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Addendum to “Morse theory of causal geodesics in a stationary spacetime via Morse theory of geodesics of a Finsler metric” [Ann. I. H. Poincaré – AN 27 (3) (2010) 857–876]

Erasmo Caponio [a,∗,1], Miguel Ángel Javaloyes [b,2], Antonio Masiello [a,1]

[a] Dipartimento di Meccanica, Matematica e Management, Politecnico di Bari, Via Orabona 4, 70125, Bari, Italy
[b] Departamento de Geometría y Topología, Facultad de Ciencias, Universidad de Granada, Campus Fuentenueva s/n, 18071 Granada, Spain

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Abstract

We give the details of the proof of equality (29) in Caponio et al. (2010) [3].

Résumé

On donne les détails de la preuve de l’équation (29) dans Caponio et al. (2010) [3].

MSC: 58E05; 53C60; 53C22

Keywords: Morse theory; Critical groups; Finsler metrics

1. Introduction

In [3, Eq. (29)], we claim that the relative homology groups $H_*(\tilde{E}^c_{|X} \cap \tilde{O}^*, \tilde{E}^c_{|X} \cap \tilde{O}^* \setminus \{0\})$ and $H_*(\tilde{E}^c \cap \tilde{O}^*, \tilde{E}^c \cap \tilde{O}^* \setminus \{0\})$ are isomorphic, where, we recall, $X = C^1_0([0,1], U)$, U is a neighbourhood of $0 \in \mathbb{R}^n$,

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∗ Corresponding author.
E-mail addresses: caponio@poliba.it (E. Caponio), majava@ugr.es, ma.javaloyes@gmail.com (M.Á. Javaloyes), masiello@poliba.it (A. Masiello).
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$\tilde{E}: H^1_0([0,1],U) \to \mathbb{R}$, $\tilde{E}(x) = \int_0^1 \tilde{G}(s,x,\dot{x})\,\mathrm{d}s$, $0 \in H^1_0([0,1],U)$ is a non-degenerate critical point of $\tilde{E}$, $c = \tilde{E}(0)$, $\tilde{E}^c = \{x \in H^1_0([0,1],U) \mid \tilde{E}(x) \leqslant c\}$ and $\tilde{O}^*$ is a neighbourhood of 0 in $H^1_0([0,1],U)$.

For this we refer to the following result by Palais [12, Theorem 16]:

Theorem 1. *(See Palais [12].) Let V_1 and V_2 be two locally convex topological vector spaces, f be a continuous linear map from V_1 onto a dense linear subspace of V_2 and let O be an open subset of V_2 and $\tilde{O} = f^{-1}(O)$. If V_1 and V_2 are metrizable then $\tilde{f} = f_{|\tilde{O}}: \tilde{O} \to O$ is a homotopy equivalence.*

As a consequence, if E is a Banach space which is dense and continuously immersed in a Hilbert space H and (A,B) is a pair of open subsets of H with $B \subset A$, then the relative homology groups $H_*(A,B)$ and $H_*(\tilde{A},\tilde{B})$, where $\tilde{A} = A \cap E$ and $\tilde{B} = B \cap E$, are isomorphic.

In this addendum we would like to make clear how the above result can be applied to get

$$H_*\big(\tilde{E}^c_{|X} \cap \tilde{O}^*, \tilde{E}^c_{|X} \cap \tilde{O}^* \setminus \{0\}\big) \cong H_*\big(\tilde{E}^c \cap \tilde{O}^*, \tilde{E}^c \cap \tilde{O}^* \setminus \{0\}\big).$$

Although it is not difficult to find some open subsets which are homotopically equivalent, with respect to the H^1 topology, to the ones involved in the computations of the critical groups (cf., for example, [5, Ch. III, Corollary 1.2]), it is not trivial to ensure, after applying Palais's result, that the intersections of these subsets with X continue to be homotopically equivalent in the C^1 topology.

Actually, the equality between the critical groups of a Dirichlet functional with respect to the H^1 and C^1 topology is not a novelty (cf. [5,6,10]). Anyway, there are some issues for the functional $\tilde{E}$ that we would like to point out. First, $\tilde{E}$ is not C^2 with respect to the H^1 topology (this is a very general phenomenon for smooth, at most quadratic in the velocities Lagrangians, cf. [1, Prop. 3.2]); moreover, as $\tilde{G}$ is not everywhere twice differentiable, $\tilde{E}$ is not also twice Gateaux differentiable at any *non-$\tilde{G}$-regular* curve (see Definition 2). Secondly, although its flow is well defined on X, the gradient of $\tilde{E}$ is not of the type identity plus a compact operator, thus we cannot immediately state that it possesses the *retractible property* in [4, §III], which ensures that the deformation retracts involved in the computation of the critical groups are also continuous in X, where the Palais–Smale condition does not hold. To overcome this problem, we extend a result in [1], constructing a smooth vector field, which is a pseudo-gradient in $\mathscr{U} \setminus \bar{B}(0,r)$, where $\mathscr{U}$ is a neighbourhood of 0 in $H^1_0([0,1],U)$ and $\bar{B}(0,r)$ is the closure of a ball, and whose flow satisfies the retractible property.

The proof we give in the next section (without Lemma 4, which becomes superfluous) also holds for any *smooth* Lagrangian on $[0,1] \times TM$, where M is a finite dimensional manifold, which is fiberwise strongly convex and has at most quadratic growth in each fibre. We can also consider, with minor modifications, more general boundary conditions as the curves joining two given submanifolds in M. The Lagrangian action functional will be then defined on the Hilbert manifold of the H^1 curves between the two submanifolds. As we have already mentioned above, such functional is in general not C^2. Assuming that at least one of the submanifolds is compact and that all the critical points are non-degenerate, we can obtain, as in [3, Theorem 9], the Morse relations for the solutions of the corresponding Lagrangian system. In this case, the number of the conjugate instants along a geodesic, counted with their multiplicity, is replaced by the number of the "focal instants" with respect to one of the two submanifold (counted with multiplicities) along a solution plus the index of a bilinear symmetric form related to the other submanifold [7]. We recall that a Morse complex for the action functional of such kind of Lagrangian, whose homology is isomorphic to the singular homology of the path space between the two submanifolds, has been obtained in [1].

