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The CR Yamabe conjecture the case $n = 1$

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Abstract. Let (M, θ) be a compact CR manifold of dimension $2n + 1$ with a contact form θ, and $L = (2 + 2/n)\Delta_b + R$ its associated CR conformal laplacien. The CR Yamabe conjecture states that there is a contact form $\tilde{\theta}$ on M conformal to θ which has a constant Webster curvature. This problem is equivalent to the existence of a function u such that

$$\begin{cases} Lu = u^{1+2/n} \\ \qquad\qquad\qquad \text{on } M \\ u > 0 \,. \end{cases}$$

D. Jerison and J.M. Lee solved the CR Yamabe problem in the case where $n \geq 2$ and (M, θ) is not locally CR equivalent to the sphere S^{2n+1} of $\mathbf{C}^n$. In a join work with R. Yacoub, the CR Yamabe problem was solved for the case where (M, θ) is locally CR equivalent to the sphere S^{2n+1} for all n. In the present paper, we study the case $n = 1$, left by D. Jerison and J.M. Lee, which completes the resolution of the CR Yamabe conjecture for all dimensions.

1. Introduction

Let (M, θ) be a compact CR manifold of dimension $2n + 1$ with a contact form θ. Let $L = (2 + 2/n)\Delta_b + R$, be the conformal laplacian associated to (M, θ).

The CR Yamabe conjecture states that there is a contact form $\tilde{\theta}$ on M conformal to θ which has a constant scalar curvature $\tilde{R}$.

The CR Yamabe problem is equivalent to the existence of a positive function u such that:

$$\begin{cases} Lu = u^{1+2/n} \\ \qquad\qquad\qquad \text{on } M \\ u > 0 \,. \end{cases} \tag{1}$$

D. Jerison and J.M. Lee formulated the CR Yamabe problem in [1], [2] and [3] and developed the analogy between it and the Yamabe problem in conformal Riemannian geometry which had already been solved by T. Aubin [10] and R. Schoen [11].

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D. Jerison and J.M. Lee introduced the CR invariant:

$$\lambda(M) = \inf_{u \in S_1^2(M)} \{A_\theta(u)/B_\theta(u) = 1\}, \text{ where } S_1^2(M)$$

is a Folland and Stein space,

$$A_\theta(u) = \int_M \big((2 + 2/n)|du|_\theta^2 + Ru^2\big)\theta \wedge d\theta^n \tag{2}$$

the functional associated to the CR Yamabe equation, here $|du|_\theta$ is the norm of the cotangent vector du (see [1]) and $B_\theta(u) = \int_M u^{2+2/n}\theta \wedge d\theta^n$. They solved the Yamabe problem in special cases as summarized in the following result.

Theorem 3.4 [1] and **Theorem A [2]:** *Let M be a compact, orientable, strictly pseudoconvex integrable CR manifold of dimension* $2n + 1, \theta$ *any contact form on M.*

(a) $\lambda(M)$ *depends of the CR structure of M, and not of the choice of* θ.
(b) $\lambda(M) \leq \lambda(S^{2n+1})$*, where* $S^{2n+1} \subset \mathbb{C}^{n+1}$ *is the unit sphere with its standard CR structure.*
(c) *If* $\lambda(M) < \lambda(S^{2n+1})$ *and* $n \geq 2$*, then equation (1) has a solution.*
(d) *If* $n \geq 2$ *and M is not locally CR equivalent to* S^{2n+1}*, then* $\lambda(M) < \lambda(S^{2n+1})$*, and thus the CR Yamabe problem can be solved on M.*

In [7], we studied the case where (M, θ) is a CR compact manifold, locally CR equivalent to the sphere of the same dimension and proved:

Theorem [7]: *Let* (M, θ) *be a CR compact* $2n + 1$*- dimensional manifold, locally CR equivalent to the sphere* S^{2n+1}*, then (1) has a solution.*

In the present paper, we will be interested in the only remaining open case of the CR Yamabe conjecture. We will prove:

Theorem 1: *Let* (M, θ) *be a compact CR 3-dimensional manifold, not locally CR equivalent to the sphere* S^3*, then (1) has a solution.*

The proof of Theorem 1 is based on a contradiction argument and involves several steps.

In Sect. 2 we recall the definition of pseudohermitian normal coordinates for an abstract CR manifold given by D. Jerison and J.M. Lee in [2]. This definition is a refined version of the notion of normal coordinates (see [1]).

Since the asymptotic expansion of the Yamabe functional J on M

$$J(u) = \frac{\int (Lu)u\theta \wedge d\theta}{(\int u^4\theta \wedge d\theta)^{1/2}}$$

is expressed in terms of pseudohermitian curvature and torsion invariants. In order to make the calculation as easy as possible, D. Jerison and J.M. Lee introduced the pseudohemitian normal coordinates to simplify these invariants at a base point and showed that the contact form θ can be chosen in a neigborhood of a base point so that, the pseudohermitian Ricci and torsion tensors and certain combinations of their covariant derivatives, vanish at this base point (see Theorem 3.1 and Proposition 3.12 of [2]).

In Sect. 3, we will be interested in the case $n = 1$, without any hypothesis of CR conformal flatness.

We recall the extremals for the Yamabe functional J on the Heisenberg group $\mathbb{H}^1$:

$$\phi(z,t) = 2|w+i|^{-1}(w = (t + i|z|^2), (z,t) \in \mathbb{H}^1).$$

For $\varepsilon > 0$, we denote $\phi_\varepsilon(z,t) = \varepsilon^{-1}\delta^*_{1/\varepsilon}\phi(z,t) = 2\varepsilon|w + i\varepsilon^2|^{-1}$.The dilation $\delta^*_{\frac{1}{\varepsilon}}$ generalizes to all functions $g(z,t)$ with $\delta^*_{\frac{1}{\varepsilon}} g(z,t) = g\left(\frac{z}{\varepsilon}, \frac{t}{\varepsilon^2}\right)$.

In pseudohermitian normal coordinates for some contact form θ near $q \in M$, for $|w| < 2r$, we define a family of test functions $f_\varepsilon(z,t) = \psi(w)\phi_\varepsilon(z,t)(\psi$ is a cut-off function used to localize our function near the point q when $\varepsilon \to 0$), and a family of "almost solutions" φ_ε to be the unique solutions on M of:

$$L\varphi_\varepsilon = (f_\varepsilon)^3.$$

We may assume that the Webster scalar curvature R, is strictly positive (one can assume that, with a conformal change of contact form, $R > 0$ or $R = 0$ or $R < 0$. The Yamabe problem is easily solved for the cases $R = 0$ and $R < 0$).

Let $H_\varepsilon = \varepsilon^{-1}(\varphi_\varepsilon - f_\varepsilon)$.

In Sect. 1, we show that

$$|H_\varepsilon| \le c(1 + |\log(\varepsilon^2 + d^2)|) \text{ on } B(q, 2r).$$

Where $d = d(x,q) = \rho(qx^{-1})$, $x \in B(q,2r)$ and ρ is the distance in $\mathbb{H}^1$.

In Sect. 2, we give bounds for the interaction $\int \varphi^3_{a,\varepsilon}\varphi_{b,\varepsilon}$ in terms of the corresponding test functions.

In Sect. 3, we expand the functional J, near the set of potential critical points $V(p,\varepsilon')$, for $\varepsilon' > 0$ and $p \in \mathbb{N}^*$. This set is defined in analogy with the Riemannian case [6]:

$$V(p,\varepsilon') = \left\{ \begin{array}{l} u \in \sum_+ \text{ such that there exists p concentration points} \\ a_1, \ldots, a_p \text{ in } M \text{ and } p \text{ concentrations } \varepsilon_1, \ldots, \varepsilon_p \in [0, 1[\\ \text{such that } \left\| u - \frac{1}{p^{\frac{1}{2}} S} \sum_{i=1}^p \varphi_{a_i,\varepsilon_i} \right\|_H < \varepsilon', \text{ with } \varepsilon_i < \varepsilon' \\ \text{and } \varepsilon_{ij} = \frac{\varepsilon_i}{\varepsilon_j} + \frac{\varepsilon_j}{\varepsilon_i} + \frac{d(a_i,a_j)^2}{\varepsilon_i\varepsilon_j} \ge \frac{1}{\varepsilon'} \text{ for } i \neq j. \end{array} \right\}$$

where

$$H = \left\{u \in S^2_1(M) / \int |du|^2_\theta < \infty \text{ and } \int u^4 < \infty\right\}$$

($S^2_1(M)$ is a Folland-Stein space see [8]),

$$||u||_H = \left(\int_M \left(4|du|^2_\theta + Ru^2\right)\theta \wedge d\theta\right)^{1/2},$$

$\sum_+ = \{u \in \sum \quad s.t \quad u \geq 0\}$, where $\sum = \{u \in H \quad s.t \quad ||u||_H = 1\}$; S the Sobolev constant of $\mathbb{H}^1$ and $d(x, y)$, if x and y are in a small ball of M of radius r is $|| \exp_x^{-1}(y)||(|| \cdot ||_{\mathbb{H}^1}$ is the norm in $\mathbb{H}^1)$, with $\exp_x$ the CR exponential map for the point x, $d(x, y)$ is equal to $r/2$ otherwise.

We prove that for $p \geq p_0$ (i.e for p large enough), the energy $J(\sum_p^{i=1} \alpha_i \varphi_{a_i,\varepsilon}) \leq p^{1/2} S$ for ε small enough. And show that the function:

$$f_p(\varepsilon) : B_p(M) \longrightarrow W_p$$

where $B_p(M) = \{\sum_{i=1}^p \alpha_i \delta_{a_i}, \sum_{i=1}^p \alpha_i = 1, a_i \in M\}$, with δ_{a_i} the Dirac mass at a_i, $B_0(M) = \emptyset$, and $W_p = \{u \in \sum_+ \quad s.t \quad J(u) \leq (p+1)^{1/2} S\}$, defined by

$$f_p(\varepsilon)(\sum_{i=1}^p \alpha_i \delta_{a_i}) = \frac{\sum_{i=1}^p \alpha_i \varphi_{a_i,\varepsilon}}{\| \sum_{i=1}^p \alpha_i \varphi_{a_{i,\varepsilon}} \|_H}$$

is homologically trivial for $p \geq p_0$, (Proposition 8). On the other hand assuming that equation (1) has no solution, we prove in Sect. 5 by using the result of [7] Sect. 6, Proposition 22 that $f_{p^*}(\varepsilon) \neq 0$ for all $p \in \mathbb{N}^*$, this gives a contradiction to Proposition 8, and hence completes the proof of Theorem 1.

I am indebted to A. Bahri who suggested to me to study the CR Yamabe conjecture. Our present paper shows how the techniques of critical points at infinity can settle the case $n = 1$, without assuming that M is locally conformally flat. The techniques apply to the other dimensions, the main observation is that the CR Yamabe case for $n = p$ is similar to the Riemannian one for $n = 2p+2$. Therefore we are not going to provide the proof here for the case $n > 1$, since the sketch is analogous to the one given for the case $n = 1$. Indeed, in order to compute the CR Yamabe functional $J(u) = \frac{(\int (Lu)u \ \theta \wedge d\theta^n)^{1+\frac{2}{n}}}{\int u^{2+\frac{2}{n}} \ \theta \wedge d\theta^n}$, we will use the same methods given by A. Bahri and H. Brezis in [5] for the Riemannian case. We need no more assumptions for the CR case $n > 1$, we have only to follow the sketch of the proof for the case $n = 1$, with introducing where it is needed some required modifications due to the dimension of the CR manifold.

The analysis of the Palais-Smale sequences for the CR Yamabe problem is slightly different from the classical Yamabe problem, because some operations (H_0^1-projections, for example) are not available at this point, in the CR framework. We have shown in [7] how to overcome this difficulty and have an analysis of the behaviour of the Palais-Smale sequences completely analogous to the classical case. The results of [7] readily extend here and we will use them directly.

Our present result completes the resolution of the CR Yamabe conjecture for all dimensions.

2. Pseudohermitian normal coordinates [2]

In order to give a precise asymptotic expression for the coresponding functional to the equation (1) near a base point, D. Jerison and J.M. Lee have refined their notion of normal coordinates defined in [1] by constructing in [2] new intrinstic

CR normal coordinates for an abstract CR manifold. These coordinates are called pseudohermitian normal coordinates.

The notions and results introduced and proved by D. Jerison and J.M. Lee are parallel, with drastically different techniques, to the ones introduced by J.M. Lee and T. Parker for the Riemannian Yamabe conjecture [4].

Let $\mathbb{H}^n = \mathbb{C}^n \times \mathbb{R}$ be the Lie group whose underlying manifold is $\mathbb{C}^n \times \mathbb{R}$ (the Heisenberg group) with coordinates (z, t). $\mathbb{H}^n$ is a pseudohermitian manifold with holomorphic tangent bundle $\mathcal{H}$ spanned by the vector fields:

$$Z_\alpha = \frac{\partial}{\partial z^\alpha} + i\bar{z}^\alpha \frac{\partial}{\partial t}, \quad \alpha = 1, \cdots, n \ ,$$

and standard contact form $\theta_0 = dt + iz^\alpha d\bar{z}^\alpha - i\bar{z}^\alpha dz^\alpha$.

The characteristic vector field of θ_0 is $T = \frac{\partial}{\partial t}$, the admissible coframe dual to Z_α is $\{dz^\alpha\}$, and the Levi-form is given by $h_{\alpha\bar{\beta}} = 2\delta_{\alpha\bar{\beta}}$, $\bar{\beta} = \beta + n$ with $\beta = 1, \cdots, n$. The natural parabolic dilations on $\mathbb{H}^n$ are the CR-automorphisms $\delta_s : \mathbb{H}^n \to \mathbb{H}^n$ such that $\delta_s(z, t) = (sz, s^2 t)$ for $s > 0$.

The infinitesimal generator of this $\mathbb{R}^+$-action on $\mathbb{H}^n$ is the vector field:

$$P_{(z,t)} = z^\alpha \frac{\partial}{\partial z^\alpha} + \bar{z}^\alpha \frac{\partial}{\partial \bar{z}^\alpha} + 2t\frac{\partial}{\partial t} = z^\alpha Z_\alpha + \bar{z}^\alpha Z_{\bar{\alpha}} + 2tT.$$

The orbits of the dilations (except for the degenerate orbits where $z = 0$ or $t = 0$) lie in parabolas through 0. For fixed $(W, c) \in \mathbb{H}^n$, considering the curve $\gamma : \mathbb{H}^n \to \mathbb{H}^n$ given by $\gamma(s) = (sW, s^2 c)$, we have $\dot{\gamma}(s) = s^{-1} P_{\gamma(s)}$ for $s \neq 0$.

