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Asymptotic stabilization of entropy solutions to scalar conservation laws through a stationary feedback law

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Abstract

In this paper, we study the problem of asymptotic stabilization by closed loop feedback for a scalar conservation law with a convex flux and in the context of entropy solutions. Besides the boundary data, we use an additional control which is a source term acting uniformly in space.

Keywords: Asymptotic stabilization; Conservation law; Entropy solution; Generalized characteristics; PDE control

1. Introduction

This paper is concerned with the asymptotic stabilization problem for a nonlinear scalar conservation law with a source term, on a bounded interval and in the framework of entropy solutions:

$$\begin{aligned} &\partial_t u + \partial_x f(u) = g(t), \\ &u(0,x) = u_0(x), \\ &u(t,0) = u_l(t), \\ &u(t,L) = u_r(t), \end{aligned} \qquad (t,x) \in (0,T) \times (0,1). \tag{1}$$

Here u is the state and u_l, u_r and g are the controls. For any regular strictly convex flux f in $C^1(\mathbb{R};\mathbb{R})$ and any state $\bar{u} \in \mathbb{R}$ we will provide explicit stationary feedback law for g, u_r and u_l such that the state $\bar{u}$ is asymptotically stable in the $L^1(0,1)$ norm and in the $L^\infty(0,1)$ norm.

1.1. Generalities and previous results

Scalar conservation laws are used for instance to model traffic flow or gas networks, but their importance also consists in being a first step in the understanding of systems of conservation laws. Those systems of equations model a huge number of physical phenomena: gas dynamics, electromagnetism, magneto-hydrodynamics, shallow water theory, combustion theory, ..., see [12, Chapter 2].

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For equations such as (1), the Cauchy problem on the whole line is well posed in small time in the framework of classical solutions and with a $\mathcal{C}^1$ initial value. However those solutions generally blow up in finite time: shock waves appear. Hence to get global in time results, a weaker notion of solution is called for. In [23] Oleinik proved that given a flux $f \in \mathcal{C}^2$ such that $f'' > 0$ and any $u_0 \in L^\infty(\mathbb{R})$ there exists a unique weak solution to:

$$u_t + \big(f(u)\big)_x = 0, \quad x \in \mathbb{R} \text{ and } t > 0, \tag{2}$$

$$u(0,.) = u_0, \tag{3}$$

satisfying the additional condition:

$$\frac{u(t,x+a) - u(t,x)}{a} \leqslant \frac{E}{t} \quad \text{for } x \in \mathbb{R},\ t > 0,\ \text{and } a > 0. \tag{4}$$

Here E depends only on the quantities $\inf(f'')$ and $\sup(f')$ taken on $[-\|u_0\|_{L^\infty}, \|u_0\|_{L^\infty}]$ and not on u_0. Later in [21], Kruzkov extended this global result to the multidimensional problem, with a C^1 flux $f:\mathbb{R} \to \mathbb{R}^n$ not necessarily convex:

$$u_t + \operatorname{div}\big(f(t,x,u)\big) = g(t,x,u), \quad \text{for } t > 0 \text{ and } x \in \mathbb{R}^n. \tag{5}$$

This time the weak entropy solution is defined as satisfying the following integral inequality:

$$\text{for all real numbers } k \text{ and all non-negative functions } \phi \text{ in } \mathcal{C}^1_c\big(\mathbb{R}^2\big), \tag{6}$$

$$\iint\limits_{\mathbb{R}^2} |u-k|\phi_t + \operatorname{sgn}(u-k)\big(f(u) - f(k)\big)\nabla\phi + \operatorname{sgn}(u-k)g(t,x,u)\phi\,dt\,dx + \int\limits_{\mathbb{R}} u_0(x)\phi(0,x)\,dx \geqslant 0. \tag{7}$$

The initial boundary value problem for Eq. (1) is also well posed as shown by Leroux in [19] for the one dimensional case with BV data, by Bardos, Leroux and Nédélec in [6] for the multidimensional case with C^2 data and later by Otto in [24] (see also [22]) for L^∞ data. However the meaning of the boundary condition is quite intricate and the Dirichlet condition may not be fulfilled pointwise a.e. in time. We will go into further details later.

Now for a general control system:

$$\begin{cases} \dot{X} = F(X,U), \\ X(t_0) = X_0 \end{cases} \tag{8}$$

(X being the state of the system belongs to the space $\mathcal{X}$ and U the so called control belongs to the space $\mathcal{U}$), we can consider two classical problems (among others) in control theory.

1. First the exact controllability problem which consists, given two states X_0 and X_1 in $\mathcal{X}$ and a positive time T, in finding a certain function $t \in [0,T] \mapsto U(t) \in \mathcal{U}$ such that the solution to (8) satisfies $X(T) = X_1$.
2. If $F(0,0) = 0$, the problem of asymptotic stabilization by a stationary feedback law asks to find a function of the state $X \in \mathcal{X} \mapsto U(X) \in \mathcal{U}$, such that for any state X_0 a maximal solution $X(t)$ of the closed loop system:

$$\begin{cases} \dot{X}(t) = F\big(X(t), U\big(X(t)\big)\big), \\ X(t_0) = X_0, \end{cases} \tag{9}$$

is global in time and satisfies additionally:

$$\forall R > 0,\ \exists r > 0 \text{ such that } \|X_0\| \leqslant r \quad \Rightarrow \quad \forall t \in \mathbb{R},\ \big\|X(t)\big\| \leqslant R, \tag{10}$$

$$X(t) \underset{t\to+\infty}{\longrightarrow} 0. \tag{11}$$

The asymptotic stabilization property might seem weaker than exact controllability: for any initial state X_0, we can find T and $U(t)$ such that the solution to (8) satisfies $X(T) = 0$ in this way we stabilize 0 in finite time. However this method suffers from a lack of robustness with respect to perturbation: with any error on the model, or on the initial state, the control may not act properly anymore. This motivates the problem of asymptotic stabilization by a stationary feedback law which is more robust. In fact in finite dimension, the asymptotic stabilization property automatically guarantees the existence of a Lyapunov function.

In the framework of entropy solutions, only a few results exist for the exact controllability problem see [2,3,1,7,14,15,17]. In all cases the control act only at the boundary points. Furthermore many of those results show that boundary controls are not sufficient to reach many states. However with an additional control $g(t)$ as in (1) and with $f(z)=\frac{z^2}{2}$ (Burgers equation), Chapouly showed in [8] that in the framework of classical solutions, any regular state is reachable from any regular initial data and in any time (note that in this context, the controls also had to prevent the blow up of the solution, which will not be a concern for entropy solutions). It was shown in [25] that the same kind of improvement also occur in the framework of entropy solutions.

The aim of this paper is to investigate the problem of asymptotic stabilization with this additional control on the right-hand side. To the author's knowledge it is the first result about asymptotic stabilization through a closed loop feedback law in the framework of entropy solutions. However, it should be noted that in the framework of classical solutions the problem has been studied extensively, see for example: [18], [4] or [5] among many others.

As for the physical significance of g, it can be seen as a pressure field in the case where the Burgers equation is considered as a one dimensional isentropic Euler compressible equation. Eq. (1) is also a toy model for the compressible Euler–Poisson system:

$$\begin{cases} \partial_t\rho+\partial_x m=0, \quad \rho(0,.)=\rho_0,\ \rho(t,0)=\rho_l(t),\ \rho(t,L)=\rho_r(t), \\ \partial_t m+\partial_x\left(\dfrac{m^2}{\rho}+p(\rho)\right)=\rho\partial_x V-m, \quad m(0,.)=m_0,\ m(t,0)=m_l(t),\ m(t,L)=m_r(t), \\ -\partial_{xx}^2 V=\rho, \quad V(t,0)=V_l(t),\ V(t,L)=V_r(t), \end{cases} \tag{12}$$

where the controls are ρ_l, ρ_r, m_l, m_r, V_l and V_r. Indeed once we take $g(t)=\frac{V_r(t)-V_l(t)}{L}$ we get $\partial_x V=g(t)+A\rho(t)$ with A the following linear integral operator:

$$A\theta(x)=\int_0^x\theta(z)\,dz-\frac{1}{L}\int_0^L\int_0^y\theta(z)\,dz\,dy.$$

So we have to deal with a hyperbolic system controlled by the boundary data and an additional source term depending only on the time variable.

1.2. Results

In this article, functions in $\mathrm{BV}(\mathbb{R})$ will be considered continuous from the left in order to prevent ambiguity on the representative of the L^1 equivalence class.

We will also suppose that the flux f is a $\mathcal{C}^1$ strictly convex function.

We will make use of the following notation:

$$\forall\alpha,\beta\in\mathbb{R},\quad I(\alpha,\beta)=\big[\min(\alpha,\beta),\max(\alpha,\beta)\big]. \tag{13}$$

We are interested in the following equation:

$$\begin{cases} \partial_t u+\partial_x\big(f(u)\big)=g(t) & \text{on } (0,+\infty)\times(0,1), \\ u(0,.)=u_0 & \text{on } (0,1), \\ \operatorname{sgn}\big(u(t,1^-)-u_r(t)\big)\big(f\big(u(t,1^-)\big)-f(k)\big)\geqslant 0 & \forall k\in I\big(u_r(t),u(t,1^-)\big),\ dt \text{ a.e.}, \\ \operatorname{sgn}\big(u(t,0^+)-u_l(t)\big)\big(f\big(u(t,0^+)\big)-f(k)\big)\leqslant 0 & \forall k\in I\big(u_l(t),u(t,0^+)\big),\ dt \text{ a.e.} \end{cases} \tag{14}$$

We recall that following [19] and [6] a function u in $L^\infty((0,+\infty);\mathrm{BV}(0,1))$ is an entropy solution of (14) when it satisfies the following inequality for every k in $\mathbb{R}$ and every non-negative function ϕ in $\mathcal{C}_c^1(\mathbb{R}^2)$:

$$\int_0^{+\infty}\int_0^1|u-k|\phi_t+\operatorname{sgn}(u-k)\big(f(u)-f(k)\big)\phi_x+\operatorname{sgn}(u-k)g(t)\phi\,dx\,dt+\int_0^1\big|u_0(x)-k\big|\phi(0,x)\,dx$$

$$+\int_0^{+\infty} \operatorname{sgn}\bigl(u_r(t)-k\bigr)\bigl(f(k)-f\bigl(u\bigl(t,1^-\bigr)\bigr)\bigr)\phi(t,1)-\operatorname{sgn}\bigl(u_l(t)-k\bigr)\bigl(f(k)-f\bigl(u\bigl(t,0^+\bigr)\bigr)\bigr)\phi(t,0)\,dt$$
$$\geqslant 0. \tag{15}$$

Remark 1.

- Let us recall that with a convex flux, an entropy solution satisfy (see [11] or [12, Chapter 11]):
$$\forall (t,x)\in(0,+\infty)\times(0,1),\quad u\bigl(t,x^-\bigr)\geqslant u\bigl(t,x^+\bigr). \tag{16}$$
- It should be noted that if u in $L^\infty((0,+\infty);\mathrm{BV}(0,1))\cap \mathrm{Lip}([0,+\infty);L^1(0,1))$ the exists a unique representative u such that:
$$u\in \mathrm{Lip}\bigl([0,+\infty);L^1(0,1)\bigr)\quad\text{and}\quad \forall t\geqslant 0,\quad u(t,.)\in \mathrm{BV}(0,1).$$
Thus the traces of u at $x=0$ and $x=1$ are taken as the limit of this representative for every time t and the boundary conditions in (14) hold almost everywhere and not necessarily everywhere. This will make the analysis more delicate as will be seen in Section 3.

Here the functions g, u_l and u_r will not depend on the time but on the state $u(t,.)$. Their value will be prescribed by a closed loop feedback law.

Consider $\bar u\in\mathbb{R}$. It is clear that if we define u by:
$$\forall (t,x)\in\mathbb{R}^+\times(0,1),\quad u(t,x)=\bar u,$$
then u is an entropy solution of (14) for initial and boundary data equal to $\bar u$. In the following we will provide two feedback laws and two corresponding results (in the respective cases $f'(\bar u)=0$ and $f'(\bar u)>0$), such that the previous stationary solution is asymptotically stable. Note that the case $f'(\bar u)<0$ can be directly deduced from the positive case using the transformation:
$$X=1-x,$$
$$F(z)=f(-z),$$
$$U(t,X)=-u(t,1-x).$$
If $f'(\bar u)>0$, we will use the following stationary feedback laws:
$$\forall W\in L^1(0,1),\quad \mathcal{G}_1(W)=\frac{f'(\bar u)}{2}\|W-\bar u\|_{L^1(0,1)}, \tag{17}$$
$$\forall W\in L^1(0,1),\quad u_l(W)=u_r(W)=\bar u. \tag{18}$$
In the system (14) we will replace: $g(t)$ by $\mathcal{G}_1(u(t,.))$, $u_l(t)$ by $u_l(u(t,.))$ and finally $u_r(t)$ by $u_r(u(t,.))$ to obtain a closed loop system.

We will need to distinguish between two possible behaviors of f as follows:

Definition 1.

- We say that f if of type I if there exists u^* such that:
$$f'\bigl(u^*\bigr)=0. \tag{19}$$
The Burgers equation has a flux of type I.
- We say that f is of type II otherwise. In this case we have either
$$\forall z\in\mathbb{R},\quad f'(z)>0, \tag{20}$$
or

$$\forall z \in \mathbb{R}, \quad f'(z) < 0. \tag{21}$$

The flux $f(z) = e^z$ is of type II.

Remark 2. If the flux f is of type I, we can deduce since it also strictly convex:

$$\lim_{z\to+\infty} f(z) = \lim_{z\to-\infty} f(z) = +\infty. \tag{22}$$

If $f'(\bar{u}) \neq 0$, guarantees the existence of $\hat{u} \neq \bar{u}$ such that $f(\bar{u}) = f(\hat{u})$. We can then reformulate the boundary condition of (14) as follows (we describe the case $f'(\bar{u}) > 0$):

$$u(t, 1^-) \in [u^*; +\infty) \quad dt \text{ a.e.}, \tag{23}$$

$$u(t, 0^+) \in (-\infty, \hat{u}] \cup \{\bar{u}\} \quad dt \text{ a.e.} \tag{24}$$

We can now announce the following result.

Theorem 1. *For any u_0 in* $\mathrm{BV}(0,1)$*, the closed loop system* (14) *where u_l, u_r and g are given by the feedback laws* (18) *and* (17) *has a unique entropy solution u. It is global in time, belongs to the space* $L^\infty((0,+\infty); \mathrm{BV}(0,1)) \cap \mathrm{Lip}([0,+\infty); L^1(0,1))$ *and continuously depends on the initial data. Furthermore if the flux f is of type I we have*:

- *There exist two positive constants C_1 and C_2 depending only on $\bar{u}$ such that u satisfies*:

$$\forall t \geqslant 0, \quad \|u(t,.) - \bar{u}\|_{L^1(0,1)} \leqslant C_1 e^{-\frac{f'(\bar{u})}{2}t} \|u_0 - \bar{u}\|_{L^1(0,1)}, \tag{25}$$

$$\forall t \geqslant 0, \quad \|u(t,.) - \bar{u}\|_{L^\infty(0,1)} \leqslant C_2 e^{-\frac{f'(\bar{u})}{2}t} \|u_0 - \bar{u}\|_{L^\infty(0,1)}. \tag{26}$$

- *There exists a certain time T depending only on $\bar{u}$ such that u is $\mathcal{C}^1$ on $(T,+\infty) \times [0,1]$.*

On the other hand if the flux f is of type II we have the following properties.

- *The solution u satisfies the following stabilization estimate*:

$$\forall t \geqslant 0, \quad \|u(t,.) - \bar{u}\|_{L^\infty(0,1)} \leqslant e^{\frac{f'(\bar{u})}{f'(\bar{u} - \|u_0 - \bar{u}\|_{L^\infty(0,1)})}} e^{-\frac{f'(\bar{u})}{2}t} \|u_0 - \bar{u}\|_{L^\infty(0,1)}. \tag{27}$$

- *There exists a certain time T' depending on $\bar{u}$ and $\|u_0 - \bar{u}\|_{L^\infty(0,1)}$ such that u is $\mathcal{C}^1$ on $(T',+\infty) \times [0,1]$.*

Remark 3.

- In Section 4, we will provide explicit formulas for C_1, C_2, C_3, T and T'.
- It is interesting to see that a feedback using the L^1 norm actually provides a control in the L^∞ norm. On the other hand a feedback relying on the L^∞ norm might be problematic due to the impossibility of taking the limit in $\|.\|_{L^\infty(0,1)}$ with only a pointwise convergence and also due to the lack of time regularity of $\|u(t,.)\|_{L^\infty(0,1)}$ for an entropy solution of the open loop system.