2. Proof of the isomorphism between the critical groups in H^1 and C^1

We recall that the Lagrangian $\tilde{G}: [0,1] \times U \times \mathbb{R}^n \to [0,+\infty)$ is given by

$$\tilde{G}(t,q,y) = F^2\big(\varphi(t,q), \mathrm{d}\varphi(t,q)\big[(1,y)\big]\big),$$

where F is a Finsler metric on the n-dimensional smooth manifold M and $\varphi: [0,1] \times U \to M$ is defined as $\varphi(t,q) = \exp_{\gamma_0(t)} P_t(q)$; here, exp is the exponential map with respect to any auxiliary Riemannian metric h on M, γ_0 is the geodesic of (M,F) in which we want to compute the critical groups, $P_t: U \to T_{\gamma_0(t)}M$ is given by $P_t(q_1,\dots,q_n) = \sum_{i=1}^n q_i E_i(t)$, where $\{E_i\}_{i\in\{1,\dots,n\}}$ are n-orthonormal smooth vector fields along γ_0 and U is the Euclidean ball of radius $\rho/2$, where ρ is the minimum of the injectivity radii (with respect to the metric h) at the points $\gamma(t)$, $t \in [0,1]$.

The set Z where $\tilde{G}$ is not twice differentiable is defined by the equation $d\varphi(t,q)[1,y]=0$ and then it corresponds to the subset of $[0,1]\times U\times\mathbb{R}^n$ where the Lagrangian $\tilde{G}(t,q,y)=F^2(\varphi(t,q),d\varphi(t,q)[(1,y)])$ vanishes. We recall also that for each $(t,q)\in[0,1]\times U$ there is only one $y\in\mathbb{R}^n$ such that $d\varphi(t,q)[(1,y)]=0$. Indeed, $d\varphi(t,q)[(1,y)]=\partial_t\varphi(t,q)+\partial_q\varphi(t,q)[y]$ and, as $\partial_q\varphi(t,q)$ is one-to-one, $y\in\mathbb{R}^n$ is the only vector such that

$$\partial_q\varphi(t,q)[y]=-\partial_t\varphi(t,q).$$

We recall also that the φ defines a smooth injective map $\varphi_*: H_0^1([0,1],U)\to\Omega_{p_0,q_0}(M)$, $\varphi_*(x)(t)=\varphi(t,x(t))$, such that $\tilde{E}=E\circ\varphi_*$, where E is the energy functional of F, i.e. $E(\gamma)=\frac{1}{2}\int_0^1F^2(\gamma,\dot\gamma)\,dt$ and Ω_{p_0,q_0} is the Hilbert manifold of the H^1 curves on M between p_0 and q_0. Observe that the curve of constant value 0 is mapped by φ_* to the geodesic γ_0 (hence 0 is a critical point of $\tilde{E}$).

From the fact that F^2 is fiberwise positively homogeneous of degree 2 and φ is a smooth map, it follows that there exists a constant c_1, depending only on U, such that

$$\left\|\tilde{G}_{qq}(s,q,y)\right\|\leqslant c_1\left(1+|y|^2\right),\qquad\left\|\tilde{G}_{qy}(s,q,y)\right\|\leqslant c_1\left(1+|y|\right),\qquad\left\|\tilde{G}_{yy}(s,q,y)\right\|\leqslant c_1,\tag{1}$$

for every $(s,q,y)\in[0,1]\times U\times\mathbb{R}^n\setminus Z$, where $|\cdot|$ and $\|\cdot\|$ are, respectively, the Euclidean norm and the norm of bilinear forms on $\mathbb{R}^n$.

Moreover, since F^2 is fiberwise strongly convex, there exists a positive constant c_2 such that

$$\tilde{G}_{yy}(s,q,y)[w,w]\geqslant c_2|w|^2,\tag{2}$$

for each $(s,q,y)\in[0,1]\times U\times\mathbb{R}^n\setminus Z$ and $w\in\mathbb{R}^n$.

Definition 2. A curve $x\in H_0^1([0,1],U)$ is said to be $\tilde{G}$-*regular* if the set of points $t\in[0,1]$ where $(t,x(t),\dot{x}(t))\in Z$ is negligible.

Let $\alpha:\mathbb{R}^n\to\mathbb{R}$ be a smooth function such that $\alpha_{|U'}=1$, $\alpha_{|U^C}=0$, where U' is an open subset of $\mathbb{R}^n$ such that $0\in U'$ and $\bar{U}'\subset U$. Consider the Lagrangian $\mathscr{L}:\mathbb{R}\times\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}$, $\mathscr{L}(t,q,y)=\alpha(q)\tilde{G}(t,q,y)+(1-\alpha(q))|y|^2$. Clearly, by the definition of α, 0 is also a critical point of the action functional $\mathscr{A}_{\mathscr{L}}(x)=\frac{1}{2}\int_0^1\mathscr{L}(s,x,\dot{x})\,ds$. Notice also that, like $\tilde{E}$, $\mathscr{A}_{\mathscr{L}}: H_0^1([0,1],\mathbb{R}^n)\to\mathbb{R}$ is a C^1 functional with locally Lipschitz differential.

Let $\mathscr{B}$ be a closed ball in $H_0^1([0,1],\mathbb{R}^n)$, centred in 0 and containing curves that have support in U'.

