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# Profile of bubbling solutions to a Liouville system

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**Abstract**

In several fields of Physics, Chemistry and Ecology, some models are described by Liouville systems. In this article we first prove a uniqueness result for a Liouville system in $\mathbb{R}^2$. Then we establish a uniform estimate for bubbling solutions of a locally defined Liouville system near an isolated blowup point. The uniqueness result, as well as the local uniform estimates are crucial ingredients for obtaining a priori estimate, degree counting formulas and existence results for Liouville systems defined on Riemann surfaces.
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**Résumé**

En plusieurs champs de Physique, Chimie et Écologie, quelques modèles sont décrits par les systèmes de Liouville. Dans cet article nous prouvons d'abord un résultat de caractère unique pour un système de Liouville dans $\mathbb{R}^2$. Alors nous établissons une estimation uniforme pour les solutions d'explosion d'un système de Liouville localement défini prés d'un point d'explosion isolé. Le résultat d'unicité, aussi bien que les estimations uniformes locales sont les ingrédients cruciaux pour obtenir a priori l'estimation, les formules comptant le degré, et l'existence pour les systèmes de Liouville définis sur des surfaces de Reimann.
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## 1. Introduction

In this article we are concerned with the following generalized Liouville system:

$$\Delta u_i + \sum_{j=1}^{n} a_{ij} h_j e^{u_j} = 0, \quad i \in I \equiv \{1, \dots, n\}, \qquad \Omega \subset \mathbb{R}^2, \tag{1.1}$$

where $\Omega$ is a subset of $\mathbb{R}^2$, $h_1, \dots, h_n$ are positive smooth functions, $A = (a_{ij})_{n\times n}$ is an invertible, symmetric and non-negative matrix.

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(1.1) is an extension of the well-known classical Liouville equation

$$\Delta u + Ve^u = 0, \qquad \Omega \subset \mathbb{R}^2,$$

which finds applications in many fields in Physics and Mathematics. For example the Liouville equation is related to finding a metric whose Gauss curvature is a prescribed function [7]. In Physics, the Liouville equation represents the electric potential induced by the charge carriers in electrolytes theory [25] and the Newtonian potential of a cluster of self-gravitation mass distribution [1,4,26,27]. Moreover, it is closely related to the abelian model in the Chern–Simons theories [16–18].

The Liouville systems are natural extensions of the Liouville equation and they also have applications in different fields of physics, chemistry and ecology. Indeed, various Liouville systems are used to describe models in the theory of chemotaxis [12,19], in the physics of charged particle beams [2,14,20] and in the theory of semi-conductors [24]. For applications of Liouville systems, see [8,13] and the references therein. Here we also note that another important extension of the Liouville equation is the Toda system, which is closely related to the non-abelian Chern–Simons theory [15,28].

Chanillo and Kiessling [8] first studied the type of Liouville systems described by (1.1) with constant coefficients in $\mathbb{R}^2$ and they proved that under certain assumptions on $A$, all the entire solutions ($\Omega = \mathbb{R}^2$) are symmetric with respect to some point. Their result was improved by Chipot, Shafrir and Wolansky [13], who proved among other things the following symmetry result:

**Theorem A** *(Chipot–Shafrir–Wolansky). Let $A = (a_{ij})_{n\times n}$ be an*

$$\textit{invertible, symmetric, non-negative and irreducible matrix,} \tag{1.2}$$

*$u = \{u_1, \dots, u_n\}$ be an entire solution of*

$$\begin{cases} \Delta u_i + \sum_{j=1}^{n} a_{ij} e^{u_j} = 0, & \mathbb{R}^2, \\ \int_{\mathbb{R}^2} e^{u_i} < \infty, & i \in I \equiv \{1, \dots, n\}. \end{cases} \tag{1.3}$$

*Then there exists $p \in \mathbb{R}^2$ such that all $u_1, \dots, u_n$ are radially symmetric and decreasing about $p$.*

Recall that a matrix $A$ is called non-negative if $a_{ij} \geqslant 0$ ($i, j \in I$), irreducible if there is no partition of $I = I_1 \cup I_2$ ($I_1 \cap I_2 = \emptyset$) such that $a_{ij} = 0$, $\forall i \in I_1$, $\forall j \in I_2$.

It turns out that the following quadratic polynomial is important to the study of (1.3):

$$\Lambda_J(\sigma) = 4\sum_{i\in J} \sigma_i - \sum_{i,j\in J} a_{ij}\sigma_i\sigma_j, \qquad J \subset I \equiv \{1, \dots, n\}, \tag{1.4}$$

where $\sigma_i = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{u_i}$, $\sigma = \{\sigma_1, \dots, \sigma_n\}$.

It was first proved by Chanillo and Kiessling [8] that entire solutions of (1.3) must satisfy a Rellich–Pohozaev identity:

$$\Lambda_I(\sigma) = 4\sum_{i\in I} \sigma_i - \sum_{i,j\in I} a_{ij}\sigma_i\sigma_j = 0. \tag{1.5}$$

Later Chipot, Shafrir and Wolansky [13] proved the necessary and sufficient condition for the existence of entire solutions to (1.3):

**Theorem B** *(Chipot–Shafrir–Wolansky). Let $A$ satisfies* (1.2). *Then $\sigma = \{\sigma_1, \dots, \sigma_n\}$ satisfies*

$$\Lambda_I(\sigma) = 0 \quad \text{and} \quad \Lambda_J(\sigma) > 0, \quad \forall \emptyset \subsetneq J \subsetneq I, \tag{1.6}$$

*if and only if there exists a solution $\{u_1, \dots, u_n\}$ of* (1.3) *such that $\frac{1}{2\pi}\int_{\mathbb{R}^2} e^{u_i} = \sigma_i$, $i \in I$.*

From now on we use $\Pi$ to represent the hyper-surface that satisfies (1.6). It is immediate to observe that for each $\sigma = \{\sigma_1, \dots, \sigma_n\}$ on $\Pi$, there is more than one solution corresponding to $\sigma$. Indeed, let $\{u_1, \dots, u_n\}$ be such a solution, then $\{v_1, \dots, v_n\}$ defined by

$$v_i(y) = u_i(x_0 + \delta y) + 2\log\delta, \quad \forall x_0 \in \mathbb{R}^2,\ \forall\delta > 0,\ i \in I,$$

clearly solves (1.3) and satisfies $\int_{\mathbb{R}^2} e^{v_i} = \int_{\mathbb{R}^2} e^{u_i}$ $(i \in I)$. A natural question is: are all the solutions corresponding to $\sigma$ obtained from $\{u_1, \dots, u_n\}$ by translations and scalings? Our first result in this paper is to give an affirmative answer to this question:

**Theorem 1.1.** *Let $A$ satisfies* (1.2)*, $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ be two radial solutions of* (1.3) *such that $\int_{\mathbb{R}^2} e^{u_i} = \int_{\mathbb{R}^2} e^{v_i}$, $i \in I$, then there exists $\delta > 0$ such that $v_i(y) = u_i(\delta y) + 2\log\delta$, $i \in I$.*

As is well known, for various equations it is important to have a classification of all the global solutions. The classification theorems of Caffarelli, Gidas and Spruck [6], Chen and Li [11], Jost and Wang [16] and Lin [22] play a central role in the blowup analysis for prescribing scalar curvature equations, prescribing Gauss curvature equations, Toda systems and prescribing $Q$-curvature equations, respectively. The existence result of Chipot–Shafrir–Wolansky (Theorem B) and the uniqueness result (Theorem 1.1) can be combined to serve as a classification theorem for the study of the blowup phenomena of Liouville systems.

In [13] Chipot, Shafrir and Wolansky also studied the Dirichlet problem for the Liouville system (1.1) on bounded domains. They considered the nonlinear functional $F$:

$$F(u) = \frac{1}{2}\sum_{i,j\in I}\int_{\Omega} a^{ij}\nabla u_i \nabla u_j - \sum_{j\in I}\rho_j \log\left(\int_{\Omega} h_j e^{u_j}\right), \quad u \in H_0^1(\Omega),$$

where $a^{ij}$ $(i, j \in I)$ are the entries of $A^{-1}$, $\rho_i$ $(i \in I)$ are constants, and $h_i$ $(i \in I)$ are positive smooth functions. Suppose the matrix $A = (a_{ij})$ is positive definite, it was shown in [13] that $F$ is bounded from below in $H_0^1(\Omega)$ if and only if $\Lambda_I(\rho) \geqslant 0$ $(\rho = (\rho_1, \dots, \rho_n))$, and a minimizer of $F(u)$ exists if $\Lambda_I(\rho) > 0$. Obviously the Euler–Lagrange equation for the functional $F$ is the following:

$$\begin{cases} \Delta u_i + \displaystyle\sum_{j=1}^{n} a_{ij}\rho_j \frac{h_j e^{u_j}}{\int_{\Omega} h_j e^{u_j}} = 0, \qquad \Omega \subset \mathbb{R}^2, \quad i \in I, \\ u_i = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.7}$$

so the existence problem for (1.7) is solved if $\Lambda_I(\rho) > 0$.

It is also natural to consider Liouville systems on Riemann surfaces. Let $(M, g)$ be a Riemann surface of volume equal to 1, then the following variational form

$$J_\rho(u) = \frac{1}{2}\sum_{i,j=1}^{n}\int_{M} a^{ij}\nabla_g u_i \nabla_g u_j + \sum_{j=1}^{n}\int_{M}\rho_j u_j - \sum_{j=1}^{n}\rho_j \log\int_{M} h_j e^{u_j}$$

corresponds to the system

$$\Delta_g u_i + \sum_{j=1}^{n}\rho_j a_{ij}\left(\frac{h_j e^{u_j}}{\int_M h_j e^{u_j}\, dV_g} - 1\right) = 0, \quad M,\ i \in I. \tag{1.8}$$

(1.7) and (1.8) are generalizations of the Liouville equation defined locally or on Riemann surfaces, respectively. For the single Liouville equation, various results on a priori estimate, degree counting formula and the existence of solutions have been obtained by Chen and Lin [9,10]. To study (1.7) and (1.8), it is important to understand the asymptotic behavior of blowup solutions.

In this article, we consider the following local estimate crucial to the study of (1.7) and (1.8): Let $u^k = \{u_1^k, \dots, u_n^k\}$ be a sequence of functions which satisfies

$$\begin{cases} \Delta u_i^k + \sum_{j=1}^n a_{ij} h_j^k e^{u_j^k} = 0, \qquad B_1 \subset \mathbb{R}^2, \quad i \in I, \\ \int_{B_1} h_i^k e^{u_i^k} \leqslant C, \quad i \in I,\ k = 1, 2, \ldots, \end{cases} \tag{1.9}$$

where $B$ is the unit ball with center 0, $\{h_i^k\}_{i \in I}$ are positive $C^1$ functions uniformly bounded away from 0:

$$c_1^{-1} \leqslant h_i^k \leqslant c_1, \qquad \max_{B_1} \left|\nabla h_i^k\right| \leqslant c_1, \quad i \in I,\ k = 1, 2, \ldots. \tag{1.10}$$

Suppose 0 is the only blow-up point for $u^k$ and each component of $u^k$ has a finite oscillation on $\partial B_1$:

$$\max_{\Omega} u_i^k \leqslant C(\Omega), \quad \forall \Omega \Subset B_1 \setminus \{0\},\ i \in I,\ k = 1, 2, \ldots, \tag{1.11}$$

$$\left|u_i^k(x) - u_i^k(y)\right| \leqslant c_0, \quad \forall x, y \in \partial B_1,\ i \in I. \tag{1.12}$$

Our main assumption on $u^k$ is that $u^k$ converges to a Liouville system of $n$ equations after scaling: Let $u_1^k(x_1^k) = \max_{B_1} u_i^k$ $(i \in I)$, $\epsilon_k = e^{-\frac{1}{2} u_1^k(x_1^k)}$ and

$$v_i^k(y) = u_i^k\left(\epsilon_k y + x_1^k\right) - u_1^k\left(x_1^k\right), \quad y \in \Omega_k,\ i \in I, \tag{1.13}$$

where $\Omega_k := \{y; e^{-\frac{1}{2} u_1^k(x_1^k)} y + x_1^k \in B_1\}$. Then

$$v^k = \left(v_1^k, \ldots, v_n^k\right) \text{ converges in } C^2_{\mathrm{loc}}\left(\mathbb{R}^2\right) \text{ to } v = (v_1, \ldots, v_n), \tag{1.14}$$

which is a solution of the Liouville system

$$\Delta v_i + \sum_{j=1}^n a_{ij} h_j e^{v_j} = 0, \quad \mathbb{R}^2, \qquad h_i = \lim_{k \to \infty} h_i^k\left(x_1^k\right), \quad i \in I.$$

Note that $v_1, \ldots, v_n$ are all radial functions because by Theorem A they are all radially symmetric with respect to a common point and 0 is the maximum of $v_1$. Our major local uniform estimate is:

**Theorem 1.2.** *Let $A$ satisfies* (1.2), *$u^k = (u_1^k, \ldots, u_n^k)$ be a sequence of solutions to* (1.9) *such that* (1.9)–(1.14) *hold. Then*:

(1) *there exists a sequence of radial solutions $V^k = (V_1^k, \ldots, V_n^k)$ of*

$$\Delta V_i^k + \sum_{j=1}^n a_{ij} h_j^k(0) e^{V_j^k} = 0, \quad \mathbb{R}^2, \qquad \int_{\mathbb{R}^2} e^{V_i^k} < \infty, \quad i \in I,$$

*such that along a subsequence*

$$\left|u_i^k(x) - V_i^k\left(x - x_1^k\right)\right| \leqslant C(A, c_0, c_1, \sigma), \quad i \in I,\ x \in B_1, \tag{1.15}$$

*where $\sigma = (\sigma_1, \ldots, \sigma_n)$, $\sigma_i = \frac{1}{2\pi} \int_{\mathbb{R}^2} h_i e^{v_i}$, $V^k$ is uniquely determined by*:
(a) $V_1^k(0) = u_1^k(x_1^k)$;
(b) $\int_{\mathbb{R}^2} h_j^k(0) e^{V_j^k} = \int_{B_1} h_j^k e^{u_j^k}$, $j = 1, \ldots, n-1$.