Let us now suppose that $f'(\bar{u}) = 0$, we introduce the following auxiliary function A:

$$\mathrm{A}(z) = \begin{cases} \frac{f(\bar{u}+z) - f(\bar{u})}{2} & \text{if } 0 \leqslant z \leqslant 1, \\ \frac{f'(\bar{u}+1)}{2}(z-1) + \frac{f(\bar{u}+1) - f(\bar{u})}{2} & \text{if } z \geqslant 1. \end{cases} \tag{28}$$

We will use once again:

$$\forall W \in L^1(0,1), \quad u_l(W) = u_r(W) = \bar{u},$$

for the boundary terms but the stationary feedback law for the source term will now be:

$$\forall W \in L^1(0,1), \quad \mathcal{G}_2(W) = \mathrm{A}(\|W - \bar{u}\|_{L^1(0,1)}), \tag{29}$$

and as before we will replace $g(t)$ by $\mathcal{G}_2(u(t,.))$ in (14). This allows us to prove the following.

Theorem 2. *The closed loop system* (14) *where* u_l, u_r *and* g *are provided by the feedback laws* (18) *and* (29) *has the following properties.*

- *For any* u_0 *in* $\mathrm{BV}(0,1)$ *there exists a unique entropy solution* u. *It is global in time, belongs to the space*
$$L^\infty\big((0,+\infty);\mathrm{BV}(0,1)\big)\cap \mathrm{Lip}\big([0,+\infty);L^1(0,1)\big),$$
and depends continuously on the initial data.
- *The solution satisfies*:
$$\big\|u(t,.)-\bar{u}\big\|_{L^\infty(0,1)}\underset{t\to+\infty}{\longrightarrow}0. \tag{30}$$
- *If additionally*
$$\alpha=\inf_{z\in\mathbb{R}}f''(z)>0,$$
then we have a globally Lipschitz function R *such that*:
$$R(0)=\frac{f'(1+\bar{u})}{2\alpha}\sqrt{\frac{2e}{e-1}}+\mathrm{A}^{-1}\left(\frac{e(f'(1+\bar{u}))^2}{4\alpha(e-1)}\right), \tag{31}$$
$$\forall t\geqslant 0,\quad \big\|u(t,.)-\bar{u}\big\|_{L^\infty(0,1)}\leqslant R\big(\|u_0-\bar{u}\|_{L^\infty(0,1)}\big). \tag{32}$$

Remark 4. The last property is weaker than stability, thus we do not have asymptotic stability of $\bar{u}$. However taking c positive and adjusting A as follows:
$$\mathrm{A}(z)=\begin{cases}\frac{f(\bar{u}+z)-f(\bar{u})}{2} & \text{if } 0\leqslant z\leqslant c,\\ \frac{f'(\bar{u}+c)}{2}(z-c)+\frac{f(\bar{u}+c)-f(\bar{u})}{2} & \text{if } z\geqslant c,\end{cases} \tag{33}$$
we can see that $\frac{f'(\bar{u}+c)}{2}$ tends to 0 with c and therefore $R(0)$ can be as small as we want.

The feedback laws (17) and (29) act in two steps. In the first step the control g uniformly increases the state $u(t,.)$ and therefore the characteristic speed $f'(u(t,.))$ (in the case where $f'(\bar{u})\geqslant 0$) to eventually reach a point where the speed is everywhere positive on $(0,1)$ (the same speed profile as the target state $\bar{u}$). It should be noted that we may potentially increase $\|u(t,.)-\bar{u}\|$ during this part. Once such a speed profile is reached the feedback loop increases the speed $f'(u)$ more than the state u and we have stabilization toward $\bar{u}$. This is the same strategy as the return method of J.-M. Coron [9,10].

The paper will be organized as follows. In Section 2, we will prove using a Banach fixed point theorem that the closed loop systems of both Theorem 1 and Theorem 2 has a unique maximal entropy solution which furthermore is global in time, one might consider looking at the articles [26] and [27] where related questions are considered (note that a Lax–Friedrichs scheme with a discrete $\|u(t,.)-\bar{u}\|_{L^1(0,1)}$ would have also provided existence). In Section 3, we will adapt the result of [11] and describe the influence of the boundary conditions on the generalized characteristics touching the boundary points. In Section 4 we prove Theorem 1. Finally in Section 5 we will prove Theorem 2.

2. Cauchy problem for the closed loop system

In this section, we will prove the following result which will imply the first part of Theorem 1 and Theorem 2 about existence uniqueness and continuous dependence on the initial data for the closed loop systems.

Proposition 2.1. *For* $\bar{u}$ *in* $\mathbb{R}$, u_0 *in* $\mathrm{BV}(0,1)$ *and* g *a* $\mathcal{C}^1$ *function on* $\mathbb{R}$ *which is globally Lipschitz* (*with constant* L_G) *and satisfies* $g(0)=0$, *there exists a unique entropy solution of*:
$$\begin{cases}\partial_t u+\partial_x\big(f(u)\big)=g\big(\big\|u(t,.)-\bar{u}\big\|_{L^1(0,1)}\big) & \text{on } (0,+\infty)\times(0,1),\\ u(0,.)=u_0 & \text{on } (0,1),\\ \operatorname{sgn}\big(u(t,1^-)-\bar{u}\big)\big(f\big(u(t,1^-)\big)-f(k)\big)\geqslant 0 & \forall k\in I\big(\bar{u},u(t,1^-)\big),\ dt \text{ a.e.},\\ \operatorname{sgn}\big(u(t,0^+)-\bar{u}\big)\big(f\big(u(t,0^+)\big)-f(k)\big)\leqslant 0 & \forall k\in I\big(\bar{u},u(t,0^+)\big),\ dt \text{ a.e.}\end{cases} \tag{34}$$

Furthermore two solutions u and v of (34) *for two initial data u_0 and v_0 satisfy*:

$$\forall t \geqslant 0, \quad \|u(t,.)-v(t,.)\|_{L^1(0,1)} \leqslant \|u_0-v_0\|_{L^1(0,1)} e^{L_G t}.$$

It is crucial that the boundary data is independent of u, so that we can use a fixed point theorem on the source term of the equation. This proposition implies the first parts of Theorem 1 and Theorem 2 because both take the form of (34).

Thanks to [19,20,6], we know that for any u_0 in $\mathrm{BV}(0,1)$ and any function h in $\mathcal{C}^0(\mathbb{R}^+)$ there exists a unique entropy solution v in $L^\infty_{\mathrm{loc}}((0,+\infty);\mathrm{BV}(0,1)) \cap \mathrm{Lip}_{\mathrm{loc}}([0,+\infty);L^1(0,1))$ to

$$\begin{cases} \partial_t v + \partial_x\big(f(v)\big) = h(t), & \text{on } (0,+\infty)\times(0,1),\\ v(0,.) = u_0, & \text{on } (0,1),\\ \operatorname{sgn}\big(v(t,1^-)-\bar u\big)\big(f\big(v(t,1^-)\big)-f(k)\big) \geqslant 0, & \forall k \in I\big(\bar u, v\big(t,1^-\big)\big),\ dt \text{ a.e.},\\ \operatorname{sgn}\big(v(t,0^+)-\bar u\big)\big(f\big(v(t,0^+)\big)-f(k)\big) \leqslant 0, & \forall k \in I\big(\bar u, v\big(t,0^+\big)\big),\ dt \text{ a.e.} \end{cases} \tag{35}$$

We now have the following key estimate, which is a classical result of Kruzkov [21] when there is no boundary.

Lemma 1. *If v and $\tilde v$ are entropy solutions of* (35) *with respective source terms h and $\tilde h$ and respective initial data u_0 and $\tilde u_0$ then we have*:

$$\forall T \geqslant 0, \quad \|v(T,.)-\tilde v(T,.)\|_{L^1(0,1)} \leqslant \|u_0-\tilde u_0\|_{L^1(0,1)} + \int_0^T \big|h(s)-\tilde h(s)\big|\,ds. \tag{36}$$

Proof. Following the method of Kruzkov [21], we take ψ a non-negative function in $\mathcal{C}^1_c(\mathbb{R}^4)$. Since for any $(\tilde t,\tilde x)$ in $\mathbb{R}^2$, $\psi(.,\tilde t,.,\tilde x)$ is in $\mathcal{C}^1_c(\mathbb{R}^2)$ we get with $k=\tilde v(\tilde t,\tilde x)$ in (15):

$$\begin{aligned}
&\int_0^{+\infty}\int_0^1 \big|v(t,x)-\tilde v(\tilde t,\tilde x)\big|\psi_t(t,\tilde t,x,\tilde x) + \operatorname{sgn}\big(v(t,x)-\tilde v(\tilde t,\tilde x)\big)\big(f\big(v(t,x)\big)-f\big(\tilde v(\tilde t,\tilde x)\big)\big)\psi_x(t,\tilde t,x,\tilde x)\\
&\quad + \operatorname{sgn}\big(v(t,x)-\tilde v(\tilde t,\tilde x)\big)h(t)\psi(t,\tilde t,x,\tilde x)\,dx\,dt + \int_0^1 \big|u_0(x)-\tilde v(\tilde t,\tilde x)\big|\psi(t,\tilde t,x,\tilde x)\,dx\\
&\quad + \int_0^{+\infty} \operatorname{sgn}\big(\bar u-\tilde v(\tilde t,\tilde x)\big)\big(f\big(\tilde v(\tilde t,\tilde x)\big)-f\big(v\big(t,1^-\big)\big)\big)\psi(t,\tilde t,1,\tilde x)\\
&\quad - \operatorname{sgn}\big(\bar u-\tilde v(\tilde t,\tilde x)\big)\big(f\big(\tilde v(\tilde t,\tilde x)\big)-f\big(v\big(t,0^+\big)\big)\big)\psi(t,\tilde t,0,\tilde x)\,dt\\
&\geqslant 0.
\end{aligned}$$

Integrating the above inequality in $\tilde t,\tilde x$ (which is possible since we have a compact support) and we get:

$$\begin{aligned}
&\int_0^{+\infty}\int_0^{+\infty}\int_0^1\int_0^1 \big|v(t,x)-\tilde v(\tilde t,\tilde x)\big|\psi_t(t,\tilde t,x,\tilde x) + \operatorname{sgn}\big(v(t,x)-\tilde v(\tilde t,\tilde x)\big)\big(f\big(v(t,x)\big)-f\big(\tilde v(\tilde t,\tilde x)\big)\big)\psi_x(t,\tilde t,x,\tilde x)\\
&\quad + \operatorname{sgn}\big(v(t,x)-\tilde v(\tilde t,\tilde x)\big)h(t)\psi(t,\tilde t,x,\tilde x)\,dx\,d\tilde x\,dt\,d\tilde t + \int_0^{+\infty}\int_0^1\int_0^1 \big|u_0(x)-\tilde v(\tilde t,\tilde x)\big|\psi(0,\tilde t,x,\tilde x)\,dx\,d\tilde x\,d\tilde t\\
&\quad + \int_0^{+\infty}\int_0^1\int_0^{+\infty} \operatorname{sgn}\big(\bar u-\tilde v(\tilde t,\tilde x)\big)\big(f\big(\tilde v(\tilde t,\tilde x)\big)-f\big(v\big(t,1^-\big)\big)\big)\psi(t,\tilde t,1,\tilde x)\\
&\quad - \operatorname{sgn}\big(\bar u-\tilde v(\tilde t,\tilde x)\big)\big(f\big(\tilde v(\tilde t,\tilde x)\big)-f\big(v\big(t,0^+\big)\big)\big)\psi(t,\tilde t,0,\tilde x)\,dt\,d\tilde x\,d\tilde t\\
&\geqslant 0.
\end{aligned} \tag{37}$$

Reversing the role of v and $\tilde{v}$ we also have:

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^1\int_0^1 \left|v(t,x)-\tilde{v}(\tilde{t},\tilde{x})\right|\psi_{\tilde{t}}(t,\tilde{t},x,\tilde{x}) + \operatorname{sgn}\bigl(v(t,x)-\tilde{v}(\tilde{t},\tilde{x})\bigr)\bigl(f\bigl(v(t,x)\bigr)-f\bigl(\tilde{v}(\tilde{t},\tilde{x})\bigr)\bigr)\psi_{\tilde{x}}(t,\tilde{t},x,\tilde{x})$$
$$+\operatorname{sgn}\bigl(\tilde{v}(\tilde{t},\tilde{x})-v(t,x)\bigr)\tilde{h}(\tilde{t})\psi(t,\tilde{t},x,\tilde{x})\,dx\,d\tilde{x}\,dt\,d\tilde{t} + \int_0^{+\infty}\int_0^1\int_0^1 \left|\tilde{u}_0(\tilde{x})-v(t,x)\right|\psi(t,0,x,\tilde{x})\,d\tilde{x}\,dx\,dt$$
$$+\int_0^{+\infty}\int_0^1\int_0^{+\infty} \operatorname{sgn}\bigl(\bar{u}-v(t,x)\bigr)\bigl(f\bigl(v(t,x)\bigr)-f\bigl(\tilde{v}(\tilde{t},1^-)\bigr)\bigr)\psi(t,\tilde{t},x,1)$$
$$-\operatorname{sgn}\bigl(\bar{u}-\tilde{v}(t,x)\bigr)\bigl(f\bigl(v(t,x)\bigr)-f\bigl(\tilde{v}(\tilde{t},0^+)\bigr)\bigr)\psi(t,\tilde{t},x,0)\,dt\,dx\,d\tilde{t}$$
$$\geqslant 0. \tag{38}$$

And finally adding (37) and (38) we get:

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^1\int_0^1 \left|v(t,x)-\tilde{v}(\tilde{t},\tilde{x})\right|\bigl(\psi_t(t,\tilde{t},x,\tilde{x})+\psi_{\tilde{t}}(t,\tilde{t},x,\tilde{x})\bigr) + \operatorname{sgn}\bigl(v(t,x)-\tilde{v}(\tilde{t},\tilde{x})\bigr)\bigl(h(t)-\tilde{h}(\tilde{t})\bigr)\psi(t,\tilde{t},x,\tilde{x})$$
$$+\operatorname{sgn}\bigl(v(t,x)-\tilde{v}(\tilde{t},\tilde{x})\bigr)\bigl(f\bigl(v(t,x)\bigr)-f\bigl(\tilde{v}(\tilde{t},\tilde{x})\bigr)\bigr)\bigl(\psi_x(t,\tilde{t},x,\tilde{x})+\psi_{\tilde{x}}(t,\tilde{t},x,\tilde{x})\bigr)\,dx\,d\tilde{x}\,dt\,d\tilde{t}$$
$$+\int_0^{+\infty}\int_0^1\int_0^1 \left|\tilde{u}_0(\tilde{x})-v(t,x)\right|\psi(t,0,x,\tilde{x})\,d\tilde{x}\,dx\,dt + \int_0^{+\infty}\int_0^1\int_0^1 \left|u_0(x)-\tilde{v}(\tilde{t},\tilde{x})\right|\psi(0,\tilde{t},x,\tilde{x})\,dx\,d\tilde{x}\,d\tilde{t}$$
$$+\int_0^{+\infty}\int_0^1\int_0^{+\infty} \operatorname{sgn}\bigl(\bar{u}-v(t,x)\bigr)\bigl(f\bigl(v(t,x)\bigr)-f\bigl(\tilde{v}(\tilde{t},1^-)\bigr)\bigr)\psi(t,\tilde{t},x,1)$$
$$-\operatorname{sgn}\bigl(\bar{u}-\tilde{v}(t,x)\bigr)\bigl(f\bigl(v(t,x)\bigr)-f\bigl(\tilde{v}(\tilde{t},0^+)\bigr)\bigr)\psi(t,\tilde{t},x,0)\,dt\,dx\,d\tilde{t}$$
$$+\int_0^{+\infty}\int_0^1\int_0^{+\infty} \operatorname{sgn}\bigl(\bar{u}-\tilde{v}(\tilde{t},\tilde{x})\bigr)\bigl(f\bigl(\tilde{v}(\tilde{t},\tilde{x})\bigr)-f\bigl(v(t,1^-)\bigr)\bigr)\psi(t,\tilde{t},1,\tilde{x})$$
$$-\operatorname{sgn}\bigl(\bar{u}-\tilde{v}(\tilde{t},\tilde{x})\bigr)\bigl(f\bigl(\tilde{v}(\tilde{t},\tilde{x})\bigr)-f\bigl(v(t,0^+)\bigr)\bigr)\psi(t,\tilde{t},0,\tilde{x})\,dt\,d\tilde{x}\,d\tilde{t}$$
$$\geqslant 0. \tag{39}$$

Now consider ϕ a non-negative function in $\mathcal{C}_c^1(\mathbb{R}^2)$ and ρ a non-negative, even, $\mathcal{C}^\infty$ function with support in $[-1;1]$ and satisfying $\int_{-1}^1 \rho(x)\,dx = 1$. We define the family (ψ_n) of non-negative functions in $\mathcal{C}_c^1(\mathbb{R}^4)$ by:

$$\psi_n(t,\tilde{t},x,\tilde{x}) = n^2\phi\left(\frac{t+\tilde{t}}{2},\frac{x+\tilde{x}}{2}\right)\rho\bigl(n(t-\tilde{t})\bigr)\rho\bigl(n(x-\tilde{x})\bigr). \tag{40}$$