As $\mathscr{L}=\tilde{G}$ on $\mathbb{R}\times U'\times\mathbb{R}^n$, we have that $\mathscr{A}_{\mathscr{L}}|_{\mathscr{B}}=\tilde{E}|_{\mathscr{B}}$. Since $\tilde{E}$ satisfies the Palais–Smale condition (see [2]), we also have that $\mathscr{A}_{\mathscr{L}}$ satisfies the Palais–Smale condition in $\mathscr{B}$.

Moreover, from (1) it follows that $\tilde{E}$ is twice Gateaux differentiable at any $\tilde{G}$-regular curve $x\in H_0^1([0,1],U)$ and then the same property is satisfied by $\mathscr{A}_{\mathscr{L}}$.

Observe that, as the endpoints of the geodesic γ_0 are not conjugate, then we can assume that $\mathscr{B}$ is an isolating neighbourhood of the critical point 0. Moreover, the non-conjugacy assumption implies also that 0 is a non-degenerate critical point of $\tilde{E}$, that is, the kernel of the operator A, which represents the second Gateaux differential at 0 of both $\tilde{E}$ and $\mathscr{A}_{\mathscr{L}}$, with respect to the scalar product $\langle\cdot,\cdot\rangle$ in $H_0^1([0,1],\mathbb{R}^n)$, is empty.

The following proposition has been obtained in [1, Lemma 4.1 and formula (4.8)] for the action functional of a C^2, time-dependent, fiberwise strongly convex, at most quadratic in the velocities, Lagrangian on TM.

Proposition 3. *There exist a neighbourhood $\mathscr{U}'$ of 0 in $H_0^1([0,1],\mathbb{R}^n)$ (that we can assume it is contained in $\mathscr{B}$) and a positive constant μ_0, such that the linear vector field $x\in\mathscr{U}'\mapsto Ax$, satisfies the inequality*

$$d\mathscr{A}_{\mathscr{L}}(x)[Ax]\geqslant\mu_0\left\|\nabla\mathscr{A}_{\mathscr{L}}(x)\right\|_0^2,\tag{3}$$

for each $x\in\mathscr{U}'$.

Here $\|\cdot\|_0$ is the H_0^1 norm. In our setting, the Lagrangian $\mathscr{L}$ is not twice differentiable on $Z\subset TM$ and this leads to some differences between the proof of [1, Lemma 4.1] and ours, which we outline in Lemmata 4, 7 and 8.

Lemma 4. *Let x be a smooth curve (non-necessarily $\tilde{G}$-regular) in $H_0^1([0,1],U)$. Then the curves $t\in[0,1]\mapsto sx(t)$ can be non-$\tilde{G}$-regular for s in a subset of $[0,1]$ which is at most countable.*

Proof. We recall that $w \in H_0^1([0,1],U)$, $w = w(t)$, is not $\tilde{G}$-regular if $(t, w(t), \dot{w}(t)) \in Z$ for each t in a subset of positive Lebesgue measure in $[0,1]$. Now, for $x:[0,1] \to U$, smooth and $x(0) = x(1) = 0$, let us consider the map $f:[0,1]\times[0,1] \to M$ defined as $f(s,t) = \varphi(t, sx(t))$. Observe that for each $\bar{t} \in [0,1]$, $s \mapsto f(s,\bar{t})$ is the affinely parametrized geodesic $\sigma_{\bar{t}}$ of the Riemannian metric h defined by $\sigma_{\bar{t}}(s) = \varphi(\bar{t}, sx(\bar{t})) = \exp_{\gamma(\bar{t})}(sx(\bar{t}))$ (for $\bar{t} = 0$ and $\bar{t} = 1$ the geodesics are constant) while, for each $\bar{s} \in [0,1]$, $t \mapsto f(\bar{s},t)$ is the curve $\gamma_{\bar{s}}$ corresponding to $\bar{s}x$ by the map φ_* (for $\bar{s} = 0$ and $\bar{s} = 1$, we get respectively γ_0, the geodesic of (M,F), and the curve $\gamma_1 = \varphi_*(x)$). Thus $f = f(s,t)$ defines a geodesic congruence and, then, $s \mapsto J_t(s) = \partial_t f(s,t) = \dot{\gamma}_s(t)$ defines a Jacobi field along σ_t for each $t \in (0,1)$ where $x(t) \neq 0$. Observe that at the instants $\bar{t}$ where $x(\bar{t}) = 0$ (if they exist), $\sigma_{\bar{t}}$ is constant and equal to $\gamma_0(\bar{t})$. Since there is only one $y \in \mathbb{R}^n$ such that $(\bar{t}, 0, y) \in Z$ and such y cannot be equal to 0 (otherwise $0 = \mathrm{d}\varphi(\bar{t},0)[1,0] = \partial_t\varphi(\bar{t},0) + \partial_q\varphi(\bar{t},0)[0] = \partial_t\varphi(\bar{t},0) = \dot{\gamma}_0(\bar{t}) \neq 0$), there can be at most one $s \in (0,1]$ such that $(\bar{t}, sx(\bar{t}), s\dot{x}(\bar{t})) = (\bar{t}, 0, s\dot{x}(\bar{t})) \in Z$. Now let us assume that for $s, s' \in (0,1]$, $s \neq s'$, the curves sx and $s'x$ are not $\tilde{G}$-regular. From what we have recalled above, this is equivalent to the fact that the curves γ_s and $\gamma_{s'}$ have velocity vector fields vanishing on, respectively, $Z_s \subset [0,1]$ and $Z_{s'} \subset [0,1]$ with $|Z_s|, |Z_{s'}| > 0$. We claim that $Z_s \cap Z_{s'} = \emptyset$. Indeed, if there exists $\bar{t} \in Z_s \cap Z_{s'}$, then $x(\bar{t})$ must be different from 0 and this implies that the Jacobi field $J_{\bar{t}}$ is well defined and equal to 0 at the instants s and s'. Thus the points $\sigma_{\bar{t}}(s)$ and $\sigma_{\bar{t}}(s')$ are conjugate along $\sigma_{\bar{t}}$, but this is impossible (see, e.g., [8, Prop. 2.2, p. 267]) because such geodesic has length less than the injectivity radius at $\gamma_0(\bar{t})$. Therefore the set $\mathscr{Z}$ of $s \in [0,1]$ such that $|Z_s| > 0$ is at most countable. Indeed, by contradiction, assume that $\mathscr{Z}$ is uncountable and consider the set $A_h = \{s \in (0,1]:\ |Z_s| > \frac{1}{h}\}$. Since $\bigcup_{h\in\mathbb{N}} A_h = \mathscr{Z}$, there must exist at least one $k \in \mathbb{N}$ such that A_k is uncountable. Thus, for infinitely many $s \in [0,1]$, we would have disjoint subsets $Z_s \subset [0,1]$ having measure greater than $\frac{1}{h}$, which is impossible. □