(2) *There exists $\delta > 0$ such that*

$$\sum_{i,j \in I} a_{ij} \int_{B_1} h_i^k e^{u_i^k} \int_{B_1} h_j^k e^{u_j^k} = 8\pi \sum_{i \in I} \int_{B_1} h_i^k e^{u_i^k} + O\left(e^{-\delta u_1^k(x_1^k)}\right).$$

First we note that since every entire solution of the Liouville system satisfies (1.5), $\int_{\mathbb{R}^2} h_n^k(0)e^{V_n^k}$ is uniquely determined by (b) and

$$\sum_{ij} a_{ij} \int_{\mathbb{R}^2} h_i^k(0)e^{V_i^k} \int_{\mathbb{R}^2} h_j^k(0)e^{V_j^k} = 8\pi \sum_i \int_{\mathbb{R}^2} h_i^k(0)e^{V_i^k}.$$

Second, one is tempted to think that (1.15) is equivalent to $|v_i^k - v_i| \leqslant C$ $(i \in I)$ in $\Omega_k$. In fact, the function $v$ may not be $V^k$ scaled according to the maximum of $u^k$ and the difference between $v^k$ and $v$ may not be uniformly bounded in $\Omega_k$. This is a special feature of Liouville systems which can be observed from the entire solutions of (1.3) as follows: Every point on $\Pi$ corresponds to an entire solution. Let $\sigma^k = (\sigma_1^k, \dots, \sigma_n^k)$ be a sequence of points on $\Pi$ that tends to $\sigma = (\sigma_1, \dots, \sigma_n)$. Let $\{w^k = (w_1^k, \dots, w_n^k)\}$ be a sequence of solutions corresponding to $\sigma_k$ which converges in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ to $w = (w_1, \dots, w_n)$, a solution corresponding to $\sigma$. By standard potential analysis (see [13])

$$w_i^k(x) = -\left(\sum_j a_{ij}\sigma_j^k\right) \ln|x| + O(1), \quad |x| > 1,$$

and

$$w_i(x) = -\left(\sum_j a_{ij}\sigma_j\right) \ln|x| + O(1), \quad |x| > 1,\ i \in I.$$

From the above we see that even though $\sigma^k \to \sigma$, the difference between $w^k$ and $w$ may not be finite at infinity. Therefore the choice of $V^k$ in the statement of Theorem 1.2 is necessary.

For Liouville equations without singular data, the type of estimate in Theorem 1.2 was first derived by Li [21]. Later Bartolucci, Chen, Lin and Tarantello [3] and Jost, Lin and Wang [18] established the same type of estimates for Liouville equations with singular data and Toda systems, respectively. The results of Li[21] and Bartolucci, Chen, Lin, Tarantello [3] have been improved by Chen and Lin [9] and Zhang [29,30] to a sharper form.

The estimates in Theorem 1.2 would be very important when a sequence of solutions $\{u^k\}$ of (1.8) has more than one blowup point. Suppose $u^k = (u_1^k, \dots, u_n^k)$ is a sequence of solutions of (1.8) with $\rho_i > 0$ $(i \in I)$. Assume that $p_1, p_2$ are two blowup points, and the assumptions of Theorem 1.2 hold in neighborhoods around $p_1$ and $p_2$. By Theorem 1.2 there exist two entire solutions obtained from the scaling of $u^k$ at $p_1$ and $p_2$. The question is whether these two entire solutions are equal. Indeed, the answer is yes when $A$ is positive definite, which is a consequence of Theorems 1.1 and 1.2 (see Section 5 for a proof of this fact). The conclusion here is crucial to proving a priori estimates for (1.7) and (1.8). In a forthcoming paper [23] we shall discuss the a priori estimates, degree counting formulas and existence results for (1.7) and (1.8).

Our next result concerns the location of blowup points for a sequence of blowup solutions. Let $\{u^k\}$ be a sequence of solutions of (1.9) that satisfies the assumptions in Theorem 1.2. Let $\{\psi_i^k\}_{i\in I}$ be the harmonic functions defined by the oscillations of $u_i^k$ on $\partial B_1$:

$$\begin{cases} \Delta\psi_i^k = 0, & B_1, \\ \psi_i^k = u_i^k - \dfrac{1}{2\pi}\displaystyle\int_{\partial B_1} u_i^k\, dS, & \text{on } \partial B_1. \end{cases}$$

By the mean value property of harmonic functions we have $\psi_i^k(0) = 0$. Also, since $\{u_i^k\}_{i\in I}$ have bounded oscillation on $\partial B_1$, all the derivatives of $\{\psi_i^k\}_{i\in I}$ on $B_{1/2}$ are uniformly bounded.

**Theorem 1.3.** *Let $h_i$, $\psi_i$ $(i \in I)$ be limits of $h_i^k$ and $\psi_i^k$, respectively, then under the same assumptions in Theorem* 1.2

$$\sum_{i\in I}\left(\frac{\nabla h_i(0)}{h_i(0)} + \nabla\psi_i(0)\right)\sigma_i = 0.$$

Theorem 1.3 can be used to determine the locations of blowup points for (1.8) in the following typical situation. Let $\{u^k\}$ be a sequence of blowup solutions to (1.8) with $\rho_i > 0$ $(i \in I)$, $A$ satisfies (1.2). In addition we assume

$A$ to be positive definite for simplicity. We can certainly assume $\int_M h_i^k e^{u_i^k}\, dV_g = 1$ $(i \in I)$ because for any solution $u = \{u_1, \dots, u_n\}$ to (1.8), adding a constant vector $\{C_1, \dots, C_n\}$ to $u$ gives another solution. Suppose $p_1, \dots, p_m$ are disjoint blowup points of $u^k$ such that around each $p_t$ $(t = 1, \dots, m)$, $u^k$ converges in $C^2_{\text{loc}}(\mathbb{R}^2)$ to a Liouville system of $n$ equations after scaling. Let $G$ be the Green's function with respect to $-\Delta_g$ on $M$:

$$-\Delta_g G(x,p) = \delta_p - 1, \qquad \int\limits_M G(x,p)\, dV_g(x) = 0.$$

Corresponding to $G$ we define

$$G^*(x,p) = G(x,p) + \frac{1}{2\pi}\chi(r)\log r$$

where $r = d_g(x,p)$, $\chi$ is a cut-off function supported in a small neighborhood of $p$. Using $G^*$, the blowup points $p_1, \dots, p_m$ are related by the following equation:

$$\sum_{i\in I}\left(\frac{\nabla_g h_i(p_s)}{h_i(p_s)} + \frac{1}{m}\left(\sum_{j\in I}\rho_j a_{ij}\right)\sum_{t=1}^{m}\nabla_1 G^*(p_s,p_t)\right) = 0, \quad s = 1,\dots,m, \tag{1.16}$$

where $\nabla_1 G^*$ means the covariant differentiation with respect to the first component.

Even though the results in this paper (Theorems 1.1–1.3) have their counterparts for the Liouville equation, there are some essential differences between the Liouville equation and the Liouville system that make the analysis for the latter harder. First, the uniqueness theorem (Theorem 1.1) for the system is generally harder to prove than one single equation, because of the lack of the Sturm–Liouville comparison theory for the linearized system. New ideas are needed to handle this difficulty. In this article, we mainly use the method of continuation to prove Theorem 1.1. Second, for the Liouville equation on $\mathbb{R}^2$

$$\Delta u + e^u = 0, \quad \mathbb{R}^2, \qquad \int\limits_{\mathbb{R}^2} e^u < \infty.$$

All the solutions satisfy $\int_{\mathbb{R}^2} e^u = 8\pi$. However, for the Liouville system (1.3), let $\sigma = (\sigma_1, \dots, \sigma_n)$ be the integration of the entire solutions, which is on $\Pi$ (see (1.6)). From Theorem B we see that under some conditions we have a continuum of solutions, as every point on $\Pi$ corresponds to a family of solutions. This difference on the structure of entire solutions exists not only between the Liouville equation and the Liouville system, but also between the Liouville system and Toda systems [18]. Finally, for the Liouville equation, the Pohozaev identity is a very useful tool, which gives a balancing condition between the interior integration and the boundary integration. However, for the Liouville system, the information from the Pohozaev identity is limited, as we have more than one equation. In this article, we use the uniqueness theorem (Theorem 1.1) to remedy what the Pohozaev identity cannot provide.

The organization of the paper is as follows: In Section 2 we prove Theorem 1.1 for two equations. We feel that the case of two equations is more explicit and represents most of the difficulties of the system. Then in Section 3 we prove the general case of Theorem 1.1 by mainly stating the difference with the proof in Section 2. In Section 4 we prove Theorem 1.2 and in Section 5 we prove Theorem 1.3 as well as (1.16). Finally in Appendix A we list a few Pohozaev identities to be used in different contexts.

## 2. Proof of Theorem 1.1 for two equations

In this section we prove Theorem 1.1 for two equations. So the system is

$$\begin{cases} \Delta u_1 + a_{11}e^{u_1} + a_{12}e^{u_2} = 0, \\ \Delta u_2 + a_{12}e^{u_1} + a_{22}e^{u_2} = 0, \quad \mathbb{R}^2, \\ \int\limits_{\mathbb{R}^2} e^{u_1} < \infty, \qquad \int\limits_{\mathbb{R}^2} e^{u_2} < \infty, \end{cases} \tag{2.1}$$

where the assumption on $A$ now becomes $a_{ii} \geqslant 0$, $i = 1, 2$, $a_{12} > 0$ and $a_{12}^2 \neq a_{11}a_{22}$. Let

$$\sigma_i = \frac{1}{2\pi} \int_{\mathbb{R}^2} e^{u_i} \quad \text{and} \quad m_i = \sum_j a_{ij}\sigma_j, \quad i \in I = \{1, 2\}.$$

By standard potential analysis (see, for example [13]) we have

$$m_i > 2, \quad i \in I = \{1, 2\}, \tag{2.2}$$

and

$$u_i(x) = -m_i \ln|x| + O(1), \quad |x| > 1, \ i \in I. \tag{2.3}$$

Let $u = \{u_1, u_2\}$ be a radial solution of (2.1) and we consider the linearized equation of (2.1) at $u$:

$$\big(r\phi_i'(r)\big)' + \sum_j a_{ij} e^{u_j} \phi_j(r) r = 0, \quad 0 < r < \infty, \ i \in I. \tag{2.4}$$

**Lemma 2.1.** *Let $\phi = (\phi_1, \phi_2)$ be a solution of* (2.4)*, then $\phi_i(r) = O(\ln r)$ at infinity for $i \in I$.*

**Proof.** Let $\psi(t) = (\psi_1(t), \psi_2(t))$ be defined as

$$\psi_i(t) = \phi_i\big(e^t\big), \quad i \in I.$$

Then $\psi$ satisfies

$$\psi_i''(t) + \sum_j a_{ij} e^{u_j(e^t)+2t} \psi_j(t) = 0, \quad -\infty < t < \infty, \ i \in I.$$

Let $\psi_3 = \psi_1'$, $\psi_4 = \psi_2'$ and $\mathbf{F} = (\psi_1, \dots, \psi_4)^T$, then $\mathbf{F}$ satisfies

$$\mathbf{F}' = \mathbf{MF}$$

where $\mathbf{M} = \begin{pmatrix} \mathbf{0} & \mathbf{I} \\ \mathbf{B} & \mathbf{0} \end{pmatrix}$. $\mathbf{B}$ is a $2 \times 2$ matrix with $\mathbf{B}_{ij} = -a_{ij} e^{u_j(e^t)+2t}$. For $t > 1$, the solution for $\mathbf{F}$ is

$$\mathbf{F}(t) = \lim_{N \to \infty} e^{\epsilon \mathbf{M}(t_N)} \dots e^{\epsilon \mathbf{M}(t_0)} \mathbf{F}(0), \tag{2.5}$$

where $t_0, \dots, t_N$ satisfy $t_j = j * \epsilon$, $j = 0, \dots, N$, $\epsilon = t/N$. Since $u_i(e^t) + 2t \sim (-m_i + 2)t$ when $t$ is large and $m_i > 2$ (see (2.2)), we have $\|\mathbf{B}\| \sim e^{-\delta t}$ for some $\delta > 0$ and $t$ large. With this property we further have

$$\|\mathbf{M}\|^k \leqslant C e^{-k\delta_1 t}, \quad k = 2, 3, \dots, \ t > 0, \tag{2.6}$$

for some $\delta_1 > 0$. Using (2.6) in (2.5) we have

$$\big\|\mathbf{F}(t)\big\| = O(t), \quad t > 1.$$

Lemma 2.1 is established. □

**Lemma 2.2.** *Let $\phi = \{\phi_1, \phi_2\}$ be a bounded solution of* (2.4)*, then $\phi = C(ru_1' + 2, ru_2' + 2)$ for some constant $C$.*

