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The boundary regularity of non-linear parabolic systems I

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Abstract

This is the first part of a work aimed at establishing that for solutions to Cauchy–Dirichlet problems involving general non-linear systems of parabolic type, almost every parabolic boundary point is a Hölder continuity point for the spatial gradient of solutions. Here we develop the basic necessary and sufficient condition for establishing the regular nature of a boundary point.

Contents

1. Introduction and results

This is the first of a series of papers devoted to study in a complete and systematic way the up to the boundary regularity of general non-linear parabolic systems. In this part we shall provide a *regularity condition* ensuring that a boundary point is regular, that is, the spatial gradient of the solutions is Hölder continuous in a relative neighborhood of such a point. In the next part [6] we shall derive further global regularity properties of the gradient ensuring that such a regularity condition is satisfied at almost every boundary point with respect to the usual boundary surface measure. As a consequence, we obtain the basic result asserting that *in the case of Cauchy–Dirichlet problems involving parabolic systems with linear growth, almost every boundary point is regular, with respect to the usual surface*

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measure of the parabolic boundary, that is, *the interior partial regularity extends up to the boundary*. To measure the progress yielded by this result we recall that while the existence of boundary irregular points is known already in the elliptic case [22], and assuming smooth boundary data, *the existence of even one regular boundary point was an open problem* when considering general non-linear parabolic systems. We recall that the corresponding interior partial regularity statement has been obtained in [20], while the $C^{0,\alpha}$ full boundary regularity has been obtained for systems with special, almost diagonal structure, i.e. p-Laplacian type systems [11].

In this first paper we study the regularity properties at the parabolic boundary of weak solutions to a non-linear parabolic system with polynomial p-growth, $p \geqslant 2$; the case of linear growth systems $p = 2$ appears therefore as a particular case. To be more precise we are dealing with systems of the following type

$$\begin{cases} u_t - \operatorname{div} a(z, u, Du) = 0 & \text{in } \Omega_T, \\ u = g & \text{on } \partial_{\mathcal{P}}\Omega_T, \end{cases} \tag{1.1}$$

under natural p-growth and ellipticity assumptions on the vector field $a : \Omega_T \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$. The parabolic system will be considered in a cylindrical domain

$$\Omega_T = \Omega \times (0, T),$$

where $\Omega \subset \mathbb{R}^n$, $n \geqslant 2$ is a bounded domain in $\mathbb{R}^n$ and $T > 0$ whose parabolic boundary consisting of the lateral and initial boundary and the edge-points will be denoted by

$$\partial_{\mathcal{P}}\Omega_T = \big(\partial\Omega \times (0, T)\big) \cup \big(\Omega \times \{0\}\big) \cup \big(\partial\Omega \times \{0\}\big).$$

In the sequel we will specify our assumptions imposed on the vector field a, the *continuous* boundary datum g and the boundary $\partial\Omega$ of the domain Ω when presenting the various results. The notion of a weak solution of (1.1) is

Definition 1.1. A map $u \in L^p(0, T; W^{1,p}(\Omega, \mathbb{R}^N))$ is called a (weak) solution to (1.1) if and only if

$$\int_{\Omega_T} u \cdot \varphi_t - \langle a(z, u, Du), D\varphi\rangle\, dz = 0$$

holds for every test-function $\varphi \in C_0^\infty(\Omega_T, \mathbb{R}^N)$, and the following boundary conditions holds:

$$u(\cdot, t) - g(\cdot, t) \in W_0^{1,p}\big(\Omega; \mathbb{R}^N\big) \quad \text{for a.e. } t \in (0, T)$$

and

$$\lim_{h \downarrow 0} \frac{1}{h} \int_0^h \int_\Omega \big|u(x, t) - g(x, 0)\big|^2\, dx\, dt = 0. \tag{1.2}$$

Here we assume that the vector field $a : \Omega_T \times \mathbb{R}^N \times \mathbb{R}^{Nn} \to \mathbb{R}^{Nn}$ fulfills the standard p-growth and ellipticity conditions; i.e. we shall assume that $(z, u, w) \mapsto a(z, u, w)$ and $(z, u, w) \mapsto \partial_w a(z, u, w)$ are continuous in $\Omega_T \times \mathbb{R}^N \times \mathbb{R}^{Nn}$ and that

$$\big|a(z, u, w)\big| \leqslant L\big(1 + |w|^{p-1}\big), \tag{1.3}$$

$$\big\langle \partial_w a(z, u, w)\widetilde{w}, \widetilde{w}\big\rangle \geqslant \nu\big(1 + |w|^{p-2}\big)|\widetilde{w}|^2, \tag{1.4}$$

for every choice of $z = (x, t) \in \Omega_T$, $u \in \mathbb{R}^N$ and $w, \widetilde{w} \in \mathbb{R}^{Nn}$. The structure constants will satisfy (unless otherwise stated)

$$p \geqslant 2, \quad 0 < \nu \leqslant 1 \leqslant L < \infty.$$

Furthermore, we shall assume that $\partial_w a$ is – not necessarily uniformly – bounded. More precisely, we assume that for given $M > 0$ there exists κ_M, such that

$$\big|\partial_w a(z, u, w)\big| \leqslant L\kappa_M, \tag{1.5}$$

for all $z \in \Omega_T$, $u \in \mathbb{R}^N$ and $w \in \mathbb{R}^{Nn}$ such that $|u| + |w| \leqslant M$. With respect to the variables (x, u) we will impose a Hölder continuity assumption on the vector field a. To be precise, we assume that

$$(x, u) \mapsto \frac{a(x, t, u, w)}{1 + |w|^{p-1}}$$

is Hölder continuous with Hölder exponent $\beta \in (0, 1)$, that is we assume

$$\big|a(x, t, u, w) - a(x_0, t, u_0, w)\big| \leqslant L\theta\big(|u| + |u_0|, |x-x_0| + |u-u_0|\big)\big(1 + |w|^{p-1}\big), \tag{1.6}$$

for every choice of $x, x_0 \in \Omega$, $t \in (0, T)$, $u, u_0 \in \mathbb{R}^N$ and $w \in \mathbb{R}^{Nn}$, where

$$\theta(y, s) = \min\big\{1, K(y)s^\beta\big\},$$

and $K : [0, \infty) \to [1, \infty)$ is a given non-decreasing function. Concerning the regularity of the lateral boundary and the Dirichlet boundary values, i.e. of $\partial\Omega$ and g, we shall assume that $g : \overline{\Omega_T} \to \mathbb{R}^N$ is a continuous function such that

$$\partial\Omega \text{ is } C^{1,\beta}, \qquad Dg \in C^{\beta,0}\big(\overline{\Omega} \times [0, T); \mathbb{R}^{Nn}\big), \qquad \partial_t g \in L^{2,2-2\beta}\big(\Omega_T; \mathbb{R}^N\big). \tag{1.7}$$

The definition of Morrey spaces of the type $L^{2,2-2\beta}$ is given in Definition 2.1 below.

Concerning the interior situation there are very recent results providing a good understanding of the regularity properties of weak solutions to non-linear parabolic systems of the type considered in (1.1). Under the assumptions explained before it is known that the space derivative Du of a weak solution u is Hölder continuous with respect to the parabolic metric with Hölder exponent β in a set of full Lebesgue measure [2,5,8,18,20]. Moreover, the size of the so-called singular set, i.e. the set on which Du is not Hölder continuous, can be further estimated in terms of the parabolic Hausdorff dimension. On the other hand, several counter-examples [10,22,36] illustrate that singularities might occur and therefore everywhere regularity cannot be achieved for non-linear parabolic systems. For results of partial regularity in the elliptic case we refer to the classical book [23] and the more recent survey paper [33]; for singular sets estimates we refer also to [31,33] as far as systems are concerned, and to [27] for the variational case.

In spite of such a more or less complete picture in the interior, for general non-linear parabolic systems the regularity theory at the parabolic boundary is widely open. Only a few results for special types of parabolic systems are known. In [12] DiBenedetto proved for p-Laplacian type systems global Hölder continuity of u. Moreover, for homogeneous Dirichlet data, i.e. $g = 0$, the Hölder continuity of Du up to the boundary was shown. For general Dirichlet data partial Hölder continuity of u up to the boundary was established for quasilinear parabolic systems by Arkhipova, [3]. In contrary to the parabolic case, the boundary regularity for general elliptic systems was treated recently in [15,16,25,26,28,29].

The main result of this paper gives a precise extension of the interior regularity criterion for weak solutions of non-linear parabolic systems with p-growth proved in [20] to the boundary case. For the sake of simplicity we shall restrict our attention to the case of homogeneous systems of the type (1.1). The non-homogeneous case is treatable with a few extensions of the techniques hereby introduced. Our result provides a characterization of regular boundary points, i.e. the set of boundary points where Du is continuous. For its formulation it is convenient to introduce the *set of regular boundary points*

$$\mathrm{Reg}_{\mathcal{P}}\, u \equiv \big\{z_0 \in \partial_{\mathcal{P}}\Omega_T\colon\ Du \in C^0\big(\overline{U \cap \Omega_T}; \mathbb{R}^{Nn}\big) \text{ for some neighborhood } U \text{ of } z_0\big\}.$$

Theorem 1.2. *Let $u \in L^p(0, T; W^{1,p}(\Omega; \mathbb{R}^N))$ be a weak solution of the non-linear parabolic system* (1.1) *in Ω_T under the assumptions* (1.4)–(1.7). *Then, there holds*

$$\partial_{\mathcal{P}}\Omega_T \setminus \mathrm{Reg}_{\mathcal{P}}\, u \subset \Sigma := \Sigma^1 \cup \Sigma^2$$

where

$$\Sigma^1 = \left\{z_0 \in \partial_{\mathcal{P}}\Omega_T\colon\ \liminf_{\varrho\downarrow 0} \fint_{\Omega_T\cap Q_\varrho(z_0)} \big|D(u-g) - \big(D(u-g)\big)_{\Omega_T\cap Q_\varrho(z_0)}\big|^p\, dz > 0\right\}$$

and

$$\Sigma^2 = \Big\{ z_0 \in \partial_{\mathcal{P}} \Omega_T \colon \limsup_{\varrho \downarrow 0} \big| \big(D(u-g) \big)_{\Omega_T \cap Q_\varrho(z_0)} \big| = \infty \Big\}.$$

Furthermore, if $z_0 \in \mathrm{Reg}_{\mathcal{P}}\, u$ *then* $Du \in C^{\beta, \frac{\beta}{2}}(\overline{U \cap \Omega_T}; \mathbb{R}^{Nn})$ *for some neighborhood* U *of* z_0.

Let us mention that near the initial boundary it would be enough to assume $(1.7)_2$ rather than $(1.7)_1$–$(1.7)_3$, as explained in Section 2.1. Moreover, taking into account the assumption $(1.7)_2$ it is clear that we could also have omited the presence of g in the definition of the singular sets Σ^1 and Σ^2 without changing them.

The previous result means in particular that in a neighborhood of a point $z_0 \in \mathrm{Reg}_{\mathcal{P}}\, u$ the spatial derivative Du is Hölder continuous up to the parabolic boundary with respect to the standard parabolic metric given by

$$\mathrm{d}_{\mathcal{P}}(z, z_0) \equiv \max\big\{ |x - x_0|, \sqrt{|t - t_0|} \big\} \approx \sqrt{|x - x_0|^2 + |t - t_0|}, \tag{1.8}$$

$z = (x, t)$, $z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$. For the proof of Theorem 1.2 we will separately treat the lateral boundary $\partial_{\mathrm{lat}} \Omega_T = \partial \Omega \times (0, T)$ (see Section 4.1), the initial boundary $\Omega_0 = \Omega \times \{0\}$ (see Section 4.2) and the edge-points $\partial \Omega \times \{0\}$ (see Section 4.3). Thereby we shall carry out much more detailed the proof for the lateral boundary situation since this is the most interesting and also most difficult one. Note that the set of edge-points already has the parabolic Hausdorff dimension $n - 1$ and therefore does not play any role in our dimension reduction results presented in [6], and in particular in the proof of the almost everywhere boundary regularity of solutions: the parabolic Hausdorff dimension of $\partial \Omega \times \{0\}$ is already strictly smaller than the one of the parabolic boundary $\partial_{\mathcal{P}} \Omega_T$. Nevertheless, in order to give a complete characterization of regular boundary points, we included also the treatment of edge-points.

One of the ingredients of the proof of Theorem 1.2 is a suitable boundary version of the method of A-caloric approximation which was introduced in [18] to treat the interior regularity problem for non-linear parabolic systems with quadratic growth, i.e. the case $p = 2$, and later extended to the case of systems with polynomial growth $p \geqslant 2$ [20,19,18,17]. Originally, this technique was used in the setting of geometric measure theory in order to prove regularity result for almost minimizing currents of elliptic integrands in the interior and at the boundary; see [21]. Later it was adapted to treat regularity issues for weak solutions of non-linear elliptic systems; first for the interior situation [13,14] and later on for the boundary situation [26,25,4]. For elliptic problems the technique goes back to the classical harmonic approximation lemma of De Giorgi (see [9,35]) and we refer to the recent survey paper [19] for a systematic presentation of such techniques. In the parabolic setting the method of A-caloric approximation allows us to approximate a weak solutions of the original problem with solutions of a linear parabolic system with constant coefficients, which are therefore used as comparison maps. This leads us to exploit good a priori estimates for solutions of linear systems, see Section 3. We just mention that such a method avoids the use of so-called higher integrability results [1,23].

As stated at the beginning Theorem 1.2 is the first step in the proof of the almost everywhere boundary regularity; it basically asserts that a boundary point $z_0 \in \partial_{\mathcal{P}} \Omega_T$ is regular if the boundary excess functional

$$\fint_{\Omega_T \cap Q_\varrho(z_0)} \big| D(u - g) - \big(D(u - g) \big)_{\Omega_T \cap Q_\varrho(z_0)} \big|^p \, dz$$

is small and $|(D(u - g))_{\Omega_T \cap Q_\varrho(z_0)}|$ stays bounded as $\varrho \downarrow 0$. In the interior situation by Lebesgue's theorem such a result already guarantees that the set of interior regular points has full $\mathcal{L}^{n+1}$-measure. However, in the boundary situation Theorem 1.2 does not guarantee the existence of even one regular boundary point. The principal difficulty concerning partial regularity at the boundary originates from the fact that $\partial_{\mathcal{P}} \Omega_T$ is a set of $\mathcal{L}^{n+1}$-measure zero – in fact it is of dimension n with respect to the Euclidean metric in $\mathbb{R}^{n+1}$, and therefore Lebesgue's theorem does not provide the existence of regular boundary points. Therefore, in order to ensure the existence of at least one regular boundary point we have to show that the regularity criterion form Theorem 1.2 is fulfilled on a larger set, or vice versa that the complement of the regular set – the so-called singular set – is small in a certain sense. This kind of problems – that is the singular set dimension reduction – will be treated in the subsequent paper [6], where we shall give conditions under which the main boundary criterium Theorem 1.2 will provide the almost everywhere regularity at the boundary. Results of this type have been obtained in the stationary case both for systems [16,31–33] and variational integrals [27–29] but are still missing in the parabolic case.