Remark 5. From Lemma 4, it also follows that any smooth non-$\tilde{G}$-regular curve $x \in H_0^1([0,1],U)$ is the limit, in the H^1 topology, of some sequence $(x_k) \subset H_0^1([0,1],U)$ of smooth $\tilde{G}$-regular curves. Indeed, it is enough to consider a sequence $(s_n) \subset [0,1]$ such that $s_n \to 1$ and $s_n x$ is $\tilde{G}$-regular.

Remark 6. From (2), the second Gateaux differential of $\tilde{E}$ at a $\tilde{G}$-regular curve x is represented by a linear bounded self-adjoint operator on $H_0^1([0,1],\mathbb{R}^n)$ of the type $A_x = B_x + K_x$ where B_x is a strictly positive definite operator and K_x is compact. Moreover from (1), if a sequence of $\tilde{G}$-regular curves $\{x_n\}$ converges to a $\tilde{G}$-regular curve x in the H^1 topology then K_{x_n} converges to K_x in the norm topology of the bounded operators and B_{x_n} converges strongly to B_x, i.e. $B_{x_n}[\xi] \to B_x[\xi]$ for each $\xi \in H_0^1([0,1],\mathbb{R}^n)$ (cf. Claims 1 and 2 of the proof of Lemma 4.1 in [1]). We recall that from [3, Lemma 2], $A \equiv A_0$ is given by $I + K$ (that is, B_0 is the identity operator).

The following two results are analogous to, respectively, Eq. (4.5) and Claim 3 in [1].

Lemma 7. *Let $(x_n) \subset H_0^1([0,1],U)$ be a sequence of smooth $\tilde{G}$-regular curves such that $x_n \to 0$ in the H^1 topology. Then*

$$\mathrm{d}\tilde{E}(x_n)[Ax_n] = \int_0^1 \big\langle \big(B_{sx_n}^{1/2} + K_{sx_n}\big)^2 x_n, x_n\big\rangle\,\mathrm{d}s + o\big(\|x_n\|_0^2\big), \quad \text{as } n \to \infty.$$

Proof. Eqs. (1)–(2) imply that $\tilde{G}(t,q,y)$ satisfies assumptions $(L1')$ and $(L2')$ on page 605 of [1], for each $(t,q,y) \in [0,1]\times U \times \mathbb{R}^n \setminus Z$. Hence the lemma follows arguing as in [1, Lemma 4.1], taking into account that

$$\mathrm{d}\tilde{E}(x)[Ax] = \big\langle \nabla\tilde{E}(x), x + K(x)\big\rangle = \left(\int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\big\langle \nabla\tilde{E}(sx), x + K(x)\big\rangle\,\mathrm{d}s\right)$$

$$= \int_0^1 \big\langle (B_{sx} + K_{sx})x, x + K(x)\big\rangle\,\mathrm{d}s. \tag{4}$$

In fact, $\frac{\mathrm{d}}{\mathrm{d}s}\nabla\tilde{E}(sx) = (B_{sx} + K_{sx})[x]$ at the points s where the curve $t \in [0,1] \mapsto sx(t)$ is $\tilde{G}$-regular. From Lemma 4, the set of points $s \in [0,1]$ where sx is not $\tilde{G}$-regular is at most countable. □

The next lemma follows as in Claim 3 of [1, Lemma 4.1], recalling Remark 6 and the fact that 0 is a non-degenerate critical point of $\tilde{E}$.

Lemma 8. *There exist a number $\mu > 0$ and a neighbourhood $\mathscr{U}''$ of 0 in $H_0^1([0,1], U)$ such that, for each smooth and $\tilde{G}$-regular curve $x \in \mathscr{U}''$, the spectrum of the self-adjoint operator $B_x^{1/2} + K_x$ is disjoint from $[-\mu, \mu]$.*

Proof of Proposition 3. Since $\mathscr{A}_{\mathscr{L}}|_{\mathscr{B}} = \tilde{E}|_{\mathscr{B}}$, it is enough to prove the proposition for the functional $\tilde{E}$. From Lemmata 7 and 8, we get that there exists a positive constant μ_1, such that

$$\mathrm{d}\tilde{E}(x)[Ax] \geqslant \mu_1 \|x\|_0^2, \tag{5}$$

for each smooth $\tilde{G}$-regular curve $x \in \mathscr{U}''$. From Remark 5 and the continuity of $\mathrm{d}\tilde{E}$ and A with respect to the H^1 topology, inequality (5) can be extended to any smooth curve in $\mathscr{U}''$ and then, since smooth curves are dense in $H_0^1([0,1], U)$, to any $x \in \mathscr{U}''$. As $\nabla\tilde{E}$ is a locally Lipschitz field and $\nabla\tilde{E}(0) = 0$, we get

$$\mathrm{d}\tilde{E}(x)[Ax] \geqslant \mu_0 \big\|\nabla\tilde{E}(x)\big\|_0^2,$$

for some positive constant μ_0 and for all x in some neighbourhood $\mathscr{U}'$ of 0. □