Using the fact that the pseudohermitian connection ∇ on $\mathbb{H}^n$ satisfies $\nabla Z_\alpha = \nabla T = 0$, we have

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 2cT \ . \tag{2.1}$$

On a manifold M, a pseudohermitian structure yields a natural splitting $TM = H \oplus \mathbb{R}T$, $H = Rel(\mathcal{H} \oplus \bar{\mathcal{H}})$, where $\mathcal{H} \subset \mathbb{C}TM$ is the holomorphic tangent bundle, satisfying $\mathcal{H} \cap \bar{\mathcal{H}} = \{0\}$ and $[\mathcal{H}, \mathcal{H}] \subset \mathcal{H}$, which determines a natural family of parabolic dilations on any tangent space $T_q M$ analogous to those on the Heisenberg group, by setting $\delta_s(W + cT) = sW + s^2 cT$ for $W \in H$, $c \in \mathbb{R}$. The curves in $T_q M$ given by $\sigma_{W,c}(s) = sW + s^2 cT$ are parabolas analogous to the curves γ.

Theorem 2.1 [2]: *Let M be nondegenerate pseudohermitian manifold and $q \in M$. For any $W \in H_q$ and $c \in \mathbb{R}$, let $\gamma = \gamma_{W,c}$ denote the solution to the ordinary differential equation (2.1) on M with initial conditions $\gamma(0) = q$ and $\dot{\gamma}(0) = W$. We call γ the parabolic geodesic determined by W and c. Define the parabolic exponential map $\psi : T_q M \to M$ by:*

$$\psi(W + cT) = \gamma_{W,c}(1) \tag{2.2}$$

when defined. Then ψ maps a neighborhood of 0 in $T_q M$ diffeomorphically to a neighborhood of q in M and sends $\sigma_{W,c}$ to $\gamma_{W,c}$.

D. Jerison et J.M. Lee defined on a strictly pseudo convex CR manifold a special frame to be a holomorphic frame $\{W_\alpha\} \in \mathcal{H}$ which is parallel along each curve $\gamma_{W,c}$ and for which $h_{\alpha\bar{\beta}} = 2\delta_{\alpha\bar{\beta}}$, they called the dual admissible coframe a special coframe. We then have

Proposition 2.3 [2]: *Any holomorphic frame at $q \in M$ for which $h_{\alpha\bar{\beta}} = 2\delta_{\alpha\bar{\beta}}$ can be extended smoothly to a special frame $\{W_\alpha\}$ in a neighborhood of q. The dual special coframe $\{\theta^\alpha\}$ is parallel along each curve $\gamma_{W,c}$, and satisfies $d\theta = 2\theta^\alpha \wedge \theta^{\bar{\alpha}}$. Any two such extentions agree on their common domain.*

The choice of a special frame $\{W_\alpha\}$ near q implies the existence of a dual special coframe $\{\theta^\alpha\}$ which determines an isomorphism:

$$\begin{array}{rcl} \lambda : TqM & \longrightarrow & \mathbb{H}^n \\ V & \longmapsto & \lambda(V) = (\theta^\alpha(V), \theta(V)) = (z^\alpha, t) \end{array}$$

and then we have a coordinate chart $\lambda o \psi^{-1}$ in a neighborhood of q we call such a chart pseudohermitian normal coordinates determined by $\{W_\alpha\}$.

Identifying a neighborhood of $q \in M$ with an open set in $\mathbb{H}^n$ by means of pseudohermitian normal coordinates chart, we can consider θ and θ^α as one forms on (a subset of) $\mathbb{H}^n$. We then have:

Definition 1 [2]: A function or tensor ω on $\mathbb{H}^n$ is homogeneous of degree m with respect to the dilations if and only if its lie derivative with respect to P satisfies $L_P(\omega) = m\omega$.

If φ is any tensor field on $\mathbb{H}^n$, we denote by $\varphi_{(m)}$ the part of its Taylor series that is homogeneous of degree m in terms of the parabolic dilations. Each term $\varphi_{(m)}$ satisfies $L_P\varphi_{(m)} = m\varphi_{(m)}$, therefore if φ is a differential form, we have:

$$\varphi_{(m)} = \frac{1}{m}(L_P\varphi)_{(m)} = \frac{1}{m}(P\rfloor d\varphi + d(P\rfloor\varphi))_{(m)} \tag{2.3}$$

D. Jerison and J.M. Lee used the relation (2.3) to compute the homogeneous parts of θ and θ^α:

Proposition 2.5 [2]: *Let $\{W_\alpha\}$ be a special frame and $\{\theta^\alpha\}$ the dual special coframe. Then in pseudohermitian normal coordinates:*

a) $\theta_{(2)} = \theta_0$; $\theta_{(3)} = 0$; $\theta_{(m)} = \frac{2}{m}(iz^\alpha\theta^{\bar{\alpha}} - i\bar{z}^\alpha\theta^\alpha)_{(m)}$, $m \geq 4$;
b) $\theta^\alpha_{(1)} = dz^\alpha$; $\theta^\alpha_{(2)} = 0$; $\theta^\alpha_{(m)} = \frac{1}{m}\big(z^\beta{\omega_\beta}^\alpha + tA_{\overline{\alpha\beta}}\theta^{\bar{\beta}} - \frac{1}{2}z^{\bar{\beta}}A_{\overline{\alpha\beta}}\theta\big)_{(m)}$,
$m \geq 3$;
c) ${\omega_\beta}^\alpha_{(1)} = 0$; ${\omega_\beta}^\alpha_{(m)} = \frac{1}{m}\big({R_\beta}^\alpha{}_{\rho\bar{\sigma}}(z^\rho\theta^{\bar{\sigma}} - z^{\bar{\sigma}}\theta^\rho) + \frac{1}{2}A_{\beta\gamma,\bar{\alpha}}(z^\gamma\theta - 2t\theta^\gamma)$
$-\frac{1}{2}A_{\overline{\alpha\gamma},\beta}(z^{\bar{\gamma}}\theta - 2t\theta^{\bar{\gamma}}) + iA_{\overline{\alpha\gamma}}(z^{\bar{\beta}}\theta^{\bar{\gamma}} - z^{\bar{\gamma}}\theta^{\bar{\beta}}) - iA_{\beta\gamma}(z^\gamma\theta^\alpha - z^\alpha\theta^\gamma)\big)_{(m)}$,
$m \geq 2$.

Where $A_{\alpha,\beta}$ are the components of the pseudohermitian torsion and ${\omega_\alpha}^\beta$ are one forms satisfying: $\nabla W_\alpha = {\omega_\alpha}^\beta \otimes W_\beta$, ∇ is the pseudohermitian connection on M.

To compute the numerator and denominator of the CR Yamabe functional we will use the approach used by D. Jerison and J.M. Lee who gave the Taylor series of a contact form θ and *a* special coframe $\{\theta^\alpha\}$ to high order at a point $q \in M$ in terms of the pseudohermitian curvature and torsion. Since the problem is CR invariant, they had to choose the contact form θ so as to simplify the curvature and torsion at q as much as possible and determined how these can be simplified by a choice of contact form and showed that a certain tensor Q constructed from the pseudohermitian Ricci and torsion tensors can be made to vanish at q, together with its symmetrized convariant derivatives of all orders.

Let θ be any contact form for M and let (z, t) be pseudohermitian normal coordinates for θ contered at q.

Write $Z_\alpha = \frac{\partial}{\partial z^\alpha} + iz^{\bar\alpha}\frac{\partial}{\partial t}$ in these coordinates and $\mathcal{L}_0 = -\frac{1}{2}(Z_\alpha Z_{\bar\alpha} + Z_{\bar\alpha} Z_\alpha)$.

Notations: D. Jerison and J.M. Lee adopted the following index conventions in [2]

$$\alpha, \beta, \gamma, \varepsilon, \rho, \sigma \in \{1, \cdots, n\}\ ;\ a, b, c \in \{1, \cdots, 2n\};\ \ j, k, \ell \in \{0, \cdots, 2n\}.$$

Let $x = (t, z, \bar z)$ with $x^0 = t, x^\alpha = z^\alpha, x^{\bar\alpha} = z^{\bar\alpha}$ and $\bar\alpha = \alpha + n$.

Let $\theta^0 = \theta$, $W_0 = T$ and $Z_0 = \frac{\partial}{\partial t}$

$$o\ (j) = \begin{cases} 2 & \text{if} \quad j = 0 \\ 1 & \text{for} \quad j \neq 0 \end{cases}.$$

For a multi-index $J = (j_1, \cdots, j_s)$ denote by $\#J = s$,

$$o(J) = o(j_1) + o(j_2) + \cdots + o(j_s), x^J = x^{j_1} \cdots x^{j_s}$$

$Z_J = Z_{j_s} \cdots Z_{j_1}$ and $\partial_J = \frac{\partial^s}{\partial x^{j_s} \cdots \partial x^{j_1}}$.

These notations lead us to the following other definition: if z^α is a coordinate, then $\bigcirc(z^\alpha) = 2$ if $\alpha = 0$ and $\bigcirc(z^\alpha) = 1$ if $\alpha > 0$. Hence $\bigcirc(z^\alpha) = o(\alpha)$. If Z_α is on the other hand a base vector field then $\bigcirc(Z_\alpha) = -2$ if $\alpha = 0$, $\bigcirc(Z_\alpha) = -1$ if $\alpha > 0$. Hence $\bigcirc(Z_\alpha) = -o(\alpha)$. For an expression combining coordinates and vector fields $\Gamma = z^{j_1}....z^{j_r} Z_{i_1}....Z_{i_l}$, the order $\bigcirc(\Gamma)$ is the sum $\sum_{k=1}^r o(j_k) - \sum_{k=1}^l o(i_k)$.

D. Jerison and J.M. Lee use these convenient notations and show that, given two operators Γ_1 and Γ_2 of degree l_0, if $\Gamma_2 = \Gamma_1 + \bigcirc(m)$, Γ_2 reads as Γ_1 to what is added a combination of $Z_{i_1}......Z_{i_{l_0}}$ affected with coefficients $a_{i_1.....i_{l_0}}$ which are (in ρ as function now, near 0 in a local chart) of the order of ρ^γ with $\gamma = m + \sum_{j=1}^{l_0} o(i_j)$.

The symmetrization of an r-tensor with components $B_J = B_{j_1,\cdots,j_r}$ is the symmetric tensor with components $B_{<J>} = \frac{1}{r!}\sum_{s \in S_r} B_{\sigma(J)}$, S_r is the symmetric group of r elements and $\sigma(J) = (j_{\sigma(1)}, ..., j_{\sigma(r)})$.

Definition 2 [2]: On a pseudhermitian manifold (M, θ), let $Q = Q_{j,k}\theta^j\theta^k$ denote the real symmetric tensor whose components with respect to any admissible coframe are:

$$Q_{\alpha\beta} = \overline{Q_{\overline{\alpha\beta}}} = (n+2)iA_{\alpha\beta},\ \ Q_{\alpha\bar\beta} = Q_{\bar\beta\alpha} = R_{\alpha\bar\beta},$$

$$Q_{o\alpha} = Q_{\alpha 0} = \bar Q_{o\bar\alpha} = \bar Q_{\bar\alpha o} = 4A_{\alpha\beta,}{}^{\beta} + \frac{2i}{n+1}R_{,\alpha},$$

$$Q_{00} = \frac{16}{n} Im A_{\alpha\beta,}{}^{\beta\alpha} - \frac{4}{n(n+1)}\Delta_b R,$$

where $R_{\alpha\bar{\beta}}$ are the components of the Webster Ricci tensor, R is the Webster scalar curvature, and the components of the pseudohermitian covariant derivatives of a tensor are denoted by indices preceded by a comma.

We have the following key result:

Theorem 3.1 [2]: *Let M be a strictly pseudo convex CR manifold . For any integer $N \geq 2$, there exists a choice of contact form θ such that all symmetrized covariant derivatives of Q with total order $\leq N$ vanish at q, that is:*

$$Q_{<jk,L>} = 0 \text{ if } o(jkL) \leq N.$$

Writing $\theta = e^{2u}\bar{\theta}$ for some fixed contact form $\bar{\theta}$, the one jet of u can be chosen arbitrarily, once it is fixed, the Taylor series of u at q is uniquely determined by this condition.

An application of this result is:

Proposition 3.12 [2]: *Suppose θ is a contact form satisfying Theorem 3.1 [2] for $N = 4$. Then the following relations hold at q:*
(a) $R = 0$; $R_{\alpha\bar{\beta}} = 0$; $A_{\alpha\beta} = 0$;
(b) $A_{\alpha\beta,\gamma} = 0$;
(c) $R_{,\alpha} = A_{\alpha\beta,}{}^{\beta} = R_{\alpha\bar{\beta},}{}^{\bar{\beta}} = 0$;
(d) $R_{\alpha\bar{\beta},}{}^{\alpha\bar{\beta}} = A_{\alpha\beta,}{}^{\alpha\beta} = \Delta_b R = R_{,0} = 0$.

