x (P_{\tau+1})^2 \right\|_{L^2} d\tau
$$

$$
\begin{aligned}
&\leqslant C\int_{\frac{t}{2}}^{t}\|P_{\tau+1}\|_{L^2}\|\partial_x P_{\tau+1}\|_{L^\infty}\,d\tau\\
&\leqslant C\int_{\frac{t}{2}}^{t}(\tau+1)^{-\frac{5}{2}}\,d\tau\\
&\leqslant Ct^{-\frac{3}{2}}.
\end{aligned}
$$

For the interval $[0,t/2]$, we have from Plancherel theorem and considering the restriction $|\xi|\leqslant t^{-1}$

$$
\begin{aligned}
\left\|\int_0^{\frac{t}{2}} II\,d\tau\right\|_{L^2} &= M^2\left\||\xi|\int_0^{\frac{t}{2}} e^{-(t-\tau)|\xi|}\int_{\mathbb{R}} e^{-(\tau+1)|\xi-\eta|}e^{-(\tau+1)|\eta|}\,d\eta\,d\tau\right\|_{L^2}\\
&\geqslant cM^2\left\||\xi|\int_0^{\frac{t}{2}} e^{-\frac{t}{2}|\xi|}e^{-c(\tau+1)|\xi|}(1+\tau)^{-1}\,d\tau\right\|_{L^2(|\xi|\leqslant t^{-1})}\\
&\geqslant cM^2\left\||\xi|\int_0^{\frac{t}{2}}(1+\tau)^{-1}\,d\tau\right\|_{L^2(|\xi|\leqslant t^{-1})}\\
&= cM^2t^{-\frac{3}{2}}\log(1+t).
\end{aligned}
$$

Therefore, (7.1) is obtained by (7.2) and the above three estimates on I, II. The proof of (1.7) is completed.

Conflict of interest statement

The author has declared no conflicts of interest.

Acknowledgement

The author is partially supported by Japan Society for the Promotion of Science, Grant-in-Aid for Young Scientists B #25800069.

References

[1] N. Alibaud, Entropy formulation for fractal conservation laws, J. Evol. Equ. 7 (1) (2007) 145–175.
[2] N. Alibaud, B. Andreianov, Non-uniqueness of weak solutions for the fractal Burgers equation, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 27 (4) (2010) 997–1016.
[3] N. Alibaud, J. Droniou, J. Vovelle, Occurrence and non-appearance of shocks in fractal Burgers equations, J. Hyperbolic Differ. Equ. 4 (3) (2007) 479–499.
[4] N. Alibaud, C. Imbert, G. Karch, Asymptotic properties of entropy solutions to fractal Burgers equation, SIAM J. Math. Anal. 42 (1) (2010) 354–376.
[5] I. Bejenaru, T. Tao, Sharp well-posedness and ill-posedness results for a quadratic non-linear Schrödinger equation, J. Funct. Anal. 233 (1) (2006) 228–259.
[6] P. Biler, G. Karch, W.A. Woyczynski, Asymptotics for multifractal conservation laws, Stud. Math. 135 (3) (1999) 231–252.
[7] P. Biler, G. Karch, W.A. Woyczynski, Multifractal and Lévy conservation laws, C. R. Acad. Sci. Paris Sér. I Math. 330 (5) (2000) 343–348.
[8] P. Biler, G. Karch, W.A. Woyczynski, Critical nonlinearity exponent and self-similar asymptotics for Lévy conservation laws, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 18 (5) (2001) 613–637.
[9] P. Biler, G. Karch, W.A. Woyczynski, Asymptotics for conservation laws involving Lévy diffusion generators, Stud. Math. 148 (2) (2001) 171–192.
[10] J.-M. Bony, Calcul symbolique et propagation des singularités pour les équations aux dérivées partielles non linéaires, Ann. Sci. Éc. Norm. Super. (4) 14 (2) (1981) 209–246.
[11] D. Chae, On the well-posedness of the Euler equations in the Triebel–Lizorkin spaces, Commun. Pure Appl. Math. 55 (5) (2002) 654–678.

[12] D. Chae, J. Lee, Local existence and blow-up criterion of the inhomogeneous Euler equations, J. Math. Fluid Mech. 5 (2003) 144–165.

[13] C.H. Chan, M. Czubak, Regularity of solutions for the critical N-dimensional Burgers' equation, Ann. Inst. Henri Poincaré, Anal. Non Linéaire 27 (2) (2010) 471–501.

[14] H. Dong, D. Du, D. Li, Finite time singularities and global well-posedness for fractal Burgers equations, Indiana Univ. Math. J. 58 (2) (2009) 807–821.

[15] J. Droniou, T. Gallouet, J. Vovelle, Global solution and smoothing effect for a non-local regularization of a hyperbolic equation, dedicated to Philippe Bénilan, J. Evol. Equ. 3 (3) (2003) 499–521.

[16] M. Escobedo, E. Zuazua, Large time behavior for convection–diffusion equations in R^n, J. Funct. Anal. 100 (1) (1991) 119–161.

[17] K. Ishige, T. Kawakami, Refined asymptotic profiles for a semilinear heat equation, Math. Ann. 353 (1) (2012) 161–192.

[18] M. Kato, Sharp asymptotics for a parabolic system of chemotaxis in one space dimension, Differ. Integral Equ. 22 (1–2) (2009) 35–51.

[19] G. Karch, C. Miao, X. Xu, On convergence of solutions of fractal Burgers equation toward rarefaction waves, SIAM J. Math. Anal. 39 (5) (2008) 1536–1549.

[20] A. Kiselev, F. Nazarov, R. Shterenberg, Blow up and regularity for fractal Burgers equation, Dyn. Partial Differ. Equ. 5 (3) (2008) 211–240.

[21] H. Kozono, T. Ogawa, Y. Taniuchi, Navier–Stokes equations in the Besov space near L^∞ and BMO, Kyushu J. Math. 57 (2003) 303–324.

[22] H. Kozono, M. Yamazaki, Semilinear heat equations and the Navier–Stokes equation with distributions in new function spaces as initial data, Commun. Partial Differ. Equ. 19 (5–6) (1994) 959–1014.

[23] C. Miao, G. Wu, Global well-posedness of the critical Burgers equation in critical Besov spaces, J. Differ. Equ. 247 (6) (2009) 1673–1693.

[24] T. Nagai, R. Syukuinn, M. Umesako, Decay properties and asymptotic profiles of bounded solutions to a parabolic system of chemotaxis in R^n, Funkc. Ekvacioj 46 (3) (2003) 383–407.

[25] T. Nagai, T. Yamada, Large time behavior of bounded solutions to a parabolic system of chemotaxis in the whole space, J. Math. Anal. Appl. 336 (1) (2007) 704–726.

[26] H.C. Pak, Y.J. Park, Existence of solution for the Euler equations in a critical Besov space $B^1_{\infty,1}(R^n)$, Commun. Partial Differ. Equ. 29 (7–8) (2004) 1149–1166.

[27] L. Silvestre, On the differentiability of the solution to the Hamilton–Jacobi equation with critical fractional diffusion, Adv. Math. 226 (2) (2011) 2020–2039.

[28] H. Triebel, Theory of Function Spaces, Birkhäuser-Verlag, Basel, 1983.

[29] M. Yamamoto, Asymptotic expansion of solutions to the dissipative equation with fractional Laplacian, SIAM J. Math. Anal. 44 (6) (2012) 3786–3805.