

Choquet simplices as spaces of invariant probability measures on post-critical sets

María Isabel Cortez^{a,1}, Juan Rivera-Letelier^{b,*,2}

^a *Departamento de Matemática y Ciencia de la Computación, Universidad de Santiago de Chile, Av. Libertador Bernardo O'Higgins 3363, Santiago, Chile*

^b *Facultad de Matemáticas, Campus San Joaquín, P. Universidad Católica de Chile, Avenida Vicuña Mackenna 4860, Santiago, Chile*

Received 19 March 2009; received in revised form 19 July 2009; accepted 20 July 2009

Available online 5 August 2009

Abstract

A well-known consequence of the ergodic decomposition theorem is that the space of invariant probability measures of a topological dynamical system, endowed with the weak* topology, is a non-empty metrizable Choquet simplex. We show that every non-empty metrizable Choquet simplex arises as the space of invariant probability measures on the post-critical set of a logistic map. Here, the post-critical set of a logistic map is the ω -limit set of its unique critical point. In fact we show the logistic map f can be taken in such a way that its post-critical set is a Cantor set where f is minimal, and such that each invariant probability measure on this set has zero Lyapunov exponent, and is an equilibrium state for the potential $-\ln|f'|$.

© 2009 L'Association Publications de l'Institut Henri Poincaré. Published by Elsevier B.V. All rights reserved.

Résumé

Une conséquence bien connue du théorème de décomposition ergodique est que l'espace des mesures de probabilité invariantes d'un système dynamique topologique est un simplexe de Choquet métrisable et non vide. On montre que tout simplexe de Choquet métrisable et non vide se réalise comme l'espace des mesures de probabilité invariantes sur l'ensemble post-critique d'une application logistique. Ici, l'ensemble post-critique d'une application logistique est l'ensemble ω -limite de son unique point critique. En effet, on démontre que l'application logistique f peut être choisie de telle façon que son ensemble post-critique soit un ensemble de Cantor où f est minimal, et tel que chaque mesure de probabilité invariante sur cet ensemble soit d'exposant de Lyapunov nul, et un état d'équilibre pour le potentiel $-\ln|f'|$.

© 2009 L'Association Publications de l'Institut Henri Poincaré. Published by Elsevier B.V. All rights reserved.

MSC: 37E05; 37A99; 37B10; 54H20

Keywords: Logistic map; Post-critical set; Invariant measures; Choquet simplices; Minimal Cantor system; Generalized odometer

* Corresponding author.

E-mail addresses: maria.cortez@usach.cl (M.I. Cortez), riveraletelier@mat.puc.cl (J. Rivera-Letelier).

¹ Partially supported by Fondecyt de Iniciación 11060002, Nucleus Millenius P04-069-F, and Research Network on Low Dimensional Dynamics, PBCT ACT-17, CONICYT, Chile.

² Partially supported by Research Network on Low Dimensional Dynamics, PBCT ACT-17, CONICYT, Chile.

1. Introduction

A well-known consequence of the ergodic decomposition theorem is that the space of invariant probability measures of a topological dynamical system, endowed with the weak* topology, is a metrizable Choquet simplex.³ The purpose of this paper is to show that every non-empty metrizable Choquet simplex arises in this way within the *logistic family* of maps $(f_\lambda)_{(0,4]}$, where for each parameter λ in $(0, 4]$ the *logistic map* $f_\lambda : [0, 1] \rightarrow [0, 1]$ is defined by

$$f_\lambda(x) = \lambda x(1 - x).$$

To make a more precise statement note that $x = \frac{1}{2}$ is the unique point in $[0, 1]$ at which the derivative of f_λ vanishes. We call $x = \frac{1}{2}$ the *critical point* of f_λ , and its ω -limit set is called the *post-critical set* of f_λ . It is a compact set that is forward invariant by f_λ .

The following is our main result. Recall that for a compact topological space X a continuous map $T : X \rightarrow X$ is *minimal*, if every forward orbit of T is dense in X .

Main Theorem. *For each non-empty metrizable Choquet simplex \mathcal{C} there is a parameter $\lambda \in (0, 4]$ such that the post-critical set of the logistic map f_λ is a Cantor set, the restriction of f_λ to this set is minimal, and such that the space of invariant probability measures supported by this set, endowed with weak* topology, is affine homeomorphic to \mathcal{C} .*

The first result of this kind was shown by Downarowicz in [11], who showed that every non-empty metrizable Choquet simplex arises, up to an affine homeomorphism, as the space of invariant probability measures of a “minimal Cantor system”; that is, a dynamical system generated by a minimal homeomorphism of a Cantor set. In fact he showed that the minimal Cantor system can be taken as a “0–1 Toeplitz flow”: A special type of subshift of $\{0, 1\}^{\mathbb{Z}}$. See also [14,26] for a different approach to this result, and see [8] for an analogous result in the case of actions of \mathbb{Z}^d .

The following corollary is a direct consequence of the Main Theorem and of the fact that for each non-empty Polish space \mathcal{P} there is a metrizable Choquet simplex whose set of extreme points is homeomorphic to \mathcal{P} , see for example [19].

Corollary 1. *For each non-empty Polish space \mathcal{P} there is a parameter $\lambda \in (0, 4]$ such that the post-critical set of the logistic map f_λ is a Cantor set, the restriction of f_λ to this set is minimal, and such that the space of ergodic and invariant probability measures supported by this set, endowed with weak* topology, is homeomorphic to \mathcal{P} .*

The special case where the Polish space \mathcal{P} is compact and totally disconnected is precisely [9, Main Theorem]. The first result in this direction was shown by Bruin, who gave an example of a parameter $\lambda \in (0, 4]$ such that the post-critical set of f_λ is a Cantor set where f_λ is minimal, but not uniquely ergodic [7, Theorem 4]. The proof of the Main Theorem is based on the tools developed by Bruin in [7], and by Bruin, Keller and St. Pierre in [4].

One of the interesting features of the Main Theorem, in contrast with the other realization results mentioned above, is that the systems we consider have a natural differentiable structure. It turns out that, for the parameters $\lambda \in (0, 4]$ given by (the proof of) the Main Theorem, the invariant measures supported by the post-critical set of f_λ correspond precisely to those invariant measures μ of f_λ whose Lyapunov exponent

$$\chi(\mu) := \int \ln |f'_\lambda| d\mu,$$

vanishes [9, Lemma 21]. It also turns out that every invariant probability measure supported on the post-critical set of f_λ is an “equilibrium state of f_λ for the potential $-\ln |f'_\lambda|$ ”. That is, if for each invariant measure μ we denote by h_μ its measure theoretic entropy, then the supremum

$$\sup\{h_\mu - \chi(\mu) \mid \mu \text{ invariant probability measure of } f_\lambda\},$$

is attained at each invariant probability measure supported by the post-critical set of f_λ , see [9, Lemma 21].

³ See for example [15, p. 95]. We recall the definition of Choquet simplex in Section 2.2.

We thus obtain the following corollary of the Main Theorem.

Corollary 2. *For each non-empty metrizable Choquet simplex \mathcal{C} there is a parameter $\lambda \in (0, 4]$ verifying the conclusions of the Main Theorem, and such that in addition the space of invariant probability measures of f_λ (resp. equilibrium states of f_λ for the potential $-\log |f'_\lambda|$) that are of zero Lyapunov exponent, endowed with the weak* topology, is affine homeomorphic to \mathcal{C} .*

This result is in sharp contrast with the fact that for a logistic map there can be at most one ergodic equilibrium state whose Lyapunov exponent is strictly positive.⁴

For future reference we state a holomorphic version of Corollary 2, shown in Appendix A. For a complex parameter $\lambda \in \mathbb{C}$ denote by P_λ the quadratic polynomial defined by

$$P_\lambda(z) = \lambda z(1 - z),$$

viewed as a dynamical system acting on \mathbb{C} .

Corollary 3. *For each non-empty metrizable Choquet simplex \mathcal{C} there is a parameter $\lambda \in (0, 4]$ verifying the conclusions of the Main Theorem, and such that in addition, if we denote by t_0 the Hausdorff dimension of the Julia set of P_λ , then the space of invariant probability measures of P_λ (resp. equilibrium states of P_λ for the potential $-t_0 \log |P'_\lambda|$) that are of zero Lyapunov exponent, endowed with the weak* topology, is affine homeomorphic to \mathcal{C} .*

We end this introduction by stating some questions that arise naturally from the Main Theorem. To do this, for each $\lambda \in (0, 4]$ we will denote by X_λ the post-critical set of f_λ . Given a compact metrizable topological space X and a continuous map $T : X \rightarrow X$, the Main Theorem implies that there is a parameter $\lambda \in (0, 4]$ and an affine homeomorphism H between the space of invariant probability measures of f_λ supported on X_λ , and the space of invariant probability measures of T supported on X . It is thus natural to ask whether the parameter $\lambda \in (0, 4]$ and H can be chosen in such a way that H is induced by a continuous map between X_λ and X . More precisely, the question is if $\lambda \in (0, 4]$ and H can be chosen in such a way that there is a continuous map $h : X_\lambda \rightarrow X$ such that for each invariant probability measure μ supported by X_λ we have $H(\mu) = h_*\mu$.

This type of problem is very well understood in the setting of minimal Cantor systems: Giordano, Putnam, and Skau have shown in [18] that for two minimal Cantor systems (X, T) and (X', T') there exists a homeomorphism $h : X \rightarrow X'$ that induces an affine homeomorphism between the corresponding spaces of invariant probability measures, if, and only if, (X, T) and (X', T') are “orbit equivalent”: There is a homeomorphism between X and X' mapping each orbit of T to an orbit of T' . Furthermore, to each minimal Cantor system (X, T) one can associate a dimension group that is a complete invariant for the orbit equivalence relation [18]: Two minimal Cantor systems are orbit equivalent if, and only if, the corresponding dimension groups are isomorphic as ordered groups with unit.⁵

It is thus natural to look for a special class of minimal Cantor systems realizing all of the orbit equivalent classes. Since the dimension group associated to each Toeplitz flow contains the dimension group of an odometer as a subgroup [14, Section 4.1], the class of Toeplitz flows is not sufficient to realize all orbit equivalence classes, in spite of the fact that this class realizes all the non-empty metrizable Choquet simplices as sets of invariant probability measures [11].

So the question remains whether minimal post-critical sets of logistic maps realize all orbit equivalence classes. In order to formulate a precise question we will consider natural extensions to stay in the class of minimal Cantor systems, and use the generalized odometer associated to a kneading map, see Sections 2.3, 4.1 for definitions.

⁴ This follows from a result of Ledrappier in [22], that every such equilibrium state is absolutely continuous with respect to the Lebesgue measure, and from the fact that logistic maps are ergodic with respect to the Lebesgue measure [5].

⁵ The dimension group is defined as the quotient of the space of continuous functions defined on X and taking values in \mathbb{Z} , by the subgroup of functions whose integral with respect to each invariant measure vanishes; the positive cone is the set of those classes containing a function taking values in \mathbb{N}_0 , and the unit is the class of the constant function equal to 1. See for example [18, Theorem 1.13].

Question 4. Does every orbit equivalence class contain the natural extension of a generalized odometer associated to a kneading map?

Question 5. Does every uniquely ergodic orbit equivalence class contain the natural extension of a generalized odometer associated to a kneading map?

It is well known that every odometer can be realized, up to a homeomorphism, as the post-critical set of an infinitely renormalizable logistic map, see also [3]. In Section 5.4 we give an example of a uniquely ergodic generalized odometer associated to a kneading map, whose natural extension is not orbit equivalent to an odometer, nor to a Toeplitz flow.

1.1. Notes and references

Although Corollary 1 is stronger than [9, Main Theorem], we use this last result in the proof of the Main Theorem to deal with case of finite dimensional Choquet simplices.

We have stated the Main Theorem and Corollary 1 for the logistic family for simplicity. We show that an analogous statement holds for each full family of unimodal maps, as well as for the family of symmetric tent maps. See Section 2.3 for definitions. In fact, for each infinitely dimensional metrizable Choquet simplex we construct kneading map Q such that the conclusions of the Main Theorem hold for each unimodal map whose kneading map is Q , see Section 3. Furthermore the kneading map satisfies for every $k \in \mathbb{N}_0$ the inequality $Q(k) \leq \max\{0, k - 2\}$ (part 1 of Lemma 11), and therefore every full family of unimodal maps, as well as the family of symmetric tent maps, contains a unimodal map whose kneading map is Q . For the case of finite dimensional Choquet simplices see [9, Remark 1].