Let $\{W_\alpha\}$ be a special frame and $\{\theta^\alpha\}$ the dual special coframe we display the Taylor series of W_j, which we write as

$$W_j = s_j^k Z_k = s_j^\beta Z_\beta + s_j^{\bar{\beta}} Z_{\bar{\beta}} + s_j^0 Z_0 \tag{2.4}$$

where $W_0 = T$, $Z_0 = \frac{\partial}{\partial t}$ and $\bar{\alpha} = \alpha + n$ and sum $k = 0, 1, \cdots, 2n$. Recalling that $W_{\alpha(-1)} = Z_\alpha$, $W_{\bar{\alpha}(-1)} = Z_{\bar{\alpha}}$ and $W_{0(-2)} = Z_0$, we find

$$\begin{cases} s^\beta_{\alpha(0)} = \delta^\beta_\alpha;\ s^{\bar{\beta}}_{\bar{\alpha}(0)} = \delta^{\bar{\beta}}_{\bar{\alpha}};\ s^0_{\alpha(1)} = s^{\bar{\beta}}_{\alpha(0)} = 0 \\ s^\beta_{\bar{\alpha}(0)} = s^0_{\bar{\alpha}(1)} = 0;\ s^\beta_{0(-1)} = s^{\bar{\beta}}_{0(-1)} = 0,\ s^0_{0(0)} = 1. \end{cases} \tag{2.5}$$

If we apply θ^ℓ to (2.4) and consider terms of homogeneity $m + o(\ell) - o(j)$for $m > 0$, we obtain:

$$s^\ell_{j(m+o(\ell)-o(j))} = -\sum_{i\geq 2} s^k_{j(m+o(k)-o(j)-1)}\theta^\ell_{(o(\ell)+i)}(Z_k). \tag{2.6}$$

As we will be interested in the case $n = 1$, let $(W_1, W_{\bar{1}}, W_0)$ be a special frame and $(\theta^{(1)}, \theta^{(\bar{1})}, \theta)$ the dual special coframe we can write (see (2.4))

$$W_1 = s_1^1 Z_1 + s_1^{\bar{1}} Z_{\bar{1}} + s_1^0 Z_0$$
$$W_{\bar{1}} = s_{\bar{1}}^1 Z_1 + s_{\bar{1}}^{\bar{1}} Z_{\bar{1}} + s_{\bar{1}}^0 Z_0.$$

By using (2.5) and (2.6), Proposition 2.5 gives the following Taylor series of W_1 and $W_{\bar{1}}$:

$$W_1 = Z + a(4)Z + b(4)\bar{Z} + c(5)Z_0$$
$$W_{\bar{1}} = \bar{Z} + a'(4)\bar{Z} + b'(4)Z + c'(5)Z_0$$

where, $a(4), b(4), a'(4)$ and $b'(4)$ respectively $c(5)$ and $c'(5)$ have Taylor coefficients of order 4 and higher, respectively of order 5 and higher. Since Z and $\bar{Z}$ count as order -1 we have:

$$\begin{cases} W_1 = Z + \bigcirc(3) \\ W_{\bar{1}} = \bar{Z} + \bigcirc(3) \end{cases} . \tag{2.7}$$

3. The case of a pseudohermitian manifold of dimension 3 not locally CR equivalent to the sphere S^3

The functions $\phi(z,t) = 2|w+i|^{-1} (w = t + i|z|^2)$ are the extemals for the Yamabe functional J on $\mathbb{H}^1$ ([3])

$$J(u) = \frac{\int \left(4|du|_\theta^2 + Ru^2\right)\theta \wedge d\theta}{(\int_M u^4 \theta \wedge d\theta)^{1/2}} \tag{3.1}$$

For each $\varepsilon > 0$, let us denote $\phi_\varepsilon = \varepsilon^{-1}\delta^*_{1/\varepsilon}\phi = 2\varepsilon|w + i\varepsilon^2|^{-1}$. It is also an extremal for J normalized so that $\int_{\mathbb{H}^1} |\phi_\varepsilon|^4 \theta \wedge d\theta$ is a constant independent of ε.

Suppose that (z, t) are pseudohermitian normal coordinates for some contact form θ near $q \in M$, defined for $|w| < 2r$ for some $r > 0$. Define a family of test functions:

$$f_\varepsilon(z,t) = \psi(w)\phi_\varepsilon(z,t) \tag{3.2}$$

where $\psi \in C_0^\infty(\mathbb{C})$ is supported in the set $\{|w| < 2r\}$ and $\psi(w) = 1$ for $|w| < r$ (a cut–off function), (Observe that f_ε has support near zero). Therefore, $Z_j f_\varepsilon$ and higher order derivatives make sence.). Define a family of "almost" solutions φ_ε to be the unique solutions on M of:

$$L\varphi_\varepsilon = (f_\varepsilon)^3 \tag{3.3}$$

let

$$H_\varepsilon = \varepsilon^{-1}(\varphi_\varepsilon - f_\varepsilon) \tag{3.4}$$

We start the proof of Theorem 1 with the following estimates:

Lemma 1: *There exists $c(r)$ independent of ε such that*

$$\left|Z_j f_\varepsilon\right|(x) \leq c(r)\phi_\varepsilon(x)\varepsilon^{-o(j)} \text{ and}$$
$$\left|Z_j Z_k f_\varepsilon\right|(x) \leq c(r)\phi_\varepsilon(x)\varepsilon^{-o(j)-o(k)}.$$

Proof: We need to prove these inequalities on ϕ_ε. We have $\phi_\varepsilon = \frac{1}{\varepsilon}\delta^*_{\frac{1}{\varepsilon}}\phi$. Hence $Z_j\phi_\varepsilon = \frac{1}{\varepsilon}\varepsilon^{-o(j)}\delta^*_{\frac{1}{\varepsilon}}(Z_j\phi)$. It is very easy to check that $\left|Z_j\phi\right|(x) \leq c\ \phi(x)$. Therefore,$\left|Z_j\phi_\varepsilon\right| \leq c\ \varepsilon^{-o(j)-1}\,|\phi_\varepsilon|\,\varepsilon \leq c\ \varepsilon^{-o(j)}\,|\phi_\varepsilon|$. The second order derivatives follow the same pattern.

In order to prove the existence of a solution for equation (1), we will use the same methods given par A. Bahri and H. Brezis in [5]. The main observation is that the CR case for $n = 1$ is similar to the Riemannian one for $n = 4$ given in Sect. 5 of [5]. We will therefore follow the line of proof of [5] and will mainly indicate the modifications to Sect. 1', 2' and 3' of [5].

The new sections are Sects. 1, 2, 3 and 4.

Section 1

Proposition 1: *H_ε has been defined in (3.4).*

There exists a positive constant C such that for any $x \in M$ and $\varepsilon \leq 1$

$$|H_\varepsilon(x)| \leq C(1 + |\log(\varepsilon^2 + d^2)|) \text{ if } x \in B(q, 2r)$$

where $d = d(x, q) = \rho(qx^{-1})$.

Proof: First we will estimate LH_ε:

$$LH_\varepsilon = (4\Delta_b + R)\varepsilon^{-1}(\varphi_\varepsilon - f_\varepsilon) = \varepsilon^{-1}(4\Delta_b + R)\varphi_\varepsilon - \varepsilon^{-1}Lf_\varepsilon.$$

Let us denote by $(W, \bar{W}, T)$ the special frame for q. For a real function f we have:

$$\Delta_b f = -\left(f_{,\alpha}{}^{\alpha} + f_{,\bar{\beta}}{}^{\bar{\beta}}\right) = -\frac{1}{2}(W\bar{W}f + \bar{W}Wf) = -\frac{1}{2}(f_{\alpha\bar{\alpha}} + f_{\bar{\alpha}\alpha})$$

$f_{,\alpha}{}^{\alpha} = h^{\alpha\bar{\beta}}f_{\alpha\bar{\beta}}$, where $h_{\alpha\bar{\beta}} = L_\theta(W_\alpha, W_{\bar{\beta}})$, we have $h_{\alpha\bar{\beta}} = 2\delta_{\alpha\bar{\beta}}$, thus $h^{\alpha\bar{\beta}} = \frac{1}{2}\delta^{\alpha\bar{\beta}}$.

If we write the Taylor serie of the Webster scalar curvature, we have $R = R_{(0)} + R_{(1)} + R_{(2)} + ...$, $R_{(0)} = R(q) = 0$ by Proposition 3.12 [2] . Identity (1.3) and Propositions 2.5 and 3.12 of [2] yield that $R_{(1)} = 0$. Thus $R = \bigcirc(2)$, where $\bigcirc(2)$ is a homogenous polynomial in ρ of degree at least 2. Since $W = Z + \bigcirc(3)$, $\bar{W} = \bar{Z} + \bigcirc(3)$ and $R = \bigcirc(2)$, we have the following expression for the conformal laplacian on M

$$L = 4\Delta_b + R = -2(Z\bar{Z} + \bar{Z}Z) + \bigcirc(2),$$

where we are using the index convention of D. Jerison and J.M. Lee this implies, using Lemma 1, that:

$$LH_\varepsilon = \varepsilon^{-1}[(f_\varepsilon)^3 - (-2(Z\bar{Z} + \bar{Z}Z) + \bigcirc(2))f_\varepsilon].$$

<u>On $B(0, r)$</u> $\psi \equiv 1$ and $f_\varepsilon = \phi_\varepsilon$,$-2(Z\bar{Z} + \bar{Z}Z)\phi_\varepsilon = (\phi_\varepsilon)^3$

Thus, $|LH_\varepsilon| \leq \varepsilon^{-1}O(\rho^2)|f_\varepsilon| \leq \inf(1, \frac{c(r)}{\rho^2+\varepsilon^2})$.

<u>On ${}^cB(0, 2r)$</u>, $\psi = 0$ thus $f_\varepsilon = 0$ $LH_\varepsilon = 0$.

<u>On</u> $B(0, 2r) - B(0, r)$, $\Delta_b f_\varepsilon = \Delta_b\psi\phi_\varepsilon = (\Delta_b\psi)\phi_\varepsilon + \psi(\Delta_b\phi_\varepsilon) + L_{\theta^*}(d\psi, d\phi_\varepsilon)$

The leading term is $\psi\Delta_b\phi_\varepsilon$ since ϕ_ε is small. Hence,

$$|\psi\Delta_b\phi_\varepsilon| \leq C(r)|\Delta_b\phi_\varepsilon| \leq C(r)(\phi_\varepsilon)^3 \leq C(r)\varepsilon^3 \leq C(r)\varepsilon$$

we have

$$\begin{cases} |\Delta_b f_\varepsilon| \leq C(r)\varepsilon \,, \qquad |d\phi_\varepsilon| \leq C\varepsilon \\ \text{and} \\ (f_\varepsilon)^3 \leq C'(r)\varepsilon. \end{cases} \tag{3.5}$$

Thus, $|LH_\varepsilon| \leq C(r)$ and we derive:

$$\begin{cases} |LH_\varepsilon| \leq \inf\left(1, \frac{C(r)}{\rho^2+\varepsilon^2}\right) & \text{on} \quad B(0, 2r) \\ \qquad |LH_\varepsilon| = 0 & \text{on} \quad {}^cB(0, 2r) \end{cases}. \tag{3.6}$$

We introduce the function W defined on M by:

$$LW(x) = \begin{cases} \frac{1}{\rho^2(q^{-1}x)} & \text{if } x \in B(q, r) \\ 0 & \text{otherwise} \end{cases}. \tag{3.7}$$

Using the maximum principle, we deduce the existence of a positive constant c' such that:

$$|H_\varepsilon(x)| \leq c' W(x). \tag{3.8}$$

We have the following estimates on W

$$\begin{cases} |W| \leq c(1 + |\log d|) & \text{on } B(q, 2r) \\ \qquad |W| \leq c & \text{on } {}^cB(q, r) \end{cases}. \tag{3.9}$$

Indeed:
W satisfies (3.7) then $W(z) \leq A \int \frac{1}{\rho^2(x)} \frac{1}{\rho^2((q^{-1}z)^{-1}x)} \rho^3 d\rho + B$ where A, B are positive constants.

We will divide the domain of integration in four parts:

(1) $\begin{cases} \rho((q^{-1}z)^{-1}x) \geq C_1\rho(x) \\ \text{and} \\ \rho(x) \geq C_2\rho(q^{-1}z) \end{cases}$ (2) $\begin{cases} \rho((q^{-1}z)^{-1}x) \geq C_1\rho(x) \\ \text{and} \\ \rho(x) \leq C_2\rho(q^{-1}z) \end{cases}$

(3) $\begin{cases} \rho((q^{-1}z)^{-1}x) \leq C_1\rho(x) \\ \text{and} \\ \rho(x) \leq C_2\rho(q^{-1}z) \end{cases}$ (4) $\begin{cases} \rho((q^{-1}z)^{-1}x) \leq C_1\rho(x) \\ \text{and} \\ \rho(x) \geq C_2\rho(q^{-1}z) \end{cases}$

where C_1 and C_2 are positive constants. We have:

$$(1) \int \frac{\theta_0 \wedge d\theta_0}{\rho^2(q^{-1}z)\rho^2((q^{-1}z)^{-1}x)} = c \int \frac{\rho(x)d\rho(x)}{\rho^2((q^{-1}z)^{-1}x)} \leq \frac{1}{C_1^2} \int_{\rho \geq C_2\rho(q^{-1}z)} \frac{d\rho}{\rho}$$
$$\sim C \log \rho(q^{-1}z).$$

In the second part (2) of the domain, we have: $\rho((q^{-1}z)^{-1}x) \geq C_3\rho(q^{-1}z)$ for suitably constants C_1, C_2 and C_3

$$(2)\int \frac{\theta_0\wedge d\theta_0}{\rho^2(x)\rho^2((q^{-1}z)^{-1}x)} = \frac{C}{C_1^2}\int_{\rho\geq C_3\rho(q^{-1}z)} \frac{\rho(x)d\rho(x)}{\rho^2((q^{-1}z)^{-1}x)}$$
$$\sim C\log\rho(q^{-1}z).$$

We observe now that, writing $\theta_0 \wedge d\theta_0 = \rho^3((q^{-1}z)^{-1}x)d\rho((q^{-1}z)^{-1}x)d\xi$, where $d\xi$ is the area element on the ρ-unit sphere in $\mathbb{H}^1$.

$$W = A\int \frac{\rho((q^{-1}z)^{-1}x)d\rho((q^{-1}z)^{-1}x)}{\rho^2(x)} + B.$$

This shows that (1) and (2) are equivalent to (3) and (4).

(3.9) is thereby established.

Observe now that $\lambda = \varepsilon^{-1}$ is large. We argue considering two cases:

First case: $d(y,q) > \varepsilon(\lambda\rho > 1)$. Using (3.9), we have:

$$|H_\varepsilon(y)| \leq C(1+\log|d|) \leq c(1+|\log(\varepsilon^2+d^2)|)$$

Second case: $d(y,q) < \varepsilon(\lambda\rho < 1)$.

We introduce $B_\varepsilon = \{y \in M \text{ such that } d(q,y) < \varepsilon < 1\}$.

$$\text{Then } \begin{cases} LH_\varepsilon \leq f \text{ on } B_\varepsilon \\ H_\varepsilon \leq g \text{ on } \partial B_\varepsilon \end{cases} \text{ where } \begin{cases} f \leq C\varepsilon^{-1} f_\varepsilon \\ \text{and} \\ |g(y)| \leq C(1+|\log d|) = C(1+\log\lambda) \end{cases}.$$

We have then to estimate the function σ which satisfies:

$$L\sigma = f\chi_{B_\varepsilon}$$

We have:

$$\sigma(x) = \int_{B_\varepsilon} G_q(x,y) f\chi_{B_\varepsilon}(y)dy$$

where $G_q(x,y)$ is the Green function for the conformal Laplacian $L = 4\Delta_b + R$ on q. G_q is positive and satisfy: $G_q(z,t) = O(\rho^{-2}(z,t))$. A finer result is established in the Appendix (Lemmas A_1 and A_2).