Now let $\eta_0 \colon H_0^1([0,1], \mathbb{R}^n) \to [0,1]$ be a smooth bump function such that $\operatorname{supp}\eta_0 \subset \mathscr{U}'$ and $\eta_0(x) = 1$, for all $x \in \mathscr{U}$, where $\mathscr{U}$ is an open neighbourhood of 0 in $H_0^1([0,1], \mathbb{R}^n)$ with $\overline{\mathscr{U}} \subset \mathscr{U}'$. Let us consider the vector field on $H_0^1([0,1], \mathbb{R}^n)$ defined as

$$Y(x) = -\eta_0(x)Ax - \big(1 - \eta_0(x)\big)\nabla\mathscr{A}_{\mathscr{L}}(x).$$

We point out that we cannot state that Y is a pseudo-gradient vector field because we are not able to prove that

$$\|Ax\|_0 \leqslant \mu_2 \big\|\mathrm{d}A_{\mathscr{L}}(x)\big\|_0, \tag{6}$$

for some constant $\mu_2 > \mu_0$ and all x in some neighbourhood of 0.[3] Anyway (3) implies that Y satisfies the inequality

$$\mathrm{d}\mathscr{A}_{\mathscr{L}}(x)\big[Y(x)\big] \leqslant -\mu \big\|\nabla\mathscr{A}_{\mathscr{L}}(x)\big\|_0^2, \tag{7}$$

for each $x \in H_0^1([0,1], \mathbb{R}^n)$, where $\mu = \min\{\mu_0, 1\}$. As we will show in Lemma 9, inequality (7) (together with the remark in footnote 3) is enough to get a deformation result as in [11, Lemma 8.3]. For all $x \in H_0^1([0,1], \mathbb{R}^n)$, let $(\omega^-(x), \omega^+(x))$ be the maximal interval of definition of the solution of

$$\begin{cases} \dot{\psi} = Y(\psi), \\ \psi(0) = x. \end{cases} \tag{8}$$

Observe that this problem is well defined because Y is a locally Lipschitz vector field in $H_0^1([0,1], \mathbb{R}^n)$, since A and $\nabla\mathscr{A}_{\mathscr{L}}$ are. Furthermore, (7) implies that $\mathscr{A}_{\mathscr{L}}$ is decreasing along the flow of Y and as, $Y_{|\mathscr{U}} = -A = -I - K$, such flow is given by

$$\psi(x,t) = e^{-t}x - \int_0^t e^{-t+s} K\big(\psi(x,s)\big)\,\mathrm{d}s \tag{9}$$

for $x \in \mathscr{U}$, whereas $\psi(x,t) \in \mathscr{U}$. The following lemma is an adaptation of Lemma 8.1 in [11] to the flow of the vector field Y.

[3] Actually using that $\mathscr{A}_{\mathscr{L}}$ satisfies the Palais–Smale condition and 0 is an isolated critical point of $\mathscr{A}_{\mathscr{L}}$, we can prove that Y satisfies (6) in any open subset $\mathscr{U} \setminus \bar{B}(0,r)$, where $B(0,r)$ is an open ball strictly contained in $\mathscr{U}$, for a constant μ_2 depending on $\mathscr{U} \setminus \bar{B}(0,r)$.

Lemma 9. *Let $\mathscr{V}$ be a closed neighbourhood of 0 contained in $\mathscr{U}$. Then there exist $\varepsilon > 0$ and an open neighbourhood $O' \subset \mathscr{V}$ of 0 in $H_0^1([0,1],\mathbb{R}^n)$ such that if $x \in O'$, then the solution $\psi(x,\cdot)$ of (8) either stays in $\mathscr{V}$ for all $t \in [0,+\infty)$ or it stays in $\mathscr{V}$ at least until $\mathscr{A}_{\mathscr{L}}(\psi(x,t))$ becomes less than $c-\varepsilon$ (where $c = \mathscr{A}_{\mathscr{L}}(0) = \tilde{E}(0)$).*

Proof. Observe that, since $Y_{|\mathscr{V}} = -A$, $\psi(x,\cdot)$ is defined for all times until it lies in $\mathscr{V}$. Let $B(0,\rho)$ be the ball of radius ρ centred at 0 such that $\bar{B}(0,\rho) \subset \mathscr{V}$ and let

$$\mathscr{C} = \left\{ x \in H_0^1\big([0,1],\mathbb{R}^n\big)\colon\ \frac{\rho}{2} \leqslant \|x\|_0 \leqslant \rho \right\}.$$

Since $\mathscr{C} \subset \mathscr{B}$, it is free of critical points and then

$$\delta = \inf_{x\in\mathscr{C}} \big\|\nabla \mathscr{A}_{\mathscr{L}}(x)\big\|_0 > 0, \tag{10}$$

because $\mathscr{A}_{\mathscr{L}}$ satisfies the Palais–Smale condition on $\mathscr{C}$. Moreover

$$\big\|Y(x)\big\|_0 = \|Ax\|_0 \leqslant \rho\|A\|_0 \leqslant \frac{\rho\|A\|_0}{\delta}\big\|\nabla \mathscr{A}_{\mathscr{L}}(x)\big\|_0, \tag{11}$$

for each $x \in \mathscr{C}$. Let $\nu := \frac{\rho\|A\|_0}{\delta}$ and $O' = B(0,\rho/2) \cap \overset{\circ}{\overbrace{\mathscr{A}_{\mathscr{L}}^{c+\frac{\mu\delta\rho}{4\nu}}}}$. If $x \in O'$ is such that $\psi(x,\bar{t})$ does not belong to $\mathscr{V}$ for some $\bar{t} > 0$, then there exist $0 < t_1 < t_2 < \omega^+(x)$ such that $\psi(x,t) \in \mathscr{C}$, for all $t \in (t_1,t_2)$ and $\|\psi(x,t_1)\|_0 = \rho/2$, $\|\psi(x,t_2)\|_0 = \rho$. It follows that