Similarly, it follows from [9, Lemma 21] that Corollary 2 holds for every full family of S-unimodal maps.

See [12, §15] for a survey on realization results concerning Toeplitz flows.

See [17] for the realization of some concrete simplices as the space of invariant measures of minimal Cantor systems.

1.2. Strategy and organization

In this section we explain the strategy of the proof of the Main Theorem and simultaneously describe the organization of the paper.

We only deal with infinitely dimensional Choquet simplices, the finite dimensional case being covered by [9, Main Theorem]. We use a result of Lazar and Lindenstrauss that characterizes infinite dimensional metrizable Choquet simplices as inverse limits of stochastic matrices, see Theorem 7 in Section 2.2.

We describe the logistic maps in the Main Theorem through their associated “kneading map”, see Section 2.3 for the definition of kneading map and further background on unimodal maps. In fact, the conclusions of the Main Theorem are valid for each unimodal map having the same kneading map as f_λ . To ensure that the post-critical set is a Cantor set where the unimodal map is minimal, it is enough to require that the kneading map diverges to $+\infty$ (Proposition 10).

In Section 3.1 we introduce a class of kneading maps that diverge to $+\infty$. In Section 3.2 we state a result describing the space of invariant measures supported on the post-critical set of a unimodal map with a kneading map in this class (Theorem A). In Section 3.2 we also give a proof of the Main Theorem assuming Theorem A.

In Section 4.1 we recall the definition and some properties of the generalized odometer associated to a kneading map, that was introduced in [4]. In Section 5.1 we show that for a unimodal map whose kneading map Q is as in Section 3.1, the space of invariant probability measures supported by the post-critical set is affine homeomorphic to that of the generalized odometer associated to Q . In turn, this space is affine homeomorphic to the corresponding space of the Bratteli–Vershik system associated to Q , introduced in [7]; see [7, Proposition 2] or Theorem 14 in Section 4.4. The advantage of this last space is that it can be described explicitly as an inverse limit of some “transition matrices”, see Section 4.3. We calculate the transition matrices and some of their products in Section 5.2, and give the proof of Theorem A in Section 5.3.

In Section 5.4 we give an example of a uniquely ergodic generalized odometer associated to a kneading map whose natural extension is not orbit equivalent to a Toeplitz flow.

In Appendix A we give the proof of Corollary 3.

2. Preliminaries

After fixing some notation in Section 2.1, we review some concepts and results about Choquet simplices (Section 2.2), and unimodal maps (Section 2.3).

Throughout the rest of this article \mathbb{N} denotes the ring of strictly positive integers and $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We will use the interval notation for subsets of \mathbb{N}_0 : For $n, n' \in \mathbb{N}_0$ such that $n' \geq n$, we put

$$[n, n'] := \{k \in \mathbb{N}_0 \mid n \leq k \leq n'\},$$

and we put $[n, n'] := \emptyset$ when $n' < n$.

2.1. Linear algebra

Given a non-empty finite set V , for each $v \in V$ we denote by $\vec{e}_v \in \mathbb{R}^V$ the vector having all of its coordinates equal to 0, except for the coordinate corresponding to v that is equal to 1. Notice in particular that $\{\vec{e}_v \mid v \in V\}$ is a base of \mathbb{R}^V . Furthermore we will denote by Δ_V the *unit simplex* in \mathbb{R}^V , which is defined as the (closed) convex hull of $\{\vec{e}_v \mid v \in V\}$ in \mathbb{R}^V , and by $\|\cdot\|_1$ the norm on \mathbb{R}^V defined by $\|\sum_{v \in V} \alpha_v \vec{e}_v\|_1 = \sum_{v \in V} |\alpha_v|$. Observe that $\|\cdot\|_1$ is constant equal to 1 on Δ_V .

Given non-empty finite sets V, V' denote by $\mathcal{M}_{V, V'}$ the group of matrices whose entries are real and indexed by $V \times V'$. For a matrix $A \in \mathcal{M}_{V, V'}$ we denote by A^t the transpose of A , and for $(v, v') \in V \times V'$ we denote by $A(v, v')$ the corresponding entry of A , and by $A(\cdot, v')$ the corresponding column vector of A . Given column vectors $\{\vec{x}_{v'} \mid v' \in V'\}$ in \mathbb{R}^V we denote by $(\vec{x}_{v'})_{v' \in V'}$ the matrix in $\mathcal{M}_{V, V'}$ whose column vector corresponding to the coordinate v' is equal to $\vec{x}_{v'}$.

We say that a matrix A is (left) *stochastic* if all of its entries are non-negative and if the sum of all the entries in each column is equal to 1. Observe that a stochastic matrix in $\mathcal{M}_{V, V'}$ maps $\Delta_{V'}$ into Δ_V , and that the product of stochastic matrices is stochastic.

Lemma 6. *Let V, V' be non-empty finite sets and let $A \in \mathcal{M}_{V, V'}$ be a stochastic matrix. Then for each $\vec{w}, \vec{w}' \in \Delta_V$ we have*

$$\|A(\vec{w}) - A(\vec{w}')\|_1 \leq \|\vec{w} - \vec{w}'\|_1.$$

Proof. Putting $\vec{w} = (w_v)_{v \in V}$ and $\vec{w}' = (w'_v)_{v \in V}$, we have

$$\|A(\vec{w}) - A(\vec{w}')\|_1 \leq \sum_{v \in V} \|(w_v - w'_v)A(\vec{e}_v)\|_1 = \sum_{v \in V} |w_v - w'_v| = \|\vec{w} - \vec{w}'\|_1. \quad \square$$

2.2. Choquet simplices

A compact, convex, and metrizable subset \mathcal{C} of a locally convex real vector space is said to be a (metrizable) *Choquet simplex*, if for each $v \in \mathcal{C}$ there is a unique probability measure μ that is supported on the set of extreme points of \mathcal{C} , and such that $\int x d\mu(x) = v$. See for example [1, §II.3] for several characterizations of Choquet simplices.

In the proof of the Main Theorem we will make use of the following characterization of infinite dimensional metrizable Choquet simplices.

Theorem 7. (See [24, Corollary, p. 186].) *Given an infinite dimensional Choquet simplex \mathcal{C} , for each $n \in \mathbb{N}$ there is a surjective affine map $A_n : \Delta_{[0, n+1]} \rightarrow \Delta_{[0, n]}$ such that $\varprojlim_n (\Delta_{[0, n+1]}, A_n)$ is affine homeomorphic to \mathcal{C} .*

The following lemma is a simple consequence of the previous theorem. We will say that a sequence $(A_n)_{n \in \mathbb{N}}$ as in the theorem is *normalized*, if for every $n \in \mathbb{N}$ and $j \in [0, n]$ we have $A_n(\vec{e}_j) = \vec{e}_j$.

Lemma 8. *For every infinite dimensional metrizable Choquet simplex \mathcal{C} there is a normalized sequence of linear maps $(A_n)_{n \in \mathbb{N}}$ such that $\varprojlim_n (\Delta_{[0, n+1]}, A_n)$ is affine homeomorphic to \mathcal{C} .*

Proof. Let $(\tilde{A}_n)_{n \in \mathbb{N}}$ be a sequence of affine maps given by Theorem 7.

For each $n \in \mathbb{N}$ define a permutation σ_n of $[0, n]$ by induction as follows. Let σ_1 be the identity, and suppose that for some $n \in \mathbb{N}$ the permutation σ_n is already defined. Since \tilde{A}_n maps $\Delta_{[0, n+1]}$ surjectively onto $\Delta_{[0, n]}$, for each $j \in [0, n]$ there is $k \in [0, n+1]$ such that $\tilde{A}_n(\tilde{e}_k) = \tilde{e}_j$. Equivalently, there is a map $\iota_n : [0, n] \rightarrow [0, n+1]$ such that for each $j \in [0, n]$ we have $\tilde{A}_n(\tilde{e}_{\iota_n(j)}) = \tilde{e}_j$. Let σ_{n+1} be the unique permutation of $[0, n+1]$ such that for each $j \in [0, n]$ we have $\sigma_{n+1}(\iota_n(j)) = \sigma_n(j)$.

For each n let $H_n : \mathbb{R}^{[0, n]} \rightarrow \mathbb{R}^{[0, n]}$ be the linear map so that for each $j \in [0, n]$ we have $H_n(\tilde{e}_j) = \tilde{e}_{\sigma_n(j)}$. Then, by the definition of $(\sigma_n)_{n \in \mathbb{N}}$ it follows that for each $n \in \mathbb{N}$ the linear map $A_n := H_n \circ \tilde{A}_n \circ H_{n+1}^{-1}$ maps $\Delta_{[0, n+1]}$ surjectively onto $\Delta_{[0, n]}$, and that for every $j \in [0, n]$ we have $A_n(\tilde{e}_j) = \tilde{e}_j$. Therefore the sequence of linear maps $(A_n)_{n \in \mathbb{N}}$ is normalized, and $(H_n)_{n \in \mathbb{N}}$ induces a linear homeomorphism between $\varprojlim_n (\Delta_{[0, n+1]}, \tilde{A}_n)$ and $\varprojlim_n (\Delta_{[0, n+1]}, A_n)$. \square

We end this section with the following general lemma.

Lemma 9. For each $n \in \mathbb{N}$ let $A_n, B_n : \mathbb{R}^{[0, n+1]} \rightarrow \mathbb{R}^{[0, n]}$ be stochastic matrices such that

$$\sum_{n \in \mathbb{N}} \sup \{ \|A_n(\tilde{v}) - B_n(\tilde{v})\|_1 \mid \tilde{v} \in \Delta_{[0, n+1]} \} < +\infty.$$

Then the inverse limits $\varprojlim_n (\Delta_{[0, n+1]}, A_n)$ and $\varprojlim_n (\Delta_{[0, n+1]}, B_n)$ are affine homeomorphic.

In particular, if both $(A_n)_{n \in \mathbb{N}}$ and $(B_n)_{n \in \mathbb{N}}$ are normalized and

$$\sum_{n \in \mathbb{N}} \|A_n(\tilde{e}_{n+1}) - B_n(\tilde{e}_{n+1})\|_1 < +\infty,$$

then the inverse limits $\varprojlim_n (\Delta_{[0, n+1]}, A_n)$ and $\varprojlim_n (\Delta_{[0, n+1]}, B_n)$ are affine homeomorphic.

Proof. Let $x := (\tilde{x}_n)_{n \in \mathbb{N}} \in \varprojlim_n (\Delta_{[0, n+1]}, A_n)$. For each $n, m \in \mathbb{N}$ such that $m \geq n$ define

$$\tilde{x}_{n,m} := B_n \cdots B_m \tilde{x}_{m+1} \in \Delta_{[0, n]}.$$

In particular we have $\tilde{x}_{n,n} = \tilde{x}_n$.

When $m > 0$ we have

$$\begin{aligned} \|\tilde{x}_n - \tilde{x}_{n,m}\|_1 &= \|A_n \cdots A_m \tilde{x}_{m+1} - B_n \cdots B_m \tilde{x}_{m+1}\|_1 \\ &\leq \|A_n \cdots A_m \tilde{x}_{m+1} - B_n A_{n+1} \cdots A_m \tilde{x}_{m+1}\|_1 + \|B_n A_{n+1} \cdots A_m \tilde{x}_{m+1} - B_n \cdots B_m \tilde{x}_{m+1}\|_1. \end{aligned}$$

From Lemma 6 we get

$$\|B_n A_{n+1} \cdots A_m \tilde{x}_{m+1} - B_n \cdots B_m \tilde{x}_{m+1}\|_1 \leq \|A_{n+1} \cdots A_m \tilde{x}_{m+1} - B_{n+1} \cdots B_m \tilde{x}_{m+1}\|_1,$$

and since

$$\|A_n \cdots A_m \tilde{x}_{m+1} - B_n A_{n+1} \cdots A_m \tilde{x}_{m+1}\|_1 \leq \sup \{ v \in \Delta_{[0, n+1]} : \|A_n v - B_n v\|_1 \},$$

we deduce, after an induction argument, that

$$\|\tilde{x}_n - \tilde{x}_{n,m}\|_1 \leq \sum_{k=n}^m \sup \{ \|A_k v - B_k v\|_1 \mid v \in \Delta_{[0, k+1]} \}. \tag{2.1}$$

By hypothesis and from Eq. (2.1), we deduce that for a fixed $n \in \mathbb{N}$, the sequence $(\tilde{x}_{n,m})_{m \geq n}$ is a Cauchy sequence in $\Delta_{[0, n]}$. We denote by $H_n(x)$ its limit. Observe that for each $n \in \mathbb{N}$, the sequence $(B_n \tilde{x}_{n+1, m})_{m \geq n+1}$ converges to both $B_n H_{n+1}(x)$ and $H_n(x)$. This implies that $(H_n(x))_{n \in \mathbb{N}} \in \varprojlim_n (\Delta_{[0, n+1]}, B_n)$. Thus the transformation

$$H : \varprojlim_n (\Delta_{[0, n+1]}, A_n) \rightarrow \varprojlim_n (\Delta_{[0, n+1]}, B_n)$$

given by $H(x) = (H_n(x))_{n \in \mathbb{N}}$ is well defined. This map is clearly affine. We will show that H is a homeomorphism.