**Proof.** Let

$$\phi^0 = \big(ru_1' + 2, ru_2' + 2\big),$$

it is easy to verify that $\phi^0$ solves (2.4) and $\phi^0$ is bounded. We prove Lemma 2.2 by contradiction. Suppose $\bar\phi = (\bar\phi_1, \bar\phi_2)$ is another bounded solution of (2.4) and is not a multiple of $\phi^0$, then $\phi^0$ and $\bar\phi$ form a basis for all the solutions of (2.4). Since $\bar\phi_1(0)$ and $\bar\phi_2(0)$ cannot both be 2, without loss of generality we assume $\bar\phi_1(0) = 0$ and $\bar\phi_2(0) = 1$. We use $E$ to denote the set of all solutions. Since every solution is a linear combination of $\phi^0$ and $\bar{\phi}$, all the solutions are bounded. Let

$$S=\left\{\alpha \,\middle|\, \ni (\phi_1,\phi_2)\in E,\ \phi_1(0)=2,\ \phi_2(0)=\alpha\leqslant 2,\ \text{such that } \int_0^r e^{u_i}\phi_i(s)s\,ds>0 \text{ for all } r>0,\ i\in I\right\}.$$

We note that if $\phi_2(0)=2$, then $\phi(r)=(ru_1'(r)+2, ru_2'(r)+2)$. It is easy to see that $2\in S$ because

$$\int_0^r e^{u_i}\big(su_i'+2\big)s\,ds=r^2e^{u_i(r)}>0,\quad i\in I.$$

Next we see that $S$ is a bounded set. Because if $\alpha<0$, let $\phi=\{\phi_1,\phi_2\}$ be the bounded solution such that $\phi_1(0)=2$, $\phi_2(0)=\alpha$. Then $\int_0^r e^{u_2(s)}\phi_2(s)s\,ds<0$ for $r$ small enough. So $\alpha\notin S$.

Set $\alpha_0=\inf_S\alpha$. Then we claim that $\alpha_0\in S$. In fact, let $\{\alpha_k\in S\}$ tend to $\alpha_0$ from above as $k\to\infty$, let $\phi^k=\{\phi_1^k,\phi_2^k\}$ correspond to $\alpha_k$. Since $\alpha_k\in S$, $\int_0^r se^{u_i}\phi_i^k(s)\,ds>0$, for all $r$. Moreover, it is easy to see that $\phi^k$ converge to a solution $\phi=(\phi_1,\phi_2)$ in $E$ because $\phi^k$'s are linear combinations of $\phi^0$ and $\bar{\phi}$. It is also immediate to observe from the convergence that

$$\int_0^r e^{u_i}\phi_i(s)s\,ds\geqslant 0,\quad \text{for all } r>0,\ i\in I.$$

Thus,

$$r\phi_i'(r)=-\sum_j a_{ij}\int_0^r e^{u_j}\phi_j(s)s\,ds\leqslant 0,\quad i\in I.$$

So both $\phi_1$ and $\phi_2$ are non-increasing functions. Since they are bounded functions, for each $i\in I$ there exist $r_l\to\infty$ such that $r_l\phi_i'(r_l)\to 0$, which leads to

$$\sum_j a_{ij}\int_0^\infty e^{u_j}\phi_j(s)s\,ds=0,\quad i\in I.$$

Then we obtain the following from the invertibility of $A$:

$$\int_0^\infty e^{u_i}\phi_i(s)s\,ds=0,\quad i\in I. \tag{2.7}$$

Since $\phi_1$ and $\phi_2$ are non-increasing functions, (2.7) implies that

$$\lim_{r\to\infty}\phi_i(r)<0,\quad i\in I.$$

Indeed, for example for $\phi_1$, $\int_0^\infty e^{u_1}\phi_1(s)s\,ds=0$ and the monotonicity or $\phi_1$ imply either $\lim_{r\to\infty}\phi_1(r)<0$ or $\phi_1\equiv 0$. Then we see immediately that the latter case does not occur, as $\phi_1(0)=2$. Similarly for $\phi_2$, the case that $\phi_2\equiv 0$ also does not happen because $\phi_1\not\equiv 0$. Another immediate observation is $\phi_2(0)>0$.

For the above, we have

$$\int_0^r e^{u_i}\phi_i s\,ds>0\quad \text{if } \phi_i(r)\geqslant 0,\quad \text{and}$$

$$\int_0^r e^{u_i}\phi_i s\,ds>\int_0^\infty e^{u_i}\phi_i s\,ds=0,\quad \text{if } \phi_i(r)<0.$$

Thus $\alpha_0\in S$.

Now we claim that for $\epsilon > 0$ small enough, $\alpha_0 - \epsilon \in S$. Indeed, consider $\phi - \epsilon\bar{\phi}$, obviously this is a solution to (2.4) and satisfies $\phi_1(0) - \epsilon\bar{\phi}_1(0) = 2$, $\psi(0) - \epsilon\bar{\psi}(0) = \alpha_0 - \epsilon$. Since $\{\phi_1 - \epsilon\bar{\phi}_1, \phi_2 - \epsilon\bar{\phi}_2\}$ is a bounded solution of (2.4) we have

$$\int_0^\infty e^{u_i}(\phi_i - \epsilon\bar{\phi}_i)s\,ds = 0, \quad i \in I.$$

For $r$ large and $\epsilon$ small, since $\phi_1(r)$ and $\phi_2(r)$ are smaller than a negative number for $r$ large, it is easy to choose $\epsilon$ small enough so that

$$\int_r^\infty e^{u_i}(\phi_i - \epsilon\bar{\phi}_i)s\,ds < 0, \quad i \in I,$$

for all large $r$ large. Consequently

$$\int_0^r e^{u_i}(\phi_i - \epsilon\bar{\phi}_i)s\,ds > 0, \quad i \in I, \tag{2.8}$$

for all large $r$. Then by possibly choosing $\epsilon > 0$ smaller, we can make (2.8) hold for all $r > 0$. $\alpha_0 - \epsilon \in S$ is proved. This is a contradiction to the definition of $\alpha_0$. Lemma 2.2 is established. □

Now we are in the position to complete the proof of Theorem 1.1 for two equations. We consider the following initial-value problem:

$$\begin{cases} u_i'' + \dfrac{u_i'}{r} + \displaystyle\sum_j a_{ij}e^{u_j} = 0, \quad i = 1, 2, \\ u_1(0) = \alpha, \qquad u_2(0) = 0. \end{cases} \tag{2.9}$$

**Case 1.** $a_{ii} > 0$, $i = 1, 2$.

Since $a_{ii} > 0$, by Lemma 3.2 in Section 3, the solution pair $u_i(r)$ exists for all $r > 0$ and $i = 1, 2$, and satisfies

$$\int_0^\infty e^{u_i(r)}r\,dr < +\infty, \quad i = 1, 2.$$

Set

$$\sigma_i(\alpha) = \int_0^\infty e^{u_i(r)}r\,dr, \quad i = 1, 2.$$

Thus $\sigma(\alpha) = (\sigma_1, \sigma_2)$ is a function of $\alpha$ and lies in $\Pi$ (defined by (1.6)), which is a curve: $\Lambda_I(\sigma) = 0$ $(\sigma_1, \sigma_2 > 0)$. We want to prove that

$$\sigma : \mathbb{R} \to \Pi$$

is a 1–1 and onto map. Since both $\mathbb{R}$ and $\Pi$ are connected, it suffices to prove $\sigma$ is an open mapping. In the following, we want to show the claim

$$\frac{\partial\sigma_1}{\partial\alpha} \neq 0 \quad \text{and} \quad \frac{\partial\sigma_2}{\partial\alpha} \neq 0 \quad \text{for all } \alpha \in \mathbb{R}^2. \tag{2.10}$$

Then the openness of $\sigma$ follows immediately.

We prove this claim by contradiction. Suppose there exists $\alpha$ such that, say, $\partial_\alpha\sigma_1 = 0$. This implies immediately that

$$\int_0^\infty re^{u_1}\phi_1 = 0, \tag{2.11}$$

where $\phi_1 = \partial_\alpha u_1$. Correspondingly we set $\phi_2 = \partial_\alpha u_2$. Then $\{\phi_1, \phi_2\}$ satisfies the linearized system (2.4). By Lemma 2.1 $\phi_i(r) = O(\ln r)$ at infinity. The Pohozaev identity for (2.4) is (see Appendix A for the proof)

$$\sum_i \left( r^2\phi_i(r)e^{u_i} - 2\int_0^r se^{u_i}\phi_i(s)\,ds \right) = -\sum_{ij} a^{ij}\bigl(r\phi_i'(r)\bigr)\bigl(ru_j'(r)\bigr). \tag{2.12}$$

The first term on the left-hand side of (2.11) tends to 0 as $r \to \infty$. To deal with the terms on the right-hand side, first we use the equation for $\phi_i$ to get

$$-r\phi_i'(r) = \sum_l a_{il}\int_0^r se^{u_l}\phi_l(s)\,ds.$$

The equation for $u_i$ gives $\lim_{r\to\infty} ru_i'(r) = -m_i$. Putting the above information together we obtain the following from (2.12):

$$\sum_i (m_i - 2)\int_0^\infty se^{u_i}\phi_i(s)\,ds = 0.$$

By (2.11), we have

$$\int_0^\infty e^{u_i}\phi_i r\,dr = 0, \quad i = 1, 2.$$

Using (2.12) for the equation for $\phi_i$ we have

$$-r\phi_i'(r) = \int_0^r \sum_j a_{ij}e^{u_j}\phi_j s\,ds = -\int_r^\infty \sum_j a_{ij}e^{u_j}\phi_j s\,ds = O\bigl(r^{-\delta}\bigr)$$

for some $\delta > 0$. Therefore $\phi_i$ $(i \in I)$ is bounded at infinity. By Lemma 2.2, there is a constant $c$ such that $\phi_1 = c(r{u_1}' + 2)$, $\phi_2 = c(r{u_2}' + 2)$. But one sees immediately that this is impossible because $\phi_1(0) = 0$, $\phi_2(0) = 1$. The claim is proved.

Theorem 1.1 for this case is implied by the claim. In fact, suppose $\{\bar{u}_1, \bar{u}_2\}$ is another pair of radial solutions of the Liouville system so that $\int_{\mathbb{R}^2} e^{\bar{u}_i} = \int_{\mathbb{R}^2} e^{u_i}$ $(i = 1, 2)$. By scaling, we may assume $u_2(0) = \bar{u}_2(0) = 0$. Since the mapping $\sigma : \mathbb{R}^{n-1} \to \Pi$ is one-to-one and onto, we have $u_1(0) = \bar{u}_1(0)$. Consequently $u_i \equiv \bar{u}_i$ $(i \in I)$, hence Theorem 1.1 is proved for the case $a_{ii} > 0$, $i = 1, 2$.

**Case 2.** There exists $i$ such that $a_{ii} = 0$.

Set

$$\Pi_1 = \bigl\{\alpha \bigm| e^{u_j} \in L^1\bigl(\mathbb{R}^2\bigr),\ j = 1, 2,\ u = (u_1, u_2) \text{ is a solution of (2.9)}\bigr\}.$$

Similar to the previous step, the map $\Pi_1 \to \Pi$ is an open mapping. Since $a_{11} = 0$ or $a_{22} = 0$, $\Pi$ is non-compact and connected. Thus $\sigma$ is 1–1 and onto from each component of $\Pi_1$ onto $\Pi$.

Now suppose $\Pi_1$ has two component, say $\Pi_1^1$ and $\Pi_1^2$. Choose any $\sigma$ of $\Pi$. Then there exist $\alpha_1 \in \Pi_1^1$, and $\alpha_2 \in \Pi_1^2$ such that $u^1 = ({u_1}^1, {u_2}^1)$ and $u^2 = ({u_1}^2, {u_2}^2)$ are the corresponding solutions of (2.9) and satisfy

$$\int_0^\infty e^{u_j^1} r\,dr = \int_0^\infty e^{u_j^2} r\,dr = \sigma_j, \quad j = 1, 2.$$

Clearly, $\exists R_0$ such that for $r \geqslant R_0$ and some $\delta > 0$,

$$\big(u_j^k\big)'(r)r \leqslant -(2+2\delta), \quad j=1,2,\ k=1,2.$$

Now consider the perturbation of (2.9):

$$\begin{cases} \Delta u_i + \sum_{j=1}^{2}(a_{ij}+\epsilon\delta_{ij})e^{u_j} = 0, \quad \mathbb{R}^2,\ i=1,2, \\ u_1(0)=\alpha, \qquad u_2(0)=0. \end{cases} \tag{2.13}$$

Here we require $\epsilon \in (0,\delta_0)$ where $\delta_0$ is so small that the matrix $(a_{ij}+\epsilon\delta_{ij})_{n\times n}$ is non-singular for all $\epsilon \in (0,\delta_0)$. Let $u^{k,\epsilon} = (u_1^{k,\epsilon}, u_2^{k,\epsilon})$ be the solution of (2.13) with respect to the initial condition $(\alpha_k, 0)$ $(k=1,2)$. For $\delta_0$ small we have

$$\big(u_j^{k,\epsilon}(r)\big)'r \leqslant -(2+\delta) \quad \text{at } r=R_0,\ 0\leqslant \epsilon \leqslant \delta_0.$$

Then by the super-harmonicity of $u_j^{k,\epsilon}$ it is easy to show

$$\big(u_j^{k,\epsilon}(r)\big)'r \leqslant -(2+\delta) \quad \text{for } r \geqslant R_0.$$

Thus, $\exists C>0$ and $R_1 \geqslant R_0$ such that

$$e^{u_j^{k,\epsilon}(r)} \leqslant Cr^{-(2+\delta)} \quad \text{for } r \geqslant R_1. \tag{2.14}$$

Hence for $k=1,2$,

$$\sigma_j{}^{\epsilon}(\alpha_k) = \int_0^\infty e^{u_j^{k,\epsilon}(r)} r\,dr = \int_0^\infty e^{u_j^{k}(r)} r\,dr + o(1) = \sigma_j + o(1), \quad j=1,2,$$

where $o(1)\to 0$ as $\epsilon \to 0$.