$$\begin{aligned}
\mathscr{A}_{\mathscr{L}}\big(\psi(x,t_2)\big) &= \mathscr{A}_{\mathscr{L}}\big(\psi(x,t_1)\big) + \int_{t_1}^{t_2} \mathrm{d}\mathscr{A}_{\mathscr{L}}\big(\psi(x,t)\big)\big[Y\big(\psi(x,t)\big)\big]\,\mathrm{d}t \\
&\leqslant \mathscr{A}_{\mathscr{L}}(x) - \mu \int_{t_1}^{t_2} \big\|\nabla \mathscr{A}_{\mathscr{L}}\big(\psi(x,t)\big)\big\|_0^2\,\mathrm{d}t \\
&\leqslant c + \frac{\mu\delta\rho}{4\nu} - \mu\delta \int_{t_1}^{t_2} \big\|\nabla \mathscr{A}_{\mathscr{L}}\big(\psi(x,t)\big)\big\|_0\,\mathrm{d}t \\
&\leqslant c + \frac{\mu\delta\rho}{4\nu} - \frac{\mu\delta}{\nu} \int_{t_1}^{t_2} \big\|Y\big(\psi(x,t)\big)\big\|_0\,\mathrm{d}t \\
&\leqslant c + \frac{\mu\delta\rho}{4\nu} - \frac{\mu\delta}{\nu}\big(\big\|\psi(x,t_2)\big\|_0 - \big\|\psi(x,t_1)\big\|_0\big) \\
&= c + \frac{\mu\delta\rho}{4\nu} - \frac{\mu\delta\rho}{2\nu} = c - \frac{\mu\delta\rho}{4\nu}.
\end{aligned} \tag{12}$$

In the first inequality above, we have used the fact that $\mathscr{A}_{\mathscr{L}}$ is decreasing in the flow of (8) and inequality (7); in the second one, the fact that $x \in O' \subseteq \mathscr{A}_{\mathscr{L}}^{c+\frac{\mu\delta\rho}{4\nu}}$ and (10); in the third one, inequality (11); in the last one, the following chain of inequalities:

$$\int_{t_1}^{t_2} \big\|Y\big(\psi(x,t)\big)\big\|_0\,\mathrm{d}t = \int_{t_1}^{t_2} \big\|\dot{\psi}(x,t)\big\|_0\,\mathrm{d}t \geqslant \left\|\int_{t_1}^{t_2} \dot{\psi}(x,t)\,\mathrm{d}t\right\|_0 \geqslant \big\|\psi(x,t_2)\big\|_0 - \big\|\psi(x,t_1)\big\|_0.$$

Thus the conclusion follows with $\varepsilon = \frac{\mu\delta\rho}{4\nu}$. □

Let V be the subset of $H_0^1([0,1],\mathbb{R}^n)$ given as $V = \bigcup_{x\in O'} \psi(x,[0,\omega^+(x)))$, where O' is the neighbourhood of 0 associated to $\mathscr{V}$ by Lemma 9. Since O' is open, from standard results in ODE theory (cf. for example [9, Corollary 4.2.10]), V is also an open subset of $H_0^1([0,1],\mathbb{R}^n)$. From Lemma 9, $\mathscr{A}_{\mathscr{L}}^{-1}((c-\varepsilon,c+\varepsilon)) \cap V \setminus \{0\}$ is contained in $\mathscr{V} \subset \mathscr{U}$ and it is free of critical points.

Lemma 10. *For every $x \in \mathscr{A}_{\mathscr{L}}^{-1}([c, c+\varepsilon)) \cap V$, either there exists a unique $T(x) \in [0, \omega^+(x))$ such that $\mathscr{A}_{\mathscr{L}}(\psi(x, T(x))) = c$ or $\omega^+(x) = +\infty$ and $\psi(x,t) \to 0$, in $H_0^1([0,1], \mathbb{R}^n)$, as $t \to +\infty$.*

Proof. If $\mathscr{A}_{\mathscr{L}}(\psi(x,t)) > c$, for all $t \in [0, \omega^+(x))$, then from Lemma 9, $\omega^+(x) = +\infty$ and $\psi(x,t) \in \mathscr{V}$, for each $t \in [0, +\infty)$. From inequality (12),

$$\int_0^{+\infty} \left\|\nabla \mathscr{A}_{\mathscr{L}}\big(\psi(x,t)\big)\right\|_0^2 \mathrm{d}t \leqslant \frac{1}{\mu}\big(\mathscr{A}_{\mathscr{L}}(x) - c\big) < +\infty,$$

hence $\liminf_{t\to+\infty} \|\nabla \mathscr{A}_{\mathscr{L}}(\psi(x,t))\|_0^2 = 0$ and the Palais–Smale condition implies the existence of a sequence $\{t_n\}$ converging to $+\infty$ such that $\psi(x, t_n) \to 0$. Hence the conclusion follows from Lemma 9. □