In order to verify that H is continuous, we just need to show that for each $m \in \mathbb{N}$ the map H_m is continuous. As $\varprojlim_n (\Delta_{[0, n+1]}, A_n)$ is metrizable we just need to show that H_m is sequentially continuous. To do this, fix $\varepsilon > 0$

and consider a sequence $(x^{(n)})_{n \in \mathbb{N}} := ((\vec{x}_m^{(n)})_{m \in \mathbb{N}})_{n \in \mathbb{N}}$ in $\varprojlim_n (\Delta_{[0, n+1]}, A_n)$ that converges to $x = (\vec{x}_n)_{n \in \mathbb{N}}$. For each $n, m, k \in \mathbb{N}$ such that $k \geq m$ we have

$$\begin{aligned} \|H_m(x^{(n)}) - H_m(x)\|_1 &\leq \|H_m(x^{(n)}) - B_m \cdots B_k \vec{x}_{k+1}^{(n)}\|_1 + \|B_m \cdots B_k \vec{x}_{k+1}^{(n)} - B_m \cdots B_k \vec{x}_{k+1}\|_1 \\ &\quad + \|B_m \cdots B_k \vec{x}_{k+1} - H_m(x)\|_1 \\ &\leq \|H_m(x^{(n)}) - B_m \cdots B_k \vec{x}_{k+1}^{(n)}\|_1 + \|\vec{x}_{k+1}^{(n)} - \vec{x}_{k+1}\|_1 \\ &\quad + \|B_m \cdots B_k \vec{x}_{k+1} - H_m(\vec{x}_m)\|_1. \end{aligned}$$

We choose k sufficiently large so that

$$\|H_m(x^{(n)}) - B_m \cdots B_k \vec{x}_{k+1}^{(n)}\|_1, \|B_m \cdots B_k \vec{x}_{k+1} - H_m(x)\|_1 \leq \frac{\varepsilon}{3},$$

and n such that $\|x_{k+1}^{(n)} - \vec{x}_{k+1}\|_1 \leq \varepsilon/3$. We get

$$\|H_m(x^{(n)}) - H_m(x)\|_1 \leq \varepsilon.$$

This shows the continuity of H_m , and hence that of H .

To show that H is a homeomorphism we define in a similar way,

$$L : \varprojlim_n (\Delta_{[0, n+1]}, B_n) \rightarrow \varprojlim_n (\Delta_{[0, n+1]}, A_n).$$

Observe that by Lemma 6 we get

$$\begin{aligned} \|L_n(H_n(x)) - \vec{x}_n\|_1 &\leq \|L_n(H_n(x)) - A_n \cdots A_m H_{m+1}(x)\|_1 + \|A_n \cdots A_m H_{m+1}(x) - A_n \cdots A_m \vec{x}_{m+1}\|_1 \\ &\leq \|L_n(H_n(x)) - A_n \cdots A_m H_{m+1}(x)\|_1 + \|H_{m+1}(x) - \vec{x}_{m+1}\|_1. \end{aligned}$$

By definition of L , we have $\lim_{m \rightarrow \infty} \|L_n(H_n(x)) - A_n \cdots A_m H_{m+1}(x)\|_1 = 0$, and from (2.1) we have $\lim_{m \rightarrow \infty} \|H_{m+1}(x) - \vec{x}_{m+1}\|_1 = 0$. Thus we conclude that L and H are inverse of each other. \square

2.3. Unimodal maps, cutting times and the kneading map

A continuous map $f : [0, 1] \rightarrow [0, 1]$ is *unimodal* if $f(0) = f(1) = 0$, and if there exists a point $c \in [0, 1]$ such that f is strictly increasing on $[0, c]$, and strictly decreasing on $[c, 1]$. The point c is called the *turning* or *critical point* of f . For each $\lambda \in (0, 4]$ the logistic map f_λ is a unimodal with critical point $x = \frac{1}{2}$.

Let f be a unimodal map with critical point c . The ω -limit of c will be called the *post-critical set* of f . When either $f(c) \leq c$ or $f^2(c) \geq c$, it is easy to see that the post-critical set of f reduces to a single point. We will thus (implicitly) assume from now on that for each unimodal map f that we consider we have $f^2(c) < c < f(c)$.

To describe the dynamics of a unimodal map f on its post-critical set, we will make the following definitions. Let c be the critical point of f and for each $n \geq 1$ put $c_n = f^n(c)$. Define the sequence of compact intervals $(D_n)_{n \geq 1}$ inductively by $D_1 = [c, c_1]$, and for each $n \geq 2$, by

$$D_n = \begin{cases} f(D_{n-1}) & \text{if } c \notin D_{n-1}, \\ [c_n, c_1] & \text{otherwise.} \end{cases}$$

An integer $n \geq 1$ will be called a *cutting time* if $c \in D_n$. We will denote by $(S_k)_{k \geq 0}$ the sequence of all cutting times. From our assumption that $f^2(c) < c < f(c)$ it follows that $S_0 = 1$ and $S_1 = 2$.

It can be shown that if S and $S' > S$ are consecutive cutting times, then $S' - S$ is again a cutting time, and that this cutting time is less than or equal to S when f has no periodic attractors, see for example [6,20]. That is, if f has no periodic attractors then for each $k \geq 1$ there is a non-negative integer $Q(k)$, such that $Q(k) \leq k - 1$, and

$$S_k - S_{k-1} = S_{Q(k)}.$$

Putting $Q(0) = 0$, the function $Q : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ so defined is called the *kneading map* of f . It follows from the recursion formula above, and from $S_0 = 1$, that the sequence $(S_k)_{k \geq 0}$ of cutting times is determined by Q .

We will say that a function $Q : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ is a *kneading map* if there is a unimodal map f with critical point c , such that $f^2(c) < c < f(c)$, such that f has no periodic attractors and such that the kneading map of f is equal to Q . If we denote by \succ the lexicographical ordering in $\mathbb{N}_0^{\mathbb{N}_0}$, then a function $Q : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ is a kneading map if and only if $Q(0) = 0$, for each $k \geq 1$ we have $Q(k) \leq k - 1$, and if for each $k \geq 1$ we have

$$\{Q(k + j)\}_{j \geq 1} \succ \{Q(Q(Q(k)) + j)\}_{j \geq 1}, \tag{2.2}$$

see [6,20]. Notice in particular that, if $Q : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ is non-decreasing, $Q(0) = 0$ and for each $k \geq 1$ we have $Q(k) \leq k - 1$, then Q is a kneading map.

We will need the following well-known facts, see for example the proof of [9, Proposition 4] for precise references.

Proposition 10. *Let f be a unimodal map whose kneading map diverges to $+\infty$. Then the post-critical set of f is a Cantor set, and the restriction of f to this set is minimal and has zero topological entropy. Furthermore, if \widehat{f} is a unimodal map having the same kneading map as f , then the space of invariant probability measures of \widehat{f} supported on the post-critical set of \widehat{f} is affine homeomorphic to that of f .*

3. Reduced statement

The purpose of this section is to prove the Main Theorem assuming a result we state as Theorem A. We start introducing a class of kneading maps in Section 3.1. In Section 3.2 we state a result describing the space of invariant probability measures supported on the post-critical set of a unimodal map with a kneading map in this class (Theorem A). In Section 3.2 we also give a proof of the Main Theorem assuming Theorem A.

3.1. Kneading maps

Throughout the rest of this paper we denote by $(r_n)_{n \in \mathbb{N}_0}$ the sequence of integers defined by $r_n = \frac{(n+1)(n+2)}{2}$. Note that for each $n \in \mathbb{N}$ we have $r_n = r_{n-1} + n + 1$.

For each $n \in \mathbb{N}$ let $\vec{a}_n := (a_{n,0}, \dots, a_{n,n}) \in \mathbb{N}^{[0,n]}$ be given and let $q := (q_r)_{r \geq 0}$ be an increasing sequence of integers such that $q_0 = 0$, and such that for each $n \in \mathbb{N}$ we have

$$q_{r_n} - q_{r_{n-1}} = a_{n,0} + \dots + a_{n,n}. \tag{3.1}$$

If we put $a := (\vec{a}_n)_{n \in \mathbb{N}}$, then we will define a kneading map $Q_{(a,q)} : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ as follows. For each $n \in \mathbb{N}_0$ put

$$I_n = [q_{r_{n-1}} + 1, q_{r_n}] \quad \text{and} \quad J_n = [q_{r_n} + 1, q_{r_{n+1}-1}],$$

and note that $\{I_n, J_n \mid n \in \mathbb{N}\}$ is a partition of \mathbb{N} . Furthermore, for each $n \in \mathbb{N}$ and $m \in [0, n]$ define

$$I_{n,m} = \left[\left(q_{r_{n-1}} + 1 + \sum_{i=0}^{m-1} a_{n,i} \right), \left(q_{r_{n-1}} + \sum_{i=0}^m a_{n,i} \right) \right]$$

and

$$J_{n,m} = [q_{r_{n+m}} + 1, q_{r_{n+m+1}}].$$

By (3.1) the collection $\{I_{n,m} \mid m \in [0, n]\}$ is a partition of I_n . Since for each $n \in \mathbb{N}$ we have $r_{n+1} = r_n + n + 2$, the collection $\{J_{n,m} \mid m \in [0, n]\}$ is a partition of J_n .

With these notations we put

$$Q_{(a,q)} := \sum_{n \in \mathbb{N}} \sum_{m=0}^n q_{r_{n-1}+m} (\mathbf{1}_{I_{n,m}} + \mathbf{1}_{J_{n,m}}).$$

Note that $Q_{(a,q)}^{-1}(0) = [0, q_2]$, $Q_{(a,q)}(\mathbb{N}_0) = \{q_r \mid r \in \mathbb{N}_0\}$ and that for each $n \in \mathbb{N}$ we have

$$Q_{(a,q)}(I_n) = Q_{(a,q)}(J_n) = \{q_{r_{n-1}}, q_{r_{n-1}+1}, \dots, q_{r_{n-1}+n}\} \subset I_{n-1} \cup J_{n-1}.$$

Lemma 11. For $a := (\vec{a}_n)_{n \in \mathbb{N}}$, $q = (q_r)_{r \in \mathbb{N}_0}$ and $Q_{(a,q)}$ as above, the following properties hold.

1. For each $k \in \mathbb{N}_0$ we have $Q_{(a,q)}(k) \leq \max\{0, k - 2\}$.
2. The function $Q_{(a,q)}$ is a kneading map and for every $k \geq q_5 + 1$ we have

$$Q_{(a,q)}(k + 1) \geq Q_{(a,q)}(Q_{(a,q)}(Q_{(a,q)}(k) + 1) + 2. \tag{3.2}$$

Proof. Put $Q := Q_{(a,q)}$.