2. Notation and preliminary material

In this paper we will generally write $x = (x_1, \ldots, x_n)$ for a point in $\mathbb{R}^n$ and $z = (x, t) = (x_1, \ldots, x_n, t)$ for a point in $\mathbb{R}^{n+1}$. By $B_\varrho(x_0) \equiv \{x \in \mathbb{R}^n\colon |x - x_0| < \varrho\}$, respectively $B^+_\varrho(x_0) \equiv B_\varrho(x_0) \cap \{x \in \mathbb{R}^n\colon x_n > 0\}$ we denote the open ball, respectively half-ball in $\mathbb{R}^n$ with center $x_0 \in \mathbb{R}^n$ and radius $\varrho > 0$. *When considering $B^+_\varrho(x_0)$, unless otherwise specified, we shall always have x_0 with $(x_0)_n = 0$.* Moreover, we write

$$\Lambda_{\varrho^2}(t_0) = \left(t_0 - \varrho^2, t_0 + \varrho^2\right)$$

for the open interval around $t_0 \in \mathbb{R}$ of length $2\varrho^2$ and

$$\Lambda^0_{\varrho^2}(t_0) = \Lambda_{\varrho^2}(t_0) \cap \{t \in \mathbb{R}\colon t > 0\}.$$

As before, *we always have $t_0 = 0$ when writing $\Lambda^0_{\varrho^2}(t_0)$, unless otherwise stated.* As basic sets for our estimates we usually take parabolic cylinders – these are essentially the balls with respect to the parabolic metric in (1.8), also called "heat balls" – respectively half-cylinders. These are denoted by $Q_\varrho(z_0) \equiv B_\varrho(x_0) \times \Lambda_{\varrho^2}(t_0)$ and $Q^+_\varrho(z_0) \equiv B^+_\varrho(x_0) \times \Lambda_{\varrho^2}(t_0)$ and $Q^0_\varrho(z_0) \equiv B_\varrho(x_0) \times \Lambda^0_{\varrho^2}(t_0)$ and $Q^*_\varrho(z_0) \equiv Q^0_\varrho(z_0) \cap Q^+_\varrho(z_0)$, where $z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}$, $\varrho > 0$. Moreover, we write

$$\Gamma_\varrho(z_0) \equiv Q_\varrho(z_0) \cap \left\{(x_1, \ldots, x_n, t) \in \mathbb{R}^{n+1}\colon x_n = 0\right\}$$

for the lateral part of the boundary of $Q^+_\varrho(z_0)$ and

$$\Gamma^0_\varrho(z_0) \equiv Q^0_\varrho(z_0) \cap \left\{(x_1, \ldots, x_n, t) \in \mathbb{R}^{n+1}\colon x_n = 0\right\}$$

for the one of $Q^*_\varrho(z_0)$. For the initial boundary of $Q^0_\varrho(z_0)$ we write

$$D_\varrho(z_0) = Q_\varrho(z_0) \cap \left\{(x, t) \in \mathbb{R}^{n+1}\colon t = 0\right\}$$

and

$$D^+_\varrho(z_0) = Q^+_\varrho(z_0) \cap \left\{(x, t) \in \mathbb{R}^{n+1}\colon t = 0\right\}$$

for the one of $Q^*_\varrho(z_0)$.

If $z_0 = 0$, a typical situation occurring when treating the regularity of lateral boundary points after "flattening the boundary", we abbreviate $B_\varrho = B_\varrho(0)$, $\Lambda_{\varrho^2} = \Lambda_{\varrho^2}(0)$, $Q_\varrho = Q_\varrho(0)$ $\Gamma_\varrho = \Gamma_\varrho(0)$ and $D_\varrho = D_\varrho(0)$.

For an integrable map $v : A \to \mathbb{R}^k$, $k \in \mathbb{N}$, we write

$$(v)_A \equiv \fint_A v\,dz = \frac{1}{|A|}\int_A v\,dz$$

for its mean-value on A, provided $|A| > 0$. If $A = Q_\varrho(z_0)$ then we write $(v)_{z_0,\varrho}$ for the mean-value of v on the parabolic cylinder $Q_\varrho(z_0)$ and $(v)^+_{z_0,\varrho}$ for the mean-value on the parabolic half-cylinder $Q^+_\varrho(z_0)$ and $(v)^0_{z_0,\varrho}$ for the mean-value on $Q^0_\varrho(z_0)$ and $(v)^*_{z_0,\varrho}$ for the mean-value on $Q^*_\varrho(z_0)$. Finally, we write $\partial_{\mathrm{lat}}\Omega_T = \partial\Omega \times (0, T)$ for the lateral boundary of Ω_T and $\Omega_0 = \Omega \times \{0\}$ for its initial boundary.

Definition 2.1. With $q \geqslant 1$, $\theta \in [0, n + 2]$ and $Q \subset \mathbb{R}^{n+1}$ being a cylinder, a measurable map $v : Q \to \mathbb{R}^k$, $k \geqslant 1$ belongs to the (parabolic) Morrey space $L^{q,\theta}(Q; \mathbb{R}^k)$ if and only if

$$\|v\|^q_{L^{q,\theta}(Q;\mathbb{R}^k)} := \sup_{z_0 \in \Omega_T,\, 0<\varrho<\mathrm{diam}(\Omega_T)} \varrho^{\theta-(n+2)} \int_{\Omega_T \cap Q_\varrho(z_0)} |v|^q\,dz < \infty.$$

The local variant is defined by saying that $v \in L^{q,\theta}_{\mathrm{loc}}(Q; \mathbb{R}^k)$ if and only if $v \in L^{q,\theta}(Q'; \mathbb{R}^k)$ for every sub-cylinder $Q' \Subset Q$.

We now summarize some easy consequences of our assumptions on the vector field a which will be used frequently in the sequel. From the ellipticity (1.4) of $\partial_w a$ we infer that a is monotone with respect to its last variable w, i.e. for all $z \in \Omega_T$, $u \in \mathbb{R}^N$ and $w, \widetilde{w} \in \mathbb{R}^{Nn}$ there holds

$$\big\langle a(z,u,w) - a(z,u,\widetilde{w}), w - \widetilde{w}\big\rangle \geqslant \frac{\nu}{c(p)}\big(1 + |w - \widetilde{w}|^{p-2}\big)|w - \widetilde{w}|^2. \tag{2.1}$$

This can be seen as follows:

$$\begin{aligned}
\big\langle a(z,u,w) - a(z,u,\widetilde{w}), w - \widetilde{w}\big\rangle &= \int_0^1 \big\langle \partial_w a\big(z,u,\widetilde{w} + s(w-\widetilde{w})\big)(w-\widetilde{w}), (w-\widetilde{w})\big\rangle\, ds \\
&\geqslant \nu \int_0^1 \big(1 + \big|\widetilde{w} + s(w - \widetilde{w})\big|^{p-2}\big)|w - \widetilde{w}|^2\, ds \\
&\geqslant \frac{\nu}{c(p)}\big(1 + |w - \widetilde{w}|^{p-2}\big)|w - \widetilde{w}|^2,
\end{aligned}$$

where in the last line we have used

$$\int_0^1 |a + sb|^{p-2}\, ds \geqslant c(p)^{-1}\big(|a|^{p-2} + |b|^{p-2}\big).$$

Moreover, we will use that θ from (1.6) is a concave function with respect to s and

$$\theta\big(|u| + |u_0|, |x - x_0| + |u - u_0|\big) \leqslant K\big(2|u_0| + 1\big)\big(|x - x_0| + |u - u_0|\big)^{\beta}. \tag{2.2}$$

This can be seen by distinguishing the cases $|u - u_0| \leqslant 1$ (then $|u| + |u_0| \leqslant 2|u_0| + 1$) and $|u - u_0| > 1$ (then the term on the right-hand side is > 1; the one on the left-hand side is always $\leqslant 1$). We further set

$$H(s) \equiv K(2s + 1)\big(1 + s^{p-1}\big).$$

Combining (1.6) and (2.2) we then have

$$\big|a(x,t,u,w) - a(x_0,t,u_0,w)\big| \leqslant LH(M)\big(|x - x_0| + |u - u_0|\big)^{\beta}, \tag{2.3}$$

provided we assume $|u_0| \leqslant M$ and $|w| \leqslant M$. By virtue of the continuity of $\partial_w a$ there exists for each $M > 0$ a modulus of continuity $\omega_M : [0,\infty) \to [0,1]$ with $\lim_{s\downarrow 0} \omega_M(s) = 0$ for all $M > 0$, such that $M \mapsto \omega_M(s)$ is non-deceasing for fixed $s \geqslant 0$ and $s \mapsto \omega_M(s)^2$ is concave and non-decreasing for fixed $M > 0$, and such that

$$\big|\partial_w a(z,u,w) - \partial_w a(z_0,u_0,w_0)\big| \leqslant 2L\kappa_M \omega_M\big(\mathrm{d}_{\mathcal{P}}(z,z_0)^p + |u - u_0|^p + |w - w_0|^p\big) \tag{2.4}$$

for all $z, z_0 \in \Omega_T$, $u, u_0 \in \mathbb{R}^N$ and $w, w_0 \in \mathbb{R}^{Nn}$ with $|u| + |w| \leqslant M$ and $|u_0| + |w_0| \leqslant M$.

2.1. Transformation to the model situation

Since our results are of local nature we are allowed to consider the lateral and the initial boundary situation separately, i.e to prove regularity for a point $z_0 = (x_0, 0) \in \Omega_0$ lying on the initial boundary it is enough to take into account parabolic cylinders $Q^0_\varrho(z_0)$ with $B_\varrho(x_0) \Subset \Omega$ and the same for points lying on the lateral boundary. When considering the *lateral boundary* we will prove our results in a model situation on the half-cylinder Q_1^+ and for boundary values $u \equiv 0$ on the lateral boundary Γ_1. Therefore, we will always refer to a Cauchy–Dirichlet problem of the following type:

$$\begin{cases} u_t - \operatorname{div} a(z,u,Du) = g_t & \text{in } Q_1^+, \\ u = 0 & \text{on } \Gamma_1, \end{cases} \tag{2.5}$$

where $\partial_t g \in L^{2,2-2\beta}(Q_1^+; \mathbb{R}^N)$. We briefly describe how to transform the Dirichlet problem (1.1) to this model situation. Let $z_0 \in \partial\Omega \times (0,T)$. Without loss of generality we can assume that $z_0 = (x_0, t_0) = 0$ and that the inward pointing unit normal to $\partial\Omega$ in x_0 is $\nu_{\partial\Omega}(x_0) = e_n$. Then, for $\varrho > 0$ sufficiently small, we can flatten the boundary $B_\varrho \cap \partial\Omega$ by a $C^{1,\beta}$-function Ψ, such that $\Psi(B_\varrho \cap \partial\Omega) \subset B_\varrho \cap \{x \in \mathbb{R}^n\colon\ x_n = 0\}$. Then it is easy to verify that the transformed map

$$\tilde{v}(y,t) \equiv u\big(\Psi^{-1}(y),t\big) - g\big(\Psi^{-1}(y),t\big), \quad (y,t) \in Q_\varrho^+$$

is a weak solution of the following Cauchy–Dirichlet problem

$$\begin{cases} \tilde{v}_t - \operatorname{div}\tilde{a}(z,\tilde{v},D\tilde{v}) = \tilde{g}_t & \text{in } Q_\varrho^+, \\ \tilde{v} = 0 & \text{on } \Gamma_\varrho, \end{cases}$$

where the vector field $\tilde{a}$ is defined by

$$\tilde{a}(y,t,\tilde{v},w) \equiv a\big(\Psi^{-1}(y),t,\tilde{v}+\tilde{g}(y,t),\big(w+D\tilde{g}(y,t)\big)D\Psi\big(\Psi^{-1}(y)\big)\big)D\Psi^t\big(\Psi^{-1}(y)\big)$$

and

$$\tilde{g}(y,t) \equiv g\big(\Psi^{-1}(y),t\big).$$

From our assumptions (1.3)–(1.6) on the vector field a, we infer that $\tilde{a}$ fulfills similar hypotheses after changing the appearing structure constants suitably. It is worth to mention that we do not have to impose any further regularity of Dg with respect to t in Theorem 1.2 since we only use – and assume – the fact that the vector field a is Hölder continuous with respect to the space variable x. However, when proving estimates for the singular set [6] we shall need to assume a certain continuity of Dg with respect to t in order to have the newly defined vector field $\tilde{a}$ to be Hölder continuous with respect to x and t.

Now, it is easy to verify the standard fact asserting that $y \in \Gamma_\varrho$ is a regular point of $D\tilde{v}$ if and only if $\Psi^{-1}(y) \in \partial\Omega \times (0,T)$ is a regular point of Du. Therefore, it suffices to prove Theorem 1.2 in the model situation (2.5) (see Proposition 4.7).

Finally, we want to comment on the change of the structure constants when passing to the model situation. The new growth constant $\widetilde{L}$ then is of the form $L \cdot c(p, \|g\|_{C^{1,\beta}}, \partial\Omega)$, while the new ellipticity constant $\tilde{\nu}$ is of the form $L/c(p, \|g\|_{C^{1,\beta}}, \partial\Omega)$, where the constant $c(p, \|g\|_{C^{1,\beta}}, \partial\Omega)$ is strictly larger then 0. Therefore, in the estimates for the original problem (1.1) the constants will depend on $L/\nu \cdot c(p, \|g\|_{C^{1,\beta}}, \partial\Omega)^2$.

In the *initial boundary* situation the procedure is simpler. Here, we shall transform the problem to the model situation where the initial values are equal to zero, i.e. we consider

$$\begin{cases} u_t - \operatorname{div} a(z,u,Du) = g_t & \text{in } \Omega_T, \\ u(\cdot,0) = 0 & \text{on } \Omega, \end{cases} \tag{2.6}$$

where $\partial_t g \in L^{2,2-2\beta}(\Omega_T;\mathbb{R}^N)$. This is achieved by subtracting the initial values, i.e. we consider the map $v(x,t) = u(x,t) - g(x,t)$. Then v is a solution to

$$\begin{cases} v_t - \operatorname{div}\tilde{a}(z,v,Dv) = g_t & \text{in } \Omega_T, \\ v(\cdot,0) = 0 & \text{on } \Omega, \end{cases}$$

where $\tilde{a}$ is defined by

$$\tilde{a}(x,t,v,w) := a\big(x,t,v+g(x,t),w+Dg(x,t)\big).$$

As before, from our assumptions (1.3)–(1.6) on the vector field a and the fact that $Dg(\cdot,t) \in C^{0,\beta}(\overline{\Omega};\mathbb{R}^{Nn})$, for any $t \in [0,T]$ we find that $\tilde{a}$ satisfies similar conditions after changing the appearing structure constants suitably. We also mention that at the initial boundary it would be possible to consider $u(x,t) - g(x,0)$ rather than $u(x,t) - g(x,t)$ which would lead us to a homogeneous model problem. Then the proof would be slightly easier and not require any regularity assumption on $\partial\Omega$ and g_t, i.e. it would be enough to assume $(1.7)_2$ rather than $(1.7)_1$–$(1.7)_3$. But for the sake of consistency we shall not follow this strategy. Indeed, when proving the characterization of regular edge-points we need to combine all possible configurations, i.e. the lateral and initial boundary situation, the interior and the edge-situation. Therefore we shall consider the same type of model problem in any case. The final outcome, i.e. the characterization of regular points given in Theorem 1.2 is the same with both strategies, since by the continuity of Dg the functions $Dg(x,0)$ and $Dg(x,t)$ have the same trace at zero.

Finally, in a situation where we are near the *edge* $\partial\Omega \times \{0\}$ we will prove our result on the set Q_1^* and for boundary values $u = 0$ on $\Gamma_1^0 \cup D_1^+$. Therefore, we consider the following Cauchy–Dirichlet problem as a model problem

$$\begin{cases} u_t - \operatorname{div} a(z, u, Du) = g_t & \text{in } Q_1^*, \\ u = 0 & \text{on } \Gamma_1^0 \cup D_1^+, \end{cases} \tag{2.7}$$

where $\partial_t g \in L^{2,2-2\beta}(Q_1^*; \mathbb{R}^N)$. The transformation made in the present situation is the same as the one for the lateral boundary situation.

2.2. Steklov averages

Since weak solutions u of parabolic systems possess only weak regularity properties with respect to the time variable t, i.e. they are not assumed to be weakly differentiable, in principle it is not possible to use the solution u itself (also disregarding boundary values) as a test-functions in the weak formulation of the parabolic system. In order to be nevertheless able to test the system properly, we smooth the solution u with respect to the time direction t using the so-called Steklov means. This also enables us to work on the time-slices $\mathbb{R}^n \times \{t\}$, even if u is only an L^2-map with respect to t.