Let $\bar{H}$ be the solution of

$$(*)\begin{cases} L\bar{H} = f \text{ on } B_\varepsilon \\ \bar{H} = g \text{ on } \partial B_\varepsilon \end{cases}$$

we have

$$\begin{cases} L(\bar{H}-\sigma) = 0 \text{ on } B_\varepsilon \text{ and} \\ \bar{H}-\sigma = h \text{ on } \partial B_\varepsilon \end{cases}$$

we deduce from the maximum principle that:

$$||\bar{H}-\sigma||_{L^\infty(B_\varepsilon)} \leq C||\bar{H}-\sigma||_{L^\infty(\partial B_\varepsilon)} = C||h||_{L^\infty(\partial B_\varepsilon)}$$
$$\leq C\sup_{\partial B_\varepsilon}|\bar{H}-\sigma|.$$

Thus:

$$\begin{aligned}||\bar{H}||_{L^\infty(B_\varepsilon)} &\leq C \sup_{\partial B_\varepsilon} |\bar{H} - \sigma| + |\sigma|_{L^\infty(B_\varepsilon)} \\ &\leq C \sup_{\partial B_\varepsilon} |\bar{H}| + |\sigma|_{L^\infty(B_\varepsilon)}\end{aligned}$$

and finally:

$$|H_\varepsilon|_{L^\infty(B_\varepsilon)} \leq |\bar{H}|_{L^\infty(B_\varepsilon)} \leq C \sup_{\partial B_\varepsilon} |\bar{H}| + |\sigma|_{L^\infty(B_\varepsilon)}.$$

Observe now that

$$G_q(x, y) = O(\rho^{-2})$$

and

$$\sigma(x) = \int_{B_\varepsilon} G_q(x, y) f \chi_{B_\varepsilon}(y) dy.$$

Hence

$$\sigma(x) = \int_{\rho < \varepsilon} O(\rho^{-2}) \frac{C(r)}{(1 + \rho^2)} \rho^3 d\rho$$

and

$$||\sigma||_{L^\infty(B_\varepsilon)} \leq C(r)(1 + \log(1 + \rho^2)) \leq C(r)(1 + \log(1 + \varepsilon^2)).$$

Hence

$$\begin{aligned}||H_\varepsilon||_{L^\infty(B\varepsilon)} &\leq C \sup_{\partial B_\varepsilon} |\bar{H}| + C(r)(1 + \log(1 + \varepsilon^2)) \\ ||H_\varepsilon||_{L^\infty(B\varepsilon)} &\leq C'(r)(1 + |\log(\varepsilon^2 + d^2)|).\end{aligned}$$

Proposition 1 follows.

Section 2

Let $\eta > 0$ be given, let $\tilde{f}_\varepsilon = f_\varepsilon + \eta\varepsilon$. (3.10)

Lemma 2: *φ_ε has been defined in (3.3)*

i) $\displaystyle\int \left(4\, |d\varphi_\varepsilon|_\theta^2 + R\varphi_\varepsilon^2\right)\theta \wedge d\theta = \int |\phi|^4\, \theta_0 \wedge d\theta_0 + O\left(\varepsilon^2 \log \frac{1}{\varepsilon}\right)$

$\displaystyle\int |\varphi_\varepsilon|^4\, \theta \wedge d\theta = \int |\phi|^4\, \theta_0 \wedge d\theta_0 + O\left(\varepsilon^2 \log \frac{1}{\varepsilon}\right).$

ii) There exists a positive constant C independant of η such that:

$$\varphi_\varepsilon \geq C\varepsilon.$$

Proof: i)

$$\int \big(4\,|d\varphi_\varepsilon|_\theta^2 + R\varphi_\varepsilon^2\big)\theta\wedge d\theta = \int L\varphi_\varepsilon\varphi_\varepsilon\theta\wedge d\theta = \int f_\varepsilon^3\varphi_\varepsilon\theta\wedge d\theta$$
$$= \int f_\varepsilon^4\theta\wedge d\theta + O(\varepsilon\int f_\varepsilon^3 H_\varepsilon\theta\wedge d\theta).$$

Observe also that

$$\int |\varphi_\varepsilon|^4\,\theta\wedge d\theta = \int f_\varepsilon^4\theta\wedge d\theta + O\big(\varepsilon\int \varphi_\varepsilon^3 H_\varepsilon\theta\wedge d\theta\big).$$

We need therefore only to estimate $\int f_\varepsilon^4\theta\wedge d\theta$, $\varepsilon\int \varphi_\varepsilon^3 H_\varepsilon\theta\wedge d\theta$ and $\varepsilon\int f_\varepsilon^3 H_\varepsilon\theta\wedge d\theta$.

Since we have, $(\theta\wedge d\theta)_{(4)} = \theta_0\wedge d\theta_0$, and $(\theta\wedge d\theta)_{(5)} = 0$, Proposition 2.5 of [2] implies that in pseudohermitian normal coordinates (z, t) we have:

$$\theta\wedge d\theta = (1 + O(\rho^2))\theta_0\wedge d\theta_0$$

where $\rho(z, t) = (|z|^4 + t^2)^{1/2}$ and $O(\rho^2)$ is a polynomial in ρ in which the terms are of order at least 2, and the coefficients are expressions of pseudohermitian curvature and torsion and their covariant derivatives. Then:

$$\int \big|f_\varepsilon^4\big|\theta\wedge d\theta = \int_{B(0,2r)} |\psi|^4|\phi_\varepsilon|^4(1 + O(\rho^2))\theta_0\wedge d\theta_0$$
$$= \int_{B(0,r)} |\phi_\varepsilon|^4\theta_0\wedge d\theta_0 + O(\int_0^r |\phi_\varepsilon|^4\rho^5 d\rho)$$
$$+ \int_{B(0,2r)-B(0,r)} |\psi|^4|\phi_\varepsilon|^4\theta_0\wedge d\theta_0$$
$$+ O(\int_{B(0,2r)-B(0,r)} |\psi|^4|\phi_\varepsilon|^4\rho^5 d\rho).$$

We have:

$$O(\int_0^r |\phi_\varepsilon|^4 d\rho) = O(\int_0^{r/\varepsilon} |\phi|^4\,\varepsilon^2\rho^5 d\rho) = O(\int_0^{r/\varepsilon} (1+\rho)^{-8}\varepsilon^2\rho^5 d\rho) = O(\varepsilon^2)$$

$$O(\int_{B(0,2r)-B(0,r)} |\psi|^4|\phi_\varepsilon|^4\rho^5 d\rho) = (O\int_{r/\varepsilon}^{2r/\varepsilon} (1+\rho)^{-8}\varepsilon^2\rho^2\rho^3 d\rho) = O(\varepsilon^2)$$

$$O(\int_{B(0,2r)-B(0,r)} |\psi|^4|\phi_\varepsilon|^4\theta_0\wedge d\theta_0 = O(\int_{r/\varepsilon}^{2r/\varepsilon} (1+\rho)^{-8}\rho^3 d\rho) = O(\varepsilon^4)$$

and finally:

$$\int_{B(0,r)} |\phi_\varepsilon|^4 \theta_0 \wedge d\theta_0 = \int_{\mathbb{H}^1} |\phi_\varepsilon|^4 \theta_0 \wedge d\theta_0 - O(\int_r^{+\infty} |\phi_\varepsilon|^4)\theta_0 \wedge d\theta_0$$

$$O(\int_{r/\varepsilon}^{+\infty} |\phi|^4 \theta_0 \wedge d\theta_0) = O(\int_{r/\varepsilon}^{+\infty} (1+\rho)^{-8} \rho^3 d\rho) = O(\varepsilon^4).$$

We estimate now, for example:

$$\varepsilon \int \varphi_\varepsilon^3 H_\varepsilon \theta \wedge d\theta \leq C\varepsilon \int \varphi_\varepsilon^3 (1 + |\log(\varepsilon^2 + d^2)|)\theta \wedge d\theta$$
$$\leq C\varepsilon \log \frac{1}{\varepsilon} \int \varphi_\varepsilon^3 \theta \wedge d\theta.$$

Clearly, after the estimates on H_ε, we can derive that $\int \varphi_\varepsilon^3 \theta \wedge d\theta$ behaves like $\int f_\varepsilon^3 \theta \wedge d\theta + O(\varepsilon)$ which behaves like $\int \phi_\varepsilon^3 \theta_0 \wedge d\theta_0 + O(\varepsilon) = O(\varepsilon)$. Thus, $\varepsilon \int \varphi_\varepsilon^3 H_\varepsilon \theta \wedge d\theta = O(\varepsilon^2 \log \frac{1}{\varepsilon})$.

ii) We have:

$$L\varphi_\varepsilon = f_\varepsilon^3$$

and $G_q \geq \gamma > 0$ where G_q is the Green function associated to the conformal Laplacian $L = 4\Delta_b + R$ on q (for a proof one can see the Appendix Lemma A.1) thus

$$\varphi_\varepsilon \geq \gamma \int_M f_\varepsilon^3 \theta \wedge d\theta \geq C' \frac{\gamma\varepsilon}{2} > 0.$$

Lemma 3: *There exist two positive constants α and β depending on η such that, for $q \in M$ and $\varepsilon > 0$ small enough, we have:*

$$\alpha \tilde{f}_\varepsilon \leq \varphi_\varepsilon \leq \beta \tilde{f}_\varepsilon.$$

Proof:

$$\varphi_\varepsilon = f_\varepsilon + \varepsilon H_\varepsilon = \tilde{f}_\varepsilon + \varepsilon \tilde{H}_\varepsilon$$
$$= \tilde{f}_\varepsilon + \varepsilon O(\inf(C, C'|\log(\varepsilon^2 + d^2)|))$$

If $\varepsilon^2 + d^2$ is small ($\varepsilon^2 + d^2 < r_0, 0 < r_0 < r$), $\varphi_\varepsilon \geq \frac{\tilde{f}_\varepsilon}{2}$.

Otherwise:

$$\varepsilon^2 + d^2 \geq r_0.$$

By Lemma 2, we know that: $\varphi_\varepsilon \geq C\frac{\gamma\varepsilon}{2}$ and $\tilde{f}_\varepsilon \leq C'\varepsilon$. Thus $\varphi_\varepsilon \geq \frac{C\gamma}{2C'} \tilde{f}_\varepsilon$.

On the other hand, we can write, $\varepsilon\tilde{H} < C_1\frac{\varepsilon}{\varepsilon^2+d^2}$ and derive the existence of β such that $\varphi_\varepsilon \le \beta\tilde{f}_\varepsilon$. q.e.d.

Notations: For every $a \in M$, indexing the function ϕ, w, ψ, ϕ_ε, f_ε, $\tilde{f}_\varepsilon$, H_ε and $\tilde{H}_\varepsilon$ by a means that we consider the pseudohermitian normal coordinates (z,t) near the base point a.

Theorem 2: *For every $\eta > 0$ there exists a constant $C(\eta)$, such that for every a and b in M, for every $0 < \varepsilon \le \frac{1}{2}$ we have:*

$$\int \varphi^3_{a,\varepsilon}\varphi_{b,\varepsilon} \ge (1 - C(\eta)\varepsilon^{1/3}) \int (\tilde{f}_{a,\varepsilon})^3\varphi_{b,\varepsilon}.$$

Proof of Theorem 2: We have:

$$\int (L\varphi_{a,\varepsilon})\varphi_{b,\varepsilon}\theta \wedge d\theta = \int (f_{a,\varepsilon})^3\varphi_{b,\varepsilon}\theta \wedge d\theta \le \int (\tilde{f}_{a,\varepsilon})^3\varphi_{b,\varepsilon}\theta \wedge d\theta,$$

and $\tilde{f}_{a,\varepsilon} = \varphi_{a,\varepsilon} - \varepsilon\tilde{H}_{a,\varepsilon}$ (see (3.4), (3.10)).

Then:

$$(\tilde{f}_{a,\varepsilon})^3 \le \varphi^3_{a,\varepsilon} + C\big(\varphi^2_{a,\varepsilon}\varepsilon|\tilde{H}_{a,\varepsilon}| + (\varepsilon|\tilde{H}_{a,\varepsilon}|)^3\big).$$

On the other hand,

$$\varphi^2_{a,\varepsilon}\varepsilon|\tilde{H}_\varepsilon| \le c(\tilde{f}_{a,\varepsilon})^2\varepsilon|\tilde{H}_{a,\varepsilon}|$$

and

$$\varepsilon|\tilde{H}_{a,\varepsilon}| \le C\tilde{f}_{a,\varepsilon} \text{ which implies } (\varepsilon|\tilde{H}_{a,\varepsilon}|)^2 \le C(\tilde{f}_{a,\varepsilon})^2$$

and

$$(\varepsilon|\tilde{H}_{a,\varepsilon}|)^3 \le C(\tilde{f}_{a,\varepsilon})^2\varepsilon|\tilde{H}_{a,\varepsilon}|.$$

This yields:

$$(\tilde{f}_{a,\varepsilon})^3 \le \varphi^3_{a,\varepsilon} + C\tilde{f}^2_{a,\varepsilon}\varepsilon|\tilde{H}_{a,\varepsilon}|$$

and

$$\int (L\varphi_{a,\varepsilon})\varphi_{b,\varepsilon} \le \int \varphi^3_{a,\varepsilon}\varphi_{b,\varepsilon} + C(\eta)\varepsilon \int (\tilde{f}_{a,\varepsilon})^2\,\tilde{f}_{b,\varepsilon}|\tilde{H}_{a,\varepsilon}|\theta \wedge d\theta$$

Let

$$R = \varepsilon \int (\tilde{f}_{a,\varepsilon})^2\,\tilde{f}_{b,\varepsilon}|\tilde{H}_{a,\varepsilon}|\theta \wedge d\theta. \tag{3.11}$$

We break R in two pieces: on one hand, the contribution for $d(x,a) < 2r$. On the other hand, the contribution when $d(x,a) \ge 2r$. This second term can be upperbounded as follows:

On the complement of $B(a,2r)$, $|\tilde{H}_{a,\varepsilon}|$ is bounded and $\tilde{f}_{a,\varepsilon}$ is bounded by ε.