1. For $k \in [0, q_2]$ we have $Q(k) = 0$ so the inequality is satisfied in this case. Let $k \in \mathbb{N}$ be such that $k \geq q_2 + 1$, so there is $n \in \mathbb{N}$ such that $k \in I_n \cup J_n = [q_{r_n-1} + 1, q_{r_{n+1}-1]$. Thus $Q(k) \leq q_{r_{n-1}+n} = q_{r_n-1}$, so when $k \neq q_{r_n-1} + 1$ we have $Q(k) \leq k - 2$. Finally observe that

$$Q(q_{r_n-1} + 1) = q_{r_n-1} \leq q_{r_{n-1}+n} - n \leq q_{r_n-1} - 1.$$

2. Let $k \in \mathbb{N}$ be such that $k \geq q_5$, so there is $n \geq 2$ such that $k + 1 \in I_n \cup J_n$. Then $Q(k + 1) \geq q_{r_{n-1}}$, $Q(k) \leq q_{r_{n-1}+n}$ and therefore we have $Q(Q(k)) \leq q_{r_{n-2}+n-1}$ and

$$Q(Q(Q(k)) + 1) \leq q_{r_{n-2}} \leq q_{r_{n-1}} - (r_{n-1} - r_{n-2}) \leq q_{r_{n-1}} - 2 \leq Q(k + 1) - 2.$$

In view of part 1 and the previous inequality, to show that Q is admissible we just need to show that for each $k \in [1, q_5 - 1]$ and $j \in [1, q_2 + 1 - k]$ we have

$$Q(k + j) \geq Q(Q(Q(k)) + j), \tag{3.3}$$

with strict inequality when $j = q_2 + 1 - k$. In fact, for each $k \in [1, q_5 - 1]$ we have $Q(Q(k)) = 0$, so for each $j \in [1, q_2 + 1 - k]$, we have $Q(Q(Q(k)) + j) = 0$ and (3.3) is satisfied. When $j = q_2 + 1 - k$ we have $Q(k + j) = Q(q_2 + 1) = q_1 > 0$, so inequality (3.3) is strict in this case. \square

3.2. Reduced statement

The purpose of this section is to give a proof of the Main Theorem assuming the following one.

Theorem A. For each $n \in \mathbb{N}$ let $\vec{a}_n \in \mathbb{N}^{[0,n]}$ be given and put $a := (\vec{a}_n)_{n \in \mathbb{N}}$. Furthermore, let $q := (q_r)_{r \in \mathbb{N}_0}$ be a strictly increasing sequence of integers such that $q_0 = 0$ and such that for each $n \in \mathbb{N}$ we have

$$q_{r_n} - q_{r_n-1} = a_{n,0} + \dots + a_{n,n},$$

and let $Q_{(a,q)}$ be the corresponding kneading map. Define $(S_k)_{k \in \mathbb{N}_0}$ recursively by $S_0 = 1$ and $S_k = S_{k-1} + S_{Q_{(a,q)}(k)}$, and assume that

$$\sum_{r \in \mathbb{N} \setminus \{r_n | n \in \mathbb{N}\}} \frac{S_{q_{r-1}}}{S_{q_r}} < +\infty. \tag{3.4}$$

Moreover, for each $n \in \mathbb{N}$ let $\mathcal{E}_n : \mathbb{R}^{[0,n+1]} \rightarrow \mathbb{R}^{[0,n]}$ be the stochastic matrix such that for each $m \in [0, n]$ we have $\mathcal{E}_n(\vec{e}_m) = \vec{e}_m$ and such that

$$\mathcal{E}_n(0, n + 1) := \frac{S_{q_{r_n-1}}}{S_{q_{r_n}}} (1 + a_{n,n}) \vec{e}_0 + \sum_{m=1}^n \frac{S_{q_{r_{n-1}+n-m}}}{S_{q_{r_n}}} a_{n,n-m} \vec{e}_m.$$

Then for each unimodal map f whose kneading map is equal to $Q_{(a,q)}$, the post-critical set of f is a Cantor set, f is minimal on this set, and the space of invariant probability measures of f supported on this set is affine homeomorphic to $\varprojlim_n (\Delta_{[0,n+1]}, \mathcal{E}_n)$.

To prove the Main Theorem, we first remark that the case where the metrizable Choquet simplex \mathcal{C} is finite dimensional is given by [9, Main Theorem]. So from now on we assume that \mathcal{C} is infinite dimensional. Then by Lemma 8 there is a sequence of normalized affine maps $(A_n)_{n \in \mathbb{N}}$ such that $\varprojlim_n (\Delta_{[0,n+1]}, A_n)$ is affine homeomorphic to \mathcal{C} . In view of Lemma 9, we just need to find a and q as in the statement of Theorem A, for which (3.4) is satisfied

and such that

$$\sum_{n \in \mathbb{N}} \|\mathcal{E}_n(0, n + 1) - A_n(0, n + 1)\|_1 < +\infty.$$

This is shown in the following lemma, thus completing the proof of the Main Theorem.

Lemma 12. *For each $n \in \mathbb{N}$ let $\vec{y}_n \in \Delta_{[0,n]}$ be given. Then there are a and q as in the statement of Theorem A for which (3.4) is satisfied, and such that*

$$\sum_{n \in \mathbb{N}} \|\mathcal{E}_n(0, n + 1) - \vec{y}_n\|_1 < +\infty. \tag{3.5}$$

Proof. Given $n \in \mathbb{N}$ and a non-zero vector $\vec{a} \in \mathbb{R}^{[0,n]}$ with non-negative coordinates, we will denote by $[[\vec{a}]]$ the unique vector in $\Delta_{[0,n]}$ proportional to \vec{a} .

We will define $a := (a_n)_{n \in \mathbb{N}}$ and $q := (q_r)_{r \in \mathbb{N}_0}$ by induction as follows. Put $q_0 = 0$, fix $q_1 \geq 1$, and assume that for some $n \in \mathbb{N}$ the numbers $q_2, \dots, q_{r_{n-1}}$ and the vectors $\vec{a}_1, \dots, \vec{a}_{n-1}$ are already defined, in such a way that for each $m \in [0, n - 1]$ we have

$$q_{r_m} - q_{r_{m-1}} = a_{m,0} + \dots + a_{m,m}.$$

For each $r \in [r_{n-1} + 1, r_{n-1} + n]$ let q_r be defined in such a way that

$$q_r \geq q_{r-1} + r^2 \prod_{s=0}^{r-2} (1 + q_{s+1} + q_s).$$

Note that these choices determine $S_0, \dots, S_{q_{r_{n-1}+n}}$.

1. We will show now that for each $r \in [r_{n-1} + 1, r_{n-1} + n]$ we have

$$\frac{S_{q_{r-1}}}{S_{q_r}} \leq r^{-2}. \tag{3.6}$$

Using the recursion formula $S_l = S_{l-1} + S_{Q(l)}$ and $Q(l) \leq l - 1$, we get by induction that for every $k, k' \in [1, q_{r_{n-1}+n}]$ such that $k' < k$, we have $S_k \leq S_{k'}(1 + k - k')$. In particular for every $r \in [1, r_{n-1} + n]$ we have

$$S_{q_r} \leq S_{q_{r-1}}(1 + q_r - q_{r-1}).$$

Since $S_{q_0} = S_1 = S_0(1 + q_1 - q_0) = 1 + q_1 - q_0$, it follows by induction that for every $r \in [1, r_{n-1} + n]$ we have

$$S_{q_r} \leq \prod_{s=0}^{r-1} (1 + q_{s+1} - q_s).$$

Hence for each $r \in [r_{n-1} + 1, r_{n-1} + n]$ we have

$$S_{q_r} \geq S_{q_{r-1}}(1 + q_r - q_{r-1}) \geq q_r - q_{r-1} \geq r^2 \prod_{s=0}^{r-2} (1 + q_{s+1} + q_s) \geq r^2 S_{q_{r-1}},$$

as wanted.

2. We will show that we can choose $\vec{a}_n := (a_{n,0}, \dots, a_{n,n}) \in \mathbb{N}^{[0,n]}$ is such a way that

$$\left\| \left[S_{q_{r_{n-1}}} (1 + a_{n,n}) \vec{e}_0 + \sum_{m=1}^n S_{q_{r_{n-1}+n-m}} a_{n,n-m} \vec{e}_m \right] - \vec{y}_n \right\|_1 \leq n^{-2}. \tag{3.7}$$

For $x \in \mathbb{R}$ we denote by $[x]$ integer part of x . Put $\vec{y}_n = (y_0, \dots, y_n)$, $N = \prod_{j=0}^n S_{q_{r_{n-1}+j}}$, $k = (n + 2)^4$, for each $j \in [0, n]$ put

$$\zeta_j = N([ky_j] + 1), \quad \text{and put} \quad \zeta = \zeta_1 + \dots + \zeta_n.$$

Since for each $j \in [0, n]$ we have $Nky_j \leq \zeta_j \leq Nky_j + N$, we obtain $kN \leq \zeta \leq (k + n + 1)N$. So for each $j \in [0, 1]$ we have

$$\left| y_j - \frac{\zeta_j}{\zeta} \right| \leq \left| y_j - \frac{kNy_j}{\zeta} \right| + \frac{N}{\zeta} \leq \frac{(n+1)N}{\zeta} + \frac{N}{\zeta} \leq \frac{n+2}{k} = \frac{1}{(n+2)^3}.$$

This shows that the vector $\vec{\zeta} := (\zeta_0, \dots, \zeta_n) \in \mathbb{N}^{[0,n]}$ satisfies $\|[\vec{\zeta}] - \vec{y}_n\|_1 \leq n^{-2}$. Thus, if for each $j \in [0, n - 1]$ we put $a_{n,j} = \frac{\zeta_{n-j}}{S_{q_{r_{n-1}+j}}}$, and if we put $a_{n,n} := \frac{\zeta_0}{S_{q_{r_{n-1}}}} - 1$, then (3.7) is satisfied for this choice of \vec{a}_n . It remains to show that each of the coordinates of \vec{a}_n belongs to \mathbb{N} . By definition for each $j \in [0, n]$ the integer ζ_j is a strictly positive multiple of N , so the coordinates of \vec{a}_n are integers and for each $j \in [0, n - 1]$ we have $a_{n,j} \geq 1$. Finally observe that by (3.6) with $r = r_{n-1} + 1$ we have

$$a_{n,n} \geq \frac{N}{S_{q_{r_{n-1}}}} - 1 \geq S_{q_{r_{n-1}+1}} - 1 \geq (r_{n-1} + 1)^2 - 1 \geq 1.$$

3. Let \vec{a}_n be given by part 2 and put $q_{r_n} := q_{r_{n-1}+n} + a_{n,0} + \dots + a_{n,n}$. This completes the inductive definition of q and a .

To finish the proof of the lemma just observe that the inequalities (3.6) imply (3.4), and the inequalities (3.7) imply (3.5). \square

4. The generalized odometer and Bratteli–Vershik system associated to a kneading map

The purpose of this section is to recall the definition of the generalized odometer and the Bratteli–Vershik system associated to a kneading map, that were introduced in [4] and [7], respectively. We start recalling the definition of the generalized odometer in Section 4.1. After briefly recalling the concepts of Bratteli diagram (Section 4.2) and Bratteli–Vershik system (Section 4.3), we define the Bratteli–Vershik system associated to a kneading map in Section 4.4. See for example [2,16] for background on generalized odometers, and [10,21] and references therein for background and further properties of Bratteli–Vershik systems.

4.1. The generalized odometer associated to a kneading map

Let $Q : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ be a kneading map and put

$$\Omega_Q := \{(x_k)_{k \geq 0} \in \{0, 1\}^{\mathbb{N}_0} \mid x_k = 1 \text{ implies that for each } j = Q(k + 1), \dots, k - 1 \text{ we have } x_j = 0\}.$$

If we denote by $(S_k)_{k \geq 0}$ the strictly increasing sequence of positive integers defined recursively by $S_0 = 1$ and $S_k = S_{k-1} + S_{Q(k)}$, it can be shown that for each non-negative integer n there is a unique sequence $\langle n \rangle := (x_k)_{k \geq 0}$ in Ω_Q , that has at most finitely many 1’s, and such that $\sum_{k \geq 0} x_k S_k = n$. The sequence $\langle n \rangle$ is also characterized as the unique sequence in $\{0, 1\}^{\mathbb{N}_0}$ with finitely many 1’s such that $\sum_{k \geq 0} x_k S_k = n$, and that it is minimal with this property with respect to the lexicographical order in $\{0, 1\}^{\mathbb{N}_0}$.