Next we claim that

$$\frac{\partial \sigma_j^\epsilon}{\partial \alpha}(\alpha_k) = \frac{\partial \sigma_j}{\partial \alpha}(\alpha_k) + o(1). \tag{2.15}$$

Indeed,

$$\frac{\partial \sigma_j^\epsilon}{\partial \alpha}(\alpha_k) = \int_0^\infty r e^{u_j^{k,\epsilon}(r)} \frac{\partial u_j^{k,\epsilon}}{\partial \alpha}(r)\,dr, \quad j=1,2,\ k=1,2. \tag{2.16}$$

$(\frac{\partial u_1^{k,\epsilon}}{\partial \alpha}, \frac{\partial u_2^{k,\epsilon}}{\partial \alpha})$ satisfies the following linearized equation:

$$-\Delta\left(\frac{\partial u_i^{k,\epsilon}}{\partial \alpha}\right) = \sum_{j=1}^{2}(a_{ij}+\epsilon\delta_{ij})e^{u_j^{k,\epsilon}}\frac{\partial u_j^{k,\epsilon}}{\partial \alpha}, \quad i=1,2.$$

Using the argument of Lemma 2.1 we have

$$\left|\frac{\partial u_i^{k,\epsilon}}{\partial \alpha}(r)\right| \leqslant C\ln r, \quad r \geqslant 2,\ i=1,2, \tag{2.17}$$

where the constant $C$ is independent of $\epsilon \in (0,\delta_0)$. Moreover, for any fixed $R>0$, $\frac{\partial u_i^{1,\epsilon}}{\partial \alpha}(r)$ converges uniformly to $\frac{\partial u_i^{1}}{\partial \alpha}(r)$ over $0<r<R$ with respect to $\epsilon$. Using the decay estimates (2.14) and (2.17) in (2.16) we obtain (2.15) by elementary analysis.

Since $\lim_{\epsilon\to 0}\frac{\partial \sigma_j{}^\epsilon}{\partial \alpha}(\alpha_1) = \frac{\partial \sigma_j}{\partial \alpha}(\alpha_1) \neq 0$, there exists $\alpha_1(\epsilon) = \alpha_1 + o(1)$ such that

$$\sigma_1{}^\epsilon\big(\alpha_1(\epsilon)\big) = \sigma_1{}^\epsilon(\alpha_2). \tag{2.18}$$

Both $(\sigma_1^\epsilon(\alpha_1(\epsilon)), \sigma_2^\epsilon(\alpha_1(\epsilon)))$ and $(\sigma_1^\epsilon(\alpha_2), \sigma_2^\epsilon(\alpha_2))$ satisfy $\Lambda_I{}^\epsilon(\sigma^\epsilon) = 0$, which reads

$$\sum_{i,j=1}^{2} (a_{ij} + \epsilon\delta_{ij})\sigma_i^\epsilon \sigma_j^\epsilon = 4\sum_{i=1}^{2} \sigma_i^\epsilon.$$

Using (2.18) in the above we have

$$\sigma_2{}^\epsilon\big(\alpha_1(\epsilon)\big) = \sigma_2{}^\epsilon(\alpha_2).$$

Since $\alpha_1(\epsilon) \neq \alpha_2$, it yields a contradiction to the uniqueness property that the system (2.13) satisfies. Hence the proof of Theorem 1.1 for two equations is complete. □

## 3. Proof of Theorem 1.1 for the general case

The proof for the general case of Theorem 1.1 is similar to the case of two equations. We mainly focus on the difference in this section.

First we point out that Lemma 2.1 still holds for the general case with the same proof. The first major result in this section is the following:

**Lemma 3.1.** *Let $\phi = (\phi_1, \ldots, \phi_n)$ be a bounded solution of*

$$\big(r\phi_i'(r)\big)' + \sum_{j=1}^{n} a_{ij} e^{u_j} r\phi_j(r) = 0, \quad 0 < r < \infty,\ i \in I = \{1, \ldots, n\}, \tag{3.1}$$

*then $\phi_i(r) = ru_i'(r) + 2$, $i \in I = \{1, \ldots, n\}$.*

**Proof.** Let $\phi^0 = (ru_1'(r) + 2, \ldots, ru_n'(r) + 2)$, then by direct computation one sees that $\phi^0$ is a solution of (3.1). Suppose there is another bounded solution $\phi^1 = (\phi_1^1, \ldots, \phi_n^1)$ different from $\phi^0$, without loss of generality we assume $\phi_1(0) = 0$, as one of $\phi_i^1(0)$ must be different from 2. To derive a contradiction we define

$$S = \left\{\alpha;\ \ni \text{a bounded solution } \phi \text{ such that } \phi_1(0) = 2,\ \phi_i(0) = \alpha_i \leqslant 3,\ i = 2, \ldots, n;\ \alpha = \min\{\alpha_2, \ldots, \alpha_n\}\right.$$
$$\left.\int_0^r e^{u_i(s)}\phi_i(s)s\,ds > 0,\ \forall r > 0,\ i \in I\right\}.$$

By direct computation $2 \in S$, which corresponds to the solution $\phi^0$. Since $\phi_i^0$ $(i \in I)$ is strictly decreasing, we can choose $t$ small enough to make all components of $\phi^0 + t\phi^1$ strictly decreasing. By choosing $t$ or $-t$ we can make $2 - \epsilon \in S$ for some $\epsilon > 0$ sufficiently small. Let $\bar\alpha$ be the infimum of $S$ and let $\alpha^k = \{\alpha_1^k, \ldots, \alpha_n^k\} \in S$ be a sequence in $S$ that tends to $\bar\alpha$ from above. Suppose $\phi^k = \{\phi_1^k, \ldots, \phi_n^k\}$ is the solution corresponding to $\alpha^k$, then we claim that $\{\phi^k\}$ converges to $\bar\phi = \{\bar\phi_1, \ldots, \bar\phi_n\}$, which is also a bounded solution with strict monotone properties described in $S$. Indeed, let $\psi^m = (\psi_1^m, \ldots, \psi_n^m)$ be the solution to (3.1) such that $\psi_j^m(0) = \delta_j^m$. By Lemma 2.1 $\psi_i^m(r) = O(\ln r)$ at infinity. $\phi^k$ can be written as

$$\phi^k = \sum_{m=1}^{n} \alpha_m^k \psi^m. \tag{3.2}$$

Since $\bar\alpha \leqslant \alpha_i^k \leqslant 3$ $(i \in I)$ for all $k$, along a subsequence, $\alpha^k$ converges to $\{\bar\alpha_1, \ldots, \bar\alpha_n\}$. As a consequence, $\phi^k$ converges to $\bar\phi = \sum_{m=1}^{n} \bar\alpha_m \psi^m$ uniformly over any compact subsets of $\mathbb{R}^2$. The monotone property of $\phi^k$ implies that

$$\int_0^r e^{u_i}\bar\phi_i(s)s\,ds \geqslant 0, \quad i \in I,\ \forall r > 0.$$

On the other hand, since $\phi^k$ are all bounded functions, for each $\phi_i^k$ we find $r_l \to \infty$ such that $r_l(\phi_i^k)'(r_l) \to 0$. This leads to

$$\int_0^\infty \sum_j a_{ij} e^{u_j(s)} \phi_i^k(s) s\,ds = 0, \quad i \in I.$$

Since $A$ is invertible we have

$$0 = \int_0^\infty e^{u_i} \phi_i^k(s) s\,ds = \sum_{m=1}^n \alpha_m^k \int_0^\infty e^{u_i(s)} \psi_i^m(s) s\,ds, \quad i \in I. \tag{3.3}$$

Since $\int_0^\infty e^{u_i} \psi_i^m(s) s\,ds$ is well defined, we let $\alpha^k \to (\bar\alpha_1, \dots, \bar\alpha_n)$ to get

$$\int_0^\infty e^{u_i(s)} \bar\phi_i(s) s = 0, \quad i \in I. \tag{3.4}$$

Using the argument for the case of two equations as well as the assumption that $A$ is irreducible we know each $\bar\phi_i$ decreases into a negative constant at infinity and $\bar\phi_i(0) > 0$. As a consequence, $\int_0^r e^{u_i(s)} \bar\phi_i(s) s\,ds > 0$ for each $r > 0$ and $\bar\alpha > 0$. Thus $\bar\alpha \in S$. Then as in the case for two equations, $\{\bar\phi + t\phi^1\}$ for $t$ small enough also satisfies the strict monotone property described in the definition of $S$. Therefore $\bar\alpha - \epsilon \in S$ for $\epsilon > 0$ small enough. This is a contradiction to the definition of $\bar\alpha$. Lemma 3.1 is established. $\square$

Now we complete the proof of Theorem 1.1 for $n$ equations. Let $u = (u_1, \dots, u_n)$ satisfy

$$\begin{cases} u_i''(r) + \dfrac{u_i'(r)}{r} + \displaystyle\sum_j a_{ij} e^{u_j} = 0, \quad 0 < r < \infty,\ i \in I, \\ \displaystyle\int_0^\infty r e^{u_i(r)}\,dr < \infty, \\ u_1(0) = \beta_1, \quad \dots, \quad u_{n-1}(0) = \beta_{n-1}, \quad u_n(0) = 0. \end{cases} \tag{3.5}$$

The following lemma is useful for the case $a_{ii} > 0$.

**Lemma 3.2.** *Let $a_{ii} > 0$ ($i \in I$), then for all $\beta = (\beta_1, \dots, \beta_{n-1}) \in \mathbb{R}^{n-1}$, there exists a solution $u = (u_1, \dots, u_n)$ to* (3.5).

**Proof.** By standard ODE existence theory we see that for $\beta = (\beta_1, \dots, \beta_{n-1}) \in \mathbb{R}^{n-1}$, there exists a radial solution $u = (u_1, \dots, u_n)$ in the neighborhood of 0. Then by writing the system as a first order ODE system we see the right-hand side always satisfies the Lipschitz property, therefore by Picard's theorem the solution exists for all $r > 0$. We are left to show that $\int_0^\infty e^{u_i(s)} s\,ds < \infty$. Let $v_i(t) = u_i(e^t) + 2t$ ($i \in I$), then $v = (v_1, \dots, v_n)$ satisfies

$$v_i''(t) + \sum_j a_{ij} e^{v_j(t)} = 0, \quad -\infty < t < \infty,\ i \in I.$$

From the equation for $u_i$ we have

$$r u_i'(r) = -\int_0^r \sum_j a_{ij} e^{u_j(s)} s\,ds < 0, \quad r > 0,\ i \in I.$$

Consequently $v_i'(t) < 2$ for $t \in \mathbb{R}$. Fix $t_0 \in \mathbb{R}$ we have, for $t > t_0$,

$$v_i'(t) = v_i'(t_0) - \int_{t_0}^t \sum_j a_{ij} e^{v_j(s)}\,ds, \quad i \in I.$$

Since $a_{ii} > 0$ and $a_{ij} \geqslant 0$, it is easy to see that there exists $t > t_0$ such that $v_i'(t) < 0$. Choose $t_1$ such that $v_i'(t_1) = -\delta < 0$ for some $\delta > 0$, then we see from the equation for $v_i$ that

$$v_i(t) \leqslant v_i(t_1) - \delta(t - t_1), \quad t > t_1,$$

which is equivalent to $u_i(r) < (-2-\delta)\ln r + C$ for $r > e^{t_1}$. Therefore $\int_0^\infty e^{u_i(s)} s\,ds < \infty$. Lemma 3.2 is established. □

Recall that $\sigma_i = \frac{1}{2\pi}\int_{\mathbb{R}^2} e^{u_i} = \int_0^\infty e^{u_i(s)} s\,ds$. $\sigma = (\sigma_1, \dots, \sigma_n) \in \Pi$. Let

$$\Pi_1 := \big\{\beta = (\beta_1, \dots, \beta_{n-1});\ (3.5) \text{ has a solution}\big\}.$$