It is clear that for all n in $\mathbb{N}$ and all $(t,\tilde{t},x,\tilde{x})$ in $\mathbb{R}^4$:

$$\partial_t\psi_n(t,\tilde{t},x,\tilde{x}) + \partial_{\tilde{t}}\psi_n(t,\tilde{t},x,\tilde{x}) = n^2\partial_1\phi\left(\frac{t+\tilde{t}}{2},\frac{x+\tilde{x}}{2}\right)\rho\bigl(n(t-\tilde{t})\bigr)\rho\bigl(n(x-\tilde{x})\bigr),$$
$$\partial_x\psi_n(t,\tilde{t},x,\tilde{x}) + \partial_{\tilde{x}}\psi_n(t,\tilde{t},x,\tilde{x}) = n^2\partial_2\phi\left(\frac{t+\tilde{t}}{2},\frac{x+\tilde{x}}{2}\right)\rho\bigl(n(t-\tilde{t})\bigr)\rho\bigl(n(x-\tilde{x})\bigr).$$

We substitute ψ_n in (39) and let n tend to infinity. We will do so term by term.

$$\int_0^{+\infty}\int_0^1\int_0^1 \left|\tilde{u}_0(\tilde{x})-v(t,x)\right|\psi_n(t,0,x,\tilde{x})\,d\tilde{x}\,dx\,dt$$
$$=\int_0^{+\infty}\int_0^1\int_0^1 \left|\tilde{u}_0(\tilde{x})-v(t,x)\right|n^2\phi\left(\frac{t}{2},\frac{x+\tilde{x}}{2}\right)\rho(nt)\rho\big(n(x-\tilde{x})\big)\,d\tilde{x}\,dx\,dt$$
$$=\int_0^{+\infty}\int_{-n}^{n}\int_0^1 \left|\tilde{u}_0\left(X-\frac{\delta_x}{2n}\right)-v\left(\frac{\delta_t}{n},X-\frac{\delta_x}{2n}\right)\right|\phi\left(\frac{\delta_t}{2n},X\right)\rho(\delta_t)\rho(\delta_x)\,dX\,d\delta_x\,d\delta_t,$$

after the change of variable $(t,x,\tilde{x})\to(\delta_t=nt,\delta_x=n(x-\tilde{x}),X=\frac{x+\tilde{x}}{2})$. And since

$$\int_0^1 \left|\tilde{u}_0\left(X-\frac{\delta_x}{2n}\right)-v\left(\frac{\delta_t}{n},X-\frac{\delta_x}{2n}\right)\right|\phi\left(\frac{\delta_t}{2n},X\right)dX \underset{n\to+\infty}{\longrightarrow} \int_0^1 \left|\tilde{u}_0(X)-u_0(X)\right|\phi(0,X)\,dX \quad d\delta_t\,d\delta_x \text{ a.e.,}$$

we obtain:

$$\int_0^{+\infty}\int_0^1\int_0^1 \left|\tilde{u}_0(\tilde{x})-v(t,x)\right|\psi_n(t,0,x,\tilde{x})\,d\tilde{x}\,dx\,dt \underset{n\to+\infty}{\longrightarrow} \frac{1}{2}\int_0^1 \left|\tilde{u}_0(X)-u_0(X)\right|\phi(0,X)\,dX. \tag{41}$$

Note that the $\frac{1}{2}$ factor comes from integrating δ_t from 0 to $+\infty$ and because ρ is even.

The same type of reasoning implies:

$$\int_0^{+\infty}\int_0^1\int_0^1 \left|u_0(x)-\tilde{v}(\tilde{t},\tilde{x})\right|\psi_n(0,\tilde{t},x,\tilde{x})\,dx\,d\tilde{x}\,d\tilde{t} \underset{n\to+\infty}{\longrightarrow} \frac{1}{2}\int_0^1 \left|\tilde{u}_0(X)-u_0(X)\right|\phi(0,X)\,dX, \tag{42}$$

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^1\int_0^1 \left|v(t,x)-\tilde{v}(\tilde{t},\tilde{x})\right|\big(\partial_t\psi_n(t,\tilde{t},x,\tilde{x})+\partial_{\tilde{t}}\psi_n(t,\tilde{t},x,\tilde{x})\big)\,dx\,d\tilde{x}\,dt\,d\tilde{t}$$
$$\underset{n\to+\infty}{\longrightarrow} \int_0^{+\infty}\int_0^1 \left|v(T,X)-\tilde{v}(T,X)\right|\partial_T\phi(T,X)\,dX\,dT, \tag{43}$$

$$\int_0^{+\infty}\int_0^{+\infty}\int_0^1\int_0^1 \operatorname{sgn}\big(v(t,x)-\tilde{v}(\tilde{t},\tilde{x})\big)\big(f\big(v(t,x)\big)-f\big(\tilde{v}(\tilde{t},\tilde{x})\big)\big)\big(\partial_t\psi_n(t,\tilde{t},x,\tilde{x})+\partial_{\tilde{x}}\psi_n(t,\tilde{t},x,\tilde{x})\big)\,dx\,d\tilde{x}\,dt\,d\tilde{t}$$
$$\underset{n\to+\infty}{\longrightarrow} \int_0^{+\infty}\int_0^1 \operatorname{sgn}\big(v(T,X)-\tilde{v}(T,X)\big)\big(f\big(v(T,X)\big)-f\big(\tilde{v}(T,X)\big)\big)\partial_X\phi(T,X)\,dX\,dT. \tag{44}$$

In the derivation of (44) the fact that $(w,z)\to\operatorname{sgn}(z-w)(f(z)-f(w))$ is Lipschitz near $z=w$ is crucial so the lack of regularity of $(w,z)\to\operatorname{sgn}(z-w)$ prevents the same argument to work for the remaining terms. However it is clear that

$$\operatorname{sgn}\big(v(t,x)-\tilde{v}(\tilde{t},\tilde{x})\big)\big(h(t)-\tilde{h}(\tilde{t})\big)\psi(t,\tilde{t},x,\tilde{x})\leqslant\left|h(t)-\tilde{h}(\tilde{t})\right|\psi(t,\tilde{t},x,\tilde{x}).$$

So we get:

$$\limsup_{n\to+\infty}\int_0^{+\infty}\int_0^{+\infty}\int_0^1\int_0^1 \operatorname{sgn}\big(v(t,x)-\tilde v(\tilde t,\tilde x)\big)\big(h(t)-\tilde h(\tilde t)\big)\psi_n(t,\tilde t,x,\tilde x)\,dx\,d\tilde x\,dt\,d\tilde t$$
$$\leqslant \int_0^{+\infty}\int_0^1 \big|h(T)-\tilde h(T)\big|\phi(T,X)\,dX\,dT. \tag{45}$$

It only remains to control the boundary terms:

$$\int_0^{+\infty}\int_0^1\int_0^{+\infty} \operatorname{sgn}\big(\bar u - v(t,x)\big)\big(f\big(v(t,x)\big)-f\big(\tilde v(\tilde t,1^-)\big)\big)\psi_n(t,\tilde t,x,1)$$
$$-\operatorname{sgn}\big(\bar u-\tilde v(t,x)\big)\big(f\big(v(t,x)\big)-f\big(\tilde v(\tilde t,0^+)\big)\big)\psi_n(t,\tilde t,x,0)\,dt\,dx\,d\tilde t,$$

and

$$\int_0^{+\infty}\int_0^1\int_0^{+\infty} \operatorname{sgn}\big(\bar u - \tilde v(\tilde t,\tilde x)\big)\big(f\big(\tilde v(\tilde t,\tilde x)\big)-f\big(v(t,1^-)\big)\big)\psi_n(t,\tilde t,1,\tilde x)$$
$$-\operatorname{sgn}\big(\bar u-\tilde v(\tilde t,\tilde x)\big)\big(f\big(\tilde v(\tilde t,\tilde x)\big)-f\big(v(t,0^+)\big)\big)\psi_n(t,\tilde t,0,\tilde x)\,dt\,d\tilde x\,d\tilde t.$$

But if ϕ has a support in $\mathbb{R}\times(0,1)$ then for n large enough we have:

$$\psi_n(.,.,.,0)=\psi_n(.,.,.,1)=\psi_n(.,.,1,.)=\psi(.,.,0,.)=0,$$

so both boundary terms tend to 0. Combining this with (41), (42), (43), (44) and (45), we see that for every non-negative function ϕ in $\mathcal{C}_c^1(\mathbb{R}\times(0,1))$:

$$\int_0^{+\infty}\int_0^1 \big|v(t,x)-\tilde v(t,x)\big|\phi_t(t,x)+\operatorname{sgn}\big(v(t,x)-\tilde v(t,x)\big)\big(f\big(v(t,x)\big)-f\big(\tilde v(t,x)\big)\big)\phi_x(t,x)$$
$$+\big|h(t)-\tilde h(t)\big|\phi(t,x)\,dx\,dt+\int_0^1\big|u_0(x)-\tilde u_0(x)\big|\phi(0,x)\,dx$$
$$\geqslant 0. \tag{46}$$

A density argument shows that the previous estimate holds for any Lipschitz function ϕ with compact support in $\mathbb{R}\times(0,1)$. Now for $T>0$ and $n\in\mathbb{N}^*$, we define α_n and β_n as follows:

$$\alpha_n(t)=\begin{cases}1 & \text{for } t\leqslant T,\\ 0 & \text{for } t\geqslant T+\frac{1}{n},\\ 1-n(t-T) & \text{for } T\leqslant t\leqslant T+\frac{1}{n},\end{cases}$$
$$\beta_n(x)=\begin{cases}1 & \text{for } x\in[\frac{1}{n},1-\frac{1}{n}],\\ 2nx-1 & \text{for } x\in[\frac{1}{2n},\frac{1}{n}],\\ 2n(1-x)-1 & \text{for } x\in[1-\frac{1}{n},1-\frac{1}{2n}]\\ 0 & \text{otherwise.}\end{cases}$$

Taking $\phi=\alpha_n(t)\beta_n(x)$ in (46) and letting n tend to infinity we end up with:

$$\|u_0-\tilde u_0\|_{L^1(0,1)}-\big\|v(T,.)-\tilde v(T,.)\big\|_{L^1(0,1)}+\int_0^T\big|h(t)-\tilde h(t)\big|\,dt$$

$$
\begin{aligned}
&+\int_0^T \operatorname{sgn}\big(v(t,0^+)-\tilde v(t,0^+)\big)\big(f\big(v(t,0^+)\big)-f\big(\tilde v(t,0^+)\big)\big)\\
&-\operatorname{sgn}\big(v(t,1^-)-\tilde v(t,1^-)\big)\big(f\big(v(t,1^-)\big)-f\big(\tilde v(t,1^-)\big)\big)\,dt\\
&\geqslant 0.
\end{aligned}
\tag{47}
$$

Now for three numbers a, b, c we have:

$$
\begin{aligned}
&\forall k\in I(a,b)\cap I(a,c)\cap I(b,c),\\
&\quad \operatorname{sgn}(c-b)\big(f(c)-f(b)\big)=\operatorname{sgn}(c-a)\big(f(c)-f(k)\big)+\operatorname{sgn}(b-a)\big(f(b)-f(k)\big).
\end{aligned}
$$

Applying this identity with $(a=\bar u, b=v(t,1^-), c=\tilde v(t,1^-))$ or $(a=\bar u, b=v(t,0^+), c=\tilde v(t,0^+))$ and using the boundary conditions of (35) we obtain:

$$
\begin{aligned}
&\operatorname{sgn}\big(v(t,0^+)-\tilde v(t,0^+)\big)\big(f\big(v(t,0^+)\big)-f\big(\tilde v(t,0^+)\big)\big)\leqslant 0 \quad dt \text{ a.e.},\\
&\operatorname{sgn}\big(v(t,1^-)-\tilde v(t,1^-)\big)\big(f\big(v(t,1^-)\big)-f\big(\tilde v(t,1^-)\big)\big)\geqslant 0 \quad dt \text{ a.e.}
\end{aligned}
$$

Finally substituting those inequalities in (47) provides (36). □

We define $\mathcal{X}$ as the following function space:

$$
\mathcal{X}=\Big\{\alpha\in\mathcal{C}^0\big(\mathbb{R}^+\big)\ \big|\ \|\alpha\|_{\mathcal{X}}:=\sup_{t\geqslant 0}\big(\big|\alpha(t)\big|e^{-2L_G t}\big)<+\infty\Big\},
$$

where L_G is the Lipschitz constant of g (see 2.1). Consider the operator $\mathcal{F}$ which to $\alpha\in\mathcal{X}$ associates the function $\|v(t,.)-\bar u\|_{L^1(0,1)}$, with v the entropy solution of:

$$
\begin{cases}
\partial_t v+\partial_x\big(f(v)\big)=g\big(\alpha(t)\big) & \text{on } (0,+\infty)\times(0,1),\\
v(0,.)=u_0 & \text{on } (0,1),\\
\operatorname{sgn}\big(v(t,1^-)-\bar u\big)\big(f\big(v(t,1^-)\big)-f(k)\big)\geqslant 0 & \forall k\in I\big(\bar u, v(t,1^-)\big),\ dt \text{ a.e.},\\
\operatorname{sgn}\big(v(t,0^+)-\bar u\big)\big(f\big(v(t,0^+)\big)-f(k)\big)\leqslant 0 & \forall k\in I\big(\bar u, v(t,0^+)\big),\ dt \text{ a.e.}
\end{cases}
\tag{48}
$$

Then we have the following result.

Lemma 2. *The operator $\mathcal{F}$ has the following properties.*

- *For any function α in $\mathcal{X}$, the function $\mathcal{F}(\alpha)$ belongs to $\mathcal{X}$.*
- *The operator $\mathcal{F}$ is $\frac{1}{2}$-Lipschitz on $\mathcal{X}$.*

Proof. We take α in $\mathcal{X}$ and v the entropy solution of (48). The constant function $\bar u$ is a solution of (35) with source term equal to 0 and initial data equal to $\bar u$. So using (36) we have:

$$
\mathcal{F}(\alpha)(T)=\big\|v(T,.)-\bar u\big\|_{L^1(0,1)}\leqslant\|u_0-\bar u\|_{L^1(0,1)}+\int_0^T\big|g\big(\alpha(t)\big)\big|\,dt.
$$

Therefore for any $T\geqslant 0$, we have:

$$
\begin{aligned}
e^{-2L_G T}\mathcal{F}(\alpha)(T)&\leqslant\|u_0-\bar u\|_{L^1(0,1)}+\int_0^T e^{-2L_G T}g\big(\alpha(t)\big)\,dt\\
&\leqslant\|u_0-\bar u\|_{L^1(0,1)}+\int_0^T L_G e^{-2L_G(T-t)}\big|\alpha(t)\big|e^{-2L_G t}\,dt
\end{aligned}
$$

$$\leqslant \|u_0 - \bar{u}\|_{L^1(0,1)} + \|\alpha\|_{\mathcal{X}} \int_0^T L_G e^{-2L_G(T-t)}\, dt \leqslant \|u_0 - \bar{u}\|_{L^1(0,1)} + \frac{\|\alpha\|_{\mathcal{X}}}{2}.$$

It follows that $\mathcal{F}(\alpha)$ is in $\mathcal{X}$.

In order to prove the second assertion let us consider α, β in $\mathcal{X}$ and v_α, v_β the corresponding entropy solutions of (48). Using (36) we see that for any non-negative T:

$$\begin{aligned}\left|\mathcal{F}(\alpha)(T) - \mathcal{F}(\beta)(T)\right| &= \left|\left\|v_\alpha(T,.) - \bar{u}\right\|_{L^1(0,1)} - \left\|v_\beta(T,.) - \bar{u}\right\|_{L^1(0,1)}\right| \\ &\leqslant \left\|v_\alpha(T,.) - v_\beta(T,.)\right\|_{L^1(0,1)} \\ &\leqslant \int_0^T \left|g\big(\alpha(t)\big) - g\big(\beta(t)\big)\right| dt.\end{aligned}$$

But for any $T \geqslant 0$:

$$\begin{aligned}e^{-2L_G T}\left|\mathcal{F}(\alpha) - \mathcal{F}(\beta)\right|(T) &\leqslant \int_0^T L_G e^{-2L_G(T-t)}\left|\alpha(t) - \beta(t)\right| e^{-2L_G t}\, dt \\ &\leqslant \|\alpha - \beta\|_{\mathcal{X}} \int_0^T L_G e^{-2L_G(T-t)}\, dt \\ &\leqslant \frac{\|\alpha - \beta\|_{\mathcal{X}}}{2}. \qquad \square\end{aligned}$$

Let us now go back to the proof of Proposition 2.1. Applying the Banach fixed point theorem to $\mathcal{F}$, we see that (34) has a unique entropy solution u such that $\|u(T,.) - \bar{u}\|_{L^1(0,1)}$ is in $\mathcal{X}$. But if v is an entropy solution of (34) and if we use (36) we have:

$$\left\|v(T,.) - \bar{u}\right\|_{L^1(0,1)} \leqslant \|u_0 - \bar{u}\|_{L^1(0,1)} + \int_0^T L_G \left\|v(t,.) - \bar{u}\right\|_{L^1(0,1)}\, dt.$$

Using Gronwall's lemma we obtain:

$$\left\|v(T,.) - \bar{u}\right\|_{L^1(0,1)} \leqslant \|u_0 - \bar{u}\|_{L^1(0,1)} e^{L_G T}.$$

Thus the application:

$$T \mapsto \left\|v(T,.) - \bar{u}\right\|_{L^1(0,1)},$$

is in $\mathcal{X}$ and therefore $v = u$. Using Lemma 1 and Gronwall's lemma we have that for u and v the entropy solutions to (34) for initial data u_0 and v_0:

$$\forall t \geqslant 0, \quad \left\|u(t,.) - v(t,.)\right\|_{L^1(0,1)} \leqslant \|u_0 - v_0\|_{L^1(0,1)} e^{L_G t}.$$

This concludes the proof of Proposition 2.1.