When Q diverges to $+\infty$ the map defined on the subset $\{\langle n \rangle \mid n \in \mathbb{N}_0\}$ of Ω_Q by $\langle n \rangle \mapsto \langle n + 1 \rangle$, extends continuously to a map $T_Q : \Omega_Q \rightarrow \Omega_Q$ which is onto, minimal, and such that T_Q^{-1} is well defined on $\Omega_Q \setminus \langle 0 \rangle$; see [4, Lemma 2]. We call (Ω_Q, T_Q) the *generalized odometer*⁶ associated to Q . Given $x = (x_k)_{k \geq 0} \in \Omega_Q$ and an integer $n \geq 0$, put $\sigma(x|n) = \sum_{k=0}^n x_k S_k$. Observe that $\sigma(x|n)$ is non-decreasing with n , and when x has infinitely many 1’s, $\sigma(x|n) \rightarrow +\infty$ as $n \rightarrow +\infty$. On the other hand, if x has at most a finite number of 1’s, then $\sigma(x) := \lim_{n \rightarrow +\infty} \sigma(x|n)$ is finite and $x = \langle \sigma(x) \rangle$.

For $x = (x_k)_{k \geq 0}$ different from $\langle 0 \rangle$ we denote by $q(x) \geq 0$ the least integer such that $x_{q(x)} \neq 0$. In [4, Theorem 1] it is shown that if $\lambda \in (0, 4]$ is a parameter such that the kneading map of the logistic map f_λ is equal to Q , then for each $x \in \Omega_Q$ with infinitely many 1’s the sequence of intervals $(D_{\sigma(x|n)})_{n \geq q(x)}$ is nested and that $\bigcap_{n \geq q(x)} D_{\sigma(x|n)}$ is reduced to a point belonging to the post-critical set X_{f_λ} of f_λ . Furthermore, if we denote this point by $\pi(x)$ and for

⁶ An odometer is a topological dynamical system defined in a similar way, in the case where the sequence of integers $(S_k)_{k \in \mathbb{N}_0}$ is such that for each $k \in \mathbb{N}_0$ we have $S_k | S_{k+1}$.

$n \geq 0$ we put $\pi(\langle n \rangle) = f_\lambda^n(c)$, then the map $\pi : \Omega_Q \rightarrow X_{f_\lambda}$ so defined is continuous and conjugates the action of T_Q on Ω_Q , to the action of f_λ on X_{f_λ} .

4.2. Bratteli diagrams

A *Bratteli diagram* is an infinite directed graph (V, E) , such that the vertex set V and the edge set E can be partitioned into finite sets

$$V = V_0 \cup V_1 \cup \dots \quad \text{and} \quad E = E_1 \cup E_2 \cup \dots$$

with the following properties:

- $V_0 = \{v_0\}$ is a singleton.
- For every $j \geq 1$, each edge in E_j starts in a vertex in V_{j-1} and arrives to a vertex in V_j .
- All vertices in V have at least one edge starting from it, and all vertices except v_0 have at least one edge arriving to it.

For a vertex $e \in E$ we will denote by $s(e)$ the vertex where e starts and by $r(e)$ the vertex to which e arrives. A *path* in (V, E) is by definition a finite (resp. infinite) sequence $e_1 e_2 \dots e_j$ (resp. $e_1 e_2 \dots$) such that for each $\ell = 1, \dots, j-1$ (resp. $\ell = 1, \dots$) we have $r(e_\ell) = s(e_{\ell+1})$. Note that for each vertex v distinct from v_0 there is at least one path starting at v_0 and arriving to v .

An *ordered Bratteli diagram* (V, E, \geq) is a Bratteli diagram (V, E) together with a partial order \geq on E , so that two edges are comparable if and only if they arrive at the same vertex. For each $j \geq 1$ and $v \in V_j$ the partial order \geq induces an order on the set of paths from v_0 to V as follows:

$$e_1 \dots e_j > f_1 \dots f_j$$

if and only there exists $j_0 \in \{1, \dots, j\}$ such that $e_{j_0} > f_{j_0}$ and such that for each $\ell \in \{j_0 + 1, \dots, j\}$ we have $e_\ell = f_\ell$.

We will say that an edge e is *maximal* (resp. *minimal*) if it is maximal (resp. minimal) with respect to the order \geq on the set of all edges in E arriving at $r(e)$. Note that for each vertex v distinct from v_0 there is precisely one path starting at v_0 and arriving to v that is maximal (resp. minimal) with respect to the order \geq . It is characterized as the unique path starting at v_0 and arriving at v consisting of maximal (resp. minimal) edges.

4.3. Bratteli–Vershik system

Fix an ordered Bratteli diagram $B := (V, E, \geq)$. We denote by X_B the set of all infinite paths in B starting at v_0 . For a finite path $e_1 \dots e_j$ starting at v_0 we denote by $U(e_1 \dots e_j)$ the subset of X_B of all infinite paths $e'_1 e'_2 \dots$ such that for all $\ell = 1, \dots, j$ we have $e'_\ell = e_\ell$. We endow X_B with the topology generated by the sets $U(e_1 \dots e_j)$. Then each of this sets is clopen, so X_B becomes a compact Hausdorff space with a countable basis of clopen sets.

We will denote by X_B^{\max} (resp. X_B^{\min}) the set of all elements $(e_j)_{j \geq 1}$ of X_B so that for each $j \geq 1$ the edge e_j is a maximal (resp. minimal). It is easy to see that each of these sets is non-empty.

From now on we assume that the set X_B^{\min} is reduced to a unique point, that we will denote by x_{\min} . We will then define the transformation $V_B : X_B \rightarrow X_B$ as follows:

- $V_B^{-1}(x_{\min}) = X_{\max}$.
- Given $x \in X_B \setminus X_{\max}$, let $j \geq 1$ be the smallest integer such that e_j is not maximal. Then we denote by f_j the successor of e_j and by $f_1 \dots f_{j-1}$ the unique minimal path starting at v_0 and arriving to $s(f_j)$. Then we put

$$V_B(x) = f_1 \dots f_{j-1} f_j e_{j+1} e_{j+2} \dots$$

The map V_B is continuous, onto and invertible except at x_{\min} .

For $j \geq 1$ and $v \in V_j$ we denote by $s_j(v) > 0$ the number of paths starting at v_0 and arriving to v , and put $\vec{s}_j := (s_j(v))_{v \in V_j} \in \mathbb{R}^{V_j}$. Let $N_j \in \mathcal{M}_{V_{j-1}, V_j}$ be the matrix such that for each $v \in V_{j-1}$ and $v' \in V_j$ the entry $N_j(v, v')$ is

equal to the number of edges starting at v and arriving to v' . Observe that $N_j^t \vec{s}_{j-1} = \vec{s}_j$, so if we put $B_0 = \{1\} \in \mathcal{M}_{V_0, V_0}$ and for each $j \geq 1$ we denote by $B_j \in \mathcal{M}_{V_j, V_j}$ the diagonal matrix defined by $B_j(v, v) = s_j(v)$, then the matrix

$$M_j := B_{j-1} N_j B_j^{-1} \in \mathcal{M}_{V_{j-1}, V_j}$$

is stochastic.

The following result is well known, see [9, Lemma 14] for a proof in the precise setting considered here. Recall that for a finite set V we denote by Δ_V the unit simplex in \mathbb{R}^V .

Lemma 13. *The space of probability measures on X_B that are invariant by V_B , endowed with the weak* topology, is affine homeomorphic to $\varinjlim_j (\Delta_{V_j}, M_j)$.*

4.4. The Bratteli–Vershik system associated to a kneading map

Given a kneading map Q we will now define an ordered Bratteli diagram $B_Q := (V, E, \leq)$ that was introduced by Bruin in [7, §4].

We start defining the Bratteli diagram (V, E) :

- $V_0 = \{0\}$, $V_1 = \{k \in \mathbb{N} \mid Q(k) = 0\}$ and for $j \geq 2$,

$$V_j := \{k \in \mathbb{N} \mid k \geq j, Q(k-1) \leq j-2\}.$$

- For $j \geq 1$,

$$E_j = \{j-1 \rightarrow j\} \cup \{j-1 \rightarrow k \mid k \in V_j \setminus V_{j-1}\} \cup \{k \rightarrow k \mid k \in V_j \cap V_{j-1}\}.$$

Note that for every $j \geq 2$, each vertex in V_j different from j has at most one edge arriving at it. Besides $\{j-1 \rightarrow j\} \in E_j$, the only edge that can arrive to $j \in V_j$ is $\{j \rightarrow j\} \in E_j$, that only exists when $j \in V_{j-1}$.

So to define the partial order \geq , we just have to define it, for each $j \geq 2$, between $\{j-1 \rightarrow j\} \in E_{j-1}$ and $\{j \rightarrow j\} \in E_{j-1}$ when both exist: we put $\{j-1 \rightarrow j\} < \{j \rightarrow j\}$. The rest of the edges are maximal and minimal at the same time.

Note that for $k \geq 1$ the set V_k is reduced to a point if and only if $Q(k) = k-1$. So, if for each large $k \geq 1$ we have $Q(k) = k-1$, then the set X_{B_Q} is finite. Otherwise, it follows that the set X_{B_Q} is a Cantor set.

It is straight forward to check that the infinite path $0 \rightarrow 1 \rightarrow 2 \rightarrow \dots$ is the unique minimal path in B_Q . Therefore there is a well-defined map $V_{B_Q} : X_{B_Q} \rightarrow X_{B_Q}$, see Section 4.3. The following is [7, Proposition 2], and the last statement follows from [4, Lemma 2].

Theorem 14. *(See [7, Proposition 2].) Let Q be a kneading map that diverges to $+\infty$, and consider the corresponding Bratteli–Vershik system (X_{B_Q}, V_{B_Q}) and generalized odometer (Ω_Q, T_Q) . Then there is a homeomorphism between X_{B_Q} and Ω_Q that conjugates the action of V_{B_Q} on X_{B_Q} to the action of T_Q on Ω_Q . In particular (X_{B_Q}, V_{B_Q}) is minimal.*

We will also need the following lemma.

Lemma 15. *Let Q be a kneading map such that for every $k \in \mathbb{N}_0$ we have $Q(k) \leq \max\{0, k-2\}$, and such that $Q(k) \rightarrow +\infty$ as $k \rightarrow +\infty$. Let $(S_k)_{k \geq 1}$ be the sequence defined recursively by $S_0 = 1$ and $S_k = S_{k-1} + S_{Q(k)}$. Then for every $j \in \mathbb{N}$ we have $j+1 \in V_j$, $s_j(j) = S_{j-1}$, and for every $k \in V_j \setminus \{j\}$ we have $s_j(k) = S_{Q(k-1)}$.*

Proof. That $j+1$ is a direct consequence of the definition of V_j and the hypothesis that $Q(j) \leq \max\{0, j-2\}$. When $j=1$, we have $Q(1) = 0$ and for all $k \in V_1$ we have $s_1(k) = S_0 = 1$. So the assertions are satisfied in this case. Suppose by induction that the assertions of the lemma hold for some $j \geq 1$. Then we have

$$s_{j+1}(j+1) = s_j(j) + s_j(j+1) = S_{j-1} + S_{Q(j)} = S_j.$$

On the other hand, for each $k \in V_{j+1} \setminus \{j+1\}$ contained in V_j we have $s_{j-1}(k) = s_j(k) = S_{Q(k-1)}$. Finally, for each $k \in V_{j+1} \setminus \{j+1\}$ not in V_j we have $Q(k-1) = j-1$ and

$$s_{j+1}(k) = s_j(j) = S_{j-1} = S_{Q(k-1)}. \quad \square$$

5. Computing the space of invariant measures

This section is devoted to the proof of Theorem A. We start by showing that for each unimodal map whose kneading map is as in Section 3.1, the space of invariant probability measures supported on its post-critical set is affine homeomorphic to the space of invariant probability measures of the corresponding generalized odometer. In order to describe this space we calculate the transition matrices associated to the corresponding Bratteli–Vershik system. The key calculation of a suitable product of these transition matrices is stated as Proposition 18 in Section 5.2. The proof of Theorem A is given in Section 5.3.

5.1. From the generalized odometer to the post-critical set

The purpose of this section is to prove the following proposition, whose proof is similar to that of [9, Theorem B].