Therefore:

$$\varepsilon \int_{\rho \geq 2r} \tilde{f}_{a,\varepsilon}^2 \tilde{f}_{b,\varepsilon} |\tilde{H}_{a,\varepsilon}| \theta \wedge d\theta \leq C\varepsilon^3 \int_{\rho \geq 2r} \tilde{f}_{b,\varepsilon} \theta \wedge d\theta$$
$$\leq C\varepsilon^3 \int_M \tilde{f}_{b,\varepsilon} \leq C\varepsilon^4.$$

Indeed,

$$\int_M \tilde{f}_{b,\varepsilon} = \int_{^c B(b,2r)} \tilde{f}_{b,\varepsilon} + \int_{B(b,2r)} \tilde{f}_{b,\varepsilon} \leq c\varepsilon + \int_{\rho \geq \frac{2r}{\varepsilon}} \frac{\varepsilon^5}{\varepsilon^2} \frac{\rho^3 d\rho}{\rho^2} \leq c\varepsilon,$$

we also know that

$$\int f_{a,\varepsilon}^3 \varphi_{b,\varepsilon} \geq c\varepsilon \int f_{a,\varepsilon}^3 \geq c\varepsilon^2.$$

Hence

$$\varepsilon \int_{\rho \geq 2r} \tilde{f}_{a,\varepsilon}^2 \tilde{f}_{b,\varepsilon} |\tilde{H}_{a,\varepsilon}| \theta \wedge d\theta \leq \varepsilon^2 \int_M f_{a,\varepsilon}^3 \varphi_{b,\varepsilon} \theta \wedge d\theta.$$

We now study the first term:

We have:

$$c_1 \frac{\varepsilon}{|w_a + i\varepsilon^2|} \leq \tilde{f}_{a,\varepsilon} \leq c_2 \frac{\varepsilon}{|w_a + i\varepsilon^2|}$$

where $c_1, c_2 > 0$. Thus, on $B(a, 2r)$, we can replace $\tilde{f}_{a,\varepsilon}$ by $\phi_{a,\varepsilon}$ and $\tilde{H}_{a,\varepsilon}$ by $(1 + \log |\frac{\phi_{a,\varepsilon}}{\varepsilon}|) c(\eta)$. We know that: $\tilde{f}_{b,\varepsilon} \leq \psi_b \varphi_{b,\varepsilon} + c\varepsilon$.

Thus,

$$R \leq c\varepsilon^2 \int f_{a,\varepsilon}^3 \varphi_{b,\varepsilon} \theta \wedge d\theta + c\varepsilon^2 \int_{B(a,2r)} \phi_{a,\varepsilon}^2 (1 + \log |\varepsilon^{-1} \phi_{a,\varepsilon}|)$$
$$+ \; c\varepsilon \int_{B(a,2r) \cap B(b,2r)} \phi_{a,\varepsilon}^2 \phi_{b,\varepsilon} \inf(\log(1 + |\varepsilon^{-1} \phi_{a,\varepsilon}|, 1)$$

Let us denote by (I) the second integral and by (II) the third integral in the right hand side:

$$\text{(I)} \quad \leq c\varepsilon^2 \varepsilon^2 \int_0^{r/\varepsilon} \phi_a^2 \left(1 + \log \frac{1}{\varepsilon^2} + \log |\phi_a|\right) \rho^3 d\rho$$
$$\leq c\varepsilon^4 \log \varepsilon^{-1} \int_0^{r/\varepsilon} \phi_a^2 \rho^3 d\rho = c\varepsilon^4 \log \varepsilon^{-1}.$$

Hence

$$\text{(I)} \quad \leq c(\varepsilon^2 \log \varepsilon^{-1}) \int_M \tilde{f}_{a,\varepsilon}^3 \varphi_{b,\varepsilon} \theta \wedge d\theta.$$

We are left with (II).

On the domain $A = \{x \ / \ \phi_{a,\varepsilon}(x) \geq \varepsilon^{2/3}, d(x,a) < 2r$ and $d(x,b) < 2r\}$ we have:

$$\text{(II)} \quad \leq \frac{\varepsilon}{\varepsilon^{2/3}} \int_{\mathbb{H}^1} \phi^3_{a,\varepsilon} \ \phi_{b,\varepsilon} \ \theta_0 \wedge d\theta_0 = \varepsilon^{1/3} \int_{\mathbb{H}^1} \phi^3_{a,\varepsilon} \ \phi_{b,\varepsilon} \ \theta_0 \wedge d\theta_0.$$

On ${}^c\!A$, we have:

$$\begin{aligned} \varepsilon \int_{{}^c\!A} \phi^2_{a,\varepsilon} \phi_{b,\varepsilon} &\leq \varepsilon \varepsilon^{4/3} \int_{{}^cA} \phi_{b,\varepsilon} \theta_0 \wedge d\theta_0 \leq c\varepsilon^2 \times \varepsilon^{4/3} \\ &\leq c\varepsilon^{4/3} \int \tilde{f}^3_{a,\varepsilon} \varphi_{b,\varepsilon} \theta \wedge d\theta. \end{aligned}$$

Clearly,

$$\int \varphi^3_{a,\varepsilon} \varphi_{b,\varepsilon} \theta \wedge d\theta \geq c\varepsilon^2.$$

If $d(a,b) < r/2$, we also can see that:

$$\int \varphi^3_{a,\varepsilon} \varphi_{b,\varepsilon} \geq \frac{1}{2} \int_{\mathbb{H}^1} \phi^3_{a,\varepsilon} \phi_{b,\varepsilon}$$

since

$$\int_{d(a,x) \geq r} \phi^3_{a,\varepsilon} \ \phi_{b,\varepsilon} \leq c\varepsilon^4 \leq c' \varepsilon^2 \int \varphi^3_{a,\varepsilon} \varphi_{b,\varepsilon}.$$

Indeed, if $d(a,b) \leq r/2$, r small,

$$\int \varphi^3_{a,\varepsilon} \varphi_{b,\varepsilon} \theta \wedge d\theta \geq c_1 \int_{d(a,x) \leq r} \phi^3_{a,\varepsilon} \phi_{b,\varepsilon} \theta_0 \wedge d\theta_0.$$

If, on the other hand, $d(a,b) \geq r/2$, then

$$\begin{aligned} \int_{\mathbb{H}^1} \phi^3_{a,\varepsilon} \phi_{b,\varepsilon} \theta_0 \wedge d\theta_0 &\leq \int_{d(a,x) \leq r/4} \phi^3_{a,\varepsilon} \phi_{b,\varepsilon} + \int_{d(b,x) \leq r/4} \phi^3_{a,\varepsilon} \phi_{b,\varepsilon} \\ &\leq c\varepsilon \int \phi^3_{a,\varepsilon} + c\varepsilon^3 \int \phi_{b,\varepsilon} \\ &\leq c\varepsilon^2 \leq c \int \varphi^3_{a,\varepsilon} \varphi_{b,\varepsilon}. \end{aligned}$$

Therefore we see that:

$$\int_{H^1} \phi^3_{a,\varepsilon} \phi_{b,\varepsilon} \theta_0 \wedge d\theta_0 \leq c \int \varphi^3_{a,\varepsilon} \varphi_{b,\varepsilon} \theta \wedge d\theta.$$

Hence

$$R \leq c\varepsilon \int \tilde{f}^3_{a,\varepsilon} \varphi_{b,\varepsilon} \theta \wedge d\theta + c\varepsilon^{1/3} \int \varphi^3_{a,\varepsilon} \varphi_{b,\varepsilon} \theta \wedge d\theta.$$

Thus

$$R \leq c' \varepsilon^{1/3} \int \tilde{f}^3_{a,\varepsilon} \varphi_{b,\varepsilon} \theta \wedge d\theta.$$

The proof of Theorem 2 is thereby complete.

Section 3

Expansion of the functional at infinity

G.B. Folland and E.M. Stein have introduced functional spaces for CR manifolds analogous to Sobolev spaces for Riemannian manifolds, $S_k^p(M)$ called Folland and Stein spaces [8].

Let $H = \{u \in S_1^2(M) \text{ s.t} \int_M |du|_\theta^2 \theta \wedge d\theta < \infty\,, \int u^4 \theta \wedge d\theta < \infty\}$.

$\sum = \{u \in H, \text{ s.t } ||u||_H = 1\}$ where $||u||_H = \Big(\int_M \big(4|du|_\theta^2 + Ru^2\big)\theta \wedge d\theta\Big)^{1/2}$ and $\sum_+ = \{u \in \sum \text{ s.t } u \geq 0\}$.

For $u \in H$, we define the following functional:

$$J(u) = \frac{\displaystyle\int Luu\theta \wedge d\theta}{(\displaystyle\int u^4 \theta \wedge d\theta)^{1/2}} = \frac{N}{D}$$

let

$$I = \frac{1}{p} \sum_{i \neq j} \int \varphi_{a_i}^3 \varphi_{a_j} \theta \wedge d\theta \tag{3.12}$$

and S the Sobolev constant for $\mathbb{H}^1$ defined by

$$S = \frac{\displaystyle\int_{\mathbb{H}^1} 4\,|Z\phi|^2\,\theta_0 \wedge d\theta_0}{\Big(\displaystyle\int_{\mathbb{H}^1} \phi^4 \theta_0 \wedge d\theta_0\Big)^{1/2}}$$

we then have:

Proposition 2: *For every p, and $\alpha_1, \cdots, \alpha_p$, for every $\varepsilon \leq \frac{1}{p^{100}}$*

$$J(\sum_{i=1}^p \alpha_i \varphi_{a_i,\varepsilon}) \leq (p+1)^{1/2} S$$

Remark: The aim of Sect. 3 is to improve the estimate provided by Proposition 2 so that, under some condition on p, we will derive that $J(\sum_{i=1}^p \alpha_i \varphi_{a_i,\varepsilon})$ is in fact upperbounded by $p^{1/2}S$.

Proof of Proposition 2: Let

$$\begin{aligned} N &= \int_M \big(\sum \alpha_i f_{a_i,\varepsilon}^3\big) \times (\sum \alpha_j \varphi_{a_j,\varepsilon}) = \int_M \big(\sum \alpha_i \varphi_{a_i,\varepsilon}^3\big)(\sum \alpha_j \varphi_{a_j,\varepsilon}) \\ &+ \int_M \sum_i \alpha_i \big(f_{a_i,\varepsilon}^3 - \varphi_{a_i,\varepsilon}^3\big)(\sum_j \alpha_j \varphi_{a_j,\varepsilon}) \\ &= \Big(\int_M \big(\sum \alpha_i \varphi_{a_i,\varepsilon}^3\big)(\sum \alpha_j \varphi_{a_j,\varepsilon})\Big)(1+R) \end{aligned}$$

where R satisfies:

$$|R| \le \frac{\int_M \big(\sum \alpha_i |f^3_{a_i,\varepsilon} - \varphi^3_{a_i,\varepsilon}|\big)(\sum_j \alpha_j \varphi_{a_j,\varepsilon})}{\int_M \big(\sum \alpha_i^2 \varphi^4_{a_i,\varepsilon}\big) + \int_M \big(\sum_{i\neq j} \alpha_i\alpha_j \varphi^3_{a_i,\varepsilon}\varphi_{a_j,\varepsilon}\big)} \; . \tag{3.13}$$

Since we have

$$\begin{aligned} \big|f^3_{a_i,\varepsilon} - \varphi^3_{a_i,\varepsilon}\big| &\le \big|\tilde f^3_{a_i,\varepsilon} - \varphi^3_{a_i,\varepsilon}\big| \le \big|\tilde f_{a_i\varepsilon} - \varphi^3_{a_i,\varepsilon}\big| \tilde f^2_{a_i,\varepsilon} \\ &\le \tilde f^2_{a_i,\varepsilon} |\tilde H_{a_i,\varepsilon}| \times \varepsilon \; . \end{aligned}$$

Thus

$$|R| \le \frac{\varepsilon \int_M \big(\sum \alpha_i \tilde f^2_{a_i,\varepsilon} |\tilde H_{a_i,\varepsilon}|\big) \sum_i \alpha_j \varphi_{a_j,\varepsilon}}{\sum \alpha_i^2 \big(\int_M \varphi^4_{a_i,\varepsilon} \theta \wedge d\theta\big) + \sum_{i\neq j} \dfrac{\alpha_i\alpha_j \int \varphi^3_{a_i,\varepsilon}\varphi_{a_j,\varepsilon}}{\sum \alpha_i^2}}$$

We know that:

$$\int \varphi^4_{a_i,\varepsilon} = S^2 + O(\varepsilon^4)(\text{ and } S > 1) \text{ and } \varphi_{a_j,\varepsilon} \le c \tilde f_{a_j,\varepsilon}.$$

Thus:

$$|R| \le \varepsilon \frac{c \int_M \big(\sum \alpha_i \tilde f^2_{a_i,\varepsilon} |\tilde H_{a_i,\varepsilon}|\big)(\sum_j \alpha_j \tilde f_{a_i,\varepsilon})}{\sum \alpha_i^2 (1 + I_1)}$$

where

$$I_1 = \frac{\sum_{i\neq j} \alpha_i\alpha_j \int \varphi^3_{a_i,\varepsilon}\varphi_{a_j,\varepsilon} \theta \wedge d\theta}{\sum_{i=1}^p \alpha_i^2}. \tag{3.14}$$

We derive from Theorem 2 that

$$\begin{aligned} \varepsilon \sum_{i\neq j} \alpha_i\alpha_j \int_M \tilde f^2_{a_i,\varepsilon} \tilde f_{a_j,\varepsilon} |\tilde H_{a_i,\varepsilon}| &\le C \sum_{i\neq j} \alpha_i\alpha_j \varepsilon^{1/3} \int_M \tilde f^3_{a_i,\varepsilon} \varphi_{a_j,\varepsilon} \\ &\le C' \varepsilon^{1/3} \sum_{i\neq j} \alpha_i\alpha_j \int_M \varphi^3_{a_i,\varepsilon}\varphi_{a_j,\varepsilon} \\ &\le I_1 O(\varepsilon^{1/3})(\sum \alpha_i^2) \end{aligned}$$

and

$$\varepsilon \int_M \tilde f^3_{a_i,\varepsilon} |\tilde H_{a_i,\varepsilon}| \le C\varepsilon \int_M \tilde f^3_{a_i,\varepsilon} = O(\varepsilon^2).$$