Note that by Lemma 3.2, $\Pi_1 = \mathbb{R}^{n-1}$ if $a_{ii} > 0$ for all $i \in I$. The mapping from $\Pi_1$ to $\Pi$ is surjective. Here we claim that it is locally one-to-one. Indeed, let $\mathbf{M}$ be the following matrix:

$$\mathbf{M} = \begin{pmatrix} \partial_{\beta_1}\sigma_1 & \dots & \partial_{\beta_{n-1}}\sigma_1 \\ \vdots & \ddots & \vdots \\ \partial_{\beta_1}\sigma_{n-1} & \dots & \partial_{\beta_{n-1}}\sigma_{n-1} \end{pmatrix}.$$

We claim that $\mathbf{M}$ is non-singular for $\beta \in \Pi_1$ and $\sigma \in \Pi$. We prove this claim by contradiction. Suppose there exists a non-zero vector $\mathbf{C} = (c_1, \dots, c_{n-1})^T$ such that $\mathbf{MC} = \mathbf{0}$. Then by setting $\beta = c_1\beta_1 + \dots + c_{n-1}\beta_{n-1}$ we have

$$\partial_\beta\sigma_1 = \partial_\beta\sigma_2 = \dots = \partial_\beta\sigma_{n-1} = 0. \tag{3.6}$$

On the other hand, $\Pi$ is defined by $\Lambda_I = 0$, which reads

$$\sum_{i,j\in I} a_{ij}\sigma_i\sigma_j = 4\sum_{i\in I}\sigma_i.$$

By differentiating both sides with respect to $\beta$ we have

$$\sum_i\Big(\sum_j a_{ij}\sigma_j - 2\Big)\partial_\beta\sigma_i = 0.$$

Since $\sum_j a_{ij}\sigma_j > 2$, (3.6) implies $\partial_\beta\sigma_n = 0$. Set $\phi_i = \partial_\beta u_i$ $(i \in I)$, then $\phi = (\phi_1, \dots, \phi_n)$ satisfies the linearized equation (3.1) and $\phi_n(0) = 0$. From $\partial_\beta\sigma_i = 0$ $(i \in I)$ we have

$$\int_0^\infty e^{u_i}\phi_i(s) s\,ds = 0, \quad i \in I,$$

which implies from (3.1) that $\phi$ is bounded at infinity. By Lemma 3.1 $\phi_i = ru_i' + 2$, then we see immediately that this is not possible as $\phi_n(0) = 0$. Therefore we have proved that $\mathbf{M}$ is non-singular for all $\beta = (\beta_1, \dots, \beta_{n-1}) \in \Pi_1$.

We further assert that there is one-to-one correspondence between $\Pi_1$ and $\Pi$. This is proved in two steps as follows.

**Case 1.** $a_{ii} > 0$, $i \in I$.

In this case, $\Pi_1 = \mathbb{R}^{n-1}$. The mapping from $\Pi_1$ to $\Pi$ is proper and locally one-to-one. Since both $\mathbb{R}^{n-1}$ and $\Pi$ are simply connected, there is a one-to-one correspondence between them. Let $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ be two radial solutions such that $u_n(0) = v_n(0) = 0$, $\int_{\mathbb{R}^2} e^{u_i} = \int_{\mathbb{R}^2} e^{v_i}$ $(i \in I)$. Then $u_i(0) = v_i(0)$ $(i = 1, \dots, n-1)$. Consequently $u_i \equiv v_i$ $(i \in I)$. Theorem 1.1 is proved for this case.

**Case 2.** There exists $i_0 \in I$ such that $a_{i_0,i_0} = 0$.

We prove this case by a contradiction. Suppose $\beta^k = (\beta_1^k, \dots, \beta_{n-1}^k) \in \Pi_1$ for $k = 1, 2$ and $\beta^1 \neq \beta^2$, let $u^k$ be the solution corresponding to $\beta^k$ such that $\int_{\mathbb{R}^2} e^{u_i^1} = \int_{\mathbb{R}^2} e^{u_i^2} = \sigma_i$ $(i \in I)$.

Just like the case for two equations, we consider the following system

$$\begin{cases} u_i''(r) + \dfrac{u_i'(r)}{r} + \displaystyle\sum_j (a_{ij} + \epsilon\delta_{ij}) e^{u_j} = 0, \quad 0 < r < \infty,\ i \in I, \\ \displaystyle\int_0^\infty e^{u_i(r)} r\,dr < \infty, \quad i \in I, \\ u_1(0) = \beta_1, \quad \ldots, \quad u_{n-1}(0) = \beta_{n-1}, \quad u_n(0) = 0. \end{cases} \tag{3.7}$$

Let $u^{k,\epsilon}$ be the solution to (3.7) that corresponds to the initial condition $\beta^k$ $(k = 1, 2)$. Let $\sigma^{k,\epsilon} = (\sigma_1^{k,\epsilon}, \ldots, \sigma_n^{k,\epsilon})$ be defined as $\sigma_i^{k,\epsilon} = \int_0^\infty r e^{u_i^{k,\epsilon}(r)}\,dr$ $(i = 1, \ldots, n)$. By the same argument as in the case of two equations, we have $\sigma^{k,\epsilon} = (\sigma_1, \ldots, \sigma_n) + o(1)$ $(k = 1, 2)$ and

$$\frac{\partial \sigma_i^{k,\epsilon}}{\partial \beta_j} = \frac{\partial \sigma_i}{\partial \beta_j} + o(1), \quad i = 1, \ldots, n,\ j = 1, \ldots, n-1,\ k = 1, 2.$$

Consequently the matrix

$$\begin{pmatrix} \partial_{\beta_1} \sigma_1^{k,\epsilon} & \ldots & \partial_{\beta_{n-1}} \sigma_1^{k,\epsilon} \\ \vdots & \ddots & \vdots \\ \partial_{\beta_1} \sigma_{n-1}^{k,\epsilon} & \ldots & \partial_{\beta_{n-1}} \sigma_{n-1}^{k,\epsilon} \end{pmatrix}$$

is non-singular at $\beta^1$ or $\beta^2$ for $\epsilon$ small. On the other hand, $\sigma^{1,\epsilon}$ and $\sigma^{2,\epsilon}$ both satisfy

$$\begin{cases} \Lambda_I^\epsilon(\sigma^{k,\epsilon}) = 4 \displaystyle\sum_{\in I} \sigma_i^{k,\epsilon} - \sum_{i,j \in I} (a_{ij} + \epsilon\delta_{ij}) \sigma_i^{k,\epsilon} \sigma_j^{k,\epsilon} = 0, \\ \Lambda_J^\epsilon > 0, \quad 0 \subsetneqq J \subsetneqq I. \end{cases} \tag{3.8}$$

We use $\Pi^\epsilon$ to represent the hyper-surface described as above. For $\sigma^{2,\epsilon} = (\sigma_1^{2,\epsilon}, \ldots, \sigma_n^{2,\epsilon}) \in \Pi^\epsilon$, we can find $\beta^{1,\epsilon} = (\beta_1^{1,\epsilon}, \ldots, \beta_{n-1}^{1,\epsilon})$ such that

$$\beta_j^{1,\epsilon} = \beta_j^1 + o(1), \quad j = 1, 2, \ldots, n-1,$$

and a solution $\bar{u}^{1,\epsilon}$ of (3.7) with the initial condition $(\beta_1^{1,\epsilon}, \ldots, \beta_{n-1}^{1,\epsilon}, 0)$ such that

$$\int_0^\infty r e^{\bar{u}_j^{1,\epsilon}}\,dr = \sigma_j^{2,\epsilon}, \quad j = 1, 2, \ldots, n-1.$$

After using $\Lambda_I^\epsilon = 0$ in (3.8) we have

$$\int_0^\infty r e^{\bar{u}_n^{1,\epsilon}}\,dr = \sigma_n^{2,\epsilon}.$$

Then the difference between $\beta^1$ and $\beta^2$ implies $\beta^{1,\epsilon} \neq \beta^2$ for $\epsilon$ small. A contradiction to the uniqueness property satisfied by the system (3.7). Theorem 1.1 is proved for all the cases. $\square$

## 4. Proof of Theorem 1.2

First we state a Brezis–Merle type lemma:

**Lemma 4.1.** *Let $\Omega$ be an open, smooth, bounded subset of $\mathbb{R}^2$. If*

$$\sum_j \int_\Omega a_{ij} h_j^k e^{u_j^k} \leqslant 4\pi - \delta, \quad i \in I = \{1, \dots, n\},$$

*for some $\delta > 0$, then for any $\Omega_1 \Subset \Omega$, there exists $C(\delta, \Omega, \Omega_1) > 0$ such that*

$$u_i^k(x) \leqslant C, \quad x \in \Omega_1 \Subset \Omega, \ i \in I.$$

**Proof.** Let $f_i^k$ $(i \in I)$ be defined as

$$\begin{cases} -\Delta f_i^k(x) = \sum_j a_{ij} h_j^k e^{u_j^k}, & \Omega, \\ f_i^k(x) = 0, & \text{on } \partial\Omega. \end{cases}$$

Then by Theorem 1 of [5], we have

$$\int_\Omega e^{(1+\delta_1) f_i^k} \, dx \leqslant C,$$

where $\delta_1 > 0$ depends on $\delta$. For any $\Omega' \Subset \Omega$, let $x \in \Omega'$, suppose $B(x, \delta_2) \subset \Omega$, we have, by the mean value property

$$\begin{aligned} u_i^k(x) - f_i^k(x) &= \frac{1}{|B(x, \delta_2)|} \int_{B(x,\delta_2)} \big(u_i^k(y) - f_i^k(y)\big) \, dy \\ &\leqslant C \int_{B(x,\delta_2)} \big(u_i^k(y) - f_i^k(y)\big)^+ \, dy \\ &\leqslant C \int_\Omega \big(e^{u_i^k} + e^{f_i^k}\big) \leqslant C, \quad i \in I. \end{aligned}$$

So by writing $u_i^k$ as $u_i^k - f_i^k + f_i^k$ we see that $e^{u_i^k} \in L^{1+\delta_1}(\Omega')$, $i \in I$. Let $\bar{f}_i^k$ be defined as

$$\begin{cases} -\Delta \bar{f}_i^k(x) = \sum_{j \in I} a_{ij} h_j^k e^{u_j^k(x)}, & \Omega', \\ \bar{f}_i^k(x) = 0, & \text{on } \partial\Omega', \ i \in I. \end{cases}$$

Then standard elliptic estimate gives $|\bar{f}_i^k| \leqslant C$ in $\Omega'$ $(i \in I)$. Let $\Omega'' \Subset \Omega'$, then for $x \in \Omega''$, as before we have

$$u_i^k(x) = u_i^k(x) - \bar{f}_i^k(x) + \bar{f}_i^k(x) \leqslant C \int_{\Omega'} \big(e^{u_i^k} + e^{\bar{f}_i^k}\big) + C \leqslant C.$$

Lemma 4.1 is established. $\square$

Recall that $\sigma_i = \frac{1}{2\pi} \int_{\mathbb{R}^2} h_i e^{v_i}$ $(i \in I)$ where $h_i = \lim_{k\to\infty} h_i^k(0)$. Since $v = (v_1, \dots, v_n)$ satisfies the Liouville system in $\mathbb{R}^2$, we have

$$\sum_{j \in I} a_{ij} \sigma_j > 2, \quad i \in I. \tag{4.1}$$

Let $\bar{\sigma}_i = \lim_{r\to 0} \lim_{k\to\infty} \frac{1}{2\pi} \int_{B_r} h_i^k e^{u_i^k}$, then the assumption in Theorem 1.2 implies

$$\bar{\sigma}_i \geqslant \sigma_i, \quad i \in I. \tag{4.2}$$

So (4.1) also holds for $\{\bar{\sigma}_i\}_{i \in I}$.

**Lemma 4.2.**

$$\sum_{i,j\in I} a_{ij}\bar{\sigma}_i\bar{\sigma}_j = 4\sum_{i\in I}\bar{\sigma}_i. \tag{4.3}$$

**Proof.** In the first step we prove that in a small neighborhood of 0, say, $B(0,r_0)$, $u_i^k|_{\partial B_R} \to -\infty$ for $i \in I$ and any fixed $0 < R < r_0$.