Proposition 16. *Let $Q_{(a,q)}$ be a kneading map defined as in Section 3.1, and let $(\Omega_{Q_{(a,q)}}, T_{Q_{(a,q)}})$ be the corresponding generalized odometer. Let f be a unimodal map whose kneading map is equal to $Q_{(a,q)}$, and denote by X_f its post-critical set. Then the space of invariant probability measures of $(X_f, f|_{X_f})$ is affine homeomorphic to that of $(\Omega_{Q_{(a,q)}}, T_{Q_{(a,q)}})$.*

The following lemma is similar to [9, Lemma 11]. Observe that, since T_Q^{-1} is well defined on $\Omega_Q \setminus \{\langle 0 \rangle\}$, if we denote by $\mathcal{O}(\langle 0 \rangle)$ the grand orbit of $\langle 0 \rangle$, then

$$T^{-1}(\Omega_Q \setminus \mathcal{O}(\langle 0 \rangle)) = \Omega_Q \setminus \mathcal{O}(\langle 0 \rangle),$$

and all negative iterates of T_Q are well defined on $\Omega_Q \setminus \mathcal{O}(\langle 0 \rangle)$.

Lemma 17. *Let $Q = Q_{(a,q)}$ be a kneading map as in Section 3.1. Let (Ω_Q, T_Q) be the corresponding generalized odometer. Then for each constant $K > 0$, and for every pair of distinct points x, x' in Ω_Q that are not in the grand orbit of $\langle 0 \rangle$, there is an integer m satisfying*

$$\max\{q(T_Q^m(x)), q(T_Q^m(x'))\} \geq K \quad \text{and} \quad Q(q(T_Q^m(x)) + 1) \neq Q(q(T_Q^m(x')) + 1).$$

Proof. Let $K \geq q_2$.

1. As in the proof of [9, Lemma 10], it can be shown that there is an integer m' such that

$$\max\{q(T_Q^{m'}(x)), q(T_Q^{m'}(x'))\} \geq K \quad \text{and} \quad q(T_Q^{m'}(x)) \neq q(T_Q^{m'}(x')).$$

2. Let m' be the integer given by part 1, and put $y = (y_k)_{k \in \mathbb{N}_0} := T_Q^{m'}(x)$ and $y' = (y'_k)_{k \in \mathbb{N}_0} := T_Q^{m'}(x')$. Assume without loss of generality that $q(y) < q(y')$, so that $q(y') \geq K \geq q_2$. If $Q(q(y) + 1) \neq Q(q(y') + 1)$ then take $m = m'$. So we assume that $Q(q(y) + 1) = Q(q(y') + 1)$. Since $q(y) \geq q_2$ we have $Q(q(y) + 1) = Q(q(y') + 1) \geq 1$, so there is $n \in \mathbb{N}$ such that $q(y) + 1$ and $q(y') + 1$ belong to $I_n \cup J_n$. The definition of Q and of Ω_Q imply that for each $k \in [q(y) + 1, q_{r_{n+1}-1} - 1]$ we have $y_k = 0$. Indeed, suppose by contradiction that for such a k we have $y_k = 1$. Then the definition of Ω_Q implies that for every $j \in [Q(k+1), k-1]$ we have $y_j = 0$. Since $k+1 \in I_n \cup J_n$, we get $Q(k+1) \leq q_{r_{n+1}-1} \leq q(y)$, which is a contradiction.

Therefore $(\hat{y}_k)_{k \geq 0} := T_Q^{-S_{q(y)}}(y)$ is such that for all $k \in [0, q_{r_{n+1}-1} - 1]$ we have $\hat{y}_k = 0$. Since y is not in the grand orbit of $\langle 0 \rangle$ this implies that $q(T_Q^{-S_{q(y)}}(y)) \geq q_{r_{n+1}-1} \geq q(y') \geq K$. Thus, from the definition of Q we have

$$Q(q(T_Q^{-S_{q(y)}}(y)) + 1) \geq q_{r_n}.$$

On the other hand, since $q(y) < q(y')$, we have $q(T_Q^{-S_{q(y)}}(y')) \leq q(y') - 1 \leq q_{r_{n+1}-1} - 2$, so

$$Q(q(T_Q^{-S_{q(y)}}(y')) + 1) \leq q_{r_{n-1}}.$$

This shows that the integer $m = m' - S_{q(y)}$ satisfies the desired properties. \square

Proof of Proposition 16. Since the logistic family is full there is a parameter $\lambda \in (0, 4]$ such that the kneading map of the logistic map f_λ is $Q_{(a,q)}$. Denote by X_{f_λ} the post-critical set of f_λ . By Proposition 10 the spaces of invariant measures of (f, X_f) and $(f_\lambda, X_{f_\lambda})$ are affine homeomorphic. So, without loss of generality we assume that f is a logistic map. This ensures the existence of the factor map $\pi : \Omega_{Q_{(a,q)}} \rightarrow X_f$ defined above. Since for every sufficiently large integer k inequality (3.2) is satisfied, [9, Lemma 11] implies there is a constant $K > 0$ such that for every pair of distinct points x, x' in $\Omega_{Q_{(a,q)}}$ that are not in the grand orbit of $\langle 0 \rangle$ and that satisfy

$$\max\{q(x), q(x')\} \geq K \quad \text{and} \quad Q_{(a,q)}(q(x) + 1) \neq Q_{(a,q)}(q(x') + 1),$$

we have $\pi(x) \neq \pi(x')$. Thus, from Lemma 17 we deduce that π is injective on $\Omega_{Q_{(a,q)}} \setminus \mathcal{O}(\langle 0 \rangle)$. The rest of the proof follows as the proof of [9, Proposition 9]. \square

5.2. Transition matrices

For each $n \in \mathbb{N}$ let $\vec{a}_n \in \mathbb{N}^{[0,n]}$ be given, and put $a := (\vec{a}_n)_{n \in \mathbb{N}}$. Furthermore, let $q := (q_r)_{r \in \mathbb{N}_0}$ be a strictly increasing sequence of integers such that $q_0 = 0$ and such that for each $n \in \mathbb{N}$ we have

$$q_{r_n} - q_{r_n-1} = a_{n,0} + \dots + a_{n,n},$$

and let $Q := Q_{(a,q)}$ be the corresponding kneading map defined in Section 3.1.

Let $B_Q = (V, E, \leq)$ be the ordered Bratteli–Vershik diagram associated to the kneading map Q . From the definition of B_Q it follows that $V_1 = Q^{-1}(0) \setminus \{0\} = [1, q_2]$, and that for each $j \in [2, q_1 + 1]$ we have $V_j = [j, q_2 + 1]$. Furthermore, for each $n \in \mathbb{N}_0, m \in [0, n]$, and $j \in [q_{r_n+m} + 2, q_{r_{n+m+1}} + 1]$ we have

$$V_j = [j, q_{r_{n+1}-1} + 1] \cup \left(1 + \bigcup_{i=0}^m (I_{n+1,i} \cup J_{n+1,i}) \right), \tag{5.1}$$

and that when $m = n + 1$ we have $r_n + n + 1 = r_{n+1} - 1$, and for $j \in [q_{r_{n+1}-1} + 2, q_{r_{n+1}} + 1]$ we have

$$V_j = [j, q_{r_{n+2}-1} + 1]. \tag{5.2}$$

Note in particular that for every $n \in \mathbb{N}$ we have

$$V_{q_{r_{n-1}+1}} = [q_{r_{n-1}} + 1, q_{r_{n+1}-1} + 1] \setminus (1 + (I_{n,n} \cup J_{n,n})), \tag{5.3}$$

$$V_{q_{r_n+1}} = [q_{r_n} + 1, q_{r_{n+1}-1} + 1]. \tag{5.4}$$

Proposition 18. Given $n \in \mathbb{N}$, for each $k \in [q_{r_n}, q_{r_{n+1}-1}]$ put

$$\vec{v}(k) := \frac{S_{q_{r_n}}}{S_k} \vec{e}_{q_{r_n}+1} + \sum_{i=q_{r_n}+1}^k \frac{S_{Q(i)}}{S_k} \vec{e}_{i+1} \in \mathbb{R}^{V_{q_{r_n}+1}}.$$

Then the columns of the matrix

$$M_{q_{r_n}+2} \cdots M_{q_{r_{n+1}+1}} \in \mathcal{M}([q_{r_n} + 1, q_{r_{n+1}-1} + 1], [q_{r_{n+1}} + 1, q_{r_{n+2}-1} + 1]),$$

are given by

$$M_{q_{r_n}+2} \cdots M_{q_{r_{n+1}+1}}(\cdot, q_{r_{n+1}} + 1) = \frac{S_{q_{r_{n+1}-1}}}{S_{q_{r_{n+1}}}} \vec{v}(q_{r_{n+1}-1}) + \sum_{m=0}^{n+1} \frac{S_{q_{r_n+m}}}{S_{q_{r_{n+1}}}} a_{n+1,m} \vec{v}(q_{r_n+m}),$$

and for every $m \in [0, n + 1]$ and $\ell \in V_{q_{r_n+1}+1}$ such that $\ell - 1 \in J_{n+1,m}$, by

$$M_{q_{r_n+2}} \cdots M_{q_{r_n+1}+1}(\cdot, \ell) = \vec{v}(q_{r_n+m}).$$

In particular the rank of the matrix $M_{q_{r_n+2}} \cdots M_{q_{r_n+1}+1}$ is equal to $n + 2$.

The proof of this proposition depends on the following lemma.

Lemma 19. Fix $n \in \mathbb{N}$ and let $m \in [0, n + 1]$. If $m \in [0, n]$ then the set $V_{q_{r_n+m+1}+1}$ is equal to the disjoint union of $V_{q_{r_n+m+1}} \setminus [q_{r_n+m} + 1, q_{r_n+m+1}]$ and of $1 + Q^{-1}(q_{r_n+m})$, and we have

$$N_{q_{r_n+m+2}} \cdots N_{q_{r_n+m+1}+1}(\cdot, \ell) = \begin{cases} \sum_{k=q_{r_n+m}}^{q_{r_n+m+1}} \vec{e}_{k+1} & \text{if } \ell = q_{r_n+m+1} + 1; \\ \vec{e}_\ell & \text{if } \ell \in V_{q_{r_n+m+1}} \setminus [q_{r_n+m} + 1, q_{r_n+m+1} + 1]; \\ \vec{e}_{q_{r_n+m+1}} & \text{if } \ell - 1 \in Q^{-1}(q_{r_n+m}). \end{cases} \tag{5.5}$$

When $m = n + 1$ we have $r_n + m = r_{n+1} - 1$, $V_{q_{r_{n+1}+1}} = [q_{r_{n+1}} + 1, q_{r_{n+1}-1} + 1]$, and

$$N_{q_{r_{n+1}-1+2}} \cdots N_{q_{r_{n+1}+1}+1}(\cdot, \ell) = \begin{cases} a_{n+1,n+1} \vec{e}_{q_{r_{n+1}-1}+1} + \sum_{k=q_{r_{n+1}-1}}^{q_{r_{n+1}}-a_{n+1,n+1}} \vec{e}_{k+1} & \text{if } \ell = q_{r_{n+1}} + 1; \\ \vec{e}_\ell & \text{if } \ell - 1 \in J_{n+1} \setminus J_{n+1,n+1}; \\ \vec{e}_{q_{r_{n+1}-1}+1} & \text{if } \ell - 1 \in J_{n+1,n+1}. \end{cases} \tag{5.6}$$

Proof. It follows from the definition of B_Q that for each $r \in \mathbb{N}$ the set V_{q_r+2} is equal to the disjoint union of $V_{q_r+1} \setminus \{q_r + 1\}$ and $1 + Q^{-1}(q_r)$, and that

$$N_{q_r+2}(\cdot, \ell) = \begin{cases} \vec{e}_{q_r+1} + \vec{e}_{q_r+2} & \text{if } \ell = q_r + 2; \\ \vec{e}_\ell & \text{if } \ell \in V_{q_r+1} \setminus \{q_r + 1, q_r + 2\}; \\ \vec{e}_{q_r+1} & \text{if } \ell - 1 \in Q^{-1}(q_r). \end{cases}$$

On the other hand, for each $j \in [q_r + 3, q_{r+1} + 1]$ we have $V_{j-1} = V_j \cup \{j - 1\}$ and

$$N_j(\cdot, \ell) = \begin{cases} \vec{e}_{j-1} + \vec{e}_j & \text{if } \ell = j; \\ \vec{e}_\ell & \text{if } \ell \in V_j \setminus \{j\}. \end{cases}$$

A direct computation using the fact that for every $j \in [q_r + 2, q_{r+1}]$ the set V_j is the disjoint union of $V_{q_{r+1}+1}$ and $[j, q_r+1]$, shows that

$$N_{q_r+2} \cdots N_{q_{r+1}+1}(\cdot, \ell) = \begin{cases} \sum_{j=q_r+2}^{q_{r+1}+1} N_{q_r+2}(\cdot, j) & \text{if } \ell = q_{r+1} + 1; \\ N_{q_r+2}(\cdot, \ell) & \text{if } \ell \in V_{q_r+2} \setminus [q_r + 2, q_{r+1} + 1]. \end{cases} \tag{5.7}$$

Fix $n \in \mathbb{N}$ and let $m \in [0, n + 1]$.