By Lemmata 9 and 10, as in [11, Lemma 8.3], we get that $\mathscr{A}_{\mathscr{L}}^c \cap V$ is a strong deformation retract of $\overset{\circ}{\widehat{\mathscr{A}_{\mathscr{L}}^{c+\varepsilon/2}}} \cap V$. Analogously, $\mathscr{A}_{\mathscr{L}}^{c-\varepsilon} \cap V$ is a strong deformation retract of both $\mathscr{A}_{\mathscr{L}}^c \cap V \setminus \{0\}$ and $\overset{\circ}{\widehat{\mathscr{A}_{\mathscr{L}}^{c-\varepsilon/2}}} \cap V$. Using that, for $A \subset B \subset C$, if B is a strong deformation retract of C, then $H_*(B,A) \cong H_*(C,A)$ and if A is a strong deformation retract of B, then $H_*(C,A) \cong H_*(C,B)$ (for the last property, see for example [13, Property H_6–β]), we obtain

$$H_*\big(\mathscr{A}_{\mathscr{L}}^c \cap V, \mathscr{A}_{\mathscr{L}}^c \cap V \setminus \{0\}\big) \cong H_*\big(\overset{\circ}{\widehat{\mathscr{A}_{\mathscr{L}}^{c+\varepsilon/2}}} \cap V, \overset{\circ}{\widehat{\mathscr{A}_{\mathscr{L}}^{c-\varepsilon/2}}} \cap V\big). \tag{13}$$

Let $O = \varphi_*(O')$ and $\gamma_0 = \varphi_*(0)$, then

$$\begin{aligned} C_*(E, \gamma_0) &= H_*\big(E^c \cap O, E^c \cap O \setminus \{\gamma_0\}\big) \cong H_*\big((E\circ\varphi_*)^c \cap O', (E\circ\varphi_*)^c \cap O' \setminus \{0\}\big) \\ &= H_*\big(\tilde{E}^c \cap O', \tilde{E}^c \cap O' \setminus \{0\}\big) = H_*\big(\mathscr{A}_{\mathscr{L}}^c \cap O', \mathscr{A}_{\mathscr{L}}^c \cap O' \setminus \{0\}\big) \\ &\cong H_*\big(\mathscr{A}_{\mathscr{L}}^c \cap V, \mathscr{A}_{\mathscr{L}}^c \cap V \setminus \{0\}\big), \end{aligned} \tag{14}$$

last equivalence, by the excision property of the singular relative homology groups. By Palais's theorem above we get

$$H_*\big(\overset{\circ}{\widehat{\mathscr{A}_{\mathscr{L}}^{c+\varepsilon/2}}} \cap V, \overset{\circ}{\widehat{\mathscr{A}_{\mathscr{L}}^{c-\varepsilon/2}}} \cap V\big) \cong H_*\big(\overset{\circ}{\widehat{\mathscr{A}_{\mathscr{L}}^{c+\varepsilon/2}}}\big|_{C_0^1([0,1],\mathbb{R}^n)} \cap V, \overset{\circ}{\widehat{\mathscr{A}_{\mathscr{L}}^{c-\varepsilon/2}}}\big|_{C_0^1([0,1],\mathbb{R}^n)} \cap V\big).$$

The above equivalence, together with (13) and (14), implies that

$$C_*(E, \gamma_0) \cong H_*\big(\overset{\circ}{\widehat{\mathscr{A}_{\mathscr{L}}^{c+\varepsilon/2}}}\big|_{C_0^1([0,1],\mathbb{R}^n)} \cap V, \overset{\circ}{\widehat{\mathscr{A}_{\mathscr{L}}^{c-\varepsilon/2}}}\big|_{C_0^1([0,1],\mathbb{R}^n)} \cap V\big).$$

It remains to prove that these last relative homology groups are isomorphic to the critical groups in $X = C_0^1([0,1], U)$. To this end, let us consider the Cauchy problem (8), with $x \in C^1([0,1], \mathbb{R}^n) \cap \mathscr{A}_{\mathscr{L}}^{-1}((c - \varepsilon/2, c + \varepsilon/2)) \cap V$. Since $\mathscr{A}_{\mathscr{L}}^{-1}((c-\varepsilon/2, c+\varepsilon/2)) \cap V \subset \mathscr{V} \subset \mathscr{U}$, it holds (9) and the orbit $\psi(x, \cdot)$, defined by x, is also in $C_0^1([0,1], \mathbb{R}^n)$.

As a consequence, the strong deformation retracts that we have considered above are well defined in $C_0^1([0,1], \mathbb{R}^n) \times [0,1]$ and by the continuity of the flow (9) with respect to the C^1 topology, we immediately deduce that they are also continuous at each point different from $(0,1)$. Clearly, the continuity at the point $(0,1)$ with respect to the product topology of $C_0^1([0,1], \mathbb{R}^n)$, with the C^1 topology, and $\mathbb{R}$, with the standard one, comes into play only for the deformation map η: $\overset{\circ}{\widehat{\mathscr{A}_{\mathscr{L}}^{c+\varepsilon/2}}} \cap V \times [0,1] \to \overset{\circ}{\widehat{\mathscr{A}_{\mathscr{L}}^{c+\varepsilon/2}}} \cap V$ of $\overset{\circ}{\widehat{\mathscr{A}_{\mathscr{L}}^{c+\varepsilon/2}}} \cap V$ in $\mathscr{A}_{\mathscr{L}}^c \cap V$, which is given by

$$\eta(x,t) = \begin{cases} \rho(x, \frac{t}{t-1}) & \text{if } t \in [0,1), \\ \lim_{s\to+\infty} \rho(x,s) & \text{if } t = 1, \end{cases}$$

where ρ: $\overset{\circ}{\widehat{\mathscr{A}_{\mathscr{L}}^{c+\varepsilon/2}}} \cap V \times [0, +\infty) \to \overset{\circ}{\widehat{\mathscr{A}_{\mathscr{L}}^{c+\varepsilon/2}}} \cap V$ is the map defined as follows: if $\mathscr{A}_{\mathscr{L}}(x) > c$ and there exists $T(x) > 0$ such that $\mathscr{A}_{\mathscr{L}}(\psi(x, T(x))) = c$, then

$$\rho(x,t) = \begin{cases} \psi(x,t) & \text{if } t \in [0, T(x)], \\ \psi(x, T(x)) & \text{if } t \in (T(x), +\infty), \end{cases}$$