Indeed, since (4.1) holds for $\bar{\sigma} = (\bar{\sigma}_1,\dots,\bar{\sigma}_n)$, we have $\sum_{j\in I} a_{ij}\bar{\sigma}_j > 2 + 3\epsilon_0$ $(i \in I)$ for some $\epsilon_0 > 0$. By the definition of $\bar{\sigma}_i$, we find $r_0$ small and $r_k \to 0$ such that $\int_{B_{r_0}\setminus B_{r_k}} e^{u_i^k} \leqslant \epsilon_0$ $(i \in I)$. Let

$$\tilde{v}_i^k(y) = u_i^k(r_k y) + 2\ln r_k, \quad |y| \leqslant r_k^{-1} r_0,\ i \in I.$$

Then the equation for $\tilde{v}_i^k$ is

$$-\Delta\tilde{v}_i^k = \sum_{j\in I} a_{ij}h_j^k(r_k\cdot)e^{\tilde{v}_j^k}, \quad |y| \leqslant r_k^{-1}r_0.$$

Let

$$\bar{v}_i^k(r) = \frac{1}{2\pi r}\int_{\partial B_r}\tilde{v}_i^k, \quad 1 \leqslant r \leqslant r_k^{-1}r_0,\ i \in I.$$

Then

$$\left(\bar{v}_i^k\right)'(r) = \frac{1}{2\pi r}\int_{B_r}\Delta\tilde{v}_i^k = -\frac{1}{2\pi r}\int_{B_r}\sum_j a_{ij}h_j^k(r_k\cdot)e^{\tilde{v}_j^k}\,dy.$$

For $r > 1$,

$$\int_{B_r}\sum_j a_{ij}h_j^k(r_k\cdot)e^{\tilde{v}_j} > 4\pi + 2\epsilon_0, \quad i \in I.$$

So by the definition of $\tilde{v}_i^k$,

$$\left(\bar{v}_i^k\right)'(r) \leqslant \left(-2 - \frac{\epsilon_0}{\pi}\right)r^{-1}, \quad r > 1,\ i \in I.$$

Consequently

$$\bar{v}_i^k\left(r_k^{-1}r_0\right) \leqslant -\left(2 + \frac{\epsilon_0}{\pi}\right)\ln r_k^{-1} + C \to -\infty, \quad i \in I.$$

For any fixed $R \in (0,r_0)$, $u_i^k$ has bounded oscillation on any $\partial B_R$, then we know $u_i^k \to -\infty$ uniformly on $\partial B_R$. As an immediate consequence, $u^k$ converges to $-\infty$ on all compact subsets of $B_1 \setminus \{0\}$ because $u^k$ is bounded above in $B_1 \setminus B_R$ and $u^k$ has bounded oscillation on $\partial B_1$.

The second step is to use the first step to evaluate all the terms in the Pohozaev identity. Let $G(x,y)$ be the Green's function with the Dirichlet condition. By the Green's representation formula we have:

$$u_i^k(x) = \int_{B_1} G(x,y)\sum_j a_{ij}h_j^k e^{u_j^k} - \int_{\partial B_1}\frac{\partial G(x,y)}{\partial\nu}u_i^k(y)\,dS_y, \quad i \in I.$$

The Pohozaev identity for the system (1.9) defined on $\Omega$ is of the following form (see Appendix A for the proof):

$$\begin{aligned}
&\sum_{i\in I}\left(\int_{\Omega}\left(x\cdot\nabla h_i^k\right)e^{u_i^k} + 2h_i^k e^{u_i^k}\right)\\
&\quad = \int_{\partial\Omega}\left(\sum_i (x\cdot\nu)h_i^k e^{u_i^k} + \sum_{i,j} a^{ij}\partial_\nu u_j^k\left(x\cdot\nabla u_i^k\right) - \frac{1}{2}a^{ij}(x\cdot\nu)\left(\nabla u_i^k\cdot\nabla u_j^k\right)\right).
\end{aligned}$$

Let $\Omega = B_R$ ($R \in (0,1)$) in the Pohozaev identity, using the fact that $u_i^k \to -\infty$ in $C^2_{\mathrm{loc}}(B_1 \setminus \{0\})$ we observe that

$$\int_{\partial B_R} \sum_i (x \cdot \nu) h_i^k e^{u_i^k} \to 0 \quad \text{and} \quad \int_{B_R} \big(x \cdot \nabla h_i^k\big) e^{u_i^k} \to 0, \quad i \in I.$$

Also we have

$$\frac{1}{2\pi} \int_{B_R} \sum_i 2 h_i^k e^{u_i^k} \to 2 \sum_i \bar{\sigma}_i.$$

For $|x| = R$,

$$\nabla u_i^k(x) = \int_{B_1} \nabla_x G(x,y) \sum_j a_{ij} h_j^k e^{u_j^k} - \int_{\partial B_1} \nabla_x \left( \frac{\partial G(x,y)}{\partial \nu} \right) u_i^k(y), \quad i \in I.$$

The second term of the above is the gradient of a harmonic function that has bounded oscillation on $\partial B_1$. Let $k \to \infty$,

$$\partial_r u_i^k(x) \to \frac{\sum_j a_{ij} \bar{\sigma}_j}{R} + O(1), \qquad \partial_\theta u_i^k(x) \to O(1), \quad i \in I, \ |x| = R. \tag{4.4}$$

Using (4.4) in the Pohozaev identity, we have

$$\sum_{ij} a_{ij} \bar{\sigma}_i \bar{\sigma}_j = 4 \sum_i \bar{\sigma}_i + O(R).$$

Lemma 4.2 is established by letting $R \to 0$. $\square$

Now we claim

$$\bar{\sigma}_i = \sigma_i, \quad i \in I. \tag{4.5}$$

To see this, let $s_i = \bar{\sigma}_i - \sigma_i$. We know from (4.2) that $s_i \geqslant 0$ ($i \in I$). Since for $\{\sigma_i\}_{i \in I}$ we also have

$$\sum_{ij} a_{ij} \sigma_i \sigma_j = 4 \sum_i \sigma_i$$

we obtain the following equation for $s_i$ from Lemma 4.2 and the above:

$$\sum_j \left( \sum_i a_{ij} \bar{\sigma}_i \right) s_j + \sum_i \left( \sum_j a_{ij} \sigma_j \right) s_i = 4 \sum_i s_i.$$

Since both $\sum_i a_{ij} \bar{\sigma}_i$ and $\sum_j a_{ij} \sigma_j$ are greater than 2, it is easy to see from the above that $s_i = 0$ ($i \in I$). (4.5) is proved.

Let $\epsilon_k = e^{-\frac{u_1^k(x_1^k)}{2}}$, $\bar{h}_i^k(y) = h_i^k(\epsilon_k y + x_1^k)$ ($i \in I$). Here we recall that $u_1^k(x_1^k) = \max_{B_1} u_i^k$ ($i \in I$). Then we have

$$-\Delta v_i^k = \sum_j a_{ij} \bar{h}_j^k e^{v_j^k}, \quad \Omega_k, \ i \in I,$$

where $\Omega_k := \{y;\ \epsilon_k y + x_1^k \in B_1\}$. Let

$$\sigma_i^k = \frac{1}{2\pi} \int_{B_1} h_i^k e^{u_i^k} \quad \text{and} \quad m_i^k = \sum_j a_{ij} \sigma_j^k, \quad i \in I. \tag{4.6}$$

We have $\sigma_i^k \to \sigma_i$ and $m_i^k \to m_i > 2$ ($i \in I$).

**Proposition 4.1.** *Given $\delta > 0$, there exists $R(\delta, A, c_0, c_1, \sigma) > 1$ such that for all large $k$,*

$$\big(-m_i^k - \delta\big) \ln|y| \leqslant v_i^k(y) \leqslant \big(-m_i^k + \delta\big) \ln|y|, \quad y \in \Omega_k \setminus B_{2R}, \ i \in I. \tag{4.7}$$

**Proof.** By the convergence of $v_i^k$ to $v_i$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ we only need to prove (4.7) for $2R < |y| \leqslant \epsilon_k^{-1}$ where $R \gg 1$. By the Green's representation formula we have, for $x \in B_1$ and $i \in I$,

$$u_i^k(x) = \int_{B_1} G(x,z)\Big(\sum_j a_{ij} h_j^k e^{u_j^k(z)}\Big) - \int_{\partial B_1} \frac{\partial G(x,z)}{\partial \nu} u_i^k(z). \tag{4.8}$$

Since the major term of the Green's function is $-\frac{1}{2\pi}\ln|x-z|$ and the oscillation of $u_i^k$ on $\partial B_1$ is bounded, we have

$$u_i^k(x) - u_i^k\big(x_i^k\big) = \frac{1}{2\pi}\int_{B_1} \ln\frac{|x_i - z|}{|x-z|}\Big(\sum_j a_{ij} h_j^k e^{u_j^k(z)}\Big)\,dz + O(1).$$

where $u_i^k(x_i^k) = \max_{B_1} u_i^k$. Since our assumption is that $u^k$ converges to $v = (v_1, \ldots, v_n)$ after scaling. The radial symmetry of $v_i$ implies

$$\big|u_i^k\big(x_i^k\big) - u_j^k\big(x_j^k\big)\big| \leqslant C, \qquad e^{-\frac{1}{2}u_1^k(x_1^k)}\big|x_i^k - x_j^k\big| \to 0, \quad i, j \in I.$$

With this observation and the definition of $v_k$ (4.8) can be rewritten as

$$v_i^k(y) = \frac{1}{2\pi}\int_{\Omega_k} \ln\frac{|z|}{|y-z|}\Big(\sum_j a_{ij}\bar{h}_j^k e^{v_j^k(z)}\Big)\,dz + O(1), \quad i \in I. \tag{4.9}$$

The proof of (4.7) can be put into two steps. First we show: For $N > 1$, there exists $R \gg 1$ such that for $|y| > 2R$ and all large $k$,

$$v_i^k(y) \leqslant -2\ln|y| - N, \quad |y| > 2R,\ i \in I. \tag{4.10}$$

To this end, we use the argument in Lemma 4.1. Since $\sigma_i^k \to \sigma_i$, for $\epsilon > 0$ small to be determined, we choose $R \gg 1$ such that

$$\int_{\Omega_k \setminus B_R} e^{v_i^k} \leqslant \epsilon, \quad i \in I.$$

Fix $r > 2R$ and set

$$\bar{v}_i(z) = v_i^k(rz) + 2\ln r + 2N, \quad \frac{1}{2} < |z| < 2,\ i \in I.$$

By letting $\bar{h}_i(z) = \bar{h}_i^k(rz)$ we have

$$-\Delta\bar{v}_i(z) = \sum_j a_{ij}\bar{h}_j(z)e^{-2N}e^{\bar{v}_j(z)}, \quad \frac{1}{2} < |z| < 2,\ i \in I.$$

Note that for simplicity we omit $k$ in $\bar{v}_i(z)$ and $\bar{h}_i$. It is readily verified that

$$\int_{\frac{1}{2}<|z|<2} e^{\bar{v}_i(z)}\,dz \leqslant e^{2N}\int_{\Omega_k\setminus B_R} e^{v_i^k(y)}\,dy, \quad i \in I.$$

Now we choose $\epsilon$ to be small enough so that

$$e^{2N}\int_{\Omega_k\setminus B_R}\sum_j a_{ij}\bar{h}_j^k e^{v_j^k} \leqslant 3\pi, \quad i \in I.$$

The inequality above implies

$$\int_{B_2\setminus B_{\frac{1}{2}}} e^{\bar{v}_i^k} \leqslant C, \quad i \in I, \tag{4.11}$$

where $C$ is independent of $N$. Using (4.11) and the argument in Lemma 4.1 we have

$$\bar{v}_i(z) \leqslant c_0, \quad |z| = 1, \ i \in I, \tag{4.12}$$

where $c_0$ is a universal constant. (4.10) follows immediately from (4.12).

In the second step we use (4.10) and (4.9) to prove (4.7). First since $|z| \sim |y - z|$ for $|z| > 2|y|$, we have

$$v_i^k(y) = \frac{1}{2\pi} \int_{B_{2|y|}} \ln \frac{|z|}{|y-z|} \left( \sum_j a_{ij} \bar{h}_j^k e^{v_j^k(z)} \right) dz + O(1).$$

Next we show that

$$\frac{1}{2\pi} \int_{B_{2|y|}} \big|\ln|z|\big| \left( \sum_j a_{ij} \bar{h}_j^k e^{v_j^k(z)} \right) dz \leqslant \frac{\delta}{10} \ln|y|, \quad |y| > R_1, \tag{4.13}$$

where $R_1$ will be chosen large in terms of $\delta$. Indeed, we can choose $R_1$ so large that

$$\frac{1}{2\pi} \int_{B_{2|y|} \setminus B_{R_1}} \sum_j a_{ij} \bar{h}_j^k e^{v_j^k(z)}\, dz < \delta/10. \tag{4.14}$$

Then the integral in (4.13) can be divided into two parts, one part is the integration over $B_{R_1}$, the other part is the integration on $B_{2|y|} \setminus B_{R_1}$. Since $e^{v_i}$ decays faster than $|y|^{-2-\delta_1}$ for some $\delta_1 > 0$, we use the convergence of $v_i^k$ to $v_i$ to obtain that the integration over $B_{R_1}$ is $O(1)$. For the other term it is easy to see from (4.14) that the integration over $B_{2|y|} \setminus B_{R_1}$ is less than $\frac{\delta}{5} \ln|y|$. The last term to deal with is

$$-\frac{1}{2\pi} \int_{B_{2|y|}} \ln|y - z| \left( \sum_j a_{ij} \bar{h}_j^k e^{v_j^k(z)} \right) dz.$$

For this we divide $B_{2|y|}$ into two sub-regions:

$$\Omega_1 = \big\{ z \in \Omega_k;\ |z| < |y|/2 \big\}, \qquad \Omega_2 := B_{2|y|} \cap \Omega_k \setminus \Omega_1.$$

Since $|y - z| \sim |y|$ for $z \in \Omega_1$ and

$$\left| \frac{1}{2\pi} \int_{\Omega_1} \sum_j a_{ij} \bar{h}_j e^{v_j^k} - m_1^k \right| \leqslant \frac{\delta}{20}$$

for $|y|$ large. We obtain immediately that

$$\left| \frac{1}{2\pi} \int_{\Omega_1} \ln|y - z| \left( \sum_j a_{ij} \bar{h}_i^k e^{v_i^k(z)} \right) dz - m_i^k \ln|y| \right| \leqslant \frac{\delta}{10} \ln|y|, \quad |y| > R_1.$$