When $m \in [0, n]$ the assertion of the lemma about $V_{q_{r_n+m+1}+1}$ follows from (5.1). On the other hand, by (5.2) if $m = 0$ and by (5.1) if $m \in [1, n]$, we have

$$[q_{r_n+m} + 2, q_{r_n+m+1} + 1] \subset V_{q_{r_n+m+1}}.$$

Then in this case the assertion of the lemma follows easily from (5.7) with $r = r_n + m$.

Suppose now that $m = n + 1$. By the definition of the sequence $(r_{n'})_{n' \in \mathbb{N}_0}$ we have $r_n + n + 1 = r_{n+1} - 1$. The assertion about $V_{q_{r_{n+1}+1}}$ is given by (5.4) with n replaced by $n + 1$. Since by (5.3) we have

$$[q_{r_{n+1}-1} + 2, q_{r_{n+1}} + 1] \cap V_{q_{r_{n+1}-1}+1} = 1 + (I_{n+1} \setminus I_{n+1,n+1}) = [q_{r_{n+1}-1} + 2, q_{r_{n+1}} + 1 - a_{n+1,n+1}]$$

and since $I_{n+1,n+1} \subset Q^{-1}(q_{r_{n+1}-1})$, we conclude from (5.7) with $r = r_{n+1} - 1$, that

$$N_{q_{r_{n+1}-1+2}} \cdots N_{q_{r_{n+1}+1}+1}(\cdot, q_{r_{n+1}} + 1) = a_{n+1,n+1} \vec{e}_{q_{r_{n+1}-1}+1} + \sum_{k=q_{r_{n+1}-1}}^{q_{r_{n+1}}-a_{n+1,n+1}} \vec{e}_{k+1}.$$

On the other hand, for

$$\ell \in [q_{r_{n+1}} + 2, q_{r_{n+2}-1} + 1] \cap V_{q_{r_{n+1}-1}+1} = 1 + (J_{n+1} \setminus J_{n+1,n+1}),$$

we have

$$N_{q_{r_{n+1}-1}+2} \cdots N_{q_{r_{n+1}}+1}(\cdot, \ell) = \vec{e}_\ell,$$

and that for

$$\ell \in [q_{r_{n+1}} + 2, q_{r_{n+2}-1} + 1] \cap (1 + Q^{-1}(q_{r_{n+1}-1})) = 1 + J_{n+1,n+1},$$

we have

$$N_{q_{r_{n+1}-1}+2} \cdots N_{q_{r_{n+1}}+1}(\cdot, \ell) = \vec{e}_{q_{r_{n+1}-1}+1}.$$

This completes the proof of the lemma. \square

Proof of Proposition 18. Fix $n \in \mathbb{N}$.

1. We will show by induction that for each $m_0 \in [1, n + 1]$ we have

$$N_{q_{r_n}+2} \cdots N_{q_{r_n+m_0}+1}(\cdot, \ell) = \begin{cases} \sum_{k=q_{r_n}}^{q_{r_n+m_0}} \vec{e}_{k+1} & \text{if } \ell = q_{r_n+m_0} + 1; \\ \vec{e}_\ell & \text{if } \ell \in [q_{r_n+m_0} + 2, q_{r_{n+1}-1} + 1]; \\ \sum_{k=q_{r_n}}^{q_{r_n+m}} \vec{e}_{k+1} & \text{if } \ell - 1 \in I_{n+1,m} \cup J_{n+1,m} \text{ and } m \in [0, m_0 - 1]. \end{cases} \quad (5.8)$$

The case $m_0 = 1$ is given by (5.5) with $m = 0$. Suppose that this holds for some $m_0 \in [1, n]$. Observe that by (5.1) the set $V_{q_{r_n+m_0}+1}$ is the disjoint union of

$$V_{q_{r_n+m_0}+1} \setminus [q_{r_n+m_0} + 1, q_{r_n+m_0} + 1]$$

and

$$1 + Q^{-1}(q_{r_n+m_0}+1) = 1 + I_{n+1,m_0} \cup J_{n+1,m_0}.$$

In view of (5.5) with $m = m_0$ we obtain

$$N_{q_{r_n}+2} \cdots N_{q_{r_n+m_0}+1}(\cdot, q_{r_n+m_0} + 1) = \sum_{k=q_{r_n+m_0}}^{q_{r_n+m_0}+1} N_{q_{r_n}+2} \cdots N_{q_{r_n+m_0}+1}(\cdot, k + 1) = \sum_{k=q_{r_n}}^{q_{r_n+m_0}+1} \vec{e}_{k+1},$$

for each $\ell \in V_{q_{r_n+m_0}+1} \setminus [q_{r_n+m_0} + 1, q_{r_n+m_0} + 1]$ we have

$$N_{q_{r_n}+2} \cdots N_{q_{r_n+m_0}+1}(\cdot, \ell) = N_{q_{r_n}+2} \cdots N_{q_{r_n+m_0}+1}(\cdot, \ell) = \begin{cases} \vec{e}_\ell & \text{if } \ell \in [q_{r_n+m_0}+2, q_{r_{n+1}-1} + 1]; \\ \sum_{k=q_{r_n}}^{q_{r_n+m}} \vec{e}_{k+1} & \text{if } \ell - 1 \in I_{n+1,m} \cup J_{n+1,m} \text{ and } m \in [0, m_0 - 1]. \end{cases}$$

Finally, for each $\ell \in V_{q_{r_n+m_0}+1}$ such that $\ell - 1 \in I_{n+1,m_0} \cup J_{n+1,m_0}$ we have

$$N_{q_{r_n}+2} \cdots N_{q_{r_n+m_0}+1}(\cdot, \ell) = N_{q_{r_n}+2} \cdots N_{q_{r_n+m_0}}(\cdot, q_{r_n+m_0} + 1) = \sum_{k=q_{r_n}}^{q_{r_n+m_0}} \vec{e}_{k+1}.$$

This completes the proof of the induction step.

2. A direct computation using (5.8) with $m_0 = n$ and (5.6), gives

$$N_{q_{r_n}+2} \cdots N_{q_{r_{n+1}}+1}(\cdot, q_{r_{n+1}} + 1) = \sum_{k=q_{r_n}}^{q_{r_{n+1}}-1} \vec{e}_{k+1} + \sum_{m=0}^{n+1} a_{n+1,m} \sum_{k=q_{r_n}}^{q_{r_n+m}} \vec{e}_{k+1},$$

and for every $m \in [0, n + 1]$ and $\ell \in V_{q_{r_{n+1}+1}}$ such that $\ell - 1 \in J_{n+1,m}$,

$$N_{q_{r_n+2}} \cdots N_{q_{r_{n+1}+1}}(\cdot, \ell) = \sum_{k=q_{r_n}}^{q_{r_n+m}} \vec{e}_{k+1}.$$

The assertion of the proposition is then a direct consequence of the definition of the matrices M_j and Lemma 15. \square

5.3. Proof of Theorem A

Let f be a unimodal having Q as kneading map. That the post-critical set of f is a Cantor set and that f is minimal on this set is given by Proposition 10. In view of Proposition 16 and Theorem 14, it is enough to prove that the space of invariant probability measures of the Bratteli–Vershik system (X_{B_Q}, V_{B_Q}) is affine homeomorphic to $\varprojlim_r (\Delta_{[0,n+1]}, \mathcal{E}_n)$.

For each $n \in \mathbb{N}$ let $\Pi_n : \mathbb{R}^{V_{q_{r_n+1}}} \rightarrow \mathbb{R}^{[0,n+1]}$ be the stochastic matrix defined by

$$\Pi_n(x_{q_{r_n+1}}, \dots, x_{q_{r_{n+1}-1+1}}) = \left(\left(\sum_{k=q_{r_n+n+2}}^{q_{r_n+n+1+1}} x_k \right), \dots, \left(\sum_{k=q_{r_n+2}}^{q_{r_n+1+1}} x_k \right), x_{q_{r_n+1}} \right).$$

Using the definition of $\vec{v}(k)$ in the statement of Proposition 18, for each $m_0 \in [0, n + 1]$ we put

$$\vec{w}_n(m_0) := \Pi_n(\vec{v}(q_{r_n+n+1-m_0})) = \sum_{m=m_0}^n \frac{S_{q_{r_n+n+1-m}} - S_{q_{r_n+n-m}}}{S_{q_{r_n+n+1-m_0}}} \vec{e}_m + \frac{S_{q_{r_n}}}{S_{q_{r_n+n+1-m_0}}} \vec{e}_{n+1}.$$

Furthermore, when $n \geq 2$, we denote by $A_n : \mathbb{R}^{[0,n+1]} \rightarrow \mathbb{R}^{[0,n]}$ the stochastic matrix defined for $m \in [0, n]$ by $A_n(\cdot, m) = \vec{w}_{n-1}(m)$ and by

$$A_n(\cdot, n + 1) = \frac{S_{q_{r_n-1}}}{S_{q_{r_n}}} \vec{w}_{n-1}(0) + \sum_{m=0}^n \frac{S_{q_{r_n-1+n-m}}}{S_{q_{r_n}}} a_{n,n-m} \vec{w}_{n-1}(m).$$

A direct computation shows that for each $n \geq 2$ we have

$$\Pi_{n-1} M_{q_{r_{n-1}+2}} \cdots M_{q_{r_n+1}} = A_n \Pi_n.$$

Therefore the sequence of maps $(\Pi_n)_{n \geq 1}$ defines a continuous linear map

$$\Pi : \varprojlim_j (\mathbb{R}^{V_j}, M_j) \rightarrow \varprojlim_n (\mathbb{R}^{[0,n+1]}, A_n),$$

mapping $\varprojlim_j (\mathbb{R}^{V_j}, M_j)$ onto $\varprojlim_n (\Delta_{[0,n+1]}, A_n)$. By Proposition 18 the rank of the matrix $M_{q_{r_{n-1}+2}} \cdots M_{q_{r_n+1}}$ is equal to $n + 1$, so Π is a homeomorphism and the inverse limits $\varprojlim_j (\mathbb{R}^{V_j}, M_j)$ and $\varprojlim_n (\Delta_{[0,n+1]}, A_n)$ are affine homeomorphic.

In view of Lemma 9, the following lemma together with the hypothesis

$$\sum_{r \in \mathbb{N} \setminus \{r_n \mid n \in \mathbb{N}\}} \frac{S_{q_{r-1}}}{S_{q_r}} < +\infty$$

imply that the inverse limit $\varprojlim_n (\Delta_{[0,n+1]}, A_n)$ is affine homeomorphic to $\varprojlim_n (\Delta_{[0,n+1]}, \mathcal{E}_n)$. This completes the proof of Theorem A.