Given a function $f \in L^1(\Omega \times (t_1, t_2))$ and $0 < |h| \leqslant \frac{1}{2}(t_2 - t_1)$, we define its *Steklov mean* by

$$[f]_h(x,t) \equiv \begin{cases} \frac{1}{|h|}\int_t^{t+h} f(x,s)\,ds, & t \in [t_1 + |h|, t_2 - |h|], \\ 0, & t \in (t_1, t_1 + |h|) \cup (t_2 - |h|, t_2). \end{cases} \tag{2.8}$$

The previous definition should be used when dealing with symmetric parabolic cylinders which are far from the initial boundary. When dealing with the initial boundary problem we shall adopt the following one, valid in the case $0 < h \leqslant t_2 - t_1$,

$$[f]_h(x,t) \equiv \begin{cases} \frac{1}{h}\int_t^{t+h} f(x,s)\,ds, & t \in (t_1, t_2 - h], \\ 0, & t \in (t_2 - h, t_2). \end{cases} \tag{2.9}$$

Rewriting system (1.1) with Steklov-means $[u]_h$ of u, we obtain the following system on the time-slices $\Omega \times \{t\}$,

$$\int_\Omega \partial_t [u]_h(\cdot,t)\cdot\varphi + \big\langle \big[a\big(\cdot, t, u(\cdot,t), Du(\cdot,t)\big)\big]_h, D\varphi \big\rangle\,dx = 0 \tag{2.10}$$

for all $\varphi \in W_0^{1,p}(\Omega; \mathbb{R}^N)$ and for a.e. $t \in (0, T)$. Note that in the model situations (2.5)–(2.7) introduced in Section 2.1 we can similarly pass to the related Steklov formulations. We only have to take into account that an additional integral of the form $\int \Omega [g_t]_h(\cdot,t)\cdot\varphi\,dx$ then appears on the right-hand side.

2.3. Preliminary lemmas

In order to show partial regularity we will have to control the oscillation of the solution in a certain sense. To this aim we will approximate the solution by an affine map $\ell : \mathbb{R}^{n+1} \to \mathbb{R}^N$ of the form $\ell(x,t) = \ell(x_n) = \xi x_n$, where $\xi \in \mathbb{R}^N$, so that $\ell \equiv 0$ on the lateral boundary Γ. The next lemma provides properties of vectors minimizing certain functionals.

Lemma 2.2. *Let $u \in L^2(Q_\varrho^+; \mathbb{R}^N)$ and $\xi_\varrho \in \mathbb{R}^N$ the unique vector minimizing $\xi \mapsto ⨍_{Q_\varrho^+} |u - \xi x_n|^2\,dz$. Then*

$$\xi_\varrho = \frac{n+2}{\varrho^2} ⨍_{Q_\varrho^+} u x_n\,dz. \tag{2.11}$$

Moreover, if $u \in L^p(Q_\varrho^+; \mathbb{R}^N)$, $p \geqslant 2$, then for any $\xi \in \mathbb{R}^N$ and $p \geqslant 2$ there holds

$$|\xi_\varrho - \xi|^p \leqslant \left(\frac{n+2}{\varrho^2}\right)^{\frac{p}{2}} ⨍_{Q_\varrho^+} |u - \xi x_n|^p\,dz.$$

Proof. First, we shall verify (2.11). Since

$$\fint_{Q_\varrho^+} |u-\xi x_n|^2\,dz = \fint_{Q_\varrho^+} |u|^2\,dz - 2\xi\cdot\fint_{Q_\varrho^+} u x_n\,dz + |\xi|^2\fint_{Q_\varrho^+} x_n^2\,dz$$

is a quadratic polynomial there exists a unique minimizing $\xi_\varrho \in \mathbb{R}^N$ which satisfies

$$\frac{d}{dt}\fint_{Q_\varrho^+} |u-\xi_\varrho x_n + t\xi x_n|^2\,dz\Big|_{t=0} = 2\fint_{Q_\varrho^+} (u-\xi_\varrho x_n)\cdot \xi x_n\,dz = 0,$$

so that

$$\left(\fint_{Q_\varrho^+} u x_n\,dz - \xi_\varrho \fint_{Q_\varrho^+} x_n^2\,dz\right)\cdot\xi = 0 \quad \text{for all } \xi\in\mathbb{R}^N.$$

Taking into account the equality

$$\fint_{Q_\varrho^+} x_n^2\,dz = \frac{\varrho^2}{n+2}, \tag{2.12}$$

we obtain the desired formula for ξ_ϱ. To prove the second assertion of the lemma we consider $\xi\in\mathbb{R}^N$. From (2.12), the Cauchy–Schwarz inequality and Hölder's inequality we obtain

$$\begin{aligned}
|\xi_\varrho - \xi|^p &= \left|\frac{n+2}{\varrho^2}\fint_{Q_\varrho^+} u x_n\,dz - \frac{n+2}{\varrho^2}\xi\fint_{Q_\varrho^+} x_n^2\,dz\right|^p \\
&\leqslant \left(\frac{n+2}{\varrho^2}\fint_{Q_\varrho^+} |u-\xi x_n| x_n\,dz\right)^p \\
&\leqslant \left(\frac{n+2}{\varrho^2}\right)^p\left(\fint_{Q_\varrho^+} x_n^2\,dz\right)^{\frac{p}{2}}\left(\fint_{Q_\varrho^+} |u-\xi x_n|^2\,dz\right)^{\frac{p}{2}} \\
&\leqslant \left(\frac{n+2}{\varrho^2}\right)^{\frac{p}{2}}\fint_{Q_\varrho^+} |u-\xi x_n|^p\,dz,
\end{aligned}$$

which is the desired estimate. □

In contrast to the interior parabolic case we have when considering the lateral boundary situation a Poincaré inequality for maps $u\in L^p(\Lambda_{\varrho^2}(t_0); W^{1,p}(B_\varrho(x_0)^+;\mathbb{R}^k))$ satisfying $u\equiv 0$ on the lateral boundary $\Gamma_\varrho(z_0)$. This inequality can be obtained applying the standard Poincaré inequality to the functions $u(\cdot,t)\in W^{1,p}(B_\varrho^+(x_0);\mathbb{R}^k)$ for a.e. $t\in\Lambda_{\varrho^2}(t_0)$ and then integrating with respect to t.

Lemma 2.3. *Let $z_0=(x_0,t_0)\in\mathbb{R}^{n+1}$ with $x_0\in\mathbb{R}^{n-1}\times\{0\}$. Then for any map $u\in L^p(\Lambda_{\varrho^2}(t_0); W^{1,p}(B_\varrho(x_0)^+;\mathbb{R}^k))$, $k\geqslant 1$ satisfying $u\equiv 0$ on $\Gamma_\varrho(z_0)$ there holds*

$$\fint_{Q_\varrho^+(z_0)} |u|^p\,dz \leqslant \frac{\varrho^p}{p}\fint_{Q_\varrho^+(z_0)} |D_n u|^p\,dz.$$

We shall also need the following standard iteration lemma, which can be found for instance in [30, Lemma 2].

Lemma 2.4. *Let $\varphi : [R_0, 2R_0] \to [0, \infty)$ be a function such that*

$$\varphi(t) \leqslant \frac{1}{2}\varphi(\varrho) + \sum_{i=1}^{k} \mathcal{B}_i(\varrho - t)^{-\beta_i} + \mathcal{K} \quad \text{for every } R_0 < t < \varrho < 2R_0,$$

with $\mathcal{B}_i, \mathcal{K} \geqslant 0$ and $\beta_i > 0$ for $i = 1, \dots, k$. Then there exists a constant $c = c(\beta_1, \dots, \beta_k)$ such that

$$\varphi(R_0) \leqslant c \sum_{i=1}^{k} \mathcal{B}_i R_0^{-\beta_i} + c\mathcal{K}.$$

3. Linear parabolic systems

From the theory for linear parabolic systems it is known that weak solutions are smooth in the interior and also up to the boundary. In order to prove our characterization for regular boundary points we shall exploit good *excess-decay estimates* for linear parabolic systems with constant coefficients. Since we are dealing with three different configurations, namely the lateral and initial boundary situation and the case of edge-points i.e. lying simultaneously on the lateral boundary and at the initial time-slice Ω_0, we will need a suitable excess-decay estimate for any of them. Our aim in this section is to provide such estimates. The precise form of the estimates found plays a crucial role in the study of partial regularity since it allows to a proper comparison argument after linearization. We consider the following linear parabolic system with constant coefficients

$$\int_Q \big(u \cdot \varphi_t - \langle ADu, D\varphi\rangle\big)\, dz = 0 \quad \text{for every } \varphi \in C_0^\infty\big(Q; \mathbb{R}^N\big), \tag{3.1}$$

where either $Q = Q_\varrho^+(z_0)$ or $Q = Q_\varrho^0(z_0)$ or $Q = Q_\varrho^+(z_0) \cap Q_\varrho^0(z_0)$. Thereby the coefficients A are supposed to satisfy the following ellipticity and boundedness conditions:

$$\langle Aw, w\rangle \geqslant \nu|w|^2, \qquad \big|\langle Aw, \widetilde{w}\rangle\big| \leqslant L|w||\widetilde{w}|, \tag{3.2}$$

whenever $w, \widetilde{w} \in \mathbb{R}^{Nn}$ where $0 < \nu \leqslant L < \infty$.

3.1. Regularity up to the lateral boundary

First we turn our attention to the lateral boundary situation, where we consider the linear parabolic system (3.1) on $Q = Q_\varrho^+(z_0) = B_\varrho^+(x_0) \times \Lambda_{\varrho^2}(t_0)$. By slight modifications of the proof of [24, Theorem 2.2], we obtain the following result for solutions of linear parabolic systems near the lateral boundary. Although the above mentioned theorem is proved under Neumann boundary conditions and on non-symmetric cylinders, the same methods also apply in our situation. Moreover, a proper investigation of the arguments also yield the asserted dependence of the constant. Then, recalling the notation fixed at the beginning of Section 2, we can show

Theorem 3.1. *Suppose that $u \in L^2(\Lambda_1;\ W^{1,2}(B_1^+, \mathbb{R}^N))$ is a weak solution in Q_1^+ of the linear parabolic system* (3.1) *with $u = 0$ on the lateral boundary Γ_1 under the assumption* (3.2). *Then u is smooth up to the lateral boundary Γ_1. Moreover, for every $z_0 = (x_0, t_0) \in \Gamma_1$ and ϱ, R such that $0 < \varrho < R < \min\{1 - |x_0|, \sqrt{1 - |t_0|}\}$ we have*

$$\int_{Q_\varrho^+(z_0)} |Du|^2\, dz \leqslant c\left(\frac{\varrho}{R}\right)^{n+2} \int_{Q_R^+(z_0)} |Du|^2\, dz$$

and

$$\int_{Q_\varrho^+(z_0)} \big|Du - (Du)_{z_0,\varrho}^+\big|^2\, dz \leqslant c\left(\frac{\varrho}{R}\right)^{n+4} \int_{Q_R^+(z_0)} \big|Du - (Du)_{z_0,R}^+\big|^2\, dz,$$

where in both estimates the constant c depends on n and L/ν only.

In our application we will need the excess-decay estimates near the lateral boundary in a slightly different form which is provided in Corollary 3.4. The following is devoted to the proof of the corollary.

Remark 3.2. Under the assumptions of Theorem 3.1 there also holds the following estimate for the weak derivatives $\partial_t^j D^k u$ of u, for all $k, j \in \mathbb{N}_0$:

$$\int_{Q_\varrho^+(z_0)} \left|\partial_t^j D^k u\right|^2 dz \leqslant \frac{c}{(R-\varrho)^{4j+2k}} \int_{Q_R^+(z_0)} |u|^2\, dz,$$

where $c = c(n, L/\nu, j, k)$.

The previous estimate for the case $\varrho = R/2$ can be found in the proof of [24, Theorem 2.2, estimate (2.20)]. The general case follows by the same arguments, but a different choice of the involved cylinders and cut-off functions. Although the precise dependence on the factor $R - \rho$ is not mentioned in [24], it can be inferred by tracing back the estimates.

Lemma 3.3. *Under the assumptions of Theorem* 3.1*, for any $\ell \in \mathbb{N}_0$ and $s > 0$ there holds*

$$\sup_{Q_{R/2}^+(z_0)} \left|D^\ell u\right|^s \leqslant c(n, \ell, L/\nu, s) \fint_{Q_R^+(z_0)} |u|^s\, dz.$$

Proof. We first infer from Theorem 3.1 that u is smooth in Q_1^+ and therefore $D^\ell u$ exists on $Q_R^+(z_0)$. Due to the Sobolev embedding theorem and Remark 3.2 we have for $R/2 \leqslant \varrho < r \leqslant R$ that

$$\sup_{Q_\varrho^+} |u| + \sup_{Q_\varrho^+} \left|D^\ell u\right| \leqslant c\|u\|_{W^{k,2}(Q_\varrho^+)} \leqslant \frac{c(n, \ell, L/\nu, R)}{(r-\varrho)^{2k}} \|u\|_{L^2(Q_r^+)},$$

where we have chosen $k \in \mathbb{N}$ large enough (i.e. $k > \ell + \frac{n+1}{2}$). In the case $s \geqslant 2$ we apply the preceding inequality with $\varrho = R/2$ and $r = R$ and then use Hölder's inequality to deduce

$$\sup_{Q_{R/2}^+} |u| + \sup_{Q_{R/2}^+} \left|D^\ell u\right| \leqslant c(n, \ell, L/\nu, R)\|u\|_{L^s(Q_R^+)}.$$

In the case $0 < s < 2$ we use Young's inequality in order to derive

$$\begin{aligned}
\sup_{Q_\varrho^+} |u| + \sup_{Q_\varrho^+} \left|D^\ell u\right| &\leqslant \frac{c}{(r-\varrho)^{2k}} \|u\|_{L^2(Q_r^+)} \\
&\leqslant \frac{c}{(r-\varrho)^{2k}} \sup_{Q_r^+} |u|^{1-\frac{s}{2}} \|u\|_{L^s(Q_r^+)}^{\frac{s}{2}} \\
&\leqslant \frac{1}{2}\Big(\sup_{Q_r^+} |u| + \sup_{Q_r^+} \left|D^\ell u\right|\Big) + \frac{c}{(r-\varrho)^{\frac{4k}{s}}} \|u\|_{L^s(Q_r^+)},
\end{aligned} \tag{3.3}$$

where $c = c(n, \ell, L/\nu, R)$. Using Lemma 2.4 with the choice

$$\varphi(t) := \sup_{Q_t^+} |u| + \sup_{Q_t^+} |D^\ell u|$$

we can absorb the first term of the right-hand side in (3.3) on the left and infer that

$$\sup_{Q_{R/2}^+} \left|D^\ell u\right|^s \leqslant c(n, \ell, L/\nu, s, R)\|u\|_{L^s(Q_R^+)}^s.$$

The dependence of the constant in front of $\int_{Q_R^+} |u|^s\, dz$ on the radius R can be easily determined by considering the scaled map $v(x,t) = R^{-1}u(Rx, R^2 t)$ on Q_1^+, applying the preceding estimate on Q_1^+ and then scaling back to Q_R^+. In this way we find the following dependence: $c(n, \ell, L/\nu, s, R) = R^{-(n+2)} c(n, \ell, L/\nu, s)$. □

Now, we are in a position to prove the excess-decay estimate for weak solutions of homogeneous linear parabolic systems with constant coefficients near the lateral boundary we are looking for:

Corollary 3.4. *Suppose that $u \in L^2(\Lambda_{\varrho^2}(t_0);\ W^{1,2}(B_\varrho^+(x_0), \mathbb{R}^N))$ is a weak solution in $Q_\varrho^+(z_0)$, with $z_0 \in \Gamma$ of the linear parabolic system* (3.1) *with $u = 0$ on the lateral boundary $\Gamma_\varrho(z_0)$ under the assumption* (3.2)*. Then u is smooth up to the lateral boundary $\Gamma_\varrho(z_0)$. Moreover, the estimate*

$$\fint_{Q^+_{\theta\varrho}(z_0)} \left|\frac{u-(D_nu)^+_{z_0,\theta\varrho}x_n}{\theta\varrho}\right|^s dz \leqslant c_{Li}\theta^s \fint_{Q^+_{\varrho}(z_0)} \left|\frac{u-(D_nu)^+_{z_0,\varrho}x_n}{\varrho}\right|^s dz$$

holds for any $\theta \in (0, 1/2)$ and $s \geqslant 1$, where $c_{Li} = c_{Li}(n, L/\nu, s)$ and

$$\left|(D_nu)^+_{z_0,\theta\varrho}\right|^2 \leqslant c(n, L/\nu) \fint_{Q^+_\varrho(z_0)} |D_nu|^2\, dz.$$