3. Generalized characteristics and boundary conditions

We begin by recalling a few definitions and results from [11]. We will refer in this section to the system:

$$\begin{cases} \partial_t u + \partial_x\big(f(u)\big) = h(t) & \text{on } (0,+\infty) \times (0,1), \\ u(0,.) = u_0 & \text{on } (0,1), \\ \operatorname{sgn}\big(u(t,1^-) - \bar{u}\big)\big(f\big(u(t,1^-)\big) - f(k)\big) \geqslant 0 & \forall k \in I\big(\bar{u}, u(t,1^-)\big),\ dt \text{ a.e.}, \\ \operatorname{sgn}\big(u(t,0^+) - \bar{u}\big)\big(f\big(u(t,0^+)\big) - f(k)\big) \leqslant 0 & \forall k \in I\big(\bar{u}, u(t,0^+)\big),\ dt \text{ a.e.}, \end{cases} \tag{49}$$

where $u_0 \in \mathrm{BV}(0,1)$, $h \in \mathcal{C}^0(\mathbb{R}^+)$, $\bar{u} \in \mathbb{R}$ and u is the unique entropy solution. Following [11] we introduce the notion of generalized characteristic.

Definition 2.

- If γ is an absolutely continuous function defined on an interval $(a,b) \subset \mathbb{R}^+$ and with values in $(0,1)$, we say that γ is a generalized characteristic of (49) if:

$$\dot{\gamma}(t) \in I\big(f'\big(u\big(t,\gamma(t)^-\big)\big), f'\big(u\big(t,\gamma(t)^+\big)\big)\big) \quad dt \text{ a.e.}$$

  This is the classical characteristic ODE taken in the weak sense of Filippov [13].
- A generalized characteristic γ is said to be genuine on (a,b) if:

$$u\big(t,\gamma(t)^+\big) = u\big(t,\gamma(t)^-\big) \quad dt \text{ a.e.}$$

We recall the following results from [11].

Theorem 3.

- *For any (t,x) in $(0,+\infty) \times (0,1)$ there exists at least one generalized characteristic γ defined on (a,b) such that $a < t < b$ and $\gamma(t) = x$.*
- *If γ is a generalized characteristics defined on (a,b) then for almost all t in (a,b):*

$$\dot{\gamma}(t) = \begin{cases} f'(u(t,\gamma(t))) & \text{if } u(t,\gamma(t)^+) = u(t,\gamma(t)^-), \\ \frac{f(u(t,\gamma(t)^+)) - f(u(t,\gamma(t)^-))}{u(t,\gamma(t)^+) - u(t,\gamma(t)^-)} & \text{if } u(t,\gamma(t)^+) \neq u(t,\gamma(t)^-). \end{cases}$$

- *If γ is a genuine generalized characteristics on (a,b), then there exists a C^1 function v defined on (a,b) such that:*

$$\begin{aligned} &u\big(b,\gamma(b)^+\big) \leqslant v(b) \leqslant u\big(b,\gamma(b)^-\big), \\ &u\big(t,\gamma(t)^+\big) = v(t) = u\big(t,\gamma(t)^-\big) \quad \forall t \in (a,b), \\ &u\big(a,\gamma(a)^-\big) \leqslant v(a) \leqslant u\big(a,\gamma(a)^+\big). \end{aligned} \tag{50}$$

  Furthermore (γ, v) satisfy the classical ODE equation:

$$\begin{cases} \dot{\gamma}(t) = f'\big(v(t)\big), \\ \dot{v}(t) = h(t), \end{cases} \quad \forall t \in (a,b). \tag{51}$$

- *Two genuine characteristics may intersect only at their endpoints.*
- *If γ_1 and γ_2 are two generalized characteristics defined on (a,b), then we have:*

$$\forall t \in (a,b) \quad \big(\gamma_1(t) = \gamma_2(t) \Rightarrow \forall s \geqslant t,\ \gamma_1(s) = \gamma_2(s)\big).$$

- *For any (t,x) in $\mathbb{R}^+ \times (0,1)$ there exist two generalized characteristics χ^+ and χ^- called maximal and minimal and associated to v^+ and v^- by (51), such that if γ is a generalized characteristic going through (t,x) then*

$$\begin{aligned} &\forall s \leqslant t, \quad \chi^-(s) \leqslant \gamma(s) \leqslant \chi^+(s), \\ &\chi^+ \text{ and } \chi^- \text{ are genuine on } \{s < t\}, \\ &v^+(t) = u\big(t,x^+\big) \quad \text{and} \quad v^-(t) = u\big(t,x^-\big). \end{aligned}$$

Note that in the previous theorem, every property dealt only with the interior of $\mathbb{R}^+ \times [0,1]$. We will now be interested in the influence of the boundary conditions on the generalized characteristics. Following the method of [11], we begin with a few technical identities.

Lemma 3.

- *If χ is a Lipschitz function defined on* $[a,b]$ *and satisfying*:

$$\forall t \in (a,b), \quad 0 \leqslant \chi(t) < 1, \tag{52}$$

we have:

$$\int_0^{\chi(b)} u(b,x)\,dx - \int_0^{\chi(a)} u(a,x)\,dx$$

$$= \int_a^b \chi(t)h(t)\,dt + \int_a^b f\big(u\big(t,0^+\big)\big) - f\big(u\big(t,\chi(t)^+\big)\big) + \dot{\chi}(t)u\big(t,\chi(t)^+\big)\,dt. \tag{53}$$

- *If χ is a Lipschitz function defined on* $[a,b]$ *and such that*:

$$\forall t \in (a,b), \quad 1 \geqslant \chi(t) > 0,$$

we have:

$$\int_{\chi(b)}^{1} u(b,x)\,dx - \int_{\chi(a)}^{1} u(a,x)\,dx$$

$$= \int_a^b \big(1-\chi(t)\big)h(t)\,dt + \int_a^b f\big(u\big(t,\chi(t)^-\big)\big) - f\big(u\big(t,1^-\big)\big) - \dot{\chi}(t)u\big(t,\chi(t)^-\big)\,dt. \tag{54}$$

- *Finally if χ_1 and χ_2 are two Lipschitz functions defined on* $[a,b]$ *and satisfying*:

$$\forall t \in (a,b), \quad 0 < \chi_1(t) < \chi_2(t) < 1,$$

the following holds:

$$\int_{\chi_1(b)}^{\chi_2(b)} u(b,x)\,dx - \int_{\chi_1(a)}^{\chi_2(a)} u(a,x)\,dx = \int_a^b h(t)\big(\chi_2(t)-\chi_1(t)\big)\,dt$$

$$+ \int_a^b f\big(u\big(t,\chi_1(t)^-\big)\big) - f\big(u\big(t,\chi_2(t)^+\big)\big) + \dot{\chi}_2(t)u\big(t,\chi_2(t)^+\big) - \dot{\chi}_1(t)u\big(t,\chi_1(t)^-\big)\,dt. \tag{55}$$

Proof. We begin with the proof of (53). We may prove the equality when (52) holds on $[a,b]$ and then extend it since both sides of (53) are continuous in a and b. For $\epsilon > 0$ we define the following two functions:

$$\psi_\epsilon(t,x) = \begin{cases} 1 & \text{when } t \in [a,b] \text{ and } 0 \leqslant x \leqslant \chi(t), \\ 1 - \frac{x-\chi(t)}{\epsilon} & \text{when } t \in [a,b] \text{ and } \chi(t) \leqslant x \leqslant \chi(t)+\epsilon, \\ 0 & \text{otherwise,} \end{cases}$$

$$\forall t \in \mathbb{R}^+, \quad \rho_\epsilon(t) = \begin{cases} 1 & \text{when } a+\epsilon \leqslant t \leqslant b, \\ \frac{b-t}{\epsilon} & \text{when } b-\epsilon \leqslant t \leqslant b, \\ \frac{t-a}{\epsilon} & \text{when } a \leqslant t \leqslant a+\epsilon, \\ 0 & \text{otherwise.} \end{cases}$$

The product $\rho_\epsilon \psi_\epsilon$ is Lipschitz, non-negative and has compact support in $\mathbb{R}^+ \times [0,1]$. Since u is a weak solution of (49) we have:

$$\begin{aligned}&\int_0^{+\infty}\int_0^1 u(t,x)\big(\partial_t\rho_\epsilon(t)\psi_\epsilon(t,x)+\rho_\epsilon(t)\partial_t\psi_\epsilon(t,x)\big)+f\big(u(t,x)\big)\rho_\epsilon(t)\partial_x\psi_\epsilon(t,x)+h(t)\rho_\epsilon(t)\psi_\epsilon(t,x)\,dx\,dt\\&\quad+\int_0^T u_0(x)\rho_\epsilon(0)\psi_\epsilon(0,x)\,dx+\int_0^{+\infty} f\big(u\big(t,0^+\big)\big)\rho_\epsilon(t)\psi_\epsilon(t,0)-f\big(u\big(t,1^-\big)\big)\rho_\epsilon(t)\psi_\epsilon(t,1)\,dt\\&=0.\end{aligned}\tag{56}$$

It is easy to see that:

- for $\epsilon>0$, $\rho_\epsilon(0)=0$,
- for ϵ small enough: $\rho(t)\psi_\epsilon(t,1)=0$ for all t,
- when $\epsilon\to 0$, $\forall t\geqslant 0$, $\rho_\epsilon(t)\psi_\epsilon(t,0)\to \mathbb{1}_{[a,b]}(t)$,
- when $\epsilon\to 0$, we have:

$$\int_0^{+\infty}\int_0^1 u(t,x)\partial_t\rho_\epsilon(t)\psi_\epsilon(t,x)\,dx\,dt\to\int_0^{\chi(a)} u(a,x)\,dx-\int_0^{\chi(b)} u(b,x)\,dx,$$
$$\int_0^{+\infty}\int_0^1 u(t,x)\rho_\epsilon(t)\partial_t\psi_\epsilon(t,x)\,dx\,dt\to\int_a^b \dot\chi(t)u\big(t,\chi(t)^+\big)\,dt,$$
$$\int_0^{+\infty}\int_0^1 f\big(u(t,x)\big)\rho_\epsilon(t)\partial_x\psi_\epsilon(t,x)\,dx\,dt\to-\int_a^b f\big(u\big(t,\chi(t)^+\big)\big)\,dt.$$

Therefore taking the limit in (56) we get:

$$\int_0^{\chi(a)} u(a,x)\,dx-\int_0^{\chi(b)} u(b,x)\,dx+\int_a^T h(t)\,dt+\int_a^T f\big(u\big(t,0^+\big)\big)-f\big(u\big(t,\chi(t)^+\big)\big)+\dot\chi(t)u\big(t,\chi(t)^+\big)\,dt=0,$$

which is exactly (53).

The proof of (54) is symmetrical and is omitted. And for (55) we use the same ideas but with the following test functions:

$$\psi_\epsilon(t,x)=\begin{cases}1 & \text{when } t\in[a,b] \text{ and } \chi_1(t)\leqslant x\leqslant\chi_2(t),\\ 1-\frac{x-\chi_2(t)}{\epsilon} & \text{when } t\in[a,b] \text{ and } \chi_2(t)\leqslant x\leqslant\chi_2(t)+\epsilon,\\ \frac{x+\epsilon-\chi_1(t)}{\epsilon} & \text{when } t\in[a,b] \text{ and } \chi_1(t)-\epsilon\leqslant x\leqslant\chi_1(t),\\ 0 & \text{otherwise,}\end{cases}$$

$$\forall t\in\mathbb{R}^+,\quad \rho_\epsilon(t)=\begin{cases}1 & \text{when } a+\epsilon\leqslant t\leqslant b-\epsilon,\\ \frac{b-t}{\epsilon} & \text{when } b-\epsilon\leqslant t\leqslant b,\\ \frac{t-a}{\epsilon} & \text{when } a\leqslant t\leqslant a+\epsilon,\\ 0 & \text{otherwise.}\end{cases}\qquad\square$$

Let us also show the additional lemma.

Lemma 4. *Consider $t>0$ and x in $\{0,1\}$ and suppose that one of the following conditions is satisfied*:

$$x=1\quad\text{and}\quad f'\big(u\big(t,x^-\big)\big)>0,$$
$$x=0\quad\text{and}\quad f'\big(u\big(t,x^+\big)\big)<0.$$

Then there is a genuine backward characteristic γ going through (t,x) and such that:

$$\dot{\gamma}(t) = \begin{cases} f'(u(t,1^-)) & \text{if } x = 1, \\ f'(u(t,0^+)) & \text{if } x = 0. \end{cases}$$

Proof. We will prove only the first case, the second one being identical. Let (x_n) be an increasing sequence in $(0,1)$ such that $x_n \underset{n\to+\infty}{\longrightarrow} 1$. We immediately see that:

$$f'\big(u(t,x_n)\big) \underset{n\to+\infty}{\longrightarrow} f'\big(u\big(t,1^-\big)\big), \tag{57}$$

and so we can suppose that:

$$\forall n \geqslant 0, \quad f'\big(u(t,x_n)\big) \geqslant \frac{f'(u(t,1^-))}{2}. \tag{58}$$

Now consider χ_n the maximal generalized backward characteristic going through (t,x_n) and v_n the function associated to χ_n by (51). Using (58) and the continuity of h, we deduce that there exists $\epsilon > 0$ independent of n such that if the functions χ_n, v_n are maximally defined on an interval I then $[t-\epsilon,t] \subset I$. Now a classical ODE result asserts that because x_n and $f'(u(t,x_n))$ converge then the functions χ_n and v_n converge uniformly toward two functions γ and v satisfying:

$$\forall s \in [t-\epsilon,t], \quad \begin{cases} \dot{v}(s) = h(s), & v(t) = u\big(t,1^-\big), \\ \dot{\gamma}(s) = f'\big(v(s)\big), & \gamma(t) = 1. \end{cases} \tag{59}$$

It is known that the uniform limit of generalized characteristics is a generalized characteristic (see [16, Chapter 1] or [11, Chapter 10]) therefore γ is a generalized characteristic. It is genuine since it satisfies (59). □

In the remaining part of this section, we will suppose:

$$f'(\bar{u}) \geqslant 0 \quad \text{and} \quad \forall t \in \mathbb{R}^+, \quad h(t) \geqslant 0. \tag{60}$$

Remark 5. Note that h being non-negative and f being convex, any genuine generalized characteristic is also convex since it satisfies ODE (51).

We will now describe the behavior of generalized characteristics at boundary points.

Proposition 3.1. *There is no genuine generalized characteristic γ defined on (a,b) with $a > 0$ and such that*:

$$\gamma(t) \underset{t\to a^+}{\longrightarrow} 1. \tag{61}$$

Proof. Let us suppose that (61) is false. (The construction is depicted in Fig. 1.) We have a genuine characteristic γ defined on (a,b) such that $\gamma(a) = 1$ and $a > 0$. Thanks to Remark 5, we know

$$f'\big(v(a)\big) < 0.$$

Since we supposed $f'(\bar{u}) \geqslant 0$ this forces f to be of type I. Thus we have a unique u^* such that $f'(u^*) = 0$ and the boundary condition at $x = 1$ becomes:

$$u\big(t,1^-\big) \geqslant u^*, \quad dt \text{ a.e.} \tag{62}$$

Now consider $\epsilon > 0$ and apply (55) with $\chi_1(t) = \gamma(t) - \epsilon$ and $\chi_2 = \gamma$. Then for T in (a,b):

$$\int_{\gamma(T)-\epsilon}^{\gamma(T)} u(T,x)\,dx - \int_{1-\epsilon}^{1} u(a,x)\,dx$$
$$= \epsilon \int_a^T h(s)\,ds + \int_a^T f\big(u\big(s,\gamma(s)-\epsilon^-\big)\big) - f\big(u\big(s,\gamma(s)^+\big)\big) - \dot{\gamma}(s)\big(u\big(s,\gamma(s)-\epsilon^-\big) - u\big(s,\gamma(s)^+\big)\big)\,ds.$$

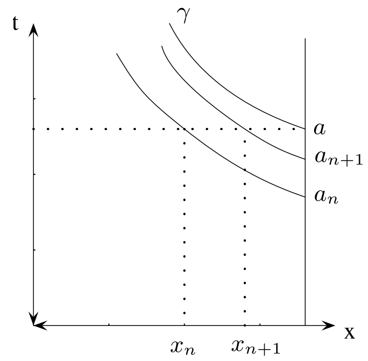

Fig. 1. Characteristic entering from the right.