We thus have:

$$N \leq \Big(\sum_i \alpha_i \int \varphi^3_{a_i,\varepsilon}\Big)\Big(\sum \alpha_j \int_M \varphi_{a_j,\varepsilon}\Big) \cdot$$
$$\cdot \left(1 + \frac{O(\varepsilon^2)\big(\sum \alpha_i^2\big) + I_1 O(\varepsilon^{1/3}) \times \big(\sum \alpha_i^2\big)}{(1+I_1)\sum \alpha_i^2}\right)$$

since $O(\varepsilon^{1/3})$ upperbounds $O(\varepsilon^2)$ we have

$$\left(1 + \frac{O(\varepsilon^2)}{1+I_1} + \frac{I_1 O(\varepsilon^{1/3})}{1+I_1}\right) \leq 1 + O(\varepsilon^{1/3})$$

Thus

$$J\Big(\sum_{i=1}^p \alpha_i \varphi_{a_i,\varepsilon}\Big) \leq \int_M \frac{\big(\sum_{i=1}^p \alpha_i \varphi^3_{a_i,\varepsilon}\big)\big(\sum_{j=1}^p \alpha_j \varphi_{a_j,\varepsilon}\big)\theta \wedge d\theta}{\|\sum_{i=1}^p \alpha_i \varphi_{a_i,\varepsilon}\|_4^2}(1 + O(\varepsilon^{1/3})).$$

On the other hand, we have the following result:

Lemma A.3 Appendix 2 [5]:

$$\int_M \frac{\big(\sum \alpha_i \varphi^3_{a_i,\varepsilon}\big)\big(\sum \alpha_j \varphi_{a_j,\varepsilon}\big)\theta \wedge d\theta}{\|\sum_i \alpha_i \varphi_{a_i,\varepsilon}\|_4^2} \leq \left(\sum_i \int \frac{\alpha_i \varphi_{a_i,\varepsilon}}{\sum \alpha_k \varphi_{a_k,\varepsilon}} \varphi^4_{a_i,\varepsilon}\right)^{1/2}.$$

Thus

$$\int_M \frac{\sum \alpha_i \varphi_{a_i,\varepsilon}}{\sum \alpha_j \varphi_{a_i,\varepsilon}} \varphi^4_{a_i,\varepsilon} \leq \sum_i \int \varphi^4_{a_i,\varepsilon} = p(S^2 + O(\varepsilon^4)).$$

Hence

$$J\Big(\sum_{i=1}^p \alpha_i \varphi_{a_i,\varepsilon}\Big) \leq p^{1/2} S(1 + O(\varepsilon^{1/3})).$$

The condition on ε implies the existence of a constant η such that

$$O(\varepsilon^{1/3}) \leq \frac{1}{10p^{1/2}}.$$

The result follows.

We now have:

Proposition 3: *There exists $\nu_0 \in (0, 1)$, $\gamma_0 > 0$, $\beta > 0$ such that for every p and for every $\alpha_i > 0$ such that $\frac{\alpha_i}{\alpha_j} \in (1 - \nu_0, 1 + \nu_0)$, for every $0 < \varepsilon \leq 1$, for every $a_1, \cdots, a_p \in M$ satisfying:*

$$I = \frac{1}{p} \sum_{i \neq j} \int \varphi^3_{a_i,\varepsilon} \varphi_{a_j,\varepsilon} \theta \wedge d\theta < \beta$$

we have:

$$J(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\varepsilon}) \le \frac{\sum\alpha_i^2}{\left(\sum\alpha_i^4\right)^{1/2}}S\left[(1+O(\varepsilon^2\log\varepsilon^{-1})-\gamma_0\frac{I}{S^2}(1+O(\varepsilon^{1/3}))\right].$$

Proof of Proposition 3: Let us denote N the numerator of $J(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\varepsilon})$. It is equal to

$$\begin{aligned}N &= \int\left(\sum\alpha_i f_{a_i,\varepsilon}^3\right)\left(\sum\alpha_j\varphi_{a_j,\varepsilon}\right)\theta\wedge d\theta\\ &= \sum\alpha_i^2\int f_{a_i,\varepsilon}^3\varphi_{a_i,\varepsilon} + \sum_{i\neq j}\alpha_i\alpha_j\int f_{a_i,\varepsilon}^3\varphi_{a_j,\varepsilon}.\end{aligned}$$

Using the estimates of Sect. 2, we give the following estimate, for the second term of the right hand side

$$\begin{aligned}\sum_{i\neq j}\alpha_i\alpha_j\int f_{a_i,\varepsilon}^3\varphi_{a_j,\varepsilon} &\le \sum_{i\neq j}\alpha_i\alpha_j\int \tilde{f}_{a_i,\varepsilon}^3\varphi_{a_j,\varepsilon}\\ &\le \sum_{i\neq j}\alpha_i\alpha_j(1-c(\eta)\varepsilon^{1/3})^{-1}\int\varphi_{a_i,\varepsilon}^3\varphi_{a_j,\varepsilon}.\end{aligned}$$

We also know that:

$$\int f_{a_i,\varepsilon}^3\varphi_{a_i,\varepsilon} = \int L\varphi_{a_i,\varepsilon}\varphi_{a_i,\varepsilon} = S^2 + O(\varepsilon^2\log\varepsilon^{-1}).$$

Therefore

$$N \le \sum\alpha_i^2(S^2+O(\varepsilon^2\log\varepsilon^{-1}))\left[1+\frac{\sum_{i\neq j}\alpha_i\alpha_j(1-c(\eta)\varepsilon^{1/3})^{-1}\int\varphi_{a_i,\varepsilon}^3\varphi_{a_j,\varepsilon}}{(1+O(\varepsilon^2\log\varepsilon^{-1}))S^2\left(\sum\alpha_i^2\right)}\right].$$

Since $\frac{\alpha_i}{\alpha_j}\in(1-\nu_0,1+\nu_0)$, we have:

$$\frac{\alpha_i}{\sum\alpha_i^2} \le \frac{1+\nu_0}{\sqrt{p}}. \tag{3.15}$$

Thus

$$N \le \sum_{i=1}^{p}\alpha_i^2 S^2(1+O(\varepsilon^2\log\varepsilon^{-1}))\left[1+\frac{1+O(\varepsilon^{1/3})}{S^2}(1+\nu_0)^2 I\right]$$

we now turn to estimate the denominator:

$$D = \left(\int_M\left(\sum\alpha_i\varphi_{a_i,\varepsilon}\right)^4\theta\wedge d\theta\right)^{1/2}.$$

We will use the:

Lemma A.2 Appendix 2 [5]: *Let $s > 2$ be given, there exists $\gamma > 1$ such that for any $r_1, \cdots, r_p > 0$, we have:*

$$(\sum_{i=1}^{p} r_i)^s \geq \sum_{i=1}^{p} (r_i)^s + \gamma \frac{s}{2} \sum_{i \neq j} (r_i)^{s-1} r_j.$$

Using Lemma A.2 [5], we have: ($s = 4$)

$$D^2 \geq \int \sum \alpha_i^4 \varphi_{a_i,\varepsilon}^4 \theta \wedge d\theta + 2\gamma \sum_{i \neq j} \int \alpha_i^3 \alpha_j \varphi_{a_i,\varepsilon}^3 \theta \wedge d\theta.$$

Using again the estimate:

$$\int_M \varphi_{a_i,\varepsilon}^4 \theta \wedge d\theta = S^2 + O(\varepsilon^4)$$

we have:

$$D^2 \geq \sum_i \alpha_i^4 (S^2 + O(\varepsilon^4)) \left(1 + 2\gamma \frac{(1 - \nu_0)^4}{S^2 + 0(\varepsilon^4)} \sum_{i \neq j} \int \varphi_{a_i,\varepsilon}^3 \varphi_{a_j,\varepsilon} \theta \wedge d\theta \right).$$

Therefore:

$$D \geq \big(\sum_i \alpha_i^4 \big)^{1/2} S(1 + O(\varepsilon^2)) \left(1 + 2\gamma (1 - \nu_0)^4 \frac{I}{S^2 (1 + O(\varepsilon^4))} \right)^{1/2}$$

and

$$J(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\varepsilon}) \leq \frac{\sum_{i=1}^{p} \alpha_i^2 S^2 (1 + O(\varepsilon^2 \log \varepsilon^{-1})) \big[1 + (1 + O(\varepsilon^{1/3}))(1 + \nu_0)^2 \frac{I}{S^2} \big]}{\big(\sum_{i=1}^{p} \alpha_i^4 \big)^{1/2} S(1 + O(\varepsilon^2)) \big[1 + 2\gamma (1 - \nu_0)^4 (1 + O(\varepsilon^4)) \frac{I}{S^2} \big]^{1/2}}.$$

Thus

$$J(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\varepsilon}) \leq \frac{\sum_{i=1}^{p} \alpha_i^2 S(1 + O(\varepsilon^2 \log \varepsilon^{-1}) \big[1 + (1 + O(\varepsilon^{1/3}))(1 + \nu_0)^2 \frac{I}{S^2} \big]}{\big(\sum \alpha_i^4 \big)^{1/2} \big[1 + 2\gamma (1 - \nu_0)^4 (1 + O(\varepsilon^{1/3})) \frac{I}{S^2} \big]^{1/2}}.$$

If u is a small number, we have:

$$\frac{1}{(1 + u)^{1/2}} \leq 1 - \frac{1}{2} u.$$

Therefore, if $I \leq \beta$ small enough, we have:

$$J(\sum \alpha_i \varphi_{a_i,\varepsilon}) \leq \frac{\sum \alpha_i^2}{\big(\sum \alpha_i^4 \big)^{1/2}} S(1 + O(\varepsilon^2 \log \varepsilon^{-1})) \left(1 - (1 + O(\varepsilon^{1/3})) \gamma_0 \frac{I}{S^2} \right)$$

where:

$$\gamma_0 = \gamma (1 - \nu_0)^4 - (1 + \nu_0)^2.$$

Since γ is larger than 1, we choose ν_0 small enough so that:

$$\gamma_0 > 0.$$

We now have:

Proposition 4: *For every $\nu_0 \in (0,1)$ there exists $\gamma_1' > 0$ such that, for every p and for every p-tuple of α_i's > 0, $i \in \{1, 2, ..., p\}$ such that $\frac{\alpha_i}{\alpha_j} \in (1-\nu_0, 1+\nu_0)$ for every (i, j), for every $\varepsilon \leq 1$, for every $a_1, \cdots, a_p \in M$, we have:*

$$J(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\varepsilon}) \leq p^{1/2} S\left(1 - \gamma_1' \frac{I}{p}(1 + O(\varepsilon^{1/3}))\right)(1 + O(\varepsilon^2 \log \varepsilon^{-1})).$$

Proof: We start the proof as for Proposition 3, we have:

$$J(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\varepsilon}) \leq \frac{\sum \alpha_i^2}{\left(\sum \alpha_i^4\right)^{\frac{1}{2}}} S(1 + O(\varepsilon^2 \log \varepsilon^{-1})) \frac{\left[1 + (1 + O(\varepsilon^{1/3}))(1+\nu_0)^2 \frac{I}{S^2}\right]}{\left[1 + (1 + O(\varepsilon^{1/3}))2\gamma(1-\nu_0)^4 \frac{I}{S^2}\right]^{1/2}}$$

we have two cases:

1. If $I < \beta$ small enough, then by Proposition 3, we have:

$$\begin{aligned} J(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\varepsilon}) &\leq \frac{\sum \alpha_i^2}{\left(\sum \alpha_i^4\right)^{1/2}} S(1 + O(\varepsilon^2 \log \varepsilon^{-1}))\left(1 - \gamma_1' I(1 + O(\varepsilon^{1/3}))\right) \\ &\leq \frac{\sum \alpha_i^2}{\left(\sum \alpha_i^4\right)^{1/2}} S(1 + O(\varepsilon^2 \log \varepsilon^{-1}))\left(1 - \gamma_1' \frac{I}{p}(1 + O(\varepsilon^{1/3}))\right). \end{aligned}$$

We note that $\gamma(1-\nu_0)^4 \geq (1+\nu_0)^2 + \theta_1$, where $\theta_1 > 0$ is a fixed constant if ν_0 is small enough ($\gamma > 1$).

2. If $I \geq \beta$. Then, expanding, we have:

$$J(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\varepsilon}) \leq \frac{\sum \alpha_i^2}{\left(\sum \alpha_i^4\right)^{1/2}} S(1 + O(\varepsilon^2 \log \varepsilon^{-1}))(1 - c(\beta)).$$

We would like to have:

$$\gamma_1' \frac{I}{p} < c(\beta).$$

Observe that $\frac{I}{p} \leq C$, where C is independent of p. Thus, we have to choose γ_1' such that $\gamma_1' \leq \frac{c(\beta)}{C}$.

Proposition 5: *Let $\nu_0 \in (0,1)$ be given. There exists a constant $C'(\nu_0)$ such that, for every $p \in N^*$, if $\varepsilon \leq \frac{C'(\nu_0)}{p^{100}}$, then, for every p-tuple of α_i's > 0, such that $\frac{\alpha_i}{\alpha_j} \in (1-\nu_0, 1+\nu_0)$ for every (i, j), and $\frac{\alpha_{i_0}}{\alpha_{j_0}} \notin (1 - \frac{\nu_0}{2}, 1 + \frac{\nu_0}{2})$ for a couple of indices (i_0, j_0), we have:*

$$J(\sum_{j=1}^{p} \alpha_i \varphi_{a_i,\varepsilon}) \leq p^{1/2} S.$$

Remark: The existence of two indices i_0, j_0 such that $\frac{\alpha_{i_0}}{\alpha_{j_0}} \notin (1-\frac{\nu_0}{2}, 1+\frac{\nu_0}{2})$ implies that $p \geq 2$.

For the proof of Proposition 5, we will use the following result

Lemma 6 [5]: *Given $\nu_0 \in (0,1)$, there exists $C(\nu_0) > 0$ such that , for every p-tuple of α_i's > 0, $i \in \{1, 2, \cdots, p\}$ satisfying $\frac{\alpha_{i_0}}{\alpha_{j_0}} \notin (1-\frac{\nu_0}{2}, 1+\frac{\nu_0}{2})$ for a couple of indices (i_0, j_0), then:*

$$\frac{\sum_{i=1}^{p} \alpha_i^2}{\left(\sum_{i=1}^{p} \alpha_i^4\right)^{1/2}} \leq p^{1/2}\left(1 - \frac{C(\nu_0)}{p}\right).$$

Proof of Proposition 5: The proof splits in two cases.