Proof. Without loss of generality we can assume that $z_0 = 0$ and $\varrho = 1$ (the general case can then be obtained by a standard translation/scaling argument). The first assertion of the corollary concerning the smoothness up to the lateral boundary of a weak solution to a linear parabolic system with constant coefficients follows from Theorem 3.1. To prove the excess-decay estimate we in turn apply the Poincaré inequality from Lemma 2.3 (which can be used here since $u - (D_nu)^+_\theta x_n \equiv 0$ on Γ_θ), the Poincaré inequality applied with respect to both the variables x and t, we note that $Q^+_\theta \subset Q^+_{1/2}$ and finally apply the standard Sobolev embedding theorem to infer

$$\begin{aligned}
\fint_{Q^+_\theta} \left|\frac{u-(D_nu)^+_\theta x_n}{\theta}\right|^s dz &\leqslant \frac{1}{s}\fint_{Q^+_\theta} \left|D_nu-(D_nu)^+_\theta\right|^s dz \\
&\leqslant c(n,s)\fint_{Q^+_\theta}\left(\theta^s|DD_nu|^s + \theta^{2s}|\partial_t D_nu|^s\right) dz \\
&\leqslant c(n,s)\Big(\theta^s \sup_{Q^+_{1/2}}\left|D^2u\right|^s + \theta^{2s}\sup_{Q^+_{1/2}}|\partial_t Du|^s\Big) \\
&\leqslant c(n,s)\theta^s\left\|u-(D_nu)^+_1x_n\right\|^s_{W^{k,2}(Q^+_{1/2})},
\end{aligned}$$

where we have chosen $k \in \mathbb{N}$ large enough (i.e. $k > 2 + \frac{n+1}{2}$). The map $u - (D_nu)^+_1 x_n$ fulfills the assumptions of Theorem 3.1 (and therefore also those of Remark 3.2), since it satisfies on the one hand the linear parabolic system (3.1) and on the other hand also $u - (D_nu)^+_1 x_n \equiv 0$ on the lateral boundary Γ_1. Therefore, the estimate from Remark 3.2 is applicable and yields

$$\fint_{Q^+_\theta} \left|\frac{u-(D_nu)^+_\theta x_n}{\theta}\right|^s dz \leqslant c\theta^s\bigg(\fint_{Q^+_{3/4}}\left|u-(D_nu)^+_1x_n\right|^2 dz\bigg)^{\frac{s}{2}}.$$

In the case $s \geqslant 2$ the assertion now follows from Hölder's inequality and by enlarging the domain of integration. In the case $1 \leqslant s < 2$ we use Lemma 3.3 and obtain

$$\begin{aligned}
\fint_{Q^+_\theta} \left|\frac{u-(D_nu)^+_\theta x_n}{\theta}\right|^s dz &\leqslant c\theta^s \sup_{Q^+_{3/4}}\left|u-(D_nu)^+_1x_n\right|^{\frac{s}{2}(2-s)}\bigg(\fint_{Q^+_{3/4}}\left|u-(D_nu)^+_1x_n\right|^s dz\bigg)^{\frac{s}{2}} \\
&\leqslant c\theta^s\fint_{Q^+_1}\left|u-(D_nu)^+_1x_n\right|^s dz.
\end{aligned}$$

This proves the first estimate in Theorem 3.1. The second one follows by a similar reasoning. Once again using the Sobolev embedding, Remark 3.2 and the Poincaré inequality from Lemma 2.3 we infer that

$$\big|(D_n u)_\theta^+\big|^2 \leqslant \sup_{Q_{1/2}^+} |D_n u|^2 \leqslant c\|Du\|^2_{W^{k,2}(Q_{1/2}^+)} \leqslant c \fint_{Q_1^+} |u|^2\,dz \leqslant c(n, L/\nu) \fint_{Q_1^+} |D_n u|^2\,dz.$$

This finishes the proof of the corollary. □

3.2. Regularity up to the initial boundary

Here we consider the linear parabolic system (3.1) with $Q = Q_\varrho^0(z_0) = B_\varrho(x_0) \times \Lambda_{\varrho^2}^0$ near the initial boundary. In this situation the excess-decay estimate essentially follows from the one in the interior case since we can extend the solution u by zero on $B_\varrho(x_0) \times (-\infty, 0)$, as it is done in [7, Lemma 5.III]. More precisely, we define

$$U(x,t) = \begin{cases} u(x,t) & \text{if } (x,t) \in B_\varrho(x_0) \times [0, \varrho^2), \\ 0 & \text{if } (x,t) \in B_\varrho(x_0) \times (-\infty, 0). \end{cases}$$

Since $u(\cdot, 0) = 0$ on $B_\varrho(x_0)$ in the usual L^2-sense, the extension U is a solution of the linear parabolic system (3.1) on $Q = B_\varrho(x_0) \times (-\infty, \varrho^2)$. Therefore the interior excess-decay estimate from [20, Lemma 4.6] is applicable to U on the cylinder $Q_\varrho(z_0) \subset B_\varrho(x_0) \times (-\infty, \varrho^2)$. When applying the lemma in our situation there is one difference compared to [20], namely we are dealing with symmetric cylinders of the type $Q_\varrho^0(z_0) = B_\varrho(x_0) \times (-\varrho^2, \varrho^2)$, whereas in [20] one-sided cylinders of the form $B_\varrho(x_0) \times (t_0 - \varrho^2, t_0)$ are considered. As already mentioned in the lateral boundary situation this is not a problem since the excess-decay estimate holds true on both types of cylinders. Having the preceding explanatory notes in mind, the application of [20, Lemma 4.6] in the case $\ell = 0$ gives for the original solution u the following

Lemma 3.5. *Suppose that $u \in L^2(\Lambda_{\varrho^2}^0; W^{1,2}(B_\varrho(x_0); \mathbb{R}^N))$ is a weak solution in $Q_\varrho^0(z_0)$, with $z_0 = (x_0, 0)$ of the linear parabolic system* (3.1) *with $u(\cdot, 0) = 0$ on $B_\varrho(x_0)$ under the assumption* (3.2)*. Then u is smooth up to the initial boundary $D_\varrho(z_0)$. Moreover, for any $\theta \in (0,1)$ and $s \geqslant 1$ there holds the following estimate*

$$\fint_{Q_{\theta\varrho}^0(z_0)} \left|\frac{u}{\theta\varrho}\right|^s dz \leqslant c_{Li}\theta^s \fint_{Q_\varrho^0(z_0)} \left|\frac{u}{\varrho}\right|^s dz, \tag{3.4}$$

where $c_{Li} = c_{Li}(n, N, L/\nu, s)$.

3.3. Regularity up to the edge

Here we consider the linear parabolic system (3.1) with $Q = Q_\varrho^*(z_0) = Q_\varrho^0(z_0) \cap Q_\varrho^+(z_0)$ near the edge $\Gamma \cap (\mathbb{R}^n \times \{0\})$. As in the initial boundary situation we can extend the solution u by zero on $B_\varrho^+(x_0) \times (-\infty, 0)$ to obtain the analogue of Corollary 3.4 for the edge-situation. This was also done in [24, Theorem 2.3] to prove that the statement of Theorem 3.1 still holds in the edge-situation. Since we already prescribed the extension procedure in the last section we now only state the result.

Lemma 3.6. *Suppose that $u \in L^2(\Lambda_{\varrho^2}^0(t_0);\ W^{1,2}(B_\varrho^+(x_0), \mathbb{R}^N))$ is a weak solution in $Q_\varrho^*(z_0)$, with $z_0 = (x_0, 0) \in \Gamma$ of the linear parabolic system* (3.1) *with $u = 0$ on $\Gamma_\varrho^0(z_0) \cup D_\varrho^+(z_0)$ under the assumption* (3.2)*. Then u is smooth up to $\Gamma_\varrho^0(z_0) \cup D_\varrho^+(z_0)$. Moreover, the estimate*

$$\fint_{Q_{\theta\varrho}^*(z_0)} \left|\frac{u - (D_n u)^*_{z_0,\theta\varrho} x_n}{\theta\varrho}\right|^s dz \leqslant c_{Li}\theta^s \fint_{Q_\varrho^*(z_0)} \left|\frac{u - (D_n u)^*_{z_0,\varrho} x_n}{\varrho}\right|^s dz$$

holds for any $\theta \in (0, 1/2)$ *and* $s \geqslant 1$, *where* $c_{Li} = c_{Li}(n, L/\nu, s)$ *and*

$$\left|(D_n u)^*_{z_0,\theta\varrho}\right|^2 \leqslant c(n, L/\nu) \fint_{Q^*_\varrho(z_0)} |D_n u|^2 \, dz.$$

4. Characterization of regular boundary points

The aim of this chapter is to prove the characterization of regular boundary points stated in Theorem 1.2. Since the arguments are different for the lateral and the initial-boundary and also for the edge-situation we separately treat the three cases. We first shall consider points $z_0 \in \partial_{\mathrm{lat}}\Omega_T$ lying on the lateral boundary and subsequently, in Sections 4.2 and 4.3 we deal with points $z_0 \in \Omega_0$ on the initial boundary, respectively $z_0 \in \partial\Omega \times \{0\}$ on the edge.

4.1. Regular points on the lateral boundary

We will prove the characterization of regular lateral boundary points in the model situation (2.5) on Q_1^+ which was explained in Section 2.1. Therefore, the statement of Theorem 1.2 concerning lateral boundary points is equivalent with Proposition 4.7.

4.1.1. A-caloric approximation

The main tool in proving partial regularity is the lemma of A-caloric approximation which states that whenever a map u is approximately a solution of a linear parabolic system with constant coefficients, then there exists a solution h of this linear system which is in some sense close to u. The following is a version of the A-caloric approximation lemma for the model situation at the lateral boundary.

Lemma 4.1. *Given* $\varepsilon > 0$, $0 < \nu \leqslant L$ *and* $p \geqslant 2$ *there exists a positive function* $\delta = \delta(n, p, \nu, L, \varepsilon) \in (0, 1]$ *with the following property*: *Whenever* A *is a bilinear form on* $\mathbb{R}^{Nn}$ *which is strongly elliptic with ellipticity constant* $\nu > 0$ *and upper bound* L, *i.e.*

$$\nu|w|^2 \leqslant \langle Aw, w\rangle \quad \text{and} \quad \langle Aw, \widetilde{w}\rangle \leqslant L|w||\widetilde{w}|$$

holds whenever $w, \widetilde{w} \in \mathbb{R}^{Nn}$ *and* $u \in L^p(\Lambda_{\varrho^2}(t_0); W^{1,p}(B^+_\varrho(x_0), \mathbb{R}^N))$ *with* $u \equiv 0$ *on the lateral boundary* $\Gamma_\varrho(z_0)$ *with* $z_0 \in \Gamma$ *and*

$$\fint_{Q^+_\varrho(z_0)} |Du|^2 + \gamma^{p-2}|Du|^p \, dz \leqslant 1,$$

where $0 < \gamma \leqslant 1$, *is approximately* A-*caloric in the sense that*

$$\left| \fint_{Q^+_\varrho(z_0)} u \cdot \varphi_t - \langle ADu, D\varphi\rangle \, dz \right| \leqslant \delta \sup_{Q^+_\varrho(z_0)} |D\varphi|, \quad \text{for every } \varphi \in C_0^\infty\big(Q^+_\varrho(z_0); \mathbb{R}^N\big),$$

then there exists an A-*caloric map* $h \in L^p(\Lambda_{(\varrho/2)^2}(t_0); W^{1,p}(B^+_{\varrho/2}(x_0); \mathbb{R}^N))$, *i.e.*

$$\int_{Q^+_{\varrho/2}(z_0)} h \cdot \varphi_t - \langle ADh, D\varphi\rangle \, dz = 0 \qquad \text{for every } \varphi \in C_0^\infty\big(Q^+_{\varrho/2}(z_0); \mathbb{R}^N\big),$$

with $h \equiv 0$ *on* $\Gamma_{\varrho/2}(z_0)$ *satisfying*

$$\fint_{Q^+_{\varrho/2}(z_0)} |Dh|^2 + \gamma^{p-2}|Dh|^p \, dz \leqslant 2 \cdot 2^{n+2}$$

and

$$\fint_{Q^+_{\varrho/2}(z_0)} \left|\frac{u-h}{\varrho/2}\right|^2 + \gamma^{p-2}\left|\frac{u-h}{\varrho/2}\right|^p dz \leqslant \varepsilon.$$

Proof. Without loss of generality we can assume that $z_0 = 0$ and $\varrho = 1$. Otherwise we rescale u to Q_1^+ via $W(x,t) \equiv \varrho^{-1}u(x_0+\varrho x, t_0+\varrho^2 t)$ to obtain the existence of an A-caloric map H on $Q^+_{1/2}$ with $H \equiv 0$ on $\Gamma_{1/2}$. Rescaling back via $h(z) = \varrho H(\frac{x-x_0}{\varrho}, \frac{t-t_0}{\varrho^2})$ to $Q^+_{\varrho/2}(z_0)$ then yields the result.

Were the lemma false, there would exist $\varepsilon > 0$ and sequences $(A_j)_{j\in\mathbb{N}}$ of bilinear forms on $\mathbb{R}^{Nn}$ with uniform ellipticity constant $\nu > 0$ and upper bound L, $(v_j)_{j\in\mathbb{N}}$ with $v_j \in L^p(\Lambda_1; W^{1,p}(B_1^+;\mathbb{R}^N))$ satisfying $v_j \equiv 0$ on Γ_1 and $\gamma_j \in (0,1]$ such that

$$\fint_{Q_1^+} |Dv_j|^2 + \gamma_j^{p-2}|Dv_j|^p\,dz \leqslant 1 \tag{4.1}$$

and

$$\left|\fint_{Q_1^+} v_j\cdot\varphi_t - \langle A_j Dv_j, D\varphi\rangle\,dz\right| \leqslant \frac{1}{j}\sup_{Q_1^+}|D\varphi|, \quad \text{for every } \varphi \in C_0^\infty\big(Q_1^+;\mathbb{R}^N\big), \tag{4.2}$$

but

$$\fint_{Q^+_{1/2}} 4|v_j - h|^2 + 2^p\gamma_j^{p-2}|v_j-h|^p\,dz > \varepsilon \tag{4.3}$$

for all A_j-caloric maps h on $Q^+_{1/2}$ with $h \equiv 0$ on $\Gamma_{1/2}$ and

$$\fint_{Q^+_{1/2}} |Dh|^2 + \gamma^{p-2}|Dh|^p\,dz \leqslant 2\cdot 2^{n+2}. \tag{4.4}$$

We let

$$\tilde v_j = \gamma_j^{\frac{p-2}{p}} v_j. \tag{4.5}$$

Then, from (4.1) and Poincaré's inequality, i.e. Lemma 2.3, we obtain that

$$\fint_{Q_1^+} |v_j|^2\,dz \leqslant \fint_{Q_1^+} |Dv_j|^2\,dz \leqslant 1 \quad \text{and} \quad \fint_{Q_1^+} |\tilde v_j|^p + |D\tilde v_j|^p\,dz \leqslant 2. \tag{4.6}$$

Passing to a subsequence (again labeled with j), we infer the existence of maps

$$v \in L^2\big(\Lambda_1; W^{1,2}\big(B_1^+;\mathbb{R}^N\big)\big) \quad \text{and} \quad \tilde v \in L^p\big(\Lambda_1; W^{1,p}\big(B_1^+;\mathbb{R}^N\big)\big),$$

of a bilinear form A on $\mathbb{R}^{Nn}$, and $\gamma \in [0,1]$, such that

$$\begin{cases} v_j \rightharpoonup v & \text{weakly in } L^2\big(Q_1^+;\mathbb{R}^N\big),\\ Dv_j \rightharpoonup Dv & \text{weakly in } L^2\big(Q_1^+;\mathbb{R}^{Nn}\big),\\ \tilde v_j \rightharpoonup \tilde v & \text{weakly in } L^p\big(Q_1^+;\mathbb{R}^N\big),\\ D\tilde v_j \rightharpoonup D\tilde v & \text{weakly in } L^p\big(Q_1^+;\mathbb{R}^{Nn}\big),\\ A_j \to A & \text{as bilinear forms on } \mathbb{R}^{Nn},\\ \gamma_j \to \gamma & \text{in } [0,1]. \end{cases} \tag{4.7}$$

Moreover, the trace theorem yields $v \equiv 0 \equiv \tilde{v}$ on Γ_1. Next, we will identify the weak limit $\tilde{v}$. To be more precise, we will show

$$\tilde{v} = \gamma^{\frac{p-2}{p}} v, \tag{4.8}$$

where we define $\gamma^{\frac{p-2}{p}} = 1$ when $p = 2$ and $\gamma = 0$. From $(4.7)_1$, $(4.7)_2$ and $(4.7)_6$ we deduce $\tilde{v}_j = \gamma_j^{(p-2)/p} v_j \rightharpoonup \gamma^{(p-2)/p} v$ weakly in $L^2(\Lambda_1; W^{1,2}(B_1^+; \mathbb{R}^N))$. Since $\tilde{v}_j \rightharpoonup \tilde{v}$ weakly in $L^p(\Lambda_1; W^{1,p}(B_1^+; \mathbb{R}^N))$ by $(4.7)_3$ and $(4.7)_4$, we conclude (4.8).