Since $u(s,\gamma(s)^+) = v(s)$, $\dot\gamma(s) = f'(v(s))$ and f is convex we obtain:

$$\int_{\gamma(T)-\epsilon}^{\gamma(T)} u(T,x)\,dx - \int_{1-\epsilon}^{1} u(a,x)\,dx \geqslant \epsilon\big(v(T) - v(a)\big).$$

Therefore after dividing by ϵ and taking the limit $\epsilon \to 0$ we arrive at:

$$v(T) - u\big(a,1^-\big) \geqslant v(T) - v(a). \tag{63}$$

And so:

$$f'\big(u\big(a,1^-\big)\big) \leqslant f'\big(v(a)\big) < 0.$$

Consider $(x_n)_{n\geqslant 0}$ satisfying:

$$\forall n \geqslant 0, \quad 0 < x_n < x_{n+1} < 1,$$
$$u(a,.) \text{ is continuous at every } x_n,$$
$$x_n \underset{n\to+\infty}{\longrightarrow} 1,$$
$$\forall n \geqslant 0, \quad f'\big(u(a,x_n)\big) \leqslant \frac{f'(u(a,1^-))}{2}.$$

This sequence exists thanks to (63) and also because $u(a,.)$ is in $\mathrm{BV}(0,1)$. Using Theorem 3 we know that for any n, there exist a unique number $a_n < a$ and two regular functions γ_n and v_n solutions of:

$$\begin{cases} \dot{v_n}(t) = h(t), \\ v_n(a) = u(a,x_n), \end{cases} \tag{64}$$

$$\begin{cases} \dot{\gamma_n}(t) = f'\big(v_n(t)\big), \\ \gamma_n(a) = x_n, \end{cases} \tag{65}$$

$$\gamma_n(a_n) = 1, \tag{66}$$

maximally defined on (a_n,a). Furthermore γ_n is a genuine generalized characteristic on (a_n,a). Using the fact that x_n is increasing and Theorem 3, we can see that a_n is non-decreasing. Furthermore using $f'(v_n(a)) \leqslant \frac{f'(u(a,1^-))}{2} < 0$, $h \geqslant 0$ and f convex, we obtain:

$$a_n \underset{n\to+\infty}{\longrightarrow} a.$$

Suppose now that given n, we have a certain T such that $a_n < T < a$ and $f'(u(T,1^-)) > 0$. Using Lemma 4, we get a time $R < T$ and a backward characteristic δ issued from $(T,1)$, defined on $[R,T]$ and genuine on (R,T). We also have $R \geqslant a_n$ and $\delta(R) = 1$ because γ_n and δ do not cross. Additionally if w is the regular function associated to δ by (51), we have $w(T) = u(T,1^-)$. It follows that:

$$
\begin{aligned}
f'\big(u\big(T,1^-\big)\big) = f'\big(w(T)\big) &= \frac{1}{T-R}\int_R^T \big(f'\big(w(T)\big) - f'\big(w(t)\big)\big)\,dt \\
&\leqslant \frac{\|f''\|_{L^\infty}}{T-R}\int_R^T\int_t^T (s)\,ds\,dt \\
&\leqslant \frac{T-R}{2}\big\|f''\big\|_{L^\infty}\|h\|_{L^\infty(R,T)} \\
&\leqslant \frac{a-a_n}{2}\big\|f''\big\|_{L^\infty}\|h\|_{L^\infty(R,T)}.
\end{aligned}
$$

Combined with (62) this implies:

$$
\operatorname*{essup}_{t\in[a-\nu,a]} f'\big(u\big(t,1^-\big)\big) \underset{\nu\to 0^+}{\longrightarrow} 0. \tag{67}
$$

Since f is convex and there is only one number u^* such that $f'(u^*) = 0$, we can deduce:

$$
\operatorname*{essup}_{t\in[a-\nu,a]} f\big(u\big(t,1^-\big)\big) \underset{\nu\to 0^+}{\longrightarrow} f\big(u^*\big). \tag{68}
$$

Let us now consider any number u_i such that $f'(u_i) < 0$ and define $\chi_\epsilon(t) = 1-\epsilon + f'(u_i)(t-a)$. Applying (54) between $a(\epsilon) = a + \frac{\epsilon}{f'(u_i)}$ and a we obtain:

$$
\begin{aligned}
\int_{1-\epsilon}^{1} u(a,x)\,dx &= \int_{a(\epsilon)}^{a}\big(1-\chi_\epsilon(t)\big)h(t)\,dt + \int_{a(\epsilon)}^{a}\big(f\big(u\big(t,\chi_\epsilon(t)^-\big)\big) - f\big(u\big(t,1^-\big)\big) - \dot\chi_\epsilon(t)u\big(t,\chi_\epsilon(t)^-\big)\big)\,dt \\
&= \int_{a(\epsilon)}^{a}\big(1-\chi_\epsilon(t)\big)h(t)\,dt + \int_{a(\epsilon)}^{a}\big(f\big(u\big(t,\chi_\epsilon(t)^-\big)\big) - f(u_i) - \dot\chi_\epsilon(t)\big(u\big(t,\chi_\epsilon(t)^-\big) - u_i\big)\big)\,dt \\
&\quad + \int_{a(\epsilon)}^{a}\big(f(u_i) - f\big(u\big(t,1^-\big)\big)\big)\,dt + \int_{a(\epsilon)}^{a}\dot\chi_\epsilon(t)u_i\,dt.
\end{aligned}
$$

Using the convexity of f and since $\dot\chi_\epsilon = f'(u_i)$ we have:

$$
\forall t\in\big(a(\epsilon),a\big),\quad f\big(u\big(t,\chi_\epsilon(t)^-\big)\big) - f(u_i) - \dot\chi_\epsilon(t)\big(u\big(t,\chi_\epsilon(t)^-\big) - u_i\big) \geqslant 0. \tag{69}
$$

We have supposed $f'(u_i) < 0$ therefore $f(u_i) > f(u^*)$ and then for ϵ small enough, $a(\epsilon)$ is close enough to a to guarantee, thanks to (68):

$$
f(u_i) \geqslant f\big(u\big(t,1^-\big)\big)\quad dt \text{ a.e. on } \big(a(\epsilon),a\big).
$$

Thus we have:

$$
\int_{1-\epsilon}^{1} u(a,x)\,dx \geqslant \int_{a(\epsilon)}^{a}\big(1-\chi_\epsilon(t)\big)h(t)\,dt + \epsilon u_i.
$$

And now dividing by ϵ and letting ϵ tend to 0 we obtain:

$$
\forall u_i \text{ s.t. } f'(u_i) < 0,\quad u\big(a,1^-\big) \geqslant u_i.
$$

In turn this implies $f'(u(a,1^-)) \geqslant 0$ and we have a contradiction. □

Proposition 3.2. *Consider a genuine generalized characteristic γ defined on (a,b) with $a > 0$ and v the regular function associated to γ by (51). If $\gamma(t) \underset{t\to a}{\longrightarrow} 0$ then $v(t) \underset{t\to a}{\longrightarrow} \bar u$.*

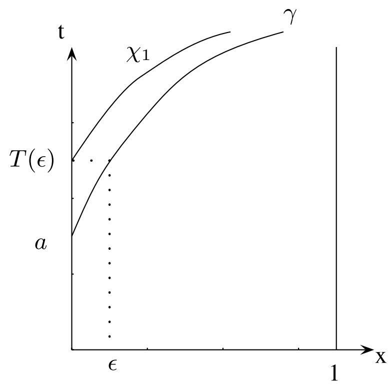

Fig. 2. Characteristic entering from the left.

Proof. We will proceed in two steps.

• First let us show that $v(a) \geqslant \bar{u}$. (The construction is depicted in Fig. 2.) Once again we consider $\epsilon > 0$ small enough and define the time

$$T(\epsilon) = \inf\bigl\{t \in (a,b) \bigm| \gamma(t) \geqslant \epsilon\bigr\}.$$

Then if we apply (55) to $\chi_1(t) = \gamma(t) - \epsilon$ and $\chi_2(t) = \gamma(t)$ on $[T(\epsilon), b]$ we get:

$$\int_{\gamma(b)-\epsilon}^{\gamma(b)} u(b,x)\,dx - \int_0^{\epsilon} u\bigl(T(\epsilon),x\bigr)\,dx$$
$$= \epsilon \int_{T(\epsilon)}^{b} h(t)\,dt + \int_{T(\epsilon)}^{b} \Bigl(f\bigl(u\bigl(t,\gamma(t)-\epsilon^-\bigr)\bigr) - f\bigl(u\bigl(t,\gamma(t)^+\bigr)\bigr) + \dot{\gamma}(t)\bigl(u\bigl(t,\gamma(t)^+\bigr) - u\bigl(t,\gamma(t)-\epsilon^-\bigr)\bigr)\Bigr)\,dt.$$

We apply (53) to $\chi(t) = \gamma(t)$ on $[a, T(\epsilon)]$ to have:

$$\int_0^{\epsilon} u\bigl(T(\epsilon),x\bigr)\,dx - \int_0^{\gamma(a)} u(a,x)\,dx$$
$$= \int_a^{T(\epsilon)} \gamma(t)h(t)\,dt + \int_a^{T(\epsilon)} \Bigl(f\bigl(u\bigl(t,0^+\bigr)\bigr) - f\bigl(u\bigl(t,\gamma(t)^+\bigr)\bigr) + \dot{\gamma}(t)u\bigl(t,\gamma(t)^+\bigr)\Bigr)\,dt.$$

Adding the two previous equalities and remembering that:

$$\gamma(a) = 0, \qquad u\bigl(t,\gamma(t)\bigr) = v(t) \quad \text{and} \quad \dot{\gamma}(t) = f'\bigl(v(t)\bigr),$$

we obtain:

$$\int_{\gamma(b)-\epsilon}^{\gamma(b)} u(b,x)\,dx = \epsilon \int_{T(\epsilon)}^{b} h(t)\,dt + \int_a^{T(\epsilon)} \gamma(t)h(t)\,dt + \int_a^{T(\epsilon)} \Bigl(f\bigl(u\bigl(t,0^+\bigr)\bigr) - f\bigl(v(t)\bigr) + f'\bigl(v(t)\bigr)v(t)\Bigr)\,dt$$
$$+ \int_{T(\epsilon)}^{b} \Bigl(f\bigl(u\bigl(t,\gamma(t)-\epsilon^-\bigr)\bigr) - f\bigl(v(t)\bigr) + f'\bigl(v(t)\bigr)\bigl(v(t) - u\bigl(t,\gamma(t)-\epsilon^-\bigr)\bigr)\Bigr)\,dt.$$

Now using the fact that $f(u(t,0^+)) \geqslant f(\bar{u})$ for almost all t (thanks to the boundary condition at $x = 0$ see (49)) and remembering that f is convex we have:

$$\begin{aligned}
&\int_{T(\epsilon)}^{b} \big(f\big(u\big(t,0^+\big)\big) - f\big(v(t)\big) + f'\big(v(t)\big)v(t)\big)\,dt \\
&\quad \geqslant \int_{T(\epsilon)}^{b} \big(f(\bar{u}) - f\big(v(t)\big) + f'\big(v(t)\big)v(t)\big)\,dt \\
&\quad \geqslant \int_{T(\epsilon)}^{b} \big(f(\bar{u}) - f\big(v(t)\big) - f'\big(v(t)\big)\big(\bar{u} - v(t)\big)\big)\,dt + \bar{u}\int_{T(\epsilon)}^{b} \dot{\gamma}(t)\,dt \\
&\quad \geqslant \epsilon\bar{u}.
\end{aligned}$$

The convexity of f also implies:

$$\int_{T(\epsilon)}^{b} f\big(u\big(t,\gamma(t)-\epsilon^-\big)\big) - f\big(v(t)\big) + f'\big(v(t)\big)\big(v(t) - u\big(t,\gamma(t)-\epsilon^-\big)\big)\,dt \geqslant 0.$$

But thanks to (51) we know that:

$$\int_{T(\epsilon)}^{b} h(t)\,dt = v(b) - v\big(T(\epsilon)\big),$$

so in the end we have for any ϵ positive and small enough:

$$\int_{\gamma(b)-\epsilon}^{\gamma(b)} u(b,x)\,dx \geqslant \int_{a}^{T(\epsilon)} \gamma(t)h(t)\,dt + \epsilon\big(v(b) - v\big(T(\epsilon)\big)\big) + \epsilon\bar{u}.$$

Dividing by ϵ and letting it tend to 0 provides (thanks to $0 \leqslant \gamma(t) \leqslant \epsilon$ and $T(\epsilon) \to 0$):

$$v(b) \geqslant v(b) - v(a) + \bar{u},$$

which is indeed $v(a) \geqslant \bar{u}$.

- Now to prove $v(a) \leqslant \bar{u}$, let us suppose $v(a) > \bar{u}$.
  For ϵ positive if we apply (55) to $\chi_1 = \gamma$ and $\chi_2 = \gamma + \epsilon$ between a and $t > a$, we have:

$$\begin{aligned}
\int_{\gamma(t)}^{\gamma(t)+\epsilon} u(t,x)\,dx - \int_{0}^{\epsilon} u(a,x)\,dx &= \epsilon\int_{a}^{t} h(s)\,ds + \int_{a}^{t} f\big(v(t)\big) - f\big(u\big(t,\big(\gamma(t)+\epsilon\big)^+\big)\big) \\
&\quad + f'\big(v(t)\big)\big(u\big(t,\big(\gamma(t)+\epsilon\big)^+\big) - v(t)\big)\,dt \\
&\leqslant \epsilon\int_{a}^{t} h(s)\,ds \\
&\leqslant v(t) - v(a).
\end{aligned}$$

Dividing by ϵ and letting it tend to 0 provides:

$$u\big(t,\gamma(t)^+\big) - u\big(a,0^+\big) \leqslant v(t) - v(a).$$

Since γ is genuine this implies:

$$\bar{u} < v(a) \leqslant u\big(a,0^+\big). \tag{70}$$

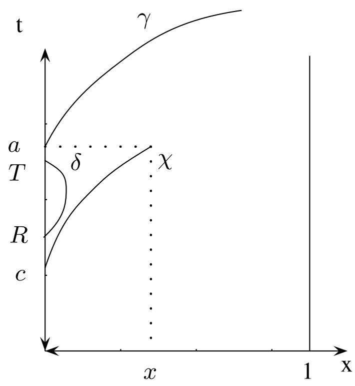

Fig. 3. Nonconvex characteristic.

Now consider x in $(0,1)$, χ the minimal generalized characteristic through (a,x) and w the function associated to it by (51). We can see that if x is close enough to 0 then thanks to $f'(u(a,0^+)) > 0$ there exists $c < a$ such that

$$\chi(c) = 0.$$

Consider T such that $c < T < a$ and suppose that $f'(u(T,0^+)) < 0$. Using Lemma 4 we get a generalized characteristic δ on $[R,T]$, genuine on (R,T) and such that $\delta(T) = 0$. But we would also have $R \geqslant c$ and $\delta(R) = 0$ because δ and γ do not cross (Theorem 3). This is impossible because thanks to Remark 5, δ is convex, therefore no such T exists. (See Fig. 3.)