To estimate the last term: $-\frac{1}{2\pi} \int_{\Omega_2} \ln|y - z| (\sum_j a_{ij} \bar{h}_j^k e^{v_j^k(z)})\, dz$, we use polar coordinates and (4.10) to obtain

$$\left| \int_{\Omega_2} \ln|y - z| \left( \sum_j a_{ij} \bar{h}_j^k e^{v_j^k(z)} \right) dz \right| \leqslant C e^{-N} \ln|y|$$

for a universal constant $C$. Choose $N$ large enough we see this term is less than $\frac{\delta}{10} \ln|y|$. Proposition 4.1 is established. □

Since $m_i^k \to m_i > 2$, $e^{v_i(y)} \sim O(|y|^{-2-\delta_2})$ for some $\delta_2 > 0$. Using this in the proof of Proposition 4.1 again we see that

$$\big| v_i^k(y) - m_i^k \ln\big(1 + |y|\big) \big| \leqslant C(A, c_0, c_1, \sigma), \quad y \in \Omega_k,\ i \in I. \tag{4.15}$$

**Proposition 4.2.**

$$\sum_{ij} a_{ij}\sigma_i^k\sigma_j^k = 4\sum_i \sigma_i^k + O\big(\epsilon_k^c\big) \tag{4.16}$$

*where $c > 0$ is a small number.*

**Remark 4.1.** Proposition 4.2 is equivalent to the second statement of Theorem 1.2.

**Proof of Proposition 4.2.** Let $m > 2$ be less than $m_i^k$ $(i \in I)$ and $L_k = \epsilon_k^{-c}$ for $c > 0$ small. We estimate each term of the Pohozaev identity on $E_k := B(0, L_k)$:

$$\sum_i \bigg( \int_{E_k} \big(y \cdot \nabla \bar{h}_i^k\big) e^{v_i^k} + 2\bar{h}_i^k e^{v_i^k} \bigg) = L_k \int_{\partial E_k} \bigg( \sum_i \bar{h}_i^k e^{v_i^k} + \sum_{ij} \bigg( a^{ij} \partial_\nu v_i^k \partial_\nu v_j^k - \frac{1}{2} a^{ij} \big(\nabla v_i^k \cdot \nabla v_j^k\big) \bigg) \bigg).$$

By the decay rate of $v_j^k$ $(j = 1, 2)$, we have

$$\int_{E_k} \big(y \cdot \nabla \bar{h}_i^k e^{v_i^k}\big) = \epsilon_k \int_{E_k} y \cdot \nabla h_i^k\big(\epsilon_k y + x_1^k\big) e^{v_i^k} = O(\epsilon_k), \quad i \in I,$$

$$\int_{E_k} 2\bar{h}_i^k e^{v_i^k} = 4\pi\sigma_i^k + O\big(L_k^{-m+2}\big), \quad i \in I.$$

Similarly

$$\int_{\partial E_k} L_k \bar{h}_i^k e^{v_i^k} = O\big(L_k^{-m+2}\big), \quad i \in I.$$

Now we estimate $\nabla v_i^k$ $(i \in I)$. By the Green's representation formula:

$$\nabla v_i^k(y) = \int_{\Omega_k} \nabla_y G(y, \eta) \bigg( \sum_j a_{ij} \bar{h}_j^k e^{v_j^k(\eta)} \bigg) d\eta - \int_{\partial\Omega_k} \nabla_y \bigg( \frac{\partial G(y, \eta)}{\partial \nu} \bigg) v_i^k(\eta)\, dS_\eta, \quad i \in I. \tag{4.17}$$

The last term above is the gradient of a harmonic function. We know that if $f$ is a harmonic function on $B_R$, then $|\nabla f(0)| \leqslant C \cdot osc(f)/R$. By this reason we know that, since $v_i^k$ has bounded oscillation on $\partial\Omega_k$ and $|y| = L_k \ll \epsilon_k^{-1}$, the last term of (4.17) is $O(\epsilon_k)$.

To estimate the first term of (4.17), we use

$$G(y, \eta) = -\frac{1}{2\pi} \ln|y - \eta| + H_k(y, \eta).$$

For $|y| = L_k$, $H_k(y, \eta)$, as a function of $\eta$, is a harmonic function of the order $O(\ln \epsilon_k^{-1})$ on $\partial\Omega_k$. So for $\eta \in E_k$, using $H_k(y, \eta) = H_k(\eta, y)$ and standard gradient estimate for harmonic functions, we have

$$\big|\nabla_y H_k(y, \eta)\big| = \big|\nabla_\eta H_k(y, \eta)\big| \leqslant C \frac{\max_{\partial\Omega_k} H_k}{\epsilon_k^{-1}} = O\big(\epsilon_k^\delta\big).$$

Consequently

$$\int_{E_k} \nabla_y H_k(y, \eta) \bigg( \sum_j a_{ij} \bar{h}_j^k e^{v_j^k} \bigg) = O\big(\epsilon_k^\delta\big)$$

for $\delta \in (0, 1)$. We are left with the estimate of the term

$$-\frac{1}{2\pi} \int_{E_k} \nabla_y \big(\ln|y - \eta|\big) \sum_j a_{ij} \bar{h}_j^k e^{v_j^k(\eta)}\, d\eta.$$

For this we use

$$\partial_{y_a}\left(-\frac{1}{2\pi}\ln|y-\eta|\right)-\partial_a\left(-\frac{1}{2\pi}\ln|y|\right)$$
$$=-\frac{1}{2\pi}\frac{-\eta_a|y|^2-y_a|\eta|^2+2y_a\sum_{t=1}^{2}y_t\eta_t}{|y-\eta|^2|y|^2},\quad a=1,2,$$

and elementary estimate to obtain

$$-\frac{1}{2\pi}\int\limits_{E_k}\nabla_y\bigl(\ln|y-\eta|-\ln|y|\bigr)\left(\sum_j a_{ij}\bar{h}_j^k e^{v_j^k(\eta)}\right)d\eta=O\bigl(L_k^{-m+1}\ln L_k\bigr).$$

Consequently

$$\partial_a v_i^k(y)=\int\limits_{\Omega_k}\partial_a\left(-\frac{1}{2\pi}\ln|y|\right)\left(\sum_j a_{ij}\bar{h}_j^k e^{v_j^k}\right)d\eta$$
$$=-m_i^k\frac{y_a}{|y|^2}+O\bigl(L_k^{-m+1}\ln L_k\bigr),\quad i\in I,\ a=1,2.$$

Using this in the computation of the Pohozaev identity we obtain (4.16). Proposition 4.2 is established. □

Now we are in the position to prove (1.15). One can find $\{\sigma_{i,k}\}_{i\in I}$ that satisfies $\Lambda_I(\sigma_{\cdot,k})=0$, which is

$$\sum_{i,j}a_{ij}\sigma_{ik}\sigma_{jk}=4\sum_i\sigma_{ik}$$

so that

$$\sigma_{i,k}=\sigma_i^k,\quad i=1,\dots,n-1,\qquad \sigma_{n,k}-\sigma_n^k=O\bigl(\epsilon_k^\delta\bigr)\tag{4.18}$$

for some $\delta>0$. For $\{\sigma_{i,k}\}_{i\in I}$ we let $\bar{V}^k=(\bar{V}_1^k,\dots,\bar{V}_n^k)$ be the unique global solution so that $\{\bar{V}_i^k\}_{i\in I}$ are radial with respect to the origin,

$$\frac{1}{2\pi}\int\limits_{\mathbb{R}^2}h_i^k(0)e^{\bar{V}_i^k}=\sigma_{i,k},\quad i\in I,\qquad \bar{V}_1^k(0)=0.$$

Note that the uniqueness is proved in Theorem 1.1. Using $\sigma_{i,k}\to\sigma_i$ $(i\in I)$ as $k\to\infty$, we assert that $\bar{V}_i^k\to v_i$ $(i\in I)$ in $C^2_{\mathrm{loc}}(\mathbb{R}^2)$ because $v=(v_1,\dots,v_n)$ is the only radial solution that satisfies $\frac{1}{2\pi}\int_{\mathbb{R}^2}h_ie^{v_i}=\sigma_i$ and $v_1(0)=0$. On the other hand, by standard potential analysis

$$\bigl|\bar{V}_i(y)+\bar{m}_{i,k}\ln|y|\bigr|\leqslant C(A,\sigma),\quad |y|>2,$$

where $\bar{m}_{i,k}=\sum_j a_{ij}\sigma_{j,k}$. (4.18) implies $|\bar{m}_{i,k}-m_i^k|=O(\epsilon_k^\delta)$. Thus by (4.15) we have

$$\bigl|v_i^k(y)-\bar{V}_i^k(y)\bigr|\leqslant C(A,c_0,c_1,\sigma),\quad y\in\Omega_k.$$

Let $V_i^k$ be defined by

$$V_i^k(\epsilon_k y)+2\log\epsilon_k=\bar{V}_i^k(y),$$

then the second statement of Theorem 1.2 is established. □

## 5. Proof of Theorem 1.3 and (1.16)

In this section we prove Theorem 1.3 and (1.16). Let

$$\tilde{u}_i^k = u_i^k - \psi_i^k, \qquad \tilde{h}_i^k = h_i^k e^{\psi_i^k}, \quad i \in I.$$

Since $\psi_i^k(0) = 0$ we have

$$\frac{\nabla \tilde{h}_i^k(0)}{\tilde{h}_i^k(0)} = \nabla \psi_i^k(0) + \frac{\nabla h_i^k(0)}{h_i^k(0)}.$$

Let $|\xi| = 1$ be a unit vector, then a Pohozaev identity for $\tilde{u}^k = (\tilde{u}_1^k, \dots, \tilde{u}_n^k)$ is of the form (see Appendix A for the proof)

$$\int_{B_R} \left( \sum_i \partial_\xi \tilde{h}_i^k e^{\tilde{u}_i^k} \right) = \int_{\partial B_R} \left( \sum_i e^{\tilde{u}_i^k} \tilde{h}_i^k (\xi \cdot \nu) + \sum_{ij} a^{ij} \left( \partial_\nu \tilde{u}_i^k \partial_\xi \tilde{u}_j^k - \frac{1}{2} (\xi \cdot \nu) \big( \nabla \tilde{u}_i^k \cdot \nabla \tilde{u}_j^k \big) \right) \right).$$

By choosing $0 < R < 1$, it is easy to see from the decay rate of $\tilde{u}_i^k$ that

$$\int_{\partial B_R} \sum_i \big( e^{\tilde{u}_i^k} \tilde{h}_i^k (\xi \cdot \nu) \big) \to 0.$$

Also, since $\tilde{h}_i^k e^{\tilde{u}_i^k} \to 2\pi \sigma_i \delta_0$ in distributional sense, the left-hand side of the Pohozaev identity tends to

$$2\pi \sum_i \frac{\partial_\xi \tilde{h}_i(0)}{\tilde{h}_i(0)} \sigma_i.$$

To consider the limit of $\nabla \tilde{u}_i^k(x)$ for $|x| = R$, we use the Green's representation formula:

$$\tilde{u}_i^k(x) = \int_{B_1} G(x, \eta) \left( \sum_j a_{ij} \tilde{h}_j^k(\eta) e^{\tilde{u}_j^k(\eta)} \right) + \text{constant}.$$

By taking the derivative on $x$ and letting $k \to \infty$, we have

$$\nabla \tilde{u}_i^k(x) \to 2\pi m_i \nabla_1 G(x, 0) = m_i \frac{x}{|x|}, \quad i \in I.$$

Using this in the computation of the Pohozaev identity we see the limit of the right-hand side is 0. Therefore we have obtained:

$$\sum_i \frac{\partial_\xi \tilde{h}_i(0)}{\tilde{h}_i(0)} \sigma_i = 0.$$

Since $\xi$ is arbitrary, Theorem 1.3 is established. □

**Proof of (1.16).** Since $\int_M h_i^k e^{u_i^k} = 1$ ($i \in I$) the equation for $\{u^k\}$ is (see (1.8))

$$\Delta_g u_i^k + \sum_{j=1}^n \rho_j a_{ij} \big( h_j^k e^{u_j^k} - 1 \big) = 0, \quad M. \tag{5.1}$$

Recall that $\{p_1, \dots, p_m\}$ are disjoint blowup points for $\{u^k\}$. Let

$$\sigma_{it} = \lim_{r \to 0} \lim_{k \to \infty} \frac{1}{2\pi} \int_{B(p_t, r)} h_i^k e^{u_i^k} \, dV_g. \tag{5.2}$$

Our assumption is that around each $p_t$, $\{u^k\}$ converges to a Liouville system of $n$ equations after scaling. Let $\delta > 0$ be small enough so that $B(p_t, \delta)$ $(t = 1, \dots, m)$ are disjoint. For each $t$, let $M_t^k$ be the maximum of $\{u_i^k\}_{i\in I}$ in $B(p_t, \delta)$. In the isothermal coordinates around $p_t$, $g = e^{\phi}\delta_0$ and $\Delta_g = e^{-\phi}\Delta$ where $\delta_0$ is the Euclidean metric. We also have $\phi(0) = |\nabla\phi(0)| = 0$. With these properties (5.1) in $B(p_t, \delta)$ becomes

$$\Delta u_i^k + \sum_{j=1}^{n} \rho_j a_{ij} e^{\phi} h_j \big(e^{u_j^k} - 1\big) = 0, \quad B_\delta,\ i \in I.$$