Since $f \mapsto ⨍_{Q_1^+} |f|^2\, dz$ is weakly lower semicontinuous with respect to weak convergence in L^2 and $f \mapsto ⨍_{Q_1^+} |f|^p\, dz$ is weakly lower semicontinuous with respect to weak convergence in L^p we obtain from (4.1) and (4.8) that

$$⨍_{Q_1^+} |Dv|^2 + \gamma^{p-2} |Dv|^p\, dz \leqslant 1.$$

Our next aim is to show that v is A-caloric on Q_1^+. To this end we observe, for $\varphi \in C_0^\infty(Q_1^+, \mathbb{R}^N)$, it holds

$$⨍_{Q_1^+} v \cdot \varphi_t - \langle ADv, D\varphi\rangle\, dz = ⨍_{Q_1^+} (v - v_j) \cdot \varphi_t - \big\langle A(Dv - Dv_j), D\varphi\big\rangle\, dz - ⨍_{Q_1^+} \big\langle (A - A_j) Dv_j, D\varphi\big\rangle\, dz$$
$$+ ⨍_{Q_1^+} v_j \cdot \varphi_t - \langle A_j Dv_j, D\varphi\rangle\, dz.$$

The first term on the right-hand side vanishes as $j \to \infty$ due to the weak convergence of v_j to v in $L^2(Q_1^+; \mathbb{R}^N)$ and Dv_j to Dv in $L^2(Q_1^+; \mathbb{R}^{Nn})$. The same holds for the second term appearing in the right-hand side in view of the convergence $A_j \to A$ and the uniform bound of Dv_j in $L^2(Q_1^+; \mathbb{R}^{Nn})$. The third term vanishes as $j \to \infty$ due to (4.2), i.e. the fact that v_j is approximately A_j-caloric. This proves that v is an A-caloric map on Q_1^+, i.e.

$$\int_{Q_1^+} v \cdot \varphi_t - \langle ADv, D\varphi\rangle\, dz = 0 \quad \text{for every } \varphi \in C_0^\infty\big(Q_1^+; \mathbb{R}^N\big), \tag{4.9}$$

satisfying $v = 0$ on the lateral boundary Γ_1. From the regularity theory for linear parabolic systems with constant coefficients developed in Section 3, i.e. from Theorem 3.1, we infer that v is smooth on any smaller half-cylinder, in particular that $v \in C^\infty(Q_1^+ \cup \Gamma_1; \mathbb{R}^N)$.

We next turn our attention to the compactness properties of $(v_j)_{j\in\mathbb{N}}$ respectively $(\tilde{v}_j)_{j\in\mathbb{N}}$ with respect to L^2 respectively L^p convergence on Q_1^+. Since v_j is possibly not differentiable with respect to t the usual compactness argument which is based on an application of Poincaré's inequality cannot be used at this stage. Instead we apply a compactness argument of J. Simon in [34]. For that purpose we have to ensure that for $s = 2$ and $s = p$,

$$\lim_{h\downarrow 0} \int_{-1}^{1-h} \gamma_j^{s-2} \big\| v_j(\cdot, t+h) - v_j(\cdot, t)\big\|_{L^s(B_1^+)}^s\, dt = 0 \tag{4.10}$$

uniformly with respect to j. To this end we estimate the first term appearing on the left-hand side of (4.2) by the remaining terms, using Hölder's inequality, (4.1), $s \geqslant 2$ and the fact that $\gamma_j \leqslant 1$,

$$\left| \gamma_j^{\frac{s-2}{s}} ⨍_{Q_1^+} v_j \cdot \varphi_t\, dz \right| \leqslant |A_j| ⨍_{Q_1^+} \gamma_j^{\frac{s-2}{s}} |Dv_j||D\varphi|\, dz + \frac{1}{j} \sup_{t\in\Lambda_1} \big\| D\varphi(\cdot, t)\big\|_{L^\infty(B_1^+)}$$

$$\leqslant |A_j| \bigg(\fint_{Q_1^+} \gamma_j^{s-2} |Dv_j|^s \, dz \bigg)^{\frac{1}{s}} \bigg(\fint_{Q_1^+} |D\varphi|^{s'} dz \bigg)^{\frac{1}{s'}} + \frac{1}{j} \sup_{t \in \Lambda_1} \big\| D\varphi(\cdot, t) \big\|_{L^\infty(B_1^+)}$$

$$\leqslant |A_j| \bigg(\int_{\Lambda_1} \|D\varphi\|_{L^{s'}(B_1^+)}^{s'} \, dt \bigg)^{\frac{1}{s'}} + \frac{1}{j} \sup_{t \in \Lambda_1} \big\| D\varphi(\cdot, t) \big\|_{L^\infty(B_1^+)},$$

where $\varphi \in C_0^\infty(Q_1^+; \mathbb{R}^N)$ and $s' = \frac{s-1}{s}$ is the Hölder conjugate to s. Given $-1 < \tau_1 < \tau_2 < 1$ and $\theta > 0$ with $\theta \leqslant \min\{1 + \tau_1, 1 - \tau_2\}$ we define

$$\zeta_\theta(t) \equiv \begin{cases} 0, & \text{for } -1 \leqslant t \leqslant \tau_1 - \theta, \\ \frac{1}{\theta}(t - \tau_1 + \theta), & \text{for } \tau_1 - \theta \leqslant t \leqslant \tau_1, \\ 1, & \text{for } \tau_1 \leqslant t \leqslant \tau_2, \\ -\frac{1}{\theta}(t - \tau_2 - \theta), & \text{for } \tau_2 \leqslant t \leqslant \tau_2 + \theta, \\ 0, & \text{for } \tau_2 + \theta \leqslant t \leqslant 1. \end{cases}$$

Then, in the preceding estimate we choose the test-function $\varphi(x,t) \equiv \psi(x)\zeta_\theta(t)$ with $\psi \in C_0^\infty(B_1^+; \mathbb{R}^N)$ and obtain

$$\bigg| \gamma_j^{\frac{s-2}{s}} \fint_{B_1^+} \frac{1}{\theta} \bigg(\int_{\tau_1 - \theta}^{\tau_1} v_j(x,t) \, dt - \int_{\tau_2}^{\tau_2 + \theta} v_j(x,t) \, dt \bigg) \cdot \psi(x) \, dx \bigg|$$

$$\leqslant |A_j| \bigg(\int_{\Lambda_1} \zeta_\theta(t)^{s'} \, dt \bigg)^{\frac{1}{s'}} \|D\psi\|_{L^{s'}(B_1^+)} + \frac{1}{j} \|D\psi\|_{L^\infty(B_1^+)} \sup_{t \in \Lambda_1} \zeta_\theta(t)$$

$$\leqslant |A_j| (\tau_2 - \tau_1 + 2\theta)^{\frac{1}{s'}} \|D\psi\|_{L^{s'}(B_1^+)} + \frac{1}{j} \|D\psi\|_{L^\infty(B_1^+)}.$$

Passing to the limit $\theta \downarrow 0$ yields

$$\bigg| \gamma_j^{\frac{s-2}{s}} \int_{B_1^+} \big(v_j(\cdot, \tau_2) - v_j(\cdot, \tau_1) \big) \cdot \psi \, dx \bigg| \leqslant |A_j| (\tau_2 - \tau_1)^{\frac{1}{s'}} \|D\psi\|_{L^{s'}} + \frac{1}{j} \|D\psi\|_{L^\infty} \tag{4.11}$$

for all $\psi \in C_0^\infty(B_1^+; \mathbb{R}^N)$ and for a.e. $-1 < \tau_1 < \tau_2 < 1$. This estimate would already imply (4.10), if on the right-hand side we would have $\|D\psi\|_{L^{s'}}$ instead of $\|D\psi\|_{L^\infty}$. In order to derive (4.10) from the weaker estimate (4.11) we use an interpolation argument. The Sobolev embedding $W_0^{\ell,s'}(B_1^+) \hookrightarrow W^{1,\infty}(B_1^+)$ for $\ell > \frac{n}{s'} + 1$ yields $\|D\psi\|_{L^\infty(B_1^+)} \leqslant c(n, \ell, p) \|\psi\|_{W_0^{\ell,s'}(B_1^+)}$, and therefore (4.11) implies

$$\bigg| \gamma_j^{\frac{s-2}{s}} \int_{B_1^+} \big(v_j(\cdot, \tau_2) - v_j(\cdot, \tau_1) \big) \cdot \psi \, dx \bigg| \leqslant \tilde{c} \bigg(|A_j| (\tau_2 - \tau_1)^{\frac{1}{s'}} + \frac{1}{j} \bigg) \|\psi\|_{W_0^{\ell,s'}(B_1^+)}$$

for a.e $\tau_1, \tau_2 \in \Lambda_1$ with a constant $\tilde{c} = \tilde{c}(n, \ell, p)$. Since $C_0^\infty(B_1^+; \mathbb{R}^N)$ is dense in $W_0^{\ell,s'}(B_1^+; \mathbb{R}^N)$, the preceding estimate also holds for all $\psi \in W_0^{\ell,s'}(B_1^+; \mathbb{R}^N)$ and implies an estimate of the $W^{-\ell,s}$-norm of $v_j(\cdot, \tau_2) - v_j(\cdot, \tau_1)$ of the form

$$\gamma_j^{\frac{s-2}{s}} \big\| v_j(\cdot, \tau_2) - v_j(\cdot, \tau_1) \big\|_{W^{-\ell,s}(B_1^+, \mathbb{R}^N)} = \sup_{\|\psi\|_{W_0^{\ell,s'}(B_1^+, \mathbb{R}^N)} \leqslant 1} \bigg| \gamma_j^{\frac{s-2}{s}} \int_{B_1^+} \big(v_j(\cdot, \tau_2) - v_j(\cdot, \tau_1) \big) \cdot \psi \, dx \bigg|$$

$$\leqslant \tilde{c} \bigg(|A_j| (\tau_2 - \tau_1)^{\frac{1}{s'}} + \frac{1}{j} \bigg).$$

Having arrived at this stage we interpolate between the spaces $W^{1,s}$ and $W^{-\ell,s}$. For $\mu > 0$ we obtain

$$\begin{aligned}
&\gamma_j^{s-2} \int_{-1}^{1-h} \|v_j(\cdot,t+h) - v_j(\cdot,t)\|^s_{L^s(B_1^+)}\,dt \\
&\leqslant \mu\gamma_j^{s-2} \int_{-1}^{1-h} \|v_j(\cdot,t+h) - v_j(\cdot,t)\|^s_{W^{1,s}(B_1^+)}\,dt + c\left(\frac{1}{\mu}\right)\gamma_j^{s-2} \int_{-1}^{1-h} \|v_j(\cdot,t+h) - v_j(\cdot,t)\|^s_{W^{-\ell,s}(B_1^+)}\,dt \\
&\leqslant 2^{s-1}\mu \int_{-1}^{1} \gamma_j^{s-2} \|v_j(\cdot,t)\|^s_{W^{1,s}(B_1^+)}\,dt + c\left(\frac{1}{\mu}\right)\tilde{c}^s \left(|A_j| h^{\frac{1}{s'}} + \frac{1}{j}\right)^s \\
&\leqslant 2^{s-1}\left[\mu + c\left(\frac{1}{\mu}\right)\tilde{c}^s\left(|A_j|^s h^{s-1} + \frac{1}{j^s}\right)\right].
\end{aligned}$$

Here we have also used (4.6). Now we show that the integral appearing on the left-hand side of the preceding inequality converges uniformly (with respect to j) to zero as $h \downarrow 0$. From the convergence of $(A_j)_{j\in\mathbb{N}}$ we infer a bound $|A_j| \leqslant a < \infty$ for all $j \in \mathbb{N}$. Now, given $\theta > 0$ we first choose $\mu = \frac{\theta}{3\cdot 2^{s-1}}$ and then $j_0 \in \mathbb{N}$ large enough to satisfy $\frac{2^{s-1}c(\mu)\tilde{c}^s}{j^s} < \frac{\theta}{3}$ for all $j \geqslant j_0$. Furthermore, we choose $h_1 > 0$ such that $2^{s-1}c(\mu)\tilde{c}^s a^s h^{s-1} < \frac{\theta}{3}$ for all $0 < h < h_1$. Finally, we choose $h_2 > 0$ in such a way that

$$\int_{-1}^{1-h} \gamma_j^{s-2} \|v_j(\cdot,t+h) - v_j(\cdot,t)\|^s_{L^s(B_1^+)}\,dt < \theta$$

holds for all $0 < h < h_2$, $j = 1, \ldots, j_0 - 1$. With these choices we infer for all $j \in \mathbb{N}$ and $0 < h < h_0 \equiv \min\{h_1, h_2\}$ that there holds

$$\int_{-1}^{1-h} \gamma_j^{s-2} \|v_j(\cdot,t+h) - v_j(\cdot,t)\|^s_{L^s(B_1^+)}\,dt < \theta,$$

yielding (4.10). Since we also know that the sequence $(\gamma_j^{(s-2)/s} v_j)_{j\in\mathbb{N}}$ is uniformly bounded in $L^1_{\mathrm{loc}}(\Lambda_1; W^{1,p}(B_1^+, \mathbb{R}^N))$, Theorem 3 from [34] applied to $X = W^{1,p}(B_1^+; \mathbb{R}^N)$, $B = L^s(B_1^+; \mathbb{R}^N)$ and $F = (\gamma_j^{(s-2)/s} v_j)_{j\in\mathbb{N}}$ ensures the existence of a subsequence $(\gamma_j^{(s-2)/s} v_j)_{j\in\mathbb{N}}$ (also labeled with j), converging strongly in $L^s(B_1^+; \mathbb{R}^N)$ for $s = 2$ and $s = p$, i.e.

$$\begin{cases} v_j \to v & \text{strongly in } L^2(Q_1^+; \mathbb{R}^N), \\ \tilde{v}_j \to \tilde{v} & \text{strongly in } L^p(Q_1^+; \mathbb{R}^N), \end{cases} \tag{4.12}$$

where we have recalled the definition of $\tilde{v}_j$, i.e. $\tilde{v}_j = \gamma_j^{(p-2)/p} v_j$.