Therefore we have:

$$\forall T \in (c,a), \quad f'\big(u\big(T,0^+\big)\big) \geqslant 0.$$

Combined with the boundary condition at $x = 0$ this implies:

$$\text{for almost all } t \text{ in } (c,a), \quad u\big(t,0^+\big) = \bar{u}.$$

Now we consider u_i larger than $\bar{u}$. This implies that $f'(u_i)$ is positive. We define γ_ϵ and a_ϵ by:

$$\gamma_\epsilon(t) = \epsilon + f'(u_i)(t-a),$$
$$a_\epsilon = \frac{\epsilon}{f'(u_i)}.$$

If we apply (53) with $\chi = \gamma_\epsilon$ for ϵ small enough (so that $a_\epsilon \geqslant c$), we obtain:

$$\begin{aligned}\int_0^\epsilon u(a,x)\,dx &= \int_{a_\epsilon}^a \gamma_\epsilon(t)h(t)\,dt + \int_{a_\epsilon}^a \Big(f\big(u\big(t,0^+\big)\big) - f\big(u\big(t,\gamma_\epsilon(t)^+\big)\big) + f'(u_i)u\big(t,\gamma_\epsilon(t)^+\big)\Big)\,dt \\ &\leqslant \int_{a_\epsilon}^a \gamma_\epsilon(t)h(t)\,dt + \int_{a_\epsilon}^a \Big(f(u_i) - f\big(u\big(t,\gamma_\epsilon(t)^+\big)\big) + f'(u_i)\big(u\big(t,\gamma_\epsilon(t)^+\big) - u_i\big)\Big)\,dt + \epsilon u_i \\ &\leqslant \int_{a_\epsilon}^a \gamma_\epsilon(t)h(t)\,dt + \epsilon u_i.\end{aligned}$$

Dividing by ϵ and letting it tend to 0 provides:

$$u\big(a,0^+\big) \leqslant u_i.$$

Since u_i can be arbitrarily close to $\bar{u}$, we end up with:

$$v(a) \leqslant u\big(a,0^+\big) \leqslant \bar{u}. \qquad \square$$

Proposition 3.3. *If γ is a genuine generalized characteristic defined on (a,b) with $a>0$ and v is the regular function associated to γ by (51). Suppose:*

$$\gamma(t)\underset{t\to b}{\longrightarrow}0 \quad \text{and} \quad v(t)\underset{t\to b}{\longrightarrow}\bar{v}, \tag{71}$$

then

$$f'(\bar{v})\leqslant 0 \quad \text{and} \quad f(\bar{v})\geqslant f(\bar{u}). \tag{72}$$

Proof. Since γ is convex and since for t in (a,b) $\gamma(t)>0=\gamma(b)$, we have:

$$\forall t\in(a,b),\quad \dot{\gamma}(t)\leqslant 0.$$

Letting t tend to b we have

$$f'(\bar{v})=\lim_{t\to b}f'\big(v(t)\big)=\lim_{t\to b}\dot{\gamma}(t)\leqslant 0.$$

Now consider ϵ positive and define:

$$T(\epsilon)=\sup\big\{t\in(a,b)\;\big|\;\gamma(t)\geqslant\epsilon\big\}.$$

We apply (55) to $\chi_1(t)=\gamma(t)-\epsilon$ and $\chi_2(t)=\gamma(t)$ on $[a,T(\epsilon)]$ and get:

$$\begin{aligned}&\int_0^{\epsilon}u\big(T(\epsilon),x\big)\,dx-\int_{\gamma(a)-\epsilon}^{\gamma(a)}u(a,x)\,dx\\&\quad=\epsilon\int_a^{T(\epsilon)}h(t)\,dt+\int_a^{T(\epsilon)}\Big(f\big(u\big(t,\big(\gamma(t)-\epsilon\big)^-\big)\big)-f\big(u\big(t,\gamma(t)^+\big)\big)+\dot{\gamma}(t)\big(u\big(t,\gamma(t)^+\big)-u\big(t,\big(\gamma(t)-\epsilon\big)^-\big)\big)\Big)\,dt.\end{aligned}$$

We apply (53) to $\chi(t)=\gamma(t)$ on $[T(\epsilon),b]$ to obtain:

$$\begin{aligned}&\int_0^{\gamma(b)}u(b,x)\,dx-\int_0^{\epsilon}u\big(T(\epsilon),x\big)\,dx\\&\quad=\int_{T(\epsilon)}^{b}\gamma(t)h(t)\,dt+\int_{T(\epsilon)}^{b}\Big(f\big(u\big(t,0^+\big)\big)-f\big(u\big(t,\gamma(t)^+\big)\big)+\dot{\gamma}(t)u\big(t,\gamma(t)^+\big)\Big)\,dt.\end{aligned}$$

As in the proof of Proposition 3.2, we add the two previous equalities and remember that:

$$\gamma(b)=0,\qquad u\big(t,\gamma(t)\big)=v(t)\quad\text{and}\quad\dot{\gamma}(t)=f'\big(v(t)\big),$$

to get:

$$\begin{aligned}-\int_{\gamma(a)-\epsilon}^{\gamma(a)}u(a,x)\,dx&=\epsilon\int_a^{T(\epsilon)}h(t)\,dt+\int_{T(\epsilon)}^{b}\gamma(t)h(t)\,dt+\int_{T(\epsilon)}^{b}\Big(f\big(u\big(t,0^+\big)\big)-f\big(v(t)\big)+f'\big(v(t)\big)v(t)\Big)\,dt\\&\quad+\int_a^{T(\epsilon)}\Big(f\big(u\big(t,\big(\gamma(t)-\epsilon\big)^-\big)\big)-f\big(v(t)\big)+f'\big(v(t)\big)\big(v(t)-u\big(t,\big(\gamma(t)-\epsilon\big)^-\big)\big)\Big)\,dt.\end{aligned}$$

Using the fact that $f(u(t,0^+))\geqslant f(\bar{u})$ for almost all t thanks to the boundary condition and remembering that f is convex, we have for any u_i such that $f(\bar{u})\geqslant f(u_i)$:

$$\int_{T(\epsilon)}^{b} \big(f\big(u\big(t,0^+\big)\big) - f\big(v(t)\big) + f'\big(v(t)\big)v(t)\big)\,dt$$
$$\geqslant \int_{T(\epsilon)}^{b} f(u_i) - f\big(v(t)\big) + f'\big(v(t)\big)v(t)\,dt$$
$$\geqslant \int_{T(\epsilon)}^{b} \big(f(u_i) - f\big(v(t)\big) - f'\big(v(t)\big)\big(u_i - v(t)\big)\big)\,dt + u_i \int_{T(\epsilon)}^{b} \dot{\gamma}(t)\,dt$$
$$\geqslant -\epsilon u_i.$$

The convexity of f also implies:

$$\int_{a}^{T(\epsilon)} \big(f\big(u\big(t,\big(\gamma(t)-\epsilon\big)^-\big)\big) - f\big(v(t)\big) + f'\big(v(t)\big)\big(v(t) - u\big(t,\big(\gamma(t)-\epsilon\big)^-\big)\big)\big)\,dt \geqslant 0.$$

Thanks to (51) we know:

$$\int_{a}^{T(\epsilon)} h(t)\,dt = v\big(T(\epsilon)\big) - v(a).$$

We deduce that for any $\epsilon > 0$ small enough:

$$-\int_{\gamma(a)-\epsilon}^{\gamma(a)} u(a,x)\,dx \geqslant \int_{T(\epsilon)}^{b} \gamma(t)h(t)\,dt + \epsilon\big(v\big(T(\epsilon)\big) - v(a)\big) - \epsilon u_i.$$

And finally dividing by ϵ and letting ϵ tend to 0 we have:

$$-v(a) \geqslant \bar{v} - v(a) - u_i.$$

In the end:

$$\forall u_i \text{ s.t. } f(u_i) \leqslant f(\bar{u}), \quad \bar{v} \leqslant u_i.$$

Thus we have proven (72). $\square$

We will now use the three previous propositions to prove two crucial estimates on the infimum and supremum of $u(t,.)$.

Proposition 3.4. *If u is the unique entropy solution of the system* (49) *then*:

$$\forall t \geqslant 0, \quad \inf_{x\in(0,1)} u(t,x) \geqslant \min\left(\bar{u}, \inf_{x\in(0,1)} u_0(x) + \int_0^t h(s)\,ds\right), \tag{73}$$

$$\forall t \geqslant 0, \quad \sup_{x\in(0,1)} u(t,x) \leqslant \max\Big(\bar{u}, \sup_{x\in(0,1)} u_0(x)\Big) + \int_0^t h(s)\,ds. \tag{74}$$

Proof. Take (t,x) in $(0,+\infty) \times (0,1)$ and consider χ^+ the maximal backward generalized characteristic going through (t,x), and v the function associated to χ^+ by (51). We suppose that χ^+ is maximally defined on $[a,b]$ for a certain a. If $\chi^+(a) = 0$ we have, thanks to Proposition 3.2:

$$v(a) \geqslant \bar{u}.$$

Therefore using the last part of Theorem 3 we get:

$$u\big(t,x^+\big)=v(t)=v(a)+\int_a^t h(s)\,ds \geqslant \bar{u}.$$

Now if $\chi^+(a)>0$ then thanks to Proposition 3.1 we get $a=0$ and using (50) we have:

$$u\big(t,x^+\big)=v(t)=v(0)+\int_0^t h(s)\,ds \geqslant u\big(0,\chi^+(0)^-\big)\geqslant \inf_{(0,1)} u_0.$$

And since u is an entropy solution $u(t,x^-)\geqslant u(t,x^+)$ for $t>0$ thus we have (73). The same kind of reasoning provides (74). □

Let us now prove a simple estimate on the characteristic speed.

Lemma 5. *Let us consider u the entropy solution of* (49) *then for any positive t and any x in* $(0,1)$ *we have*:

$$f'\big(u\big(t,x^-\big)\big)\geqslant f'\big(u\big(t,x^+\big)\big)\geqslant \frac{x-1}{t}. \tag{75}$$

Proof. Let χ^+ be the maximal backward generalized characteristic going through (t,x). Then thanks to Theorem 3 we know that:

$$\dot{\chi}^+(t)=f'\big(u\big(t,x^+\big)\big).$$

But thanks to Remark 5, χ^+ is convex and so we get:

$$\forall s\leqslant t,\quad \dot{\chi}^+(s)\leqslant f'\big(u\big(t,x^+\big)\big).$$

Finally thanks to Proposition 3.1, χ^+ cannot cross $x=1$ at a positive time, which implies the right side of (75). The left side is just the Lax entropy inequality. □

4. Proof of Theorem 1

In this section we will prove the remaining parts of Theorem 1 in the case $f'(\bar{u})>0$. The first point was already proven in Section 2, (25) will be proven in Proposition 4.2, (26) and (27) will be proven in Proposition 4.3 and finally the regularization property will be proven in Proposition 4.1.

Since the feedback law (17) satisfies:

$$\forall z\in L^1(0,1),\quad \mathcal{G}_1(z)\geqslant 0, \tag{76}$$

we can apply all the results of the preceding section to the entropy solution u of (14), (17) and (18). We begin the proof of Theorem 1 with the following geometric lemma.

Lemma 6. *Let us define the time T_2 in two ways depending on the type of the flux function f introduced in Definition* 1:

$$T_2=\begin{cases}\frac{1}{2}\frac{2\bar{u}-\hat{u}-u^*}{f(\bar{u})-f(\frac{u^*+\hat{u}}{2})}-\frac{1}{f'(\frac{u^*+\hat{u}}{2})} & \text{if } f \text{ is of type I},\\ \frac{1}{f'(\bar{u}-\|u_0-\bar{u}\|_{L^\infty(0,1)})} & \text{if } f \text{ is of type II}.\end{cases}$$

Then for any t larger than T_2 and any x in $(0,1)$ *if χ^- is the minimal backward generalized characteristic going through* (t,x) *there exists a positive time a such that*:

$$\chi^-(a)=0.$$

Proof. We begin with the case where f is of type I. Let us define:

$$T_1 = -\frac{1}{f'(\frac{u^*+\hat{u}}{2})}.$$

Thanks to the hypothesis on f, we have $0 < T_1 < +\infty$. Using Lemma 5 we see:

$$\forall x \in (0,1),\ \forall t \geqslant T_1, \quad f'\big(u(t,x)\big) \geqslant -\frac{1}{T_1} = f'\left(\frac{\hat{u}+u^*}{2}\right).$$

Since f is strictly convex we deduce:

$$\forall x \in (0,1),\ \forall t \geqslant T_1, \quad u(t,x) \geqslant \frac{\hat{u}+u^*}{2}. \tag{77}$$

Looking at the boundary condition (24) we see that this also implies:

$$\text{for almost all } t \text{ in } [T_1,+\infty), \quad u\big(t,0^+\big) = \bar{u}.$$

Consider $b > T_1$ and such that $u(b,0^+) = \bar{u}$. Then for x sufficiently close to 0 we have:

$$f'\big(u(b,x)\big) \geqslant \frac{f'(\bar{u})}{2} > 0. \tag{78}$$

Let χ be the minimal backward characteristic going through (b,x), a the time such that χ is maximally defined on $[a,b]$, and v the function associated to χ by (51). For x sufficiently close to 0 we have thanks to (78), (51) and Proposition 3.2:

$$a \geqslant T_1, \quad v(a) = \bar{u}. \tag{79}$$

Now let γ be the forward characteristic going through (b,x) and maximally defined on $[b,c)$ for a certain c (possibly infinite). Thanks to (79), we see that for any t in (b,c), the minimal backward characteristic through $(t,\gamma(t))$ cross $x=0$ at a time a_1 such that $a_1 \geqslant a \geqslant T_1 > 0$. Using (51) and Proposition 3.2 we deduce:

$$\forall t \in (b,c), \quad u\big(t,\gamma(t)^-\big) \geqslant \bar{u}.$$

Using (77) we obtain:

$$\text{for almost all } t \text{ in } (b,c), \quad \dot{\gamma}(t) \geqslant 2\frac{f(\bar{u}) - f(\frac{\hat{u}+u^*}{2})}{2\bar{u} - \hat{u} - u^*} > 0.$$

This implies that c is finite and that $\gamma(c) = 1$. Consequently if $c \leqslant T_2$ we have finished, in the other case:

$$\gamma(T_2) \geqslant 2(T_2 - b)\frac{f(\bar{u}) - f(\frac{\hat{u}+u^*}{2})}{2\bar{u} - \hat{u} - u^*}.$$

The number b can be chosen as close to T_1 as we want and

$$2(T_2 - T_1)\frac{f(\bar{u}) - f(\frac{\hat{u}+u^*}{2})}{2\bar{u} - \hat{u} - u^*} = 1.$$

This concludes the proof in the case of a flux f of type I.

Now we suppose that f is of type II. Using Proposition 3.4 we see that:

$$\begin{aligned}
\forall x \in (0,1),\ \forall t \geqslant 0, \quad u(t,x) &\geqslant \min\Big(\bar{u}, \inf_{y\in(0,1)}\big(u_0(y)\big)\Big) \\
&\geqslant \bar{u} + \min\Big(0, \inf_{y\in(0,1)}\big(u_0(y) - \bar{u}\big)\Big) \\
&\geqslant \bar{u} - \|u_0 - \bar{u}\|_{L^\infty(0,1)}.
\end{aligned}$$

Obviously since f is of type II and since $f'(\bar{u}) > 0$ we have:

$$f'\big(\bar{u} - \|u_0 - \bar{u}\|_{L^\infty(0,1)}\big) > 0 \quad \text{and thus} \quad T_2 < +\infty.$$

Now for t larger than T_2 and for x in $(0,1)$ consider χ the minimal backward characteristic through (t,x) and a the non-negative number such that χ is maximally defined on $[a,t]$. Using (51) we see

$$\forall s \in (a,t), \quad \dot{\chi}(s) \geqslant f'\big(\bar{u} - \|u_0 - \bar{u}\|_{L^\infty(0,1)}\big),$$

and since

$$T_2 f'\big(\bar{u} - \|u_0 - \bar{u}\|_{L^\infty(0,1)}\big) = 1,$$

we deduce $a > 0$, which concludes the proof. □

Proposition 4.1. *The unique entropy solution u of* (14), (17) *and* (18) *is $\mathcal{C}^1$ on $(T_2,+\infty)\times(0,1)$ and satisfies*:

$$\forall t \geqslant T_2, \quad \big\|u(t,.) - \bar{u}\big\|_{L^1(0,1)} \leqslant \big\|u(T_2,.) - \bar{u}\big\|_{L^1(0,1)} e^{-\frac{f'(\bar{u})}{2}(t-T_2)}. \tag{80}$$

Proof. Let us take t larger than T_2 and x, y in $(0,1)$ with $x<y$. Consider γ_1 and γ_2 the minimal backward generalized characteristics going through (t,x) and (t,y) and v_1, v_2 the functions associated to them by (51). Thanks to Lemma 6, we get two times a_1 and a_2 positive such that:

$$\gamma_1(a_1) = \gamma_2(a_2) = 0.$$

Furthermore since two genuine characteristics may cross only at their endpoints we have $a_2 \leqslant a_1$. But using (51) we have:

$$\begin{aligned} u(t,y) = v_2(t) &= \bar{u} + \int_{a_2}^{t} \mathcal{G}_1\big(u(s,.)\big)\,ds \\ &\geqslant \bar{u} + \int_{a_1}^{t} \mathcal{G}_1\big(u(s,.)\big)\,ds = v_1(t) = u(t,x). \end{aligned} \tag{81}$$

So for any time t larger than T_2, $u(t,.)$ is non-decreasing on $(0,1)$, using (16) this implies that it is continuous on $(0,1)$. The previous calculation also shows that:

$$\forall t \geqslant T_2,\ \forall x \in (0,1), \quad u(t,x) \geqslant \bar{u}. \tag{82}$$

Since $f'(\bar{u}) > 0$, we can see that as y tends to 0, a_2 tends to t and therefore using (51) we obtain:

$$\forall t \geqslant T_2, \quad u\big(t,0^+\big) = \bar{u}. \tag{83}$$

Let us now prove the regularity of u. For the sake of convenience let us put:

$$\forall t \geqslant 0, \quad g(t) = \mathcal{G}_1\big(u(t,.)\big).$$

Using the definition of $\mathcal{G}_1$ and the result of Section 2 we already know that g is continuous and non-negative. We introduce the following auxiliary function B:

$$\forall t \geqslant 0,\ e \geqslant 0,\ x \in (0,1),\ u \in \mathbb{R}, \quad \mathrm{B}(t,x,e,u) = \left(u - \bar{u} - \int_e^t g(s)\,ds,\ x - \int_e^t f'\left(\bar{u} + \int_e^s g(r)\,dr\right) ds\right).$$

For t larger than T_2 and x in $(0,1)$, let $e(t,x)$ be the time for which the genuine backward characteristic γ going through (t,x) satisfies:

$$\gamma\big(e(t,x)\big) = 0.$$

Using (51) and Proposition 3.2 we can see that the following holds:

$$\mathrm{B}\big(t,x,e(t,x),u(t,x)\big) = (0,0).$$

It is clear that:

$$\partial_u \mathrm{B}(t,x,e,u) = (1,0),$$
$$\partial_e \mathrm{B}(t,x,e,u) = \left(g(e),\, f'(\bar u)\left(1 + \int_e^t f''\left(\bar u + \int_e^s g(r)\,dr\right) ds\right)\right).$$

And since $f'(\bar u) > 0$ and $f'' \geqslant 0$, the regularity of u comes as a consequence of the implicit function theorem.