First case: $I < \beta$

We then use Proposition 3 and Lemma 6, Sect. 3 of [5]. We derive that:

$$J(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\varepsilon}) \leq p^{1/2} S\left(1 - \frac{C(\nu_0)}{p}\right)\left[1 + O(\varepsilon^2 \log \varepsilon^{-1}) - \gamma_0 \frac{I}{S^2}(1 + O(\varepsilon^{1/3}))\right]$$

$$J(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\varepsilon}) \leq p^{1/2} S\left(1 - \frac{C(\nu_0)}{p}\right)(1 + O(\varepsilon^2 \log \varepsilon^{-1})) \ .$$

The result follows in this case.

Second case: $I \geq \beta$

We then apply Proposition 4, we thus have:

$$J(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\varepsilon}) \leq p^{1/2} S\left(1 - \gamma_1' \frac{\beta}{S^2}(1 + O(\varepsilon^{1/3}))\right)(1 + O(\varepsilon^2 \log \varepsilon^{-1}))$$

$$J(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\varepsilon}) \leq p^{1/2} S \ \text{ if } \varepsilon < \frac{1}{p^{100}}.$$

The proof of Proposition 5 is thereby complete.

Proposition 6: *Let $\nu_0 \in (0,1)$ be given. For every $p \in N^*$ and for every $(\alpha_1, \cdots, \alpha_p)$ satisfying $\frac{\alpha_{i_0}}{\alpha_{j_0}} \notin (1 - \frac{\nu_0}{2}, 1 + \frac{\nu_0}{2})$ for one couple of indices (i_0, j_0). If $\varepsilon < p^{-100}$, we have:*

$$J(\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\varepsilon}) \leq p^{1/2} S.$$

To prove Proposition 6, we need the following two results.

Lemma 4: *There exists $\gamma_1 > 0$, such that for every $\alpha_i \geq 0$, $1 \leq i \leq p$, for every $p \in N^*$ and $\varepsilon \leq 1$*

$$J(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\varepsilon}) \leq \frac{\sum\alpha_i^2}{\left(\sum\alpha_i^4\right)^{1/2}}S(1+O(\varepsilon^2\log\varepsilon^{-1})+\gamma_1 I_1).$$

Where:

$$I_1 = \sum_{i\neq j}\frac{\alpha_i\alpha_j\int\varphi^3_{a_i,\varepsilon}\varphi_{a_j,\varepsilon}}{\sum\alpha_i^2}$$

The proof of Lemma 4 is similar to the proof of Lemma 7, Sect. 3 of [5] replacing O_n with $O(\varepsilon^2\log\varepsilon^{-1})$, n by 4 and S by S^2.

Lemma 5: *There exists $\gamma_1' > 0$, such that for every $\alpha_i \geq 0$, $i \in \{1, 2, \cdots, p\}$, for every p*

$$J(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\varepsilon}) \leq p^{1/2}S(1+O(\varepsilon^{1/3}))\left(1-\frac{\gamma_1'}{p^2}I_1\right)^{1/2}.$$

Proof of Lemma 5: The proof is similar of that Lemma 8, Sect. 3 of [5] replacing $(1+O_n')$ by $(1+O(\varepsilon^{1/3}))$, S by S^2 and n by 4, the only modification with respect to [5] occurs in the beginning when we come back to the proof of Proposition 2, which gives here:

$$\begin{aligned}J(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\varepsilon}) &\leq \int_M \frac{\left(\sum_i\alpha_i\varphi^3_{a_i,\varepsilon}\right)(\sum_j\alpha_j\varphi_{a_j,\varepsilon})\theta\wedge d\theta}{||\sum_{i=1}^p\alpha_i\varphi_{a_i,\varepsilon}||_4^2}(1+O(\varepsilon^{1/3}))\\ &\leq \left(\sum_i\int_M\frac{\alpha_i\varphi_{a_i,\varepsilon}}{\sum\alpha_k\varphi_{a_k,\varepsilon}}\varphi^4_{a_i,\varepsilon}\right)^{1/2}(1+O(\varepsilon^{1/3}))\end{aligned}$$

Proof of Proposition 6: We use Lemma 5, Proposition 6 follows if I_1 is larger than $C''(\nu_0)/p$. Indeed:

$$J(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\varepsilon}) \leq p^{1/2}S(1+O(p^{-100/3}))\times\left(1-\frac{C}{p^3}\right) < p^{1/2}S.$$

We thus assume in the sequel that: $I_1 \leq C''(\nu_0)/p$. We then apply Lemma 6, Sect. 3 of [5] and Lemma 4.

By Lemma 4, we have

$$J(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\varepsilon}) \leq \frac{\sum\alpha_i^2}{\left(\sum\alpha_i^4\right)^{1/2}}S(1+O(\varepsilon^2\log\varepsilon^{-1})+\gamma_1 I_1)$$

and by Lemma 6, Sect. 3 of [5], we have:

$$J(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\varepsilon}) \leq p^{1/2}S\left(1-\frac{C(\nu_0)}{p}\right)(1+O(\varepsilon^2\log\varepsilon^{-1})+\gamma_1 I_1)$$
$$\leq p^{1/2}S\left(1-\frac{C(\nu_0)}{p}\right)\left(1+O(\varepsilon^2\log\varepsilon^{-1})+\gamma_1\frac{C''(\nu_0)}{p}\right)$$

we choose $C''(\nu_0)$ small enough so that:

$$O(\varepsilon^2\log\varepsilon^{-1})+\gamma_1\frac{C''(\nu_0)}{p} < \frac{C(\nu_0)}{2p}$$

which ends the proof of Proposition 6.

Proposition 7: *For every $p \in \mathbb{N}^*$, for every $A < p^{-100}$, there exists $\varepsilon_0(p, A)$ such that, for every μ satisfying $A \leq \mu \leq p^{-100}$, for every $\alpha_i > 0$, $1 \leq i \leq p$, we have*

$$J(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\mu}) \leq p^{1/2}S$$

provided $Min_{i\neq j} d(a_i, a_j) < \varepsilon_0(p, A) < 2r$.

Proof: In the proof of Lemma 8 of [5], A. Bahri and H. Brezis proved that:

$$J(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\mu})$$
$$\leq p^{1/2}S(1+O(\mu^2\log\mu^{-1}))\left(1-\frac{C'}{p}\sup_{i\neq j}\int\varphi^3_{a_i,\mu}\varphi_{a_j,\mu}\theta\wedge d\theta\right)^{1/2}$$
$$\sup_{i\neq j}\int\varphi^3_{a_i,\mu}\varphi_{a_j,\mu} \geq \int\varphi^3_{a_{i_0},\mu}\varphi_{a_{j_0},\mu},$$

where a_{i_0} and a_{j_0} are such that $d(a_{i_0}, a_{j_0}) < \varepsilon_0(p, A)$.

Arguing as in the proof of Theorem 2, we have:

When $\varepsilon_0(p, A)$ goes to zero, the distance of a_{i_0} to a_{j_0} goes to zero with respect to μ.

Thus, $\int_{B(a_{i_0},2r)}\phi^3_{a_{i_0},\mu}\phi_{a_{j_0},\mu}\theta_0\wedge d\theta_0$ is lowerbounded by a constant α_0 and:

$$J(\sum_{i=1}^{p}\alpha_i\varphi_{a_i,\mu}) \leq p^{1/2}S(1+O(\mu^2\log\mu^{-1}))\left(1-\frac{C'}{p}\alpha_0\right)^{1/2}.$$

Proposition 7 follows if $\mu < p^{-100}$.

Section 4

For every p in $\mathbb{N}^*$, let $\Delta_{p-1} = \{(\alpha_1, \cdots, \alpha_p), \alpha_i \geq 0 \sum_{i=1}^{p} \alpha_i = 1\}$.

For $0 < \nu < 1$, let $\Delta_{p-1}^{\nu} = \{(\alpha_1, \cdots, \alpha_p) \in \Delta_{p-1}/\frac{\alpha_i}{\alpha_j} \in [1-\nu, 1+\nu]$ $\forall\, i, \forall\, j\}$, $\partial\Delta_{p-1}^{\nu}$ is the boundary of Δ_{p-1}^{ν}.

Let $B_p(M) = \{\sum_{i=1}^{p} \alpha_i \delta_{a_i}, \alpha_i \geq 0, \sum_{i=1}^{p} \alpha_i = 1, a_i \in M\}$, where δ_{a_i} is the Dirac mass at a_i, with the convention $B_0(M) = \emptyset$.

$F_p(M) = \{(a_1, \cdots, a_p) \in M^p / \exists i \neq j$ with $a_j = a_j\}$, and σ_p the symmetric group of order p, σ_p acts on F_p.

Let T_p, T_p^1 be two σ_p-equivariant tubular neighborhoods of F_p in M^p, such that $\bar{T}_p \subset \dot{T}_p^1$, let $V_p = \overline{M^p - T_p}$ and $V_p^1 = \overline{M^p - T_p^1}$, ∂V_p the boundary of V_p (the existence of T_p and T_p^1 is derived in the book by G. Bredon [9]) and $W_p = \left\{u \in \sum_+ / \; J(u) \leq (p+1)^{\frac{1}{2}} S\right\}$.

Let $f_p(\varepsilon)$ for $\varepsilon > 0$ and $p \in \mathbb{N}^*$, denote the map:

$$f_p(\varepsilon) : B_p(M) \longrightarrow \sum{}_+$$
$$\sum_{i=1}^{p} \alpha_i \delta_{a_i} \longmapsto \frac{\sum_{i=1}^{p} \alpha_i \varphi_{a_i,\varepsilon}}{\left\| \sum_{i=1}^{p} \alpha_i \varphi_{a_i,\varepsilon} \right\|_H}.$$

By Proposition 2, we know that if $\varepsilon \leq p^{-100}$, then $f_p(\varepsilon)$ maps $B_p(M)$ in W_p, hence $(B_p(M), B_{p-1}(M))$ into (W_p, W_{p-1}). Let ν_0 be given by Proposition 3.

For this ν_0 we consider $C \leq \min(C(\nu_0), C'(\nu_0))$, where $C(\nu)$ (respectively $C'(\nu_0)$) is given in Proposition 5 (respectively Lemma 6, Sect. 3 of [5]).

Proposition 8: *There exists $p_0 \in \mathbb{N}^*$ such that for any $p \geq p_0$ and any $\varepsilon < Cp^{-100}$ the map:*

$$f_p(\varepsilon) : (B_p(M), B_{p-1}(M)) \to (W_p, W_{p-1})$$

is homotopic to a map valued in (W_{p-1}, W_{p-1}) and is therefore homologically trivial.

Proof of Proposition 8: For more details about this construction, one can see Sect. 4 of [5]. A. Bahri and H. Brezis constructed, for p large enough and for $\varepsilon < Cp^{-100}$, a homotopy U such that:

$$\begin{cases} U : [0,1] \times B_p(M) \to W_p \text{ continuous} \\ U(t, B_{p-1}) \subset W_{p-1}, \forall\, t \in [0,1], U(0, \cdot) = f_p(\varepsilon)(\cdot) \\ U(1, B_p) \subset W_{p-1}. \end{cases}$$

We will give here the same construction of [5], adapted to our case.

We consider in $B_p(M)$ three sets A, B and C defined as follows: let $a = (a_1, \cdots, a_p)$, $\alpha = (\alpha_1, \cdots, \alpha_p)$ and $\nu_0 \in (0, 1)$

$$A = \Big\{ \sum_{i=1}^{p} \alpha_i \delta_{a_i} / (a, \alpha) \in V_p \times \Delta_{p-1}^{\nu_0} \Big\}$$

$$B = {}^cA = \Big\{ \sum_{i=1}^{p} \alpha_i \delta_{a_i} / (a, \alpha) \in \big(T_p \times \Delta_{p-1} \cup V_p \times \big(\Delta_{p-1} - \Delta_{p-1}^{\nu_0}\big)\big) \Big\}$$

$$C \subset \dot{A}, \quad C = \Big\{ \sum_{i=1}^{p} \alpha_i \delta_{a_i} / (a, \alpha) \in V_p^1 \times \Delta_{p-1}^{\nu_0/2} \Big\}.$$

Observe that B is a neighborhood of $B_{p-1}(M)$ in $B_p(M)$.

Let $\varepsilon_p < Cp^{-100}$ be given, where $C \leq \min(C(\nu_0), C'(\nu_0))$.

Let for $(a, \alpha) \in M^p \times \Delta_{p-1}$, $\psi(a, \alpha)$ be a continuous function valued in $[0, 1]$, equal to 1 on C to 0 on ${}^cA$.

$U(t, .)$ is defined as follows:

1. If $\sum_{i=1}^{p} \alpha_i \delta_{a_i} \in C$, $\quad U(t, \sum_{i=1}^{p} \alpha_i \delta_{a_i}) = \frac{\sum_{i=1}^{p} \alpha_i \varphi_{a_i, (1-t)\varepsilon + t\varepsilon_p}}{|| \sum_{i=1}^{p} \alpha_i \varphi_{a_i, (1-t)\varepsilon + t\varepsilon_p} ||_H}$.
2. If $\sum_{i=1}^{p} \alpha_i \delta_{a_i} \in B$, $\quad U(t, \sum_{i=1}^{p} \alpha_i \delta_{a_i}) = \frac{\sum_{i=1}^{p} \alpha_i \varphi_{a_i, \varepsilon}}{|| \sum_{i=1}^{p} \alpha_i \varphi_{a_i, \varepsilon} ||_H} = f_p(\varepsilon)(\sum_{i=1}^{p} \alpha_i \delta_{a_i})$.
3. If $\sum_{i=1}^{p} \alpha_i \delta_{a_i} \in A - C$, $\quad U(t, \sum_{i=1}^{p} \alpha_i \delta_{a_i}) = \frac{\sum_{i=1}^{p} \alpha_i \varphi_{a_i, (1-t\psi(a,\alpha))\varepsilon + t\psi(a,\alpha)\varepsilon_p}}{|| \sum_{i=1}^{p} \alpha_i \varphi_{a_i, (1-t\psi(a,\alpha))\varepsilon + t\psi(a,\alpha)\varepsilon_p} ||_H}$.