From (4.9) we already know that v is an A-caloric map which by Theorem 3.1 is smooth on $Q_1^+ \cup \Gamma_1$. In the following we will derive the contradiction by constructing appropriate A_j-caloric maps from v. This is done as follows: by $w_j \in L^2(\Lambda_{(3/4)^2}; W^{1,2}(B_{3/4}^+, \mathbb{R}^N))$ we denote the unique solution of the following Cauchy–Dirichlet problem

$$\begin{cases} \displaystyle\int_{Q_{3/4}^+} w_j \cdot \varphi_t - \langle A_j Dw_j, D\varphi\rangle\,dz = 0 & \text{for every } \varphi \in C_0^\infty(Q_{3/4}^+, \mathbb{R}^N), \\ w_j = v & \text{on } \partial_{\mathcal{P}} Q_{3/4}^+. \end{cases}$$

Since w_j is a solution of a linear parabolic system with $w_j = v = 0$ on $\Gamma_{3/4}$ we infer from Corollary 3.4 that $w_j \in C^\infty(Q_{3/4}^+ \cup \Gamma_{3/4}, \mathbb{R}^N)$. Our next aim is to prove that

$$Dw_j \to Dv \quad \text{strongly in } L^2(Q_{3/4}^+, \mathbb{R}^{Nn}). \tag{4.13}$$

For this we shall exploit that the difference $w_j - v$ is a weak solution of the following inhomogeneous parabolic system

$$\int_{Q^+_{3/4}} (w_j - v) \cdot \varphi_t - \langle A_j(Dw_j - Dv), D\varphi\rangle \, dz = \int_{Q^+_{3/4}} \langle (A_j - A)Dv, D\varphi\rangle \, dz,$$

for all $\varphi \in C_0^\infty(Q^+_{3/4}, \mathbb{R}^N)$. Since w_j and v agree on the parabolic boundary of $Q^+_{3/4}$ we can formally test the last relation by $\varphi = w_j - v$. Note that this procedure can be made rigorous by the use of Steklov averages. Exploiting the ellipticity of A_j we infer from the preceding equality in a standard way that

$$\begin{aligned} \nu \int_{Q^+_{3/4}} |Dw_j - Dv|^2 \, dz &\leqslant |A_j - A| \int_{Q^+_{3/4}} |Dv||Dw_j - Dv| \, dz \\ &\leqslant |A_j - A| \Big(\int_{Q^+_{3/4}} |Dv|^2 \, dz \Big)^{\frac{1}{2}} \Big(\int_{Q^+_{3/4}} |Dw_j - Dv|^2 \, dz \Big)^{\frac{1}{2}} \end{aligned}$$

from which (4.13) immediately follows since $A_j \to A$. Since w_j and v agree on the parabolic boundary of $Q^+_{3/4}$ we can apply Poincaré's inequality slicewise to find that also

$$w_j \to v \quad \text{strongly in } L^2\big(Q^+_{3/4}, \mathbb{R}^N\big). \tag{4.14}$$

Next we show that

$$Dw_j \to Dv \quad \text{and} \quad w_j \to v \quad \text{strongly in } L^p\big(Q^+_{1/2}\big). \tag{4.15}$$

This is a consequence of the a priori estimates up to the lateral boundary for linear parabolic systems from Remark 3.2 which gives together with the Sobolev embedding (with $k \in \mathbb{N}$ such that $k > \frac{n+1}{2}$)

$$\begin{aligned} \sup_{Q^+_{1/2}} \big(|w_j| + |Dw_j|\big) &\leqslant c\big(\|w_j\|_{W^{k,2}(Q^+_{1/2})} + \|Dw_j\|_{W^{k,2}(Q^+_{1/2})}\big) \\ &\leqslant c\|w_j\|_{L^2(Q^+_{3/4})} \\ &\leqslant c\big(\|v\|_{L^2(Q^+_{3/4})}, n, N, p, \nu, L\big). \end{aligned}$$

Now, (4.15) is a consequence of (4.13) and (4.14), interpolating L^p between L^2 and L^q for some $q > p$.

At this stage it is worth mentioning that the convergence $D^\ell w_j \to D^\ell v$ indeed is uniform for any $\ell \in \mathbb{N}_0$. This could be inferred by the use of a finer estimate for the non-homogeneous linear parabolic system for $w_j - v$ from above. More precisely, by an L^∞ estimate for non-homogeneous linear parabolic systems (instead of the one for homogeneous systems form Remark 3.2) we could estimate the supremum of $|D^\ell w_j - D^\ell v|$ on $Q^+_{1/2}$ in terms of $\|w_j - v\|_{L^2(Q^+_{3/4})}$ and the right-hand side both vanishing in the limit $j \to \infty$. But, since we do need the uniform convergence in the following we shall not accomplish the argument in detail here.

We now have

$$⨍_{Q^+_{1/2}} |v_j - w_j|^2 \, dz \leqslant 2\Big[⨍_{Q^+_{1/2}} |v_j - v|^2 \, dz + ⨍_{Q^+_{1/2}} |v - w_j|^2 \, dz \Big] \to 0.$$

Moreover, due to the definition of $\tilde v_j$, (4.8), the strong convergence $\tilde v_j \to \tilde v$ from (4.12) and the convergence $\gamma_j^{\frac{p-2}{2}} w_j \to \gamma^{\frac{p-2}{2}} v$ in $L^p(Q^+_{1/2}, \mathbb{R}^N)$ we obtain

$$\fint_{Q_{1/2}^+} \gamma_j^{p-2}|v_j - w_j|^p\,dz = \fint_{Q_{1/2}^+} \big|\tilde v_j - \gamma_j^{\frac{p-2}{2}} w_j\big|^p\,dz$$
$$\leqslant 2^{p-1}\bigg[\fint_{Q_{1/2}^+} |\tilde v_j - \tilde v|^p\,dz + \fint_{Q_{1/2}^+} \big|\gamma_j^{\frac{p-2}{2}} v - \gamma_j^{\frac{p-2}{2}} w_j\big|^p\,dz\bigg] \to 0$$

in the limit $j \to \infty$, which in combination yields

$$\lim_{j\to\infty} \fint_{Q_{1/2}^+} 4|v_j - w_j|^2 + 2^p\gamma_j^{p-2}|v_j - w_j|^p\,dz = 0. \tag{4.16}$$

Finally, we have

$$\lim_{j\to\infty} \fint_{Q_{1/2}^+} |Dw_j|^2 + \gamma_j^{p-2}|Dw_j|^p\,dz = \fint_{Q_{1/2}^+} |Dv|^2 + \gamma^{p-2}|Dv|^p\,dz$$
$$\leqslant 2^{n+2}\fint_{Q_1^+} |Dv|^2 + \gamma^{p-2}|Dv|^p\,dz$$
$$\leqslant 2^{n+2},$$

and therefore for $j \gg 1$ large enough there holds

$$\fint_{Q_{1/2}^+} |Dw_j|^2 + \gamma_j^{p-2}|Dw_j|^p\,dz \leqslant 2\cdot 2^{n+2}.$$

Hence, for j large enough w_j is an A_j-caloric map on $Q_{1/2}^+$ with $w_j = 0$ on the lateral boundary $\Gamma_{1/2}$ satisfying (4.4) and (4.16). Since (4.16) contradicts (4.3) for large j, we have constructed the desired contradiction, proving the assertion of the lemma. □

4.1.2. Caccioppoli inequality

As usual we need a suitable Caccioppoli inequality. In the lateral boundary situation it is convenient to approximate u by a linear map which is zero on Γ_1 and therefore of the form ξx_n with $\xi \in \mathbb{R}^N$.

Lemma 4.2. *Suppose that $u \in L^p(\Lambda_1; W^{1,p}(B_1^+;\mathbb{R}^N))$ is a weak solution of the non-linear parabolic system* (2.5) *with $u = 0$ on the lateral boundary Γ_1, where the structure conditions* (1.4)–(1.6) *are in force. Moreover let $M > 0$. Then, for any $\xi \in \mathbb{R}^N$ with $|\xi| \leqslant M$, $z_0 \in \Gamma_1$ and $\varrho \in (0,1)$ such that $Q_\varrho(z_0) \subset Q_1$ there holds*

$$\fint_{Q_{\varrho/2}^+(z_0)} |Du - \xi\otimes e_n|^2 + |Du - \xi\otimes e_n|^p\,dz \leqslant c_{Cac}\bigg(\fint_{Q_\varrho^+(z_0)} \bigg|\frac{u-\xi x_n}{\varrho}\bigg|^2 + \bigg|\frac{u-\xi x_n}{\varrho}\bigg|^p\,dz + \varrho^{2\beta}\bigg),$$

where $c_{Cac} = (1 + \|g_t\|^2_{L^{2,2-2\beta}(Q_1^+)})c(n,p,L/\nu,M,H(M),\kappa_{M+1})$.

Proof. The following calculations will be somehow formal; they can be made rigorous using a mollifying procedure in time, e.g. via Steklov averages. Since this is a standard procedure and for the sake of brevity we will proceed formally. Without loss of generality we can assume that $z_0 = 0$. We choose two cut-off functions $\eta \in C_0^\infty(B_\varrho)$ and $\zeta \in C_0^1(\Lambda_{\varrho^2})$ such that $\eta \equiv 1$ on $B_{\varrho/2}$, $0 \leqslant \eta \leqslant 1$, $|D\eta| \leqslant c/\varrho$, $\zeta \equiv 1$ on $\Lambda_{(\varrho/2)^2}$, $0 \leqslant \zeta \leqslant 1$ and $|\zeta_t| \leqslant 2/\varrho^2$. Choosing the test-function $\varphi(x,t) = \eta^p(x)\zeta^2(t)(u(x,t) - \xi x_n)$ in (2.5) we obtain

$$\int_{Q_\varrho^+} \big\langle a(z,u,Du),(Du-\mathfrak{X})\big\rangle \eta^p\zeta^2\,dz$$
$$= -p\int_{Q_\varrho^+} \big\langle a(z,u,Du),\eta^{p-1}D\eta\otimes(u-\xi x_n)\big\rangle\zeta^2\,dz + \int_{Q_\varrho^+}(u\cdot\varphi_t - g_t\cdot\varphi)\,dz,$$

where we have abbreviated $\mathfrak{X} = D(\xi x_n) = \xi\otimes e_n$. Moreover, we have

$$\int_{Q_\varrho^+} \big\langle a(z,u,\mathfrak{X}),Du-\mathfrak{X}\big\rangle\eta^p\zeta^2\,dz = \int_{Q_\varrho^+}\big\langle a(z,u,\mathfrak{X}),D\varphi\big\rangle\,dz - p\int_{Q_\varrho^+}\big\langle a(z,u,\mathfrak{X}),\eta^{p-1}D\eta\otimes(u-\xi x_n)\big\rangle\zeta^2\,dz$$

and $\int_{Q_\varrho^+}\langle a((0,t),0,\mathfrak{X}),D\varphi\rangle\,dz = 0$ for all $t\in\Lambda_{\varrho^2}$. Adding the preceding identities and using also that

$$\int_{Q_\varrho^+}\xi x_n\partial_t\varphi\,dz = 0,$$

we deduce

$$\int_{Q_\varrho^+}\big\langle a(z,u,Du)-a(z,u,\mathfrak{X}),Du-\mathfrak{X}\big\rangle\eta^p\zeta^2\,dz$$
$$= -p\int_{Q_\varrho^+}\big\langle a(z,u,Du)-a(z,u,\mathfrak{X}),\eta^{p-1}D\eta(u-\xi x_n)\big\rangle\zeta^2\,dz$$
$$- \int_{Q_\varrho^+}\big\langle a\big((x,t),u,\mathfrak{X}\big)-a\big((0,t),0,\mathfrak{X}\big),D\varphi\big\rangle\,d(x,t) + \int_{Q_\varrho^+} g_t\cdot\varphi\,dz + \int_{Q_\varrho^+}(u-\xi x_n)\partial_t\varphi\,dz$$
$$=: I + II + III + IV, \tag{4.17}$$

with the obvious meaning of I–IV and $z=(x,t)$. In the sequel we shall derive estimates for I–IV. Thereby we take $\mu\in(0,1]$.

Estimate for I: We first rewrite $I = I_1 + I_2 + I_3$ with

$$I_1 := -p\int_{Q_\varrho^+}\big\langle a(z,u,Du)-a(z,\xi x_n,Du),\eta^{p-1}D\eta\otimes(u-\xi x_n)\big\rangle\zeta^2\,dz,$$
$$I_2 := -p\int_{Q_\varrho^+}\big\langle a(z,\xi x_n,Du)-a(z,\xi x_n,\mathfrak{X}),\eta^{p-1}D\eta\otimes(u-\xi x_n)\big\rangle\zeta^2\,dz,$$
$$I_3 := -p\int_{Q_\varrho^+}\big\langle a(z,\xi x_n,\mathfrak{X})-a(z,u,\xi\otimes e_n),\eta^{p-1}D\eta\otimes(u-\xi x_n)\big\rangle\zeta^2\,dz.$$

Estimate for I_1: To estimate I_1 we use (1.6), $|\mathfrak{X}|\leqslant M$ and $|D\eta|\leqslant c/\varrho$ to obtain

$$|I_1| \leqslant cL\int_{Q_\varrho^+}\theta\big(|u|+|\xi x_n|,|u-\xi x_n|\big)\big(1+M^{p-1}+|Du-\mathfrak{X}|^{p-1}\big)\eta^{p-1}\left|\frac{u-\xi x_n}{\varrho}\right|\zeta^2\,dz$$
$$=: I_{1,1} + I_{1,2},$$

where $c = c(p)$ and $I_{1,1}$ respectively $I_{1,2}$ is the integral obtained by replacing $|Du-\mathfrak{X}|^{p-1}$ respectively $1+M^{p-1}$ by zero. To estimate $I_{1,1}$ we use (2.2), $|\xi x_n|\leqslant|\xi|\leqslant M$ on B_ϱ (note that $\varrho\leqslant 1$ by assumption) and Young's inequality to obtain

$$I_{1,1} \leqslant cL \int_{Q_\varrho^+} \left|\frac{u-\xi x_n}{\varrho}\right|^{1+\beta} \varrho^\beta \eta^{p-1}\zeta^2\,dz \leqslant cL\biggl(\int_{Q_\varrho^+}\left|\frac{u-\xi x_n}{\varrho}\right|^2 dz + \varrho^{\frac{2\beta}{1-\beta}}\bigl|Q_\varrho^+\bigr|\biggr),$$

where $c = c(p)K(2M+1)(1+M^{p-1}) = c(p)H(M)$. The estimate for $I_{1,2}$ is achieved by using $\theta \leqslant 1$ instead of (2.2) and Young's inequality: for $c = c(p)$ we have

$$\begin{aligned} I_{1,2} &\leqslant cL \int_{Q_\varrho^+} |Du-\mathfrak{X}|^{p-1}\left|\frac{u-\xi x_n}{\varrho}\right|\eta^{p-1}\zeta^2\,dz \\ &\leqslant \mu \int_{Q_\varrho^+} |Du-\mathfrak{X}|^p\eta^p\zeta^2\,dz + cL^p\mu^{1-p}\int_{Q_\varrho^+}\left|\frac{u-\xi x_n}{\varrho}\right|^p dz. \end{aligned}$$

Estimate for I_2: Here we decompose $Q_\varrho^+ = S_1 \cup S_2$ where

$$S_1 \equiv \bigl\{z \in Q_\varrho^+ \colon \bigl|Du(z)-\mathfrak{X}\bigr| \leqslant 1\bigr\}, \qquad S_2 \equiv \bigl\{z \in Q_\varrho^+ \colon \bigl|Du(z)-\mathfrak{X}\bigr| > 1\bigr\}$$

and rewrite I_2 as follows:

$$\begin{aligned} I_2 &\leqslant p\int_{Q_\varrho^+} \bigl|a(z,\xi x_n,Du)-a(z,\xi x_n,\mathfrak{X})\bigr|\eta^{p-1}|D\eta||u-\xi x_n|\zeta^2\,dz \\ &= p\int_{S_1}(\ldots)\,dz + p\int_{S_2}(\ldots)\,dz \\ &=: I_{2,1}+I_{2,2}. \end{aligned}$$

For the estimate of $I_{2,1}$ we use (1.5), Young's inequality, $|\xi x_n| \leqslant M$ for $x \in B_\varrho^+$ and $|\mathfrak{X}+s(Du(z)-\mathfrak{X})| \leqslant M+1$ for $z \in S_1$ and $0 \leqslant s \leqslant 1$ in order to obtain

$$\begin{aligned} I_{2,1} &\leqslant p\int_{S_1}\left|\int_0^1 \partial_w a\bigl(z,\xi x_n,\mathfrak{X}+s(Du-\mathfrak{X})\bigr)(Du-\mathfrak{X})\,ds\right|\eta^{p-1}|D\eta||u-\xi x_n|\zeta^2\,dz \\ &\leqslant cL\kappa_{M+1}\int_{S_1}|Du-\mathfrak{X}|\eta^{p-1}\left|\frac{u-\xi x_n}{\varrho}\right|\zeta^2\,dz \\ &\leqslant \mu\int_{Q_\varrho^+}|Du-\mathfrak{X}|^2\eta^p\zeta^2\,dz + c(p)\mu^{-1}L^2\kappa_{M+1}^2\int_{Q_\varrho^+}\left|\frac{u-\xi x_n}{\varrho}\right|^2 dz. \end{aligned}$$

To derive the estimate for $I_{2,2}$ we use the growth condition (1.3) instead of (1.5), the assumption $|\xi| \leqslant M$, the fact that $|Du-\mathfrak{X}| > 1$ on S_2 and Young's inequality; for a constant $c = c(p)(1+M^{p-1})^p$ we have

$$\begin{aligned} I_{2,2} &\leqslant pL\int_{S_2}\bigl(2+M^{p-1}+|Du|^{p-1}\bigr)\eta^{p-1}\left|\frac{u-\xi x_n}{\varrho}\right|\zeta^2\,dz \\ &\leqslant c(p)L\bigl(1+M^{p-1}\bigr)\int_{S_2}|Du-\mathfrak{X}|^{p-1}\eta^{p-1}\left|\frac{u-\xi x_n}{\varrho}\right|\zeta^2\,dz \\ &\leqslant \mu\int_{Q_\varrho^+}|Du-\mathfrak{X}|^p\eta^p\zeta^2\,dz + c\mu^{1-p}L\int_{Q_\varrho^+}\left|\frac{u-\xi x_n}{\varrho}\right|^p dz. \end{aligned}$$

Estimate for I_3: Here we use (2.3) (note that $|\xi x_n| \leqslant M$ and $|\mathfrak{X}| \leqslant M$ by assumption) and Young's inequality, to get, for a constant $c \equiv c(p)H(M)$,

$$I_3 \leqslant pL \int_{Q_\varrho^+} \theta\big(|u| + |\xi x_n|, |u - \xi x_n|\big)\big(1 + |\mathfrak{X}|^{p-1}\big)\eta^{p-1}|D\eta||u - \xi x_n|\zeta^2\,dz$$
$$\leqslant cL\bigg(\int_{Q_\varrho^+} \left|\frac{u - \xi x_n}{\varrho}\right|^2 dz + \varrho^{\frac{2\beta}{1-\beta}}\big|Q_\varrho^+\big|\bigg).$$

Combining the previous estimates we arrive at the final estimate for I

$$I \leqslant 2\mu \int_{Q_\varrho^+} \big(|Du - \mathfrak{X}|^2 + |Du - \mathfrak{X}|^p\big)\eta^p\zeta^2\,dz$$
$$+ c\big(L + \mu^{-1}L^2 + \mu^{1-p}L^p\big)\bigg(\int_{Q_\varrho^+} \left|\frac{u - \xi x_n}{\varrho}\right|^2 + \left|\frac{u - \xi x_n}{\varrho}\right|^p dz + \varrho^{\frac{2\beta}{1-\beta}}\big|Q_\varrho^+\big|\bigg),$$

where c depends on p, $H(M)$ and κ_{M+1}.