To show (80) let us consider s and t satisfying $T_2 < s < t$. Using (82), Lemma 3 and the convexity of f, we get:

$$\begin{aligned}
\|u(t,.)-\bar u\|_{L^1(0,1)} - \|u(s,.)-\bar u\|_{L^1(0,1)} &= \int_0^1 \left(\left|u(t,x)-\bar u\right| - \left|u(s,x)-\bar u\right|\right) dx \\
&= \int_0^1 \left(u(t,x)-u(s,x)\right) dx \\
&= \int_s^t \mathcal{G}_1\left(u(r,.)\right) dr + \int_s^t \left(f\left(u\left(r,0^+\right)\right) - f\left(u\left(r,1^-\right)\right)\right) dr \\
&= \int_s^t \mathcal{G}_1\left(u(r,.)\right) dr + \int_s^t \left(f(\bar u) - f\left(u\left(r,1^-\right)\right)\right) dr \\
&\leqslant \int_s^t \mathcal{G}_1\left(u(r,.)\right) dr + \int_s^t f'(\bar u)\left(\bar u - u\left(r,1^-\right)\right) dr.
\end{aligned}$$

Thanks to (81) and (83) we also have:

$$\forall x \in (0,1), \quad \left|u(r,x)-\bar u\right| = u(r,x)-\bar u \leqslant u\left(r,1^-\right) - \bar u,$$

and therefore

$$\int_0^1 \left|u(r,x)-\bar u\right| dx \leqslant u\left(r,1^-\right) - \bar u.$$

Using the two previous inequalities and the definition of $\mathcal{G}_1$ we then deduce:

$$\begin{aligned}
&\|u(t,.)-\bar u\|_{L^1(0,1)} - \|u(s,.)-\bar u\|_{L^1(0,1)} \\
&\quad\leqslant \frac{f'(\bar u)}{2}\int_s^t \|u(r,.)-\bar u\|_{L^1(0,1)}\,dr - f'(\bar u)\int_s^t \|u(r,.)-\bar u\|_{L^1(0,1)}\,dr \\
&\quad\leqslant -\frac{f'(\bar u)}{2}\int_s^t \|u(r,.)-\bar u\|_{L^1(0,1)}\,dr.
\end{aligned}$$

Applying Gronwall's lemma we obtain (80). □

We can now complete the estimate of the previous estimate.

Proposition 4.2. *If u is the entropy solution associated to the initial data u_0 we have*:

$$\forall t \geqslant 0, \quad \|u(t,.)-\bar u\|_{L^1(0,1)} \leqslant e^{f'(\bar u)T_2} e^{-\frac{f'(\bar u)}{2}t} \|u_0-\bar u\|_{L^1(0,1)}. \tag{84}$$

Proof. The constant function $\bar{u}$ is the unique entropy solution of (14), (17), (18) associated to the constant initial data $\bar{u}$. Therefore comparing u and $\bar{u}$ using Lemma 1 gives us:

$$\forall t \geqslant 0, \quad \big\|u(t,.) - \bar{u}\big\|_{L^1(0,1)} \leqslant \|u_0 - \bar{u}\|_{L^1(0,1)} + \int_0^t \mathcal{G}_1\big(u(s,.)\big) - \mathcal{G}_1(\bar{u})\,ds$$
$$\leqslant \|u_0 - \bar{u}\|_{L^1(0,1)} + \frac{f'(\bar{u})}{2}\int_0^t \big\|u(s,.) - \bar{u}\big\|_{L^1(0,1)}\,ds.$$

Using Gronwall's lemma we get:

$$\forall t \in [0,T_2], \quad \big\|u(t,.) - \bar{u}\big\|_{L^1(0,1)} \leqslant \|u_0 - \bar{u}\|_{L^1(0,1)} e^{\frac{f'(\bar{u})}{2}t} \leqslant \|u_0 - \bar{u}\|_{L^1(0,1)} e^{\frac{f'(\bar{u})}{2}T_2}. \tag{85}$$

Combining the last estimate with (80) we obtain indeed (84). □

We end this section with the last remaining estimate of Theorem 1.

Proposition 4.3. *The state $\bar{u}$ is asymptotically stable in $L^\infty(0,1)$ for the system* (14), (17) *and* (18)*, and if u is the entropy solution associated to the initial data u_0 we have*:

$$\forall t \geqslant 0, \quad \big\|u(t,.) - \bar{u}\big\|_{L^\infty(0,1)} \leqslant e^{f'(\bar{u})T_2+1}\|u_0 - \bar{u}\|_{L^\infty(0,1)} e^{-\frac{f'(\bar{u})}{2}t}. \tag{86}$$

Proof. Using Proposition 3.4 we have:

$$\forall t \in [0,T_2], \quad \inf_{x\in(0,1)}\big(u(t,x) - \bar{u}\big) \geqslant \min\Big(0, \inf_{x\in(0,1)}\big(u_0(x) - \bar{u}\big)\Big) \geqslant -\|u_0 - \bar{u}\|_{L^\infty(0,1)}.$$

Using Proposition 3.4 and (85) we obtain:

$$\forall t \in [0,T_2], \quad \sup_{x\in(0,1)}\big(u(t,x) - \bar{u}\big) \leqslant \max\left(0, \sup_{x\in(0,1)}\big(u_0(x) - \bar{u}\big) + \int_0^t \mathcal{G}_1\big(u(s,.)\big)\,ds\right)$$
$$\leqslant \|u_0 - \bar{u}\|_{L^\infty(0,1)} + \int_0^t \frac{f'(\bar{u})}{2} e^{\frac{f'(\bar{u})}{2}s}\|u_0 - \bar{u}\|_{L^1(0,1)}\,ds$$
$$\leqslant e^{\frac{f'(\bar{u})}{2}T_2}\|u_0 - \bar{u}\|_{L^\infty(0,1)}.$$

Thus we get:

$$\forall t \in [0,T_2], \quad \big\|u(t,.) - \bar{u}\big\|_{L^\infty(0,1)} \leqslant e^{\frac{f'(\bar{u})}{2}T_2}\|u_0 - \bar{u}\|_{L^\infty(0,1)}. \tag{87}$$

Thanks to (81) and (82) we have:

$$\forall t \geqslant T_2, \quad \big\|u(t,.) - \bar{u}\big\|_{L^\infty(0,1)} = u\big(t,1^-\big) - \bar{u}.$$

If we take χ_t the minimal backward generalized characteristic going through $(t,1)$ and v_t the function associated to it by (51) we know that, thanks to Proposition 3.2, there exists a_t positive such that:

$$\chi_t(a_t) = 0 \quad \text{and} \quad v_t(a_t) = \bar{u}.$$

Now using (82) and (51) we see:

$$\chi_t(t) - \chi_t(a_t) = \int_{a_t}^t f'\big(v_t(s)\big)\,ds$$
$$\geqslant (t - a_t)f'(\bar{u}).$$

In turn this shows:

$$a_t \geqslant t - \frac{1}{f'(\bar u)} \underset{t\to+\infty}{\longrightarrow} +\infty. \tag{88}$$

Thanks to estimate (84) and to the definition of $\mathcal{G}_1$ we also have:

$$\begin{aligned}\forall T \geqslant 0, \quad \int_T^{+\infty} \mathcal{G}_1\big(u(s,.)\big)\,ds &\leqslant \int_T^{+\infty} \frac{f'(\bar u)}{2} e^{-\frac{f'(\bar u)}{2}s} \|u_0 - \bar u\|_{L^1(0,1)} e^{f'(\bar u)T_2}\,ds \\ &\leqslant e^{f'(\bar u)T_2} \|u_0 - \bar u\|_{L^\infty(0,1)} e^{-\frac{f'(\bar u)}{2}T}.\end{aligned}$$

So using the last two estimates we obtain:

$$\begin{aligned}\forall t \geqslant T_2, \quad \big\|u(t,.) - \bar u\big\|_{L^\infty(0,1)} = u\big(t,1^-\big) - \bar u = v_t(t) - \bar u &= \int_{a_t}^{t} \mathcal{G}_1\big(u(s,.)\big)\,ds \\ &\leqslant \int_{a_t}^{+\infty} \mathcal{G}_1\big(u(s,.)\big)\,ds \\ &\leqslant e^{f'(\bar u)T_2} \|u_0 - \bar u\|_{L^\infty(0,1)} e^{-\frac{f'(\bar u)}{2}a_t} \\ &\leqslant e^{1+f'(\bar u)T_2} \|u_0 - \bar u\|_{L^\infty(0,1)} e^{-\frac{f'(\bar u)}{2}t}.\end{aligned} \tag{89}$$

Combining (87) and (89) we obtain (86). □

5. Proof of Theorem 2

In this section we will prove the remaining parts of Theorem 2, so from now on $f'(\bar u) = 0$. Therefore f is necessarily of type I and $\bar u = u^* = \hat u$. We will consider in the following the unique entropy solution u of (14), (17) and (18). We begin by proving the following lemma which describes two alternative behaviors for u.

Lemma 7. *If the following condition holds*:

$$\int_0^{+\infty} \mathcal{G}_2\big(u(t,.)\big)\,dt \leqslant \bar u - \inf_{x\in(0,1)} u_0(x), \tag{90}$$

then we have both:

$$\big\|u(t,.) - \bar u\big\|_{L^\infty(0,1)} \underset{t\to+\infty}{\longrightarrow} 0, \tag{91}$$

and

$$\forall t \geqslant 0, \quad \big\|u(t,.) - \bar u\big\|_{L^\infty(0,1)} \leqslant 2\|u_0 - \bar u\|_{L^\infty(0,1)}. \tag{92}$$

Otherwise with T_1 the smallest time such that:

$$\int_0^{T_1} \mathcal{G}_2\big(u(t,.)\big)\,dt = \max\Big(0, \bar u - \inf_{x\in(0,1)} u_0(x)\Big), \tag{93}$$

we have:

$$\forall t \geqslant T_1,\ \forall x \in (0,1), \quad u(t,x) \geqslant \bar u. \tag{94}$$

Proof. First let assume that condition (90) does not hold. Then (94) is a simple consequence of (93) and of Proposition 3.4. On the other hand if (90) does hold, then using Proposition 3.4 we have:

$$\forall t \geqslant 0, \quad \inf_{x\in(0,1)} u(t,x) - \bar{u} \geqslant \min\left(0, \inf_{x\in(0,1)} \big(u_0(x) - \bar{u}\big) + \int_0^t \mathcal{G}_2\big(u(s,.)\big)\,ds\right)$$
$$\geqslant \min\Big(0, \inf_{x\in(0,1)} \big(u_0(x) - \bar{u}\big)\Big) \geqslant -\|u_0 - \bar{u}\|_{L^\infty(0,1)}.$$

Moreover:

$$\forall t \geqslant 0, \quad \sup_{x\in(0,1)} \big(u(t,x) - \bar{u}\big) \leqslant \max\Big(0, \sup_{x\in(0,1)} \big(u_0(x) - \bar{u}\big)\Big) + \int_0^t \mathcal{G}_2\big(u(s,.)\big)\,ds$$
$$\leqslant \|u_0 - \bar{u}\|_{L^\infty(0,1)} + \int_0^{+\infty} \mathcal{G}_2\big(u(s,.)\big)\,ds$$
$$\leqslant 2\|u_0 - \bar{u}\|_{L^\infty(0,1)}.$$

Thus we have (92). In order to prove (91) let us consider T such that:

$$\int_T^{+\infty} \mathcal{G}_2\big(u(s,.)\big)\,ds > 0$$

(this is possible otherwise $u_0 = \bar{u}$ and everything is obvious) and define u_i by:

$$u_i = \bar{u} + \int_T^{+\infty} \mathcal{G}_2\big(u(s,.)\big)\,ds > \bar{u}. \tag{95}$$

Let us also define δ_T by:

$$\delta_T = \frac{1}{f'(u_i)}.$$

This is a finite number because $u_i > \bar{u}$, $f'(\bar{u}) = 0$ and f strictly convex.

For t larger than $T + \delta_T$ and x in $(0,1)$, consider γ the minimal backward characteristic going through (t,x) and the number a such that γ is maximally defined on $[a,t]$. Consider also the function v associated to γ by (51) we have:

$$\forall s \in \big[\max(a,T), t\big], \quad v(s) = v(t) - \int_s^t \mathcal{G}_2\big(u(r,.)\big)\,dr$$
$$\geqslant u\big(t,x^-\big) - \int_T^{+\infty} \mathcal{G}_2\big(u(r,.)\big)\,dr$$
$$\geqslant u\big(t,x^-\big) + \bar{u} - u_i.$$

We also have:

$$\forall s \in \big[\max(a,T), t\big], \quad \gamma(t) - \gamma(s) = \int_s^t f'\big(v(r)\big)\,dr$$
$$\geqslant \int_s^t f'\big(u\big(t,x^-\big) + \bar{u} - u_i\big)\,dr$$
$$\geqslant (t-s) f'\big(u\big(t,x^-\big) + \bar{u} - u_i\big). \tag{96}$$

Let us now suppose that the following holds:

$$u(t, x^-) \geqslant 2u_i - \bar{u}. \tag{97}$$

Then if we suppose $a \leqslant T$, we have thanks to (96) and since we took t larger than $T + \delta_T$:

$$\gamma(T) \leqslant x - \delta T f'(u_i) < 0,$$

which is not possible, therefore $a > T$. Thanks to Proposition 3.2 we deduce:

$$\gamma(a) = 0 \quad \text{and} \quad v(a) = \bar{u}.$$

Using (51) we can deduce:

$$u(t, x^-) = v(t) = v(a) + \int_a^t \mathcal{G}_2(u(r,.))\,dr \leqslant \bar{u} + \int_T^{+\infty} \mathcal{G}_2(u(r,.))\,dr \leqslant \bar{u} + u_i - \bar{u} \leqslant u_i.$$

However this contradicts (95) and (97) therefore:

$$\forall t \geqslant T + \delta T, \qquad \sup_{x \in (0,1)} u(t,x) \leqslant 2u_i - \bar{u} \underset{T \to +\infty}{\longrightarrow} \bar{u}.$$

Now Lemma 5 provides:

$$\liminf_{t \to +\infty} \inf_{x \in (0,1)} f'(u(t,x)) \geqslant 0,$$

and we know thanks to the strict convexity of f that $\bar{u}$ is the only number such that $f'(\bar{u}) = 0$. This ends the proof of Lemma 7. $\square$

Lemma 8. *If there exists a time $T_1 < +\infty$ as in (93), then for any time $a \geqslant T_1$, there exists a generalized characteristic γ going through $(a, 0)$, defined on $[a, b]$ for a certain b, and which satisfies*:

$$\forall t \in [a,b], \qquad \gamma(t) \geqslant \int_a^t f'\left(\bar{u} + \int_a^s \mathcal{G}_2(u(r,.))\,dr\right) ds. \tag{98}$$

Proof. Consider a time a larger or equal to T_1, the numbers b_n (larger than a and possibly infinite) and the functions γ_n such that γ_n is the unique forward generalized characteristic going through $(a, \frac{1}{n})$ and it is maximally defined on $[a, b_n]$. Thanks to Theorem 3 we have:

$$\text{for almost all } t \text{ in } (a, b_n), \qquad \dot{\gamma}_n(t) = \begin{cases} f'(u(t, \gamma_n(t))) & \text{if } u(t,\gamma_n(t)^-) = u(t, \gamma_n(t)^+), \\ \frac{f(u(t,\gamma_n(t)^-)) - f(u(t,\gamma_n(t)^+))}{u(t,\gamma_n(t)^-) - u(t,\gamma_n(t)^+)} & \text{otherwise.} \end{cases} \tag{99}$$

As we recalled in (16), u satisfies:

$$\forall (t,x) \in (0,+\infty) \times (0,1), \qquad u(t,x^-) \geqslant u(t,x^+).$$

Therefore we deduce:

$$\text{for almost all } t \text{ in } (a, b_n), \qquad \dot{\gamma}_n(t) \geqslant f'(u(t, \gamma_n(t)^+)). \tag{100}$$

For any t in (a, b_n) the maximal backward generalized characteristic going through $(t, \gamma_n(t))$ is necessarily defined at least on (a, t). Indeed it cannot cross $x = 0$ at a time $s > a$ because it is maximal, and it cannot cross $x = 1$ because of Proposition 3.1. Since $a \geqslant T_1$ we have, using (51):

$$u(t, \gamma_n(t)^+) \geqslant \bar{u} + \int_a^t \mathcal{G}_2(u(s,.))\,ds.$$

After substituting in (100) we get:

$$\forall t \in (a, b_n), \quad \gamma_n(t) = \frac{1}{n} + \int_a^t \dot{\gamma}_n(s)\,ds \tag{101}$$

$$\geqslant \int_a^t f'\left(\bar{u} + \int_a^s \mathcal{G}_2\big(u(r,.)\big)\,dr\right) ds. \tag{102}$$

Since $a \geqslant T_1$ and thanks to the choice of T_1, the characteristics γ_n may leave $(0, 1)$ only at $x = 1$. Thanks to Theorem 3 there is only one forward generalized characteristic going through a point of $(0, +\infty) \times (0, 1)$. So the sequence b_n is non-decreasing. We can choose b larger than a such that all characteristics γ_n are defined on $[a, b]$.