Let $f_i$ satisfies

$$\Delta f_i = \sum_j \rho_j a_{ij} e^{\phi} h_j, \quad B_\delta,\ i \in I,$$

and $f_i = 0$ on $\partial B_\delta$, then the equation for $u_i^k$ can further be written as

$$\Delta\big(u_i^k + f_i\big) + \sum_j \rho_j a_{ij} e^{\phi - f_i} h_j e^{u_j^k + f_i} = 0, \quad B_\delta,\ i \in I. \tag{5.3}$$

Let

$$\sigma_{it}^k = \frac{1}{2\pi} \int_{B_\delta} h_i^k e^{\phi} e^{u_i^k}, \qquad m_{it}^k = \sum_j a_{ij} \sigma_{jt}^k.$$

By Theorem 1.2 and $\phi(0) = 0$ the limit of $\sigma_{it}^k$ is $\sigma_{it}$ (defined in (5.2)). Let $m_{it} > 2$ be the limit of $m_{it}^k$, then from Theorem 1.2 we have, for $x \in \partial B(p_t, \delta)$:

$$u_i^k(x) = -\frac{m_{it}^k - 2}{2} M_t^k + O(1), \quad x \in \partial B(p_t, \delta),\ i \in I,\ t = 1, \dots, m. \tag{5.4}$$

From the Green's representation of $u_i^k$ it is easy to see that the difference between $u_i^k(x)$ and $u_i^k(y)$ for $x, y$ away from the blowup set is uniformly bounded. Therefore for fixed $t_1$ and $t_2$, using $M_t^k \to \infty$ we obtain from (5.4) that

$$\frac{m_{it_1} - 2}{m_{it_2} - 2} = \lambda_{t_1 t_2}, \quad i \in I. \tag{5.5}$$

We claim that $\lambda_{t_1t_2} = 1$. Indeed, $\{\sigma_{it}\}_{i\in I}$ satisfies

$$\sum_{ij} a_{ij}\sigma_{it}\sigma_{jt} = 4 \sum_i \sigma_{it},$$

which can be written as

$$\sum_{ij} a^{ij} m_{it} m_{jt} = 4 \sum_{ij} a^{ij} m_{jt}.$$

The above is equivalent to

$$\sum_{ij} a^{ij}(m_{it} - 2)(m_{jt} - 2) = 4 \sum_{ij} a^{ij}.$$

Replacing $m_{it}$ by $m_{it_1}$ and $m_{it_2}$ respectively in the above, we have

$$\big(1 - \lambda_{t_1t_2}^2\big) \sum_{ij} a^{ij} = 0.$$

Recall that $A$ is assumed to be positive definite. So $\sum_{ij} a^{ij} > 0$, we have $\lambda_{t_1t_2} = 1$ $(t_1, t_2 = 1, \dots, m)$.

We can further claim that

$$\sigma_{it} = \frac{1}{2\pi m}, \quad i \in I,\ t = 1, \dots, m, \tag{5.6}$$

because $\int_M h_i^k e^{u_i^k} \equiv 1$ ($i \in I$), $m_{it_1} = m_{it_2}$ ($i \in I$) and

$$\int\limits_{M \setminus \bigcup_{t=1}^m B(p_t,\delta)} e^{u_i^k}\, dV_g \to 0, \quad i \in I.$$

The Green's representation for $u_i^k$ is

$$u_i^k(x) = \bar{u}_i^k + \int\limits_M G(x,\eta) \sum_j \rho_j a_{ij} h_j e^{u_j^k}\, dV_g. \tag{5.7}$$

The last term of the above tends to

$$\sum_{t=1}^m G(x,p_t)\Big(\sum_j \rho_j a_{ij}\Big)/m. \tag{5.8}$$

Recall that

$$G(x,\eta) = -\frac{1}{2\pi}\chi \ln d(x,\eta) + G^*(x,\eta). \tag{5.9}$$

For $x \in \partial B(p_s,\delta)$, by choosing the support of $\chi$ possibly smaller, we observe that $G(x,p_t) = G^*(x,p_t)$ for $t \neq s$. Therefore, let $\phi_k$ be the harmonic function on $B(p_s,\delta)$ defined by the oscillation of $u_i^k$ on $\partial B(p_s,\delta)$, using (5.7)–(5.9) we have

$$\lim_{k\to\infty} \nabla_g \phi_k(p_s) = \sum_{t=1}^m \nabla_1 G^*(p_s,p_t)\Big(\sum_j \rho_j a_{ij}\Big)/m.$$

Then (1.16) is a consequence of Theorem 1.3 and the above. □

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## Appendix A. The Pohozaev identity for the Liouville system

In this section we derive the Pohozaev identity for the Liouville system

$$-\Delta u_i = \sum_{j=1}^n a_{ij} h_j e^{u_j}, \qquad \Omega \Subset \mathbb{R}^2, \quad i \in I. \tag{A.1}$$

The Pohozaev identity for (A.1) is

$$\begin{aligned} &\sum_{i\in I}\Big(\int\limits_\Omega (x\cdot\nabla h_i)e^{u_i} + 2h_i e^{u_i}\Big) \\ &\quad = \int\limits_{\partial\Omega}\Big(\sum_i (x\cdot\nu)h_i e^{u_i} + \sum_{i,j} a^{ij}\Big(\partial_\nu u_j (x\cdot\nabla u_i) - \frac{1}{2}(x\cdot\nu)(\nabla u_i\cdot\nabla u_j)\Big)\Big). \end{aligned} \tag{A.2}$$

**Proof of (A.2).** We write (A.1) as

$$-\sum_j a^{ij}\Delta u_j = h_i e^{u_i}, \quad \Omega,\ i \in I. \tag{A.3}$$

By multiplying $x \cdot \nabla u_i$ to the right-hand side of (A.3) and integration by parts, we obtain the following terms:

$$\int_{\partial\Omega}(x\cdot\nu)h_i e^{u_i} - 2\int_{\Omega} h_i e^{u_i} - \int_{\Omega}(x\cdot\nabla h_i)e^{u_i}.$$

Multiply $x \cdot \nabla u_i$ to the left-hand side of (A.3) and use integration by parts, we have, after taking the summation on $i$:

$$-\sum_{ij}\int_{\partial\Omega} a^{ij}\partial_\nu u_j x\cdot\nabla u_i + \int_{\Omega}\sum_{ij} a^{ij}\nabla u_i\nabla u_j + \sum_{ij}\int_{\Omega}\sum_{a=1}^{n}\sum_{b=1}^{n} a^{ij}x_b\partial_a u_j\partial_{ab}u_i.$$

Using the symmetry of $a^{ij}$ and integration by parts again the left-hand side is equal to

$$-\sum_{ij}\int_{\partial\Omega} a^{ij}\partial_\nu u_j x\cdot\nabla u_i + \frac{1}{2}\sum_{ij}\int_{\partial\Omega} a^{ij}(x\cdot\nu)(\nabla u_i\cdot\nabla u_j).$$

Then (A.2) follows. □

A different version of the Pohozaev identity is as follows. Let $\xi$ be a unit vector, then we have

$$\sum_i\int_{\Omega}\partial_\xi h_i e^{u_i} = \int_{\partial\Omega}\sum_i e^{u_i}h_i(\xi\cdot\nu) + \sum_{i,j} a^{ij}\left(\partial_\nu u_i\partial_\xi u_j - \frac{1}{2}(\xi\cdot\nu)(\nabla u_i\cdot\nabla u_j)\right). \tag{A.4}$$

The third Pohozaev identity is for the linearized system:

$$\big(r\phi_i'(r)\big)' + \sum_j a_{ij}e^{u_j}r\phi_j(r) = 0, \quad 0 < r < \infty,\ i \in I.$$

The Pohozaev identity is:

$$\sum_i\left(r^2\phi_i(r)e^{u_i} - 2\int_0^r se^{u_i}\phi_i\,ds\right) = -\sum_{i,j} a^{ij}\big(r\phi_j'(r)\big)\big(ru_i'(r)\big). \tag{A.5}$$

To derive (A.5) we just need to write the linear system as

$$-\sum_j a^{ij}\big(r\phi_j'(r)\big)' = e^{u_i}\phi_i(r)r, \quad i \in I.$$

Multiply $ru_i'(r)$ to both sides of the above and use integration by parts, we obtain (A.5).

## References


[1] J.J. Aly, Thermodynamics of a two-dimensional self-gravitating system, Phys. Rev. A 49 (5) (1994) 3771–3783.

[2] W.H. Bennet, Magnetically self-focusing streams, Phys. Rev. 45 (1934) 890–897.

[3] D. Bartolucci, C.C. Chen, C.S. Lin, G. Tarantello, Profile of blow-up solutions to mean field equations with singular data, Comm. Partial Differential Equations 29 (7–8) (2004) 1241–1265.

[4] P. Biler, T. Nadzieja, Existence and nonexistence of solutions of a model of gravitational interactions of particles, I & II, Colloq. Math. 66 (1994) 319–334; Colloq. Math. 67 (1994) 297–309.

[5] H. Brezis, F. Merle, Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions, Comm. Partial Differential Equations 16 (8–9) (1991) 1223–1253.

[6] L.A. Caffarelli, B. Gidas, J. Spruck, Asymptotic symmetry and local behavior of semilinear elliptic equations with critical Sobolev growth, Comm. Pure Appl. Math. 42 (3) (1989) 271–297.

[7] S.A. Chang, P. Yang, Conformal deformations of metrics on $\mathbb{S}^2$, J. Differential Geom. 27 (1988) 256–296.

[8] S. Chanillo, M.K.-H. Kiessling, Conformally invariant systems of nonlinear PDE of Liouville type, Geom. Funct. Anal. 5 (6) (1995) 924–947.
[9] C.C. Chen, C.S. Lin, Sharp estimates for solutions of multi-bubbles in compact Riemann surfaces, Comm. Pure Appl. Math. 55 (6) (2002) 728–771.
[10] C.C. Chen, C.S. Lin, Topological degree for a mean field equation on Riemann surfaces, Comm. Pure Appl. Math. 56 (12) (2003) 1667–1727.
[11] W.X. Chen, C.M. Li, Classification of solutions of some nonlinear elliptic equations, Duke Math. J. 63 (3) (1991) 615–622.
[12] S. Childress, J.K. Percus, Nonlinear aspects of chemotaxis, Math. Biosci. 56 (1981) 217–237.
[13] M. Chipot, I. Shafrir, G. Wolansky, On the solutions of Liouville systems, J. Differential Equations 140 (1) (1997) 59–105.
[14] P. Debye, E. Huckel, Zur Theorie der Electrolyte, Phys. Z. 24 (1923) 305–325.
[15] G. Dunne, Self-Dual Chern–Simons Theories, Lecture Notes in Phys., vol. m36, Springer-Verlag, Berlin, 1995.
[16] J. Jost, G. Wang, Classification of solutions of a Toda system in $\mathbb{R}^2$, Int. Math. Res. Not. 6 (2002) 277–290.
[17] J. Jost, G. Wang, Analytic aspects of the Toda system, I. A Moser–Trudinger inequality, Comm. Pure Appl. Math. 54 (11) (2001) 1289–1319.
[18] J. Jost, C.S. Lin, G. Wang, Analytic aspects of the Toda system, II. Bubbling behavior and existence of solutions, Comm. Pure Appl. Math. 59 (4) (2006) 526–558.
[19] E.F. Keller, L.A. Segel, Traveling bands of chemotactic bacteria: A theoretical analysis, J. Theoret. Biol. 30 (1971) 235–248.
[20] M.K.-H. Kiessling, J.L. Lebowitz, Dissipative stationary plasmas: Kinetic modeling Bennet pinch, and generalizations, Phys. Plasmas 1 (1994) 1841–1849.
[21] Y.Y. Li, Harnack type inequality: The method of moving planes, Comm. Math. Phys. 200 (2) (1999) 421–444.
[22] C.S. Lin, A classification of solutions of a conformally invariant fourth order equation in $\mathbb{R}^n$, Comment. Math. Helv. 73 (2) (1998) 206–231 (English summary).
[23] C.S. Lin, L. Zhang, Topological degree for some Liouville systems on Riemann surfaces, in preparation.
[24] M.S. Mock, Asymptotic behavior of solutions of transport equations for semiconductor devices, J. Math. Anal. Appl. 49 (1975) 215–225.
[25] I. Rubinstein, Electro-Diffusion of Ions, SIAM Stud. Appl. Math., vol. 11, SIAM, Philadelphia, PA, 1990.
[26] G. Wolansky, On steady distributions of self-attracting clusters under friction and fluctuations, Arch. Ration. Mech. Anal. 119 (1992) 355–391.
[27] G. Wolansky, On the evolution of self-interacting clusters and applications to semi-linear equations with exponential nonlinearity, J. Anal. Math. 59 (1992) 251–272.
[28] Y. Yang, Solitons in Field Theory and Nonlinear Analysis, Springer-Verlag, 2001.
[29] L. Zhang, Blowup solutions of some nonlinear elliptic equations involving exponential nonlinearities, Comm. Math. Phys. 268 (1) (2006) 105–133 (English summary).
[30] L. Zhang, Asymptotic behavior of blowup solutions for elliptic equations with exponential nonlinearity and singular data, Commun. Contemp. Math. 11 (3) (2009) 395–411.