Since ε_p is upperbounded by Cp^{-100}, any barycenter of ε and ε_p is upperbounded by Cp^{-100} and by Proposition 2 we derive that $U(t, .)$ is valued in W_p; $B_{p-1}(M)$ is contained in B and $U(t, B_{p-1}) = f_p(\varepsilon)(B_{p-1}) \subset W_{p-1}$, thus $U(t, .)$ maps for any t, (B_p, B_{p-1}) into (W_p, W_{p-1}).

To complete the proof of Proposition 8, we need to check that $U(1, .)$ is valued in W_{p-1}.

For this purpose we distinguish three cases:

1) $\sum_{i=1}^{p} \alpha_i \delta_{a_i} \in A - C$ $\ i.e : (a, \alpha) \in V_p \times \Delta_{p-1}^{\nu_0}$ and either $a \in T_p^1$ or $\alpha \notin \Delta_{p-1}^{\nu_0/2}$.

We choose T_p^1 so that: if $(a_1, \cdots, a_p) \in T_p^1$ $Min_{i \neq j} d(a_i, a_j) < \varepsilon_0(p, A)$, where $A = Min(\varepsilon, \varepsilon_p)$ and $\varepsilon_0(p, A)$ is given in Proposition 7.

We have $A < Cp^{-100}$ and Proposition 7 holds, thus:

$$J(\sum_{i=1}^{p} \alpha_i \varphi_{a_i, \mu}) < p^{1/2} S$$

for any μ satisfying $A \leq \mu < p^{-100}$. Here we choose $\mu = (1 - t\psi(a, \alpha))\varepsilon + t\psi(a, \alpha)\varepsilon_p$. Then the result follows if $(a_1, \cdots, a_p) \in T_p^1$.

If $\alpha = (\alpha_1, \cdots, \alpha_p) \notin \Delta_{p-1}^{\nu_0/2}$, we apply Proposition 5 with

$$\varepsilon = \mu = (1 - t\psi(a, \alpha))\varepsilon + t\psi(a, \alpha)\varepsilon_p \leq Cp^{-100} .$$

2) $\sum \alpha_i \delta_{a_i} \in C$, we distinguish two subcases.

2.1) $I = \frac{1}{p} \sum_{i \neq j} \int \varphi^3_{a_i,\varepsilon_p} \varphi_{a_j,\varepsilon_p} < \beta$, where β has been defined in Proposition 3.

2.2) or $I \geq \beta$.

In case 2.1 holds, we have by Proposition 3

$J(\sum \alpha_i \varphi_{a_i,\varepsilon_p}) \leq \frac{\sum \alpha_i^2}{(\sum \alpha_i^4)^{1/2}} S\big(1 + O\big(\varepsilon_p^2 \log \varepsilon_p^{-1}\big) - \frac{\gamma_0 I}{S^2}(1 + O(\varepsilon^{1/3}))\big)$ since $\displaystyle\int_M \varphi^3_{a_i,\varepsilon_p} \varphi_{a_j,\varepsilon_p} \geq C\varepsilon_p^2$, thus $I \geq pC\varepsilon_p^2$ and $\dfrac{\sum \alpha_i^2}{(\sum \alpha_i^4)^{1/2}} \leq p^{1/2}$, we obtain $J(\sum_{i=1}^p \alpha_i \varphi_{a_i,\varepsilon_p}) < p^{1/2} S(1 + O(\varepsilon_p^2 \log \varepsilon_p^{-1}) - \gamma_0 p \frac{C}{S^2} \varepsilon_p^2 (1 + O(\varepsilon_p^{1/3})))$, we have to choose ε_p such that

$$C\big(\varepsilon_p^2 \log \varepsilon_p^{-1}\big) - p\gamma_0 \varepsilon_p^2 < 0$$

provided $\varepsilon_p < Cp^{-100}(\varepsilon_p \cong e^{-C_0 p})$ for p large enough.

In case 2.2) holds $I \geq \beta$, we apply Proposition 4 with $\frac{\alpha_i}{\alpha_j} \in (1 - \frac{\nu_0}{2}, 1 + \frac{\nu_0}{2})$, thus

$$J(\sum \alpha_i \varphi_{a_i,\varepsilon_p}) < p^{1/2} S\Bigg(1 - \gamma_1' \frac{\beta}{pS^2}\big(1 + O\big(\varepsilon_p^{1/3}\big)\big)\Bigg)\big(1 + O\big(\varepsilon_p^2 \log \varepsilon_p^{-1}\big)\big).$$

Thus

$$J(\sum \alpha_i \varphi_{a_i,\varepsilon_p}) < p^{1/2} S, \text{ if } \varepsilon_p < Cp^{-100}.$$

3) $\sum_{i=1}^p \alpha_i \delta_{a_i} \in B$, we consider two subscases.

3.1) $(a, \alpha) \in T_p \times \Delta_{p-1}$.

We use Proposition 7 with $A = \mu = \varepsilon$.

Observe that $T_p \subset T_p^1$ then if T_p^1 is chosen small enough we have

$Min_{i \neq j} d(a_i, a_i) < \varepsilon_0(p, A), \forall\ (a_1, \cdots, a_p) \in \bar{T}_p$ thus

$$J(\sum_{i=1}^p \alpha_i \varphi_{a_i,\varepsilon}) < p^{1/2} S.$$

3.2) $(a_1, \cdots, a_p, \alpha_1, \cdots, \alpha_p) \in V_p \times (\Delta_{p-1} - \Delta_{p-1}^{\nu_0})$ there are at least two indices i_0 and j_0 such that $\frac{\alpha_{j_0}}{\alpha_{j_0}} \notin (1 - \nu_0, 1 + \nu_0)$

$$U(1, \sum_{i=1}^p \alpha_i \delta_{a_i}) = \frac{\sum \alpha_i \varphi_{a_i,\varepsilon}}{|| \sum \alpha_i \varphi_{a_i,\varepsilon} ||_H}.$$

We can apply Proposition 6, since $\varepsilon < cp^{-100}$. This ends the proof of Proposition 7.

Section 5

Topological Argument (see [5], [7]*)*

Proof of Theorem 1: We prove Theorem 1 by contradiction, thus we suppose that equation (1) has no solution.

The topological argument is based on a comparison of $B_p(M)$ and the level sets W_p of the functionnal J via $f_p(\varepsilon)$. We know that:

$$f_p(\varepsilon) : (B_p, B_{p-1}) \to (W_p, W_{p-1})$$

is homologically trivial for p large enough (Proposition 8).

On the other hand, if there is no solution of the equation (1), the pair (W_p, W_{p-1}) retracts by deformation on $(W_{p-1} \cup A_p, W_{p-1})$ with $A_p \subset V(p, \varepsilon')$, where

$$V(p,\varepsilon') = \left\{ \begin{array}{l} u \in \sum_+ \text{ such that there exists p concentration points} \\ a_1, \ldots, a_p \text{ in } M \text{ and } p \text{ concentrations } \varepsilon_1, \ldots, \varepsilon_p \in [0, 1[\\ \text{such that } \left\| u - \frac{1}{p^{\frac{1}{2}} S} \sum_{i=1}^{p} \varphi_{a_i,\varepsilon_i} \right\|_H < \varepsilon', \text{ with } \varepsilon_i < \varepsilon' \\ \text{and } \varepsilon_{ij} = \frac{\varepsilon_i}{\varepsilon_j} + \frac{\varepsilon_j}{\varepsilon_i} + \frac{d(a_i,a_j)^2}{\varepsilon_i \varepsilon_j} \geq \frac{1}{\varepsilon'} \text{ for } i \neq j. \end{array} \right\}$$

where $d(x, y)$, if x and y are in a small ball of M of radius r, is $|| \exp_x^{-1}(y)||_{\mathbb{H}^1}$ ($|| \cdot ||_{\mathbb{H}^1}$ is the norm in $\mathbb{H}^1$), with $\exp_x$ the CR exponential map for the point x, and $d(x, y)$ is equal to $r/2$ otherwise. $V(p, \varepsilon')$ is a neighborhood of critical points at infinity.

Thus, the functions of A_p are of the form $\Sigma\alpha_i\varphi_{a_i,\varepsilon_i} + v$, v small in the $\|\|_H$ and we can define a natural map of $(W_{p-1} \cup A_p) - W_{p-1} = A_p$, which can be thought of as $W_p - W_{p-1}$, into $V_p \times_{\sigma_p} \Delta^{\nu}_{p-1}$, for a suitable choice of T_p and ν (see Sect. 4 of [7]).

Therefore the model $\cdots \subset B_{p-1}(M) \subset B_p(M) \subset \cdots$ can be compared via $f_p(\varepsilon)$ to $\cdots \subset W_{p-1} \subset W_p \cdots$ If the equation (1) has no solution, we can derive the:

Lemma 22 Sect. 5 [7]: *For any $p \in \mathbb{N}^*$ and $\varepsilon < 1$, let ω_p denote the homology orientation class (modulo 2) of the pair $(B_p(M), B_{p-1}(M))$. We have:*

$$f_{p^*}(\varepsilon)(\omega_p) \neq 0$$

where

$$f_{p^*}(\varepsilon) : H_*(B_p(M), B_{p-1}(M)) \to H_*(W_p, W_{p-1})$$

with $H_*(\cdot)$ denoting the homology group with Z_2 coefficients.

This result contradicts Proposition 8. Thus, we derive that the equation (1) has a solution, which proves Theorem 1.

Appendix

Let M be a compact strictly pseudo-convex CR manifold of dimension $2n+1$.

Lemma A.1: *Suppose $\lambda(M) > 0$. Then at each point $q \in M$ the Green function G_q for L exists and is strictly positive.*

Proof: Consider for $2 \leq s < r = 2 + 2/n$ the extremal problem

$$\lambda_s = \inf \left\{ A_\theta(\phi),\ \phi \in S_1^2(M), B_{\theta,s}(\phi) = 1 \right\}$$

in which A_θ is as in (2) and $B_{\theta,s} = \int_M |\phi|^s \theta \wedge d\theta^n$.

By Theorem 6.2 [1], we know that there exists a positive C^∞, solution u_s to the equation:

$$(2+2/n)\Delta_b u_s + R u_s = \lambda_s u_s^{s-1} \tag{A.1}$$

satisfying $A_\theta(u_s) = \lambda_s$ and $B_{\theta,s}(u_s) = 1$.

Let then $u > 0$ be the smooth positive solution to the subcritical equation (A.1) for s, $2 \leq s < r = 2 + 2/n$; and define a new contact form on M, $\theta' = u^{2/n}\theta$. The Webster scalar curvature R' of θ' is given by:

$$R' = u^{1-r} L u, \quad L = r\Delta_b + R\ . \tag{A.2}$$

Since $\lambda(M) > 0$, implies $\lambda_s > 0$, R' is strictly positive. Thus the conformal Laplacian $L' = r\Delta_b + R'$ is invertible and then the Green function G'_q for L' exists, (on all M: one can see Proposition 5.17 [1]).

If at its minimum $G'_q \leq 0$, then G'_q would be constant by the maximum principle, which is impossible since $L' G'_q = \delta_q$ in the distributional sense where δ_q is the Dirac measure at q.

Therefore G'_q is strictly positive. And if we set $G_q(x) = u(q)u(x)G'_q(x)$ then G_q is strictly positive and by the transformation law of the laplacian (one can see [1]) $L'(v') = u^{-(r-1)} L v$, where $v' = u^{-1} v$; we have for any $f \in C_0^\infty(M)$

$$\begin{aligned} u^{-1}(q) f(q) &= \int_M G'_q(x) L'(u^{-1}(x) f(x)) \theta' \wedge d\theta'^n \\ &= \int_M u^{-1}(q) u^{-1}(x) G_q(x) u^{1-r} (Lf) u^r \theta \wedge d\theta^n \\ &= u^{-1}(q) \int_M G_q(x) L f \theta \wedge d\theta^n. \end{aligned}$$

This is equivalent to $LG_q = \delta_q$. Thus G_q is the Green function for L.

Lemma A.2: *Let $n = 1$, with the choice of θ such that Proposition 3.12 [2] be satisfied. Let G_q be the Green function for L at q as in Lemma A.1. Then in pseudohermitian normal coordinates (z, t) near q:*

$$G_q(z,t) = C\rho^{-2}(z,t) + A + O(\rho(z,t)). \tag{A.3}$$

Where A and C are constants.

Proof: Let $\varepsilon > 0$ be small enough such that $B_\varepsilon(q) = \{\rho(z,t) < \varepsilon\}$ be contained in the pseudohermitian normal coordinates (z,t) near q.

Let $\Psi \in C^\infty(M)$ be a cut-off function such that:

$$\Psi(z,t) = \begin{cases} 1 \text{ for } (z,t) B_{\varepsilon/2}(q) \\ 0 \text{ outside } B_\varepsilon(q) \end{cases} .$$

And consider the function:

$$\tilde{G}_q(z,t) = \Psi(z,t) G_q(z,t)$$

Since $C\rho^{-2}(z,t)$ is a fundamental solution of $L_0 = -2(Z\bar{Z} + \bar{Z}Z)$ in $\mathbb{H}^1$, and $L = -2(Z\bar{Z} + \bar{Z}Z) + \bigcirc(2)$, in $B_{\varepsilon/2}(q)$ we have $L(\tilde{G}_q(z,t) - C\rho^{-2}(z,t)) \in L^\infty(B_{\varepsilon/2}(q))$ and subelliptic regularity (one can see [8]) implies that

$$\tilde{G}_q(z,t) - C\rho^{-2}(z,t) \in S_2^k(B_{\varepsilon/2}(q)), \quad \forall k > 0,$$

where S_2^k is a Folland-Stein space ([8]).

For $\beta > 0$ let $\Gamma_\beta(B_{\varepsilon/2}(q))$ be the Folland-Stein Hölder space. By Folland-Stein Sobolev embedding theorem (Theorem 21.1 [8]). We have $S_2^k(B_{\varepsilon/2}(q)) \hookrightarrow \Gamma_\beta(B_{\varepsilon/2}(q)), \quad \forall \beta$ such that $\beta = 2 - \frac{2n+2}{k} > 0$. Hence $\tilde{G}_q(z,t) - C\rho^{-2}(z,t) \in \Gamma_\beta(B_{\varepsilon/2}(q)), \quad \forall \beta < 2$, which implies the existence of a constant A such that

$$G_p(z,t) = C\rho^{-2}(z,t) + A + O(\rho).$$

The Webster scalar curvature $R > 0$ implies $\lambda(M) > 0$,and Lemma A.1 implies that G_p is strictly positive, thus we can consider that the constant C is positive.

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