Furthermore using (99) we see that the family γ_n is uniformly Lipschitz on $[a, b]$ with values in $[0, 1]$. Using the Arzela–Ascoli theorem we can suppose that there is an absolutely continuous curve γ defined on $[a, b]$ and such that:

$$\sup_{t\in[a,b]} \left|\gamma(t) - \gamma_n(t)\right| \underset{n\to+\infty}{\longrightarrow} 0.$$

But it is known [16, Chapter 1] that the uniform limit of generalized characteristics is a generalized characteristic. Therefore γ is a generalized characteristic and as the limit of the curves γ_n it also satisfies:

$$\forall t \in [a, b], \quad \gamma(t) \geqslant \int_a^t f'\left(\bar{u} + \int_a^s \mathcal{G}_2\big(u(r,.)\big)\,dr\right) ds,$$

$$\gamma(a) = 0. \qquad \square$$

We now prove a first asymptotic result in the case where we have a time $T_1 < +\infty$.

Proposition 5.1. *We have*:

$$\left\|u(t,.) - \bar{u}\right\|_{L^\infty(0,1)} \underset{t\to+\infty}{\longrightarrow} 0.$$

Proof. Let us first remark that should $u(t,.)$ be equal to $\bar{u}$ for some t, it remains at $\bar{u}$ thanks to the uniqueness of the constant solution $\bar{u}$ of the system (14), (29), (18) and the proof is finished. Otherwise we have:

$$\forall t \geqslant 0, \quad \left\|u(t,.) - \bar{u}\right\|_{L^1(0,1)} > 0. \tag{103}$$

Thanks to the definition of $\mathcal{G}_2$ this implies that the function

$$t \mapsto \int_{T_1}^t \mathcal{G}_2\big(u(s,.)\big)\,ds$$

is positive and non-decreasing on $(0, +\infty)$. Since f is strictly convex and $f'(\bar{u}) = 0$, we know that f' is positive and increasing on $(\bar{u}, +\infty)$. Thus we obtain:

$$\int_{T_1}^T f'\left(\bar{u} + \int_{T_1}^t \mathcal{G}_2\big(u(s,.)\big)\,ds\right) dt \underset{T\to+\infty}{\longrightarrow} +\infty. \tag{104}$$

Let us take T_2 the smallest time such that:

$$\int_{T_1}^{T_2} f'\left(\bar{u} + \int_{T_1}^t \mathcal{G}_2\big(u(s,.)\big)\,ds\right) dt = 1.$$

Thanks to Lemma 8, we see that the generalized characteristic γ going through $(T_1, 0)$ has reached $x = 1$ by T_2 at the latest. Therefore for any (t, x) in $[T_2, +\infty) \times (0, 1)$, if γ is the minimal backward characteristic through (t, x) there is a time a which is at least equal to T_1 and such that:

$$\gamma(a) = 0.$$

Consider $0 < x < y < 1$ and $t > T_2$. Let χ_1, χ_2 be the minimal generalized characteristics going through (t, x) and (t, y), and v_1, v_2 the functions associated to them by (51). Thanks to the choice of T_2 and since genuine characteristics may cross only at their endpoints (Theorem 3), we have a_1 and a_2 such that:

$$T_1 \leqslant a_2 \leqslant a_1, \qquad \chi_1(a_1) = 0, \qquad \chi_2(a_2) = 0.$$

Using Proposition 3.2 we also get:

$$v_1(a_1) = v_2(a_2) = \bar{u}.$$

But using (51), Proposition 3.2 and the positivity of $\mathcal{G}_2(u(s, .))$ we can see that:

$$u(t, x) = v_1(t) = \bar{u} + \int_{a_1}^{t} \mathcal{G}_2\big(u(s, .)\big)\, ds \leqslant \bar{u} + \int_{a_2}^{t} \mathcal{G}_2\big(u(s, .)\big)\, ds = v_2(a_2) = u(t, y). \tag{105}$$

Thus $u(t, .)$ is non-decreasing on $(0, 1)$, using additionally (94), we arrive at:

$$\forall t \geqslant T_2, \quad \big\|u(t, .) - \bar{u}\big\|_{L^1(0,1)} = \int_0^1 \big|u(t, x) - \bar{u}\big|\, dx = \int_0^1 \big(u(t, x) - \bar{u}\big)\, dx \tag{106}$$

$$\leqslant u\big(t, 1^-\big) - \bar{u}. \tag{107}$$

And now for $t \geqslant T_2$ and $h > 0$ we get thanks to (106) and Lemma 3:

$$\big\|u(t + h, .) - \bar{u}\big\|_{L^1(0,1)} - \big\|u(t, .) - \bar{u}\big\|_{L^1(0,1)} = \int_0^1 u(t + h, x) - u(t, x)\, dx \tag{108}$$

$$= \int_t^{t+h} \mathcal{G}_2\big(u(s, .)\big)\, ds + \int_t^{t+h} f\big(u\big(s, 0^+\big)\big) - f\big(u\big(s, 1^-\big)\big)\, ds. \tag{109}$$

Recalling the definition (28) of A we see that:

$$\forall z \geqslant 0, \quad f'(\bar{u} + z) - f'(\bar{u}) \geqslant 2A(z),$$

so using (29), (107) and the facts that $u(s, .)$ is non-decreasing on $(0, 1)$, f is convex and $f'(\bar{u}) = 0$, we end up with:

$$\begin{aligned} f\big(u\big(s, 1^-\big)\big) - f\big(u\big(s, 0^+\big)\big) &= f\big(\bar{u} + u\big(s, 1^-\big) - \bar{u}\big) - f(\bar{u}) \\ &\geqslant f\big(\bar{u} + \big\|u(s, .) - \bar{u}\big\|_{L^1(0,1)}\big) - f(\bar{u}) \geqslant 2\mathcal{G}_2\big(u(s, .)\big). \end{aligned}$$

Combining the two previous estimates we have:

$$\big\|u(t + h, .) - \bar{u}\big\|_{L^1(0,1)} - \big\|u(t, .) - \bar{u}\big\|_{L^1(0,1)} \leqslant -\int_t^{t+h} \mathcal{G}_2\big(u(s, .)\big)\, ds.$$

If we denote by Q the function

$$Q : t \mapsto \big\|u(t, .) - \bar{u}\big\|_{L^1(0,1)}, \tag{110}$$

this implies that for any t larger than T_2:

$$\dot{Q}(t) \leqslant -\mathrm{A}\big(Q(t)\big). \tag{111}$$

Therefore if we introduce the solution Q_1 of:

$$\begin{cases} \dot{Q}_1(t) = -\mathrm{A}\big(Q_1(t)\big), \\ Q_1(T_2) = Q(T_2), \end{cases} \tag{112}$$

the comparison principle provides:

$$\forall t \geqslant T_2, \quad 0 \leqslant Q(t) \leqslant Q_1(t).$$

Finally since f is strictly increasing on $(\bar{u}, +\infty)$, so is A on $(0, +\infty)$. Therefore Q_1 is strictly decreasing on $(T_2, +\infty)$. In turn this implies that $\dot{Q}_1$ is increasing on $(T_2, +\infty)$. This means that Q_1 is strictly convex, decreasing and positive therefore $\dot{Q}_1(t) \to 0$ when $t \to +\infty$. Since 0 is the only equilibrium of (112) we can deduce $Q_1(t) \to 0$ and:

$$0 \leqslant \left\| u(t,.) - \bar{u} \right\|_{L^1(0,1)} = Q(t) \leqslant Q_1(t) \underset{t \to +\infty}{\longrightarrow} 0.$$

We recall that thanks to the choices of T_1 and T_2 we have (94), (105) and then:

$$\forall t \geqslant T_2, \quad \left\| u(t,.) - \bar{u} \right\|_{L^\infty(0,1)} = u\left(t, 1^-\right) - \bar{u}.$$

For $t \geqslant T_2$ consider the number a_t and the function χ such that χ is the minimal backward characteristic through $(t, 1)$, maximally defined on $[a_t, t]$. Using (51) we have:

$$u\left(t, 1^-\right) - \bar{u} = \int_{a_t}^{t} \mathcal{G}_2\left(u(s,.)\right) ds.$$

We have seen that if Q_2 is the solution to:

$$\begin{cases} \dot{Q}_2(s) = -\mathrm{A}\left(Q_2(s)\right), \\ Q_2(a_t) = Q(a_t), \end{cases}$$

we can deduce:

$$\forall s \geqslant T_2, \quad \mathcal{G}_2\left(u(s,.)\right) = \mathrm{A}\left(Q(s)\right) \leqslant \mathrm{A}\left(Q_2(s)\right) = -\dot{Q}_2(s).$$

So we see that:

$$0 \leqslant u\left(t, 1^-\right) - \bar{u} \leqslant \int_{a_t}^{t} -\dot{Q}_2(s)\, ds = Q_2(a_t) - Q_2(t) \leqslant Q_2(a_t) = Q(a_t).$$

Thanks to Lemma 8, we see that for any time $a_1 \geqslant T_1$, we have a time $c_1 > a_1$ a generalized characteristic γ_1 maximally defined on $[a_1, c_1]$ such that $\gamma_1(a_1) = 0$ and:

$$\forall s \in (a_1, c_1), \quad \gamma_1(s) \geqslant \int_{a_1}^{s} f'\left(\bar{u} + \int_{a_1}^{r} \mathcal{G}_2\left(u(\omega,.)\right) d\omega\right) dr.$$

Combining this estimate with (103) and using the same reasoning as the one leading to (104), we obtain $c_1 < +\infty$. Therefore we get:

$$a_t \underset{t \to +\infty}{\longrightarrow} +\infty. \qquad \square$$

This concludes the proof of the first part of Theorem 2. The remaining part is proven in the next proposition.

Proposition 5.2. *If we suppose additionally that there exists a positive number α such that*:

$$\forall z \in \mathbb{R} \quad f''(z) \geqslant \alpha, \tag{113}$$

then we have:

$$\forall t \geqslant 0, \quad \left\| u(t,.) - \bar{u} \right\|_{L^\infty(0,1)} \leqslant 2\|u_0 - \bar{u}\|_{L^\infty(0,1)} + \mathrm{A}^{-1}\left(\frac{e}{e-1}\left(\frac{(f'(\bar{u}+1))^2}{4\alpha} + \mathrm{A}\left(2\|u_0 - \bar{u}\|_{L^\infty(0,1)}\right)\right)\right) + \sqrt{\frac{2e}{\alpha(e-1)}\left(\frac{(f(1+\bar{u}))^2}{4\alpha} + \mathrm{A}\left(2\|u_0 - \bar{u}\|_{L^\infty(0,1)}\right)\right)}. \tag{114}$$

Proof. Looking at (92) in Lemma 7, it is clear that we only need to prove the result in the cases where we have $T_1 < +\infty$ such that:

$$\int_0^{T_1} \mathcal{G}_2\big(u(s,.)\big)\,ds = \bar{u} - \inf_{x\in(0,1)} u_0(x).$$

Using Proposition 3.4 as in the proof of Lemma 7 we have:

$$\forall t \in [0, T_1], \quad \big\|u(t,.) - \bar{u}\big\|_{L^\infty(0,1)} \leqslant 2\|u_0 - \bar{u}\|_{L^\infty(0,1)}. \tag{115}$$

Looking at the proof of Proposition 5.1 we see that:

$$\forall t \geqslant T_2, \quad \big\|u(t,.) - \bar{u}\big\|_{L^1(0,1)} \leqslant \big\|u(T_2,.) - \bar{u}\big\|_{L^1(0,1)}.$$

Therefore we will need to estimate $\|u(t,.) - \bar{u}\|_{L^1(0,1)}$ on $[T_1, T_2]$ before concluding using the feedback (29).

Thanks to (113) we have:

$$\forall z \geqslant 0, \quad f'(\bar{u} + z) \geqslant \alpha z. \tag{116}$$

As in the proof of Proposition 5.1 we take Q given by (110) and additionally:

$$\forall t \geqslant 0, \quad \mathcal{I}(t) = \mathrm{A}\big(Q(t)\big).$$

Now for t and h non-negative and using Lemma 1 with the initial data $u(t,.)$ and $\bar{u}$ we get:

$$\big\|u(t+h,.) - \bar{u}\big\|_{L^1(0,1)} \leqslant \big\|u(t,.) - \bar{u}\big\|_{L^1(0,1)} + \int_t^{t+h} A\big(Q(s)\big)\,ds,$$

and from this we can deduce

$$\dot{Q}(t) \leqslant A\big(Q(t)\big).$$

Therefore using (29) we arrive at:

$$\forall t \geqslant 0, \quad \dot{\mathcal{I}}(t) = \dot{Q}(t)\mathrm{A}'\big(Q(t)\big) \leqslant \frac{f'(\bar{u}+1)}{2}\mathcal{I}(t). \tag{117}$$

In the following we will also use the notation:

$$L = \frac{f'(\bar{u}+1)}{2}.$$

Thanks to the definition of T_2 we have for any T in $(T_1, T_2]$:

$$\int_{T_1}^{T} f'\left(\bar{u} + \int_{T_1}^{t} \mathcal{I}(s)\,ds\right) dt \leqslant 1.$$

So using (116) we have:

$$\alpha \int_{T_1}^{T}\int_{T_1}^{t} \mathcal{I}(s)\,ds\,dt \leqslant 1. \tag{118}$$

Using (117) and Gronwall's lemma we see that

$$\forall s \in (T_1, T], \quad \mathcal{I}(s) \geqslant \mathcal{I}(T)e^{-LT}e^{Ls}.$$

Therefore we have:

$$\forall T \in [T_1, T_2], \quad \alpha\mathcal{I}(T)\int_{T_1}^{T}\int_{T_1}^{t} e^{L(s-T)}\,ds\,dt \leqslant 1,$$

which becomes:

$$\frac{\alpha}{L^2}\big(\mathcal{I}(T)\big(1-L(T-T_1)e^{-L(T-T_1)}\big)-\mathcal{I}(T)e^{-L(T-T_1)}\big)\leqslant 1.$$

We also have:

$$\forall T\geqslant T_1,\quad L(T-T_1)e^{-L(T-T_1)}\leqslant \frac{1}{e}.$$

Thus we get:

$$\mathcal{I}(T)\left(1-\frac{1}{e}\right)\leqslant \frac{L^2}{\alpha}+\mathcal{I}(T)e^{-L(T-T_1)}.$$

Finally using (117), the fact that A is increasing, Gronwall's lemma and (115) we obtain:

$$\forall T\in[T_1,T_2],\quad Q(T)\leqslant \mathrm{A}^{-1}\left(\frac{e}{e-1}\left(\frac{L^2}{\alpha}+\mathcal{I}(T_1)\right)\right)\leqslant \mathrm{A}^{-1}\left(\frac{e}{e-1}\left(\frac{L^2}{\alpha}+\mathrm{A}\big(2\|u_0-\bar{u}\|_{L^\infty(0,1)}\big)\right)\right). \tag{119}$$

Let us now introduce the constant K and the function $\mathcal{J}$ by:

$$K=\frac{e}{e-1}\left(\frac{L^2}{\alpha}+\mathrm{A}\big(2\|u_0-\bar{u}\|_{L^\infty(0,1)}\big)\right)\quad \forall t\in[T_1,T_2],$$

$$\mathcal{J}(t)=\int_{T_1}^{t}\mathcal{I}(s)\,ds.$$

We have thanks to (118):

$$\int_{T_1}^{T_2}\mathcal{J}(s)\,ds\leqslant\frac{1}{\alpha}. \tag{120}$$

Using the non-negativity of $\mathcal{I}$ and estimate (119) we can deduce:

$$\forall t\in[T_1,T_2],\quad \mathcal{J}(t)\geqslant\begin{cases}\mathcal{J}(T_2)-K(T_2-t) & \text{if } t\geqslant T_2-\frac{\mathcal{J}(T_2)}{K},\\ 0 & \text{otherwise.}\end{cases}$$

Substituting in (120) provides:

$$\mathcal{J}(T_2)\leqslant\sqrt{\frac{2K}{\alpha}}.$$

Combining with Proposition 3.4 we get:

$$\forall t\in[0,T_2],\quad \big\|u(t,.)-\bar{u}\big\|_{L^\infty(0,1)}\leqslant 2\|u_0-\bar{u}\|_{L^\infty(0,1)}+\sqrt{\frac{2K}{\alpha}}.$$

But using (111) we have:

$$\int_{T_2}^{+\infty}\mathcal{I}(t)\,dt\leqslant\int_{T_2}^{+\infty}-\dot{Q}(s)\,ds=Q(T_2).$$

Therefore we obtain:

$$\forall t\geqslant T_2,\quad \big\|u(t,.)-\bar{u}\big\|_{L^\infty(0,1)}\leqslant 2\|u_0-\bar{u}\|_{L^\infty(0,1)}+\sqrt{\frac{2K}{\alpha}}+\mathrm{A}^{-1}(K), \tag{121}$$

which concludes the proof. □

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