Estimate for II: Using the assumptions (1.6), (2.3) (note that $|\xi x_n| \leqslant M$ for $x \in B_\varrho^+$ and $|\mathfrak{X}| \leqslant M$), $|D\eta| \leqslant c/\varrho$, Young's inequality and $\varrho \leqslant 1$ we deduce

$$|II| \leqslant L\big(1 + M^{p-1}\big) \int_{Q_\varrho^+} \theta\big(|u|, \varrho + |u|\big)|D\varphi|\,dz$$
$$\leqslant cLK(1) \int_{Q_\varrho^+} \big(|u - \xi x_n|^\beta + \big(1+M^\beta\big)\varrho^\beta\big)\bigg(|Du - \mathfrak{X}|\eta^p + \left|\frac{u - \xi x_n}{\varrho}\right|\eta^{p-1}\bigg)\zeta^2\,dz$$
$$\leqslant \mu \int_{Q_\varrho^+} |Du - \mathfrak{X}|^2\eta^p\zeta^2\,dz + c\big(L + \mu^{-1}L^2\big)\bigg(\int_{Q_\varrho^+} \left|\frac{u - \xi x_n}{\varrho}\right|^2 dz + \varrho^{2\beta}\big|Q_\varrho^+\big|\bigg),$$

where the constant c depends on $p, M, K(1)$.

Estimate for III: Using Young's inequality and taking also the assumption $g_t \in L^{2,2-2\beta}(Q_1^+;\mathbb{R}^N)$ into account we obtain

$$III \leqslant \int_{Q_\varrho^+} |g_t||u - \xi x_n|\,dz \leqslant \int_{Q_\varrho^+} \left|\frac{u - \xi x_n}{\varrho}\right|^2 dz + \varrho^2 \int_{Q_\varrho^+} |g_t|^2\,dz$$
$$\leqslant \int_{Q_\varrho^+} \left|\frac{u - \xi x_n}{\varrho}\right|^2 dz + c(n)\varrho^{2\beta}\big|Q_\varrho^+\big|\varrho^{2-2\beta-(n+2)} \int_{Q_\varrho^+} |g_t|^2\,dz$$
$$\leqslant \int_{Q_\varrho^+} \left|\frac{u - \xi x_n}{\varrho}\right|^p dz + c(n)\|g_t\|^2_{L^{2,2-2\beta}(Q_1^+)}\varrho^{2\beta}\big|Q_\varrho^+\big|.$$

Estimate for IV: The integral IV can be rewritten as follows:

$$IV = \int_{Q_\varrho^+} |u - \xi x_n|^2\eta^p\partial_t\zeta^2\,dz + \frac{1}{2}\int_{Q_\varrho^+} \partial_t|u - \xi x_n|^2\eta^p\zeta^2\,dz$$
$$= \frac{1}{2}\int_{Q_\varrho^+} |u - \xi x_n|^2\eta^p\partial_t\zeta^2\,dz \leqslant 2L \int_{Q_\varrho^+} \left|\frac{u - \xi x_n}{\varrho}\right|^2 dz,$$

where we have used that $|\partial_t\zeta^2| = 2\zeta|\zeta_t| \leqslant 4/\varrho^2$ and $L \geqslant 1$ in the last line.

Lower bound for the left-hand side of (4.17): To estimate the left-hand side of (4.17) from below we use the monotonicity (2.1) of the vector field a. We find

$$\int_{-\varrho^2}^{t}\int_{B_\varrho^+}\langle a(z,u,Du)-a(z,u,\mathfrak{X}),(Du-\mathfrak{X})\rangle\eta^p\zeta^2\,dx\,d\tau \geqslant \frac{\nu}{c(p)}\int_{-\varrho^2}^{t}\int_{B_\varrho^+}\big(|Du-\mathfrak{X}|^2+|Du-\mathfrak{X}|^p\big)\eta^p\zeta^2\,dx\,d\tau.$$

Using the preceding estimates obtained for I–IV in (4.17) we arrive at

$$\begin{aligned}&\left(\frac{\nu}{c(p)}-3\mu\right)\int_{Q_\varrho^+}\big(|Du-\mathfrak{X}|^2+|Du-\mathfrak{X}|^p\big)\eta^p\zeta^2\,dz\\ &\quad\leqslant c\left(L+\frac{L^2}{\mu}+\frac{L^p}{\mu^{p-1}}\right)\left(\int_{Q_\varrho^+}\left|\frac{u-\xi x_n}{\varrho}\right|^2+\left|\frac{u-\xi x_n}{\varrho}\right|^p dz+\varrho^{2\beta}\big|Q_\varrho^+\big|\right),\end{aligned}$$

where $c=(1+\|g_t\|^2_{L^{2,2-2\beta}(Q_1^+)})c(n,p,M,H(M),\kappa_{M+1})$. Choosing μ as usual small enough (i.e. $\mu=\frac{\nu}{6c(p)}$) and recalling the choices of ζ and η respectively, the preceding estimate implies

$$\int_{Q_{\varrho/2}^+}\big(|Du-\mathfrak{X}|^2+|Du-\mathfrak{X}|^p\big)\,dz\leqslant c\left(\int_{Q_\varrho^+}\left|\frac{u-\xi x_n}{\varrho}\right|^2+\left|\frac{u-\xi x_n}{\varrho}\right|^p dz+\varrho^{2\beta}\big|Q_\varrho^+\big|\right),$$

where $c=(1+\|g_t\|^2_{L^{2,2-2\beta}(Q_1^+)})c(n,p,L/\nu,M,H(M),\kappa_{M+1})$. Finally, taking mean-values and enlarging the constant by a factor 2^{n+2} yields the complete assertion. $\Box$

4.1.3. Linearization

Here we prove that every weak solution of the non-linear parabolic system (2.5) with $u=0$ on the lateral boundary Γ_1 is an approximate solution of constant coefficient parabolic system, in a certain sense. This property is needed to apply the A-caloric approximation lemma later. For $s\geqslant 1$, $\xi\in\mathbb{R}^N$, $z_0\in\Gamma_1$ and a parabolic cylinder $Q_\varrho(z_0)\subset Q_1$ we define the excess functional by

$$\phi_s^+(z_0,\varrho,\xi):=\fint_{Q_\varrho^+(z_0)}|Du-\xi\otimes e_n|^s\,dz,\qquad \phi_s^+=\phi_s^+(z_0,\varrho,\xi).$$

Lemma 4.3. *Suppose that $u\in L^p(\Lambda_1;W^{1,p}(B_1^+;\mathbb{R}^N))$ is a weak solution of* (2.5) *satisfying $u=0$ on the lateral boundary Γ_1, where the structure conditions* (1.4)–(1.6) *are in force and let $M>0$. Then we have*

$$\begin{aligned}&\left|\fint_{Q_\varrho^+(z_0)}(u-\xi x_n)\cdot\varphi_t-\big\langle\partial_w a(z_0,0,\xi\otimes e_n)(Du-\xi\otimes e_n),D\varphi\big\rangle dz\right|\\ &\quad\leqslant c_{Eu}\Big[\omega_{M+1}\big(\phi_p^++\varrho^p\big)\sqrt{\phi_2^+}+\phi_p^++\varrho^\beta\Big]\sup_{Q_\varrho^+(z_0)}|D\varphi|,\end{aligned}$$

for any $\varphi\in C_0^\infty(Q_\varrho^+(z_0);\mathbb{R}^N)$, $\xi\in\mathbb{R}^N$ with $|\xi|\leqslant M$, $z_0\in\Gamma_1$ and $Q_\varrho(z_0)\subset Q_1$. The constant c_{Eu} is of the form $c_{Eu}=L(1+\|g_t\|_{L^{2,2-2\beta}(Q_1^+)})c(n,p,M,K(1),\kappa_{M+1})$.

Proof. Without loss of generality we may assume $\sup_{Q_\varrho(z_0)}|D\varphi|\leqslant 1$ and $z_0=0$. Again we abbreviate $\mathfrak{X}=D(\xi x_n)=\xi\otimes e_n$. Using

$$\fint_{Q_\varrho^+}\xi x_n\cdot\varphi_t\,dz=0\quad\text{and}\quad\fint_{Q_\varrho^+}\big\langle a\big((0,t),0,\mathfrak{X}\big),D\varphi\big\rangle dz=0,$$

we obtain from (2.5)

$$\begin{aligned}
&\fint_{Q_\varrho^+} \big((u-\xi x_n)\cdot\varphi_t - \big\langle \partial_w a(0,0,\mathfrak{X})(Du-\mathfrak{X}), D\varphi\big\rangle\big)\,dz \\
&\quad = \fint_{Q_\varrho^+} \big\langle a\big((0,t),0,Du\big) - a\big((0,t),0,\mathfrak{X}\big) - \partial_w a(0,0,\mathfrak{X})(Du-\mathfrak{X}), D\varphi\big\rangle\,dz \\
&\qquad + \fint_{Q_\varrho^+} \big\langle a\big((x,t),u,Du\big) - a\big((0,t),0,Du\big), D\varphi\big\rangle\,dz + \fint_{Q_\varrho^+} g_t\cdot\varphi\,dz \\
&\quad =: I + II + III,
\end{aligned} \tag{4.18}$$

with the obvious meaning of I–III. In the following we will derive estimates for I–III.

Estimate for I: First, we decompose Q_ϱ^+ into

$$S_1 \equiv \big\{z\in Q_\varrho^+ \colon \big|Du(z)-\mathfrak{X}\big|\leqslant 1\big\} \quad\text{and}\quad S_2 \equiv \big\{z\in Q_\varrho^+ \colon \big|Du(z)-\mathfrak{X}\big| > 1\big\}$$

and rewrite I as follows

$$I = \frac{1}{|Q_\varrho^+|}\int_{S_1}(\cdots)\,dz + \frac{1}{|Q_\varrho^+|}\int_{S_2}(\cdots)\,dz =: I_1 + I_2.$$

For the integrand of I_1 we have

$$\begin{aligned}
&\big|a\big((0,t),0,Du\big) - a\big((0,t),0,\mathfrak{X}\big) - \partial_w a(0,0,\mathfrak{X})(Du-\mathfrak{X})\big| \\
&\quad\leqslant \int_0^1 \big|\big(\partial_w a\big((0,t),0,\mathfrak{X}+s(Du-\mathfrak{X})\big) - \partial_w a(0,0,\mathfrak{X})\big)\cdot(Du-\mathfrak{X})\big| \\
&\quad\leqslant 2L\kappa_{M+1}\omega_{M+1}\big(\varrho^p + |Du-\mathfrak{X}|^p\big)|Du-\mathfrak{X}|.
\end{aligned}$$

Here we have used (2.4), the fact that $|\mathfrak{X}+s(Du(z)-\mathfrak{X})|\leqslant M+1$ for $z\in S_1$, $0\leqslant s\leqslant 1$ and $|\mathfrak{X}|\leqslant M$. Therefore, using Hölder's inequality and Jensen's inequality and taking into account that $s\mapsto \omega_{M+1}^p(s)$ is concave and $p\geqslant 2$ we obtain

$$\begin{aligned}
|I_1| &\leqslant \frac{2L\kappa_{M+1}}{|Q_\varrho^+|}\int_{S_1}\omega_{M+1}\big(\varrho^p + |Du-\mathfrak{X}|^p\big)|Du-\mathfrak{X}|\,dz \\
&\leqslant 2L\kappa_{M+1}\bigg(\fint_{Q_\varrho^+}\omega_{M+1}^p\big(\varrho^p + |Du-\mathfrak{X}|^p\big)\,dz\bigg)^{\frac{1}{p}}\bigg(\fint_{Q_\varrho^+}|Du-\mathfrak{X}|^{\frac{p}{p-1}}\,dz\bigg)^{\frac{p}{p-1}} \\
&\leqslant 2L\kappa_{M+1}\omega_{M+1}\big(\varrho^p+\phi_p^+\big)\sqrt{\phi_2^+}.
\end{aligned}$$

The integrand of I_2 is estimated by the use of (1.3) and (1.5), noting once again that $|\mathfrak{X}|\leqslant M$, as well as $|Du(z)-\mathfrak{X}|>1$ for $z\in S_2$ to obtain

$$\begin{aligned}
&\big|a\big((0,t),0,Du\big) - a\big((0,t),0,\mathfrak{X}\big) - \partial_w a(0,0,\mathfrak{X})(Du-\mathfrak{X})\big| \\
&\quad\leqslant L\big(1+|Du|^{p-1}\big) + L\big(1+|\mathfrak{X}|^{p-1}\big) + L\kappa_{M+1}|Du-\mathfrak{X}| \\
&\quad\leqslant c(p)L\big(1+\kappa_{M+1}+M^{p-1}\big)|Du-\mathfrak{X}|^p,
\end{aligned}$$

which directly implies

$$|I_2| \leqslant c(p)L\big(1+\kappa_{M+1}+M^{p-1}\big)\phi_p^+.$$

Combining this with the estimate for I_1 we find

$$|I| \leqslant cL\Big(\omega_{M+1}\big(\varrho^p + \phi_p^+\big)\sqrt{\phi_2^+} + \phi_p^+\Big),$$

where $c = c(p)(1 + \kappa_{M+1} + M^{p-1})$.