if $\psi(x,t) \to 0$ as $t \to +\infty$, then $\rho(x,t) = \psi(x,t)$ and if $\mathscr{A}_{\mathscr{L}}(x) \leqslant c$, then $\rho(x,t) = x$, for all $t \in [0,+\infty)$. Since the flow ψ_1 of the linear vector field $x \mapsto -Ax = -Ix - Kx$ is given by (9) and K is bounded from $H_0^1([0,1],\mathbb{R}^n)$ to $C_0^1([0,1],\mathbb{R}^n)$, we have

$$\left\| \int_0^t e^{-t+s} K\big(\psi_1(x,s)\big)\,\mathrm{d}s \right\|_{C^1} \leqslant e^{-t} \int_0^t e^s \big\| K\big(\psi_1(x,s)\big)\big\|_{C^1}\,\mathrm{d}s \leqslant Ce^{-t} \int_0^t e^s \big\|\psi_1(x,s)\big\|_0\,\mathrm{d}s.$$

Thus, if $\psi(x,t) \to 0$ in H^1, as $t \to +\infty$, then, from Lemmata 9 and 10, $\psi(x,t) = \psi_1(x,t)$. Hence, for every $\varepsilon > 0$, there exists $\bar{t} > 0$ such that for all $t > \bar{t}$, $\|\psi(x,t)\|_0 < \varepsilon$ and then the last function in the above inequalities can be estimated, for $t > \bar{t}$, as

$$\begin{aligned} e^{-t} \int_0^t e^s \big\|\psi(x,s)\big\|_0\,\mathrm{d}s &= e^{-t} \int_0^{\bar{t}} e^s \big\|\psi(x,s)\big\|_0\,\mathrm{d}s + e^{-t} \int_{\bar{t}}^t e^s \big\|\psi(x,s)\big\|_0\,\mathrm{d}s \\ &\leqslant e^{-t} \int_0^{\bar{t}} e^s \big\|\psi(x,s)\big\|_0\,\mathrm{d}s + \varepsilon\big(1 - e^{-t}e^{\bar{t}}\big). \end{aligned}$$

Thus $\psi(x,t) \to 0$ also with respect to the C^1 topology, giving the continuity of the map η at the point $(0,1)$ also with respect to the product of such a topology and the Euclidean one on the interval $[0,1]$.

In conclusion we have that the following groups are isomorphic

$$\begin{aligned} &H_*\big(\widehat{\mathscr{A}_{\mathscr{L}}^{c+\varepsilon/2}}\big|_{C_0^1([0,1],\mathbb{R}^n)} \cap V, \widehat{\mathscr{A}_{\mathscr{L}}^{c-\varepsilon/2}}\big|_{C_0^1([0,1],\mathbb{R}^n)} \cap V\big) \\ &\quad\cong H_*\big(\mathscr{A}_{\mathscr{L}}^c\big|_{C_0^1([0,1],\mathbb{R}^n)} \cap V, \mathscr{A}_{\mathscr{L}}^c\big|_{C_0^1([0,1],\mathbb{R}^n)} \cap V \setminus \{0\}\big). \end{aligned}$$

By excision, these last relative homology groups are isomorphic to

$$H_*\big(\mathscr{A}_{\mathscr{L}}^c\big|_{C_0^1([0,1],\mathbb{R}^n)} \cap O', \mathscr{A}_{\mathscr{L}}^c\big|_{C_0^1([0,1],\mathbb{R}^n)} \cap O' \setminus \{0\}\big)$$

and then, since the curves in O' have their support in U, to

$$H_*\big(\tilde{E}^c\big|_{C_0^1([0,1],U)} \cap O', \tilde{E}^c\big|_{C_0^1([0,1],U)} \cap O' \setminus \{0\}\big).$$

References

[1] A. Abbondandolo, M. Schwarz, A smooth pseudo-gradient for the Lagrangian action functional, Adv. Nonlinear Stud. 9 (2009) 597–623.
[2] E. Caponio, M.A. Javaloyes, A. Masiello, On the energy functional on Finsler manifolds and applications to stationary spacetimes, Math. Ann. 351 (2011) 365–392.
[3] E. Caponio, M.A. Javaloyes, A. Masiello, Morse theory of causal geodesics in a stationary spacetime via Morse theory of geodesics of a Finsler metric, Ann. Inst. H. Poincaré Anal. Non Linéaire 27 (2010) 857–876.
[4] K.-C. Chang, A variant mountain pass lemma, Sci. Sinica Ser. A 26 (1983) 1241–1255.
[5] K.-C. Chang, Infinite-Dimensional Morse Theory and Multiple Solution Problems, Birkhäuser, Boston, MA, 1993.
[6] K.-C. Chang, H^1 versus C^1 isolated critical points, C. R. Acad. Sci. Paris Sér. I Math. 319 (1994) 441–446.
[7] M. Crampin, The Morse index theorem for general end conditions, Houston J. Math. 27 (2001) 807–821.
[8] M.P. do Carmo, Riemannian Geometry, Birkhäuser, Boston, MA, 1992.
[9] S. Lang, Differential and Riemannian Manifolds, Grad. Texts in Math., Springer-Verlag, New York, 1995.
[10] C. Li, S. Li, J. Liu, Splitting theorem, Poincaré–Hopf theorem and jumping nonlinear problems, J. Funct. Anal. 221 (2005) 439–455.
[11] J. Mawhin, M. Willem, Critical Point Theory and Hamiltonian Systems, Appl. Math. Sci., Springer-Verlag, New York, 1989.
[12] R.S. Palais, Homotopy theory of infinite dimensional manifolds, Topology 5 (1966) 1–16.
[13] E.H. Rothe, Morse theory in Hilbert space, Rocky Mountain J. Math. 3 (1973) 251–274.