Lemma 20. For each $n \in \mathbb{N}$ and $m_0 \in [0, n - 1]$ we have

$$\|A_n(\cdot, m_0) - \mathcal{E}_n(\cdot, m_0)\|_1 = 2 \frac{S_{q_{r_{n-1}+n-m_0-1}}}{S_{q_{r_{n-1}+n-m_0}}}.$$

Furthermore $A_n(\cdot, n) = \mathcal{E}_n(\cdot, n)$, and

$$\|A_n(\cdot, n + 1) - \mathcal{E}_n(\cdot, n + 1)\|_1 \leq 2 \sum_{m=1}^n \frac{S_{q_{r_{n-1}+m-1}}}{S_{q_{r_{n-1}+m}}}.$$

Proof. By definition we have $A_n(\cdot, n) = \vec{w}_{n-1}(n) = \vec{e}_n = \mathcal{E}_n(\cdot, n)$, and for each $m_0 \in [0, n - 1]$ we have

$$\begin{aligned} \|A_n(\cdot, m_0) - \mathcal{E}_n(\cdot, m_0)\|_1 &= \|\vec{w}_{n-1}(m_0) - \vec{e}_{m_0}\|_1 \\ &= \left| \frac{S_{q_{r_{n-1}+n-m_0}} - S_{q_{r_{n-1}+n-m_0-1}}}{S_{q_{r_{n-1}+n-m_0}}} - 1 \right| \\ &\quad + \sum_{m=m_0+1}^{n-1} \frac{S_{q_{r_{n-1}+n-m}} - S_{q_{r_{n-1}+n-m-1}}}{S_{q_{r_{n-1}+n-m}}} + \frac{S_{q_{r_{n-1}}}}{S_{q_{r_{n-1}+n-m_0}}} \\ &= 2 \frac{S_{q_{r_{n-1}+n-m_0-1}}}{S_{q_{r_{n-1}+n-m_0}}}. \end{aligned} \tag{5.9}$$

On the other hand,

$$\|A_n(\cdot, n + 1) - \mathcal{E}_n(\cdot, n + 1)\|_1 \leq \sum_{m_0=1}^n \|\vec{w}_{n-1}(m_0) - \vec{e}_{m_0}\|_1,$$

so the final assertion follows from (5.9). \square

5.4. Example

Given $\beta \in \mathbb{R} \setminus \mathbb{Q}$ put

$$G(\beta) := \mathbb{Z} + \beta\mathbb{Z} \quad \text{and} \quad G_+(\beta) = \{m + \beta n \geq 0 \mid m, n \in \mathbb{Z}\}.$$

For each such β we will construct a kneading map Q such that the dimension group associated to the generalized odometer (Ω_Q, T_Q) , and hence to its natural extension, is isomorphic to $(G(\beta), G_+(\beta), 1)$. Thus we deduce that every simple dimension group which is free of rank 2 is isomorphic as ordered group to the dimension group associated to (the natural extension of) a generalized odometer associated to a kneading map. Since the rational subdimension group of $(G(\beta), G_+(\beta), 1)$ is $(\mathbb{Z}, \mathbb{N}_0, 1)$, it follows from [14, §4.1] that the dimension group $(G(\beta), G_+(\beta), 1)$ is not isomorphic to the dimension group associated to a Toeplitz flow, nor to that of an odometer.

Before defining the kneading map Q , note that the dimension groups $(G(\beta), G_+(\beta), 1)$, $(G(\beta + 1), G_+(\beta + 1), 1)$, and $(G(1 - \beta), G_+(1 - \beta), 1)$ are isomorphic to each other. So we can restrict to the case where $\beta \in (0, \frac{1}{2})$. Let $k \geq 2$ be the integer determined by $\beta \in (\frac{1}{k+1}, \frac{1}{k})$, and let $[0, a_1, a_2, a_3, \dots]$ be the continued fraction expansion of $\alpha := \frac{1}{\beta} - k \in [0, 1] \setminus \mathbb{Q}$. Consider the function $Q : \mathbb{N}_0 \rightarrow \mathbb{N}_0$ defined by

$$Q(l) = \begin{cases} 0 & \text{if } l \in [0, k]; \\ k - 1 & \text{if } l \in [k + 1, k + a_1]; \\ k - 1 + \sum_{i=1}^n a_i & \text{if } l \in [k + 1 + \sum_{i=1}^n a_i, k + \sum_{i=1}^{n+1} a_i]. \end{cases}$$

It is non-decreasing and such that for every $l \geq 1$ we have $Q(l) \leq l - 1$. So Q is a kneading map. If $(M_j)_{j \in \mathbb{N}}$ is the corresponding sequence of transition matrices, then it is easy to see that for each $n \in \mathbb{N}$ we have

$$A_n := M_{k+1+a_1+\dots+a_{n-1}} \cdots M_{k+a_1+\dots+a_n} = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}.$$

By considering a Bratteli diagram isomorphic to B_Q , that only differs with it in the first $k + 1$ levels, we obtain

$$A_0 := M_2 \cdots M_k = \begin{pmatrix} k - 1 & 1 \\ 1 & 0 \end{pmatrix},$$

so the dimension group associated to (Ω_Q, T_Q) is isomorphic to direct limit

$$\mathbb{Z} \xrightarrow{(1,1)^T} \mathbb{Z}^2 \xrightarrow{A_0} \mathbb{Z}^2 \xrightarrow{A_1} \mathbb{Z}^2 \xrightarrow{A_2} \dots,$$

see for example [18, Theorem 3.7], which by [13, Theorem 4.8] is isomorphic to $(G(\beta), G_+(\beta), 1)$.

Acknowledgements

We are grateful to Christian Skau for his help with dimension groups of Toeplitz flows, and an anonymous referee for his/her suggestions to improve the presentation of the paper.

Appendix A. Measures of zero Lyapunov exponent of complex maps

The purpose of this appendix is to prove Corollary 3. As the parameters given by (the proof of) the Main Theorem are such that the kneading map of corresponding logistic map diverges to $+\infty$, this result is a direct consequence of the following lemma.

Lemma 21. *Let $\lambda \in (0, 4]$ be a parameter such that the kneading map of f_λ diverges to $+\infty$. Consider the corresponding quadratic polynomial P_λ , and denote by t_0 the Hausdorff dimension of the Julia set of P_λ . Then, for an invariant probability measure of P_λ that is supported on the Julia set of P_λ , the following properties are equivalent.*

1. *It is supported on the post-critical set of f_λ .*
2. *Its Lyapunov exponent is zero.*
3. *It is an equilibrium state of P_λ for the potential $-t_0 \log |P'_\lambda|$, whose Lyapunov exponent is zero.*

Proof. As the restriction of P_λ to $[0, 1]$ is the logistic map f_λ , the implication $1 \Rightarrow 2$ is given by [9, Lemma 21]. The proof of the implication $2 \Rightarrow 1$ is analogous to the corresponding implication of the same lemma.

The implication $3 \Rightarrow 2$ being trivial we just need to prove the implication $2 \Rightarrow 3$ to complete the proof of the lemma. We will show that for each invariant measure μ that is supported on the Julia set of P_λ , we have $h_\mu(P_\lambda) - t_0 \int \log |P'_\lambda| d\mu \leq 0$ with equality when the Lyapunov exponent of μ is zero. For such a measure we have $\int \log |P'_\lambda| d\mu \geq 0$ by [27], so there are two cases. If $\int \log |P'_\lambda| d\mu = 0$, then μ is supported on the post-critical set of P_λ and therefore we have $h_\mu(P_\lambda) = 0$ by the variational principle and the fact that the topological entropy of P_λ restricted to its post-critical set is zero (Proposition 10). So we have $h_\mu(P_\lambda) - t_0 \int \log |P'_\lambda| d\mu = 0$ in this case. Suppose now that $\int \log |P'_\lambda| d\mu > 0$. Then by [23,25] it follows that, if we denote by $\text{HD}(\mu)$ the Hausdorff dimension of μ , then $h_\mu(P_\lambda) = \text{HD}(\mu) \int \log |P'_\lambda| d\mu$. As $\text{HD}(\mu) \leq t_0$, we obtain $h_\mu(P_\lambda) - t_0 \int \log |P'_\lambda| d\mu \leq 0$ in this case. \square

References

- [1] Erik M. Alfsen, Compact Convex Sets and Boundary Integrals, *Ergeb. Math. Grenzgeb.*, vol. 57, Springer-Verlag, New York, 1971.
- [2] Guy Barat, Tomasz Downarowicz, Pierre Liardet, Dynamiques associées à une échelle de numération, *Acta Arith.* 103 (1) (2002) 41–78.
- [3] Louis Block, James Keesling, Michał Misiurewicz, Strange adding machines, *Ergodic Theory Dynam. Systems* 26 (3) (2006) 673–682.
- [4] Henk Bruin, Gerhard Keller, Matthias St. Pierre, Adding machines and wild attractors, *Ergodic Theory Dynam. Systems* 17 (6) (1997) 1267–1287.
- [5] A.M. Blokh, M.Yu. Lyubich, Measurable dynamics of S -unimodal maps of the interval, *Ann. Sci. École Norm. Sup.* (4) 24 (5) (1991) 545–573.
- [6] H. Bruin, Combinatorics of the kneading map, in: *Thirty Years After Sharkovskii's Theorem: New Perspectives*, Murcia, 1994, in: *World Sci. Ser. Nonlinear Sci. Ser. B Spec. Theme Issues Proc.*, vol. 8, World Sci. Publ., River Edge, NJ, 1995, pp. 77–87, reprint of the paper reviewed in MR1361922 (96k:58070).
- [7] H. Bruin, Minimal Cantor systems and unimodal maps, *J. Difference Equ. Appl.* 9 (3–4) (2003) 305–318, dedicated to Professor Alexander N. Sharkovsky on the occasion of his 65th birthday.
- [8] María Isabel Cortez, Realization of a Choquet simplex as the set of invariant probability measures of a tiling system, *Ergodic Theory Dynam. Systems* 26 (5) (2006) 1417–1441.
- [9] María Isabel Cortez, Juan Rivera-Letelier, Invariant measures of minimal post-critical sets of logistic maps, *Israel. J. Math.*, in press, arXiv: 0804.4550v1.
- [10] F. Durand, B. Host, C. Skau, Substitutional dynamical systems, Bratteli diagrams and dimension groups, *Ergodic Theory Dynam. Systems* 19 (4) (1999) 953–993.

- [11] Tomasz Downarowicz, The Choquet simplex of invariant measures for minimal flows, *Israel J. Math.* 74 (2–3) (1991) 241–256.
- [12] Tomasz Downarowicz, Survey of odometers and Toeplitz flows, in: *Algebraic and Topological Dynamics*, in: *Contemp. Math.*, vol. 385, Amer. Math. Soc., Providence, RI, 2005, pp. 7–37.
- [13] Edward G. Effros, *Dimensions and C^* -Algebras*, CBMS Reg. Conf. Ser. Math., vol. 46, Conference Board of the Mathematical Sciences, Washington, DC, 1981.
- [14] Richard Gjerde, Ørjan Johansen, Bratteli–Vershik models for Cantor minimal systems: Applications to Toeplitz flows, *Ergodic Theory Dynam. Systems* 20 (6) (2000) 1687–1710.
- [15] Eli Glasner, *Ergodic Theory via Joinings*, *Math. Surveys Monogr.*, vol. 101, Amer. Math. Soc., Providence, RI, 2003.
- [16] Peter J. Grabner, Pierre Liardet, Robert F. Tichy, Odometers and systems of numeration, *Acta Arith.* 70 (2) (1995) 103–123.
- [17] Jean-Marc Gambaudo, Marco Martens, Algebraic topology for minimal Cantor sets, *Ann. Henri Poincaré* 7 (3) (2006) 423–446.
- [18] Thierry Giordano, Ian F. Putnam, Christian F. Skau, Topological orbit equivalence and C^* -crossed products, *J. Reine Angew. Math.* 469 (1995) 51–111.
- [19] Richard Haydon, A new proof that every Polish space is the extreme boundary of a simplex, *Bull. London Math. Soc.* 7 (1975) 97–100.
- [20] Franz Hofbauer, The topological entropy of the transformation $x \mapsto ax(1-x)$, *Monatsh. Math.* 90 (2) (1980) 117–141.
- [21] Richard H. Herman, Ian F. Putnam, Christian F. Skau, Ordered Bratteli diagrams, dimension groups and topological dynamics, *Internat. J. Math.* 3 (6) (1992) 827–864.
- [22] François Ledrappier, Some properties of absolutely continuous invariant measures on an interval, *Ergodic Theory Dynam. Systems* 1 (1) (1981) 77–93.
- [23] François Ledrappier, Quelques propriétés ergodiques des applications rationnelles, *C. R. Acad. Sci. Paris Sér. I Math.* 299 (1) (1984) 37–40.
- [24] A.J. Lazar, J. Lindenstrauss, Banach spaces whose duals are L_1 spaces and their representing matrices, *Acta Math.* 126 (1971) 165–193.
- [25] Ricardo Mañé, The Hausdorff dimension of invariant probabilities of rational maps, in: *Dynamical Systems*, Valparaiso, 1986, in: *Lecture Notes in Math.*, vol. 1331, Springer, Berlin, 1988, pp. 86–117.
- [26] Nicholas S. Ormes, Strong orbit realization for minimal homeomorphisms, *J. Anal. Math.* 71 (1997) 103–133.
- [27] Feliks Przytycki, Lyapunov characteristic exponents are nonnegative, *Proc. Amer. Math. Soc.* 119 (1) (1993) 309–317.