Estimate for II: To estimate II we use (1.6), (2.2) with $u_0 = 0$ and $|\xi x_n| \leqslant \varrho M$ on B_ϱ^+ and $|\mathfrak{X}| \leqslant M$ in order to obtain

$$\begin{aligned}
|II| &\leqslant L \fint_{Q_\varrho^+} \theta\big(|u|, \varrho + |u|\big)\big(1 + |Du|^{p-1}\big)\,dz \\
&\leqslant LK(1) \fint_{Q_\varrho^+} \big(|u|^\beta + \varrho^\beta\big)\big(1 + |Du|^{p-1}\big)\,dz \\
&\leqslant LK(1)\varrho^\beta \fint_{Q_\varrho^+} \bigg(\bigg|\frac{u - \xi x_n}{\varrho}\bigg|^\beta + M^\beta + 1\bigg)\big(1 + M^{p-1} + |Du - \mathfrak{X}|^{p-1}\big)\,dz \\
&\leqslant c\varrho^\beta \fint_{Q_\varrho^+} \bigg(\bigg|\frac{u - \xi x_n}{\varrho}\bigg|^\beta + \bigg|\frac{u - \xi x_n}{\varrho}\bigg|^\beta |Du - \mathfrak{X}|^{p-1} + |Du - \mathfrak{X}|^{p-1}\bigg)\,dz + c\varrho^\beta \\
&=: c(II_1 + II_2 + II_3) + c\varrho^\beta
\end{aligned}$$

with the obvious meaning of II_1–II_3 and $c = Lc(p, M, K(1))$. We now in turn estimate these terms. For the estimate of II_1 we use Young's inequality, the fact $\varrho \leqslant 1$ and the Poincaré inequality from Lemma 2.3. This leads us to

$$|II_1| \leqslant \fint_{Q_\varrho^+} \bigg|\frac{u - \xi x_n}{\varrho}\bigg|^p dz + \varrho^{\frac{p\beta}{p-\beta}} \leqslant \fint_{Q_\varrho^+} |D_n u - \xi|^p\,dz + \varrho^\beta \leqslant \phi_p^+ + \varrho^\beta.$$

To get an estimate for II_2 we use Hölder's inequality, Poincaré's inequality from Lemma 2.3, Young's inequality and $\varrho \leqslant 1$ to infer

$$\begin{aligned}
|II_2| &\leqslant \varrho^\beta \big(\phi_p^+\big)^{1-\frac{1}{p}} \bigg(\fint_{Q_\varrho^+} \bigg|\frac{u - \xi x_n}{\varrho}\bigg|^{\beta p} dz\bigg)^{\frac{1}{p}} \leqslant \varrho^\beta \big(\phi_p^+\big)^{1-\frac{1}{p}} \bigg(\fint_{Q_\varrho^+} \bigg|\frac{u - \xi x_n}{\varrho}\bigg|^p dz\bigg)^{\frac{\beta}{p}} \\
&\leqslant \varrho^\beta \big(\phi_p^+\big)^{1-\frac{1}{p}} \bigg(\fint_{Q_\varrho^+} |D_n u - \xi|^p\,dz\bigg)^{\frac{\beta}{p}} \leqslant \varrho^\beta \big(\phi_p^+\big)^{1-\frac{1-\beta}{p}} \leqslant \phi_p^+ + \varrho^{\frac{p\beta}{1-\beta}} \leqslant \phi_p^+ + \varrho^\beta.
\end{aligned}$$

Finally, we estimate II_3 with Young's inequality and recall that $\varrho \leqslant 1$ to deduce

$$|II_3| \leqslant \fint_{Q_\varrho^+} \big(|Du - \mathfrak{X}|^p + \varrho^{p\beta}\big)\,dz \leqslant \phi_p^+ + \varrho^\beta.$$

Estimate for III: Since $|D\varphi| \leqslant 1$ on Q_ϱ^+ and $\varphi \in C_0^\infty(Q_\varrho^+; \mathbb{R}^N)$, we have $|\varphi(z)| \leqslant \varrho$ for $z \in Q_\varrho^+$ and therefore obtain

$$|III| \leqslant \fint_{Q_\varrho^+} |g_t||\varphi|\,dz \leqslant \varrho \fint_{Q_\varrho^+} |g_t|\,dz \leqslant c(n)\varrho^\beta \bigg(\varrho^{2-2\beta-(n+2)} \int_{Q_\varrho^+} |g_t|^2\,dz\bigg)^{\frac{1}{2}} \leqslant c(n)\varrho^\beta \|g_t\|_{L^{2,2-2\beta}(Q_1^+)}.$$

Now the desired result follows by inserting the estimates for I–III into (4.18). □

4.1.4. A decay estimate at the lateral boundary

In Lemma 4.5 we will derive an excess-decay estimate valid for boundary points $z_0 \in \Gamma_1$, which is proved in three steps. In the first step we will use the A-caloric approximation lemma to show that – under certain smallness assumptions – the excess $\widetilde{E}_{\mathrm{lat}}$ of u fulfills a suitable growth estimate when we enlarge the half-cylinder by a constant factor. Afterwards we iterate this excess estimate by showing that the smallness assumptions are also fulfilled on the smaller half-cylinder (under the condition that they are fulfilled on the larger one). From this we finally conclude the excess-decay estimate for Du, in those points z_0, where the smallness assumptions are fulfilled.

Throughout this section we shall consider a weak solution $u \in L^p(\Lambda_1; W^{1,p}(B_1^+; \mathbb{R}^N))$ of the non-linear parabolic system (2.5) satisfying $u = 0$ on the lateral boundary Γ_1, where the structure conditions (1.4)–(1.6) are in force. Moreover, by

$$\xi_{z_0,\varrho} = \frac{n+2}{\varrho^2} \fint_{Q_\varrho^+(z_0)} u x_n \, dz, \tag{4.19}$$

with $z_0 \in \Gamma_1$ we denote the vector minimizing the mapping $\xi \mapsto \fint_{Q_\varrho^+(z_0)} |u - \xi x_n|^2 \, dz$ (see Lemma 2.2). For points lying near the lateral boundary we shall use the following excess functional. For $\xi \in \mathbb{R}^N$, $z_0 \in \Gamma_1 \cup Q_1^+$, $Q_\varrho^+(z_0) \subset Q_1^+$ and $s = 2$, respectively $s = p$, we define

$$\psi_s^+ \equiv \psi_s^+(z_0, \varrho; \xi) := \fint_{Q_\varrho^+(z_0)} \left| \frac{u - \xi x_n}{\varrho} \right|^s dz,$$

as well as

$$E_{\mathrm{lat}} \equiv E_{\mathrm{lat}}(z_0, \varrho; \xi) := \psi_2^+(z_0, \varrho; \xi) + \psi_p^+(z_0, \varrho; \xi),$$

and finally

$$\widetilde{E}_{\mathrm{lat}} \equiv \widetilde{E}_{\mathrm{lat}}(z_0, \varrho; \xi) := E_{\mathrm{lat}}(z_0, \varrho; \xi) + \varrho^{2\beta}.$$

Then we have the following *excess-decay estimate* at the lateral boundary:

Lemma 4.4. *Given $M > 0$ and $\alpha \in (\beta, 1)$, there exist $\vartheta \in (0, \frac{1}{4})$ and $\delta \in (0, 1]$ and $c_1 \geqslant 1$, depending on n, N, p, ν, L, M, $H(M)$, κ_{M+1}, α, β and $\|g_t\|_{L^{2,2-2\beta}(Q_1^+)}$, such that if*

$$\omega_{M+1}^2\big(\widetilde{E}_{\mathrm{lat}}(z_0, \varrho; \xi_\varrho)\big) + \widetilde{E}_{\mathrm{lat}}(z_0, \varrho; \xi_\varrho) \leqslant \frac{1}{2}\delta^2, \tag{4.20}$$

$$2c_1\sqrt{E_{\mathrm{lat}}(z_0, \varrho; \xi_\varrho) + \delta^{-2}\varrho^{2\beta}} \leqslant 1 \tag{4.21}$$

and

$$|\xi_\varrho| \leqslant M, \tag{4.22}$$

on $Q_\varrho^+(z_0)$ with $z_0 \in \Gamma_1$ and $Q_\varrho^+(z_0) \subset Q_1^+$. Then with, $c_2 \equiv 1 + \delta^{-2}$, there holds

$$\widetilde{E}_{\mathrm{lat}}(z_0, \vartheta\varrho, \xi_{\vartheta\varrho}) \leqslant \vartheta^{2\alpha} \widetilde{E}_{\mathrm{lat}}(z_0, \varrho, \xi_\varrho) + c_2\varrho^{2\beta}.$$

Proof. Without loss of generality we can assume that $z_0 = 0$. We consider $\xi \in \mathbb{R}^N$ satisfying $|\xi| \leqslant M$ and abbreviate $\mathfrak{X} \equiv D(\xi x_n) = \xi \otimes e_n$. From Caccioppoli's inequality Lemma 4.2 we infer

$$\phi_2^+(0, \varrho/2, \xi) + \phi_p^+(0, \varrho/2, \xi) \leqslant c_{Cac}\big(\psi_2^+(0, \varrho, \xi) + \psi_p^+(0, \varrho, \xi) + \varrho^{2\beta}\big) = c_{Cac}\widetilde{E}_{\mathrm{lat}}(0, \varrho, \xi), \tag{4.23}$$

where $c_{Cac} = c_{Cac}(n, p, L/\nu, M, H(M), \kappa_{M+1}, \|g_t\|_{L^{2,2-2\beta}(Q_1^+)}) \geqslant 1$. We now apply Lemma 4.3 to the map $v = u - \xi x_n$. Note that this is possible since $|\xi| \leqslant M$. Using also the fact that $\omega_{M+1}(cs) \leqslant c\omega_{M+1}(s)$ for $c \geqslant 1$ (since $s \mapsto \omega_{M+1}(s)$ is concave and $\omega_{M+1}(0) = 0$) we obtain

$$\left|\fint_{Q^+_{\varrho/2}} v\cdot\varphi_t - \big\langle \partial_w a(0,0,\mathfrak{X})Dv, D\varphi\big\rangle\,dz\right| \leqslant c_{Eu}\Big(\omega_{M+1}\big(\varrho^p+\phi_p^+\big)\sqrt{\phi_2^+}+\phi_p^+ +\varrho^\beta\Big)\sup_{Q^+_{\varrho/2}}|D\varphi|$$
$$\leqslant c_1\Big(\omega_{M+1}(\widetilde{E}_{\mathrm{lat}})\sqrt{\widetilde{E}_{\mathrm{lat}}}+\widetilde{E}_{\mathrm{lat}}+\varrho^\beta\Big)\sup_{Q^+_{\varrho/2}}|D\varphi|,$$

for all $\varphi\in C_0^\infty(Q^+_{\varrho/2};\mathbb{R}^N)$, where the constant c_1 is given by $c_{Eu}c_{Cac}$. Here we have also used $\varrho\leqslant 1$ and $2\beta\leqslant p$ in order to have $\varrho^p\leqslant\varrho^{2\beta}$. Now, we define

$$\langle \mathcal{A}w,\widetilde{w}\rangle \equiv \big\langle \partial_w a(0,0,\mathfrak{X})w,\widetilde{w}\big\rangle,$$

whenever $w,\widetilde{w}\in\mathbb{R}^{Nn}$. From (1.4) and (1.5) and the assumption $|\xi|\leqslant M$ we find that

$$\langle \mathcal{A}w,\widetilde{w}\rangle \leqslant L\kappa_{M+1}|w||\widetilde{w}|,\qquad \langle \mathcal{A}w,w\rangle\geqslant \nu|w|^2\quad \forall w,\widetilde{w}\in\mathbb{R}^{Nn},$$

i.e. $\mathcal{A}$ fulfills the hypotheses of Lemma 4.1 with ellipticity constant ν and upper bound $L\kappa_{M+1}$. For given $\varepsilon>0$ (which will be chosen later) we therefore determine $\delta=\delta(n,N,p,\nu,L\kappa_{M+1},\varepsilon)\in(0,1]$, accordingly to Lemma 4.1. Furthermore, we define

$$w:=\gamma^{-1}v=\gamma^{-1}(u-\xi x_n),\quad \text{where } \gamma:=2c_1\sqrt{E_{\mathrm{lat}}+\delta^{-2}\varrho^{2\beta}}.$$

Then, for the map w we have (note that $\widetilde{E}_{\mathrm{lat}}=E_{\mathrm{lat}}+\varrho^{2\beta}$)

$$\left|\fint_{Q^+_{\varrho/2}} w\cdot\varphi_t - \big\langle \partial_w a(0,0,\mathfrak{X})Dw, D\varphi\big\rangle\,dz\right| \leqslant \left[\omega^2_{M+1}(\widetilde{E}_{\mathrm{lat}})+\widetilde{E}_{\mathrm{lat}}+\frac{1}{2}\delta^2\right]^{\frac12}\sup_{Q^+_{\varrho/2}}|D\varphi|$$

for all $\varphi\in C_0^\infty(Q^+_{\varrho/2};\mathbb{R}^N)$. Moreover, from (4.23) we infer that

$$\fint_{Q^+_{\varrho/2}}\big(|Dw|^2+\gamma^{p-2}|Dw|^p\big)\,dz\leqslant \frac{c_{Cac}\widetilde{E}_{\mathrm{lat}}}{4c_1^2(E_{\mathrm{lat}}+\delta^{-2}\varrho^{2\beta})}\leqslant\frac{c_{Cac}}{4c_1^2}\leqslant 1.$$

Therefore, we are in a position to apply the lemma about A-caloric approximation, i.e. Lemma 4.1, to the map w on the cylinder $Q^+_{\varrho/2}$, provided the smallness conditions

$$\omega^2_{M+1}(\widetilde{E}_{\mathrm{lat}})+\widetilde{E}_{\mathrm{lat}}\leqslant\frac12\delta^2 \tag{4.24}$$

is satisfied and

$$\gamma=2c_1\sqrt{E_{\mathrm{lat}}+\delta^{-2}\varrho^{2\beta}}\leqslant 1. \tag{4.25}$$

Lemma 4.1 provides us with an $\mathcal{A}$-caloric map $h\in L^p(\Lambda_{(\varrho/4)^2};W^{1,p}(B^+_{\varrho/4};\mathbb{R}^N))$ satisfying $h=0$ on $\Gamma_{\varrho/4}$ and satisfying also

$$\fint_{Q^+_{\varrho/4}}\big(|Dh|^2+\gamma^{p-2}|Dh|^p\big)\,dz\leqslant 2\cdot 2^{n+2} \tag{4.26}$$

and

$$\fint_{Q^+_{\varrho/4}}\left|\frac{w-h}{\varrho/4}\right|^2+\gamma^{p-2}\left|\frac{w-h}{\varrho/4}\right|^p dz\leqslant\varepsilon. \tag{4.27}$$

In order to obtain an estimate for the excess E_{lat} of u we exploit the excess-decay estimate from Corollary 3.4 for the $\mathcal{A}$-caloric map h in the cases $s=2$ and $s=p$. Also using the Poincaré inequality from Lemma 2.3 and (4.26) we find

$$\begin{aligned}\gamma^{s-2}\left(\frac{\theta\varrho}{4}\right)^{-s} \fint_{Q^+_{\theta\varrho/4}} \left|h-(D_nh)^+_{\theta\varrho/4}x_n\right|^s dz &\leqslant c_{Li}\theta^s\gamma^{s-2}\left(\frac{\varrho}{4}\right)^{-s} \fint_{Q^+_{\varrho/4}} \left|h-(D_nh)^+_{\varrho/4}x_n\right|^s dz\\ &\leqslant c_{Li}\theta^s\gamma^{s-2}s^{-1} \fint_{Q^+_{\varrho/4}} \left|D_nh-(D_nh)^+_{\varrho/4}\right|^s dz\\ &\leqslant 2^{s-1}c_{Li}\theta^s\gamma^{s-2} \fint_{Q^+_{\varrho/4}} |D_nh|^s\,dz\\ &\leqslant c_{Li}2^{n+2+s}\theta^s,\end{aligned}$$

where $c_{Li} = c(n, N, p, L\kappa_{M+1}/\nu)$. Combining this with (4.27) we obtain the following excess improvement for w:

$$\begin{aligned}\gamma^{s-2}\left(\frac{\theta\varrho}{4}\right)^{-s} \fint_{Q^+_{\theta\varrho/4}} \left|w-(D_nh)^+_{\theta\varrho/4}x_n\right|^s dz &\leqslant 2^{s-1}\gamma^{s-2}\left(\frac{\theta\varrho}{4}\right)^{-s} \fint_{Q^+_{\theta\varrho/4}} |w-h|^s+\left|h-(D_nh)^+_{\theta\varrho/4}x_n\right|^s dz\\ &\leqslant 2^{s-1}\left(\theta^{-n-2-s}\varepsilon+c_{Li}2^{n+2+s}\theta^s\right)\\ &\leqslant c_{Li}2^{n+1+2s}\left(\theta^{-n-2-s}\varepsilon+\theta^s\right).\end{aligned}$$

Rescaling back to u via $w := \gamma^{-1}(u-\xi x_n)$ then implies for $s = 2$ respectively $s = p$ that

$$(\theta\varrho/4)^{-s} \fint_{Q^+_{\theta\varrho/4}} |u-\widetilde{\xi}x_n|^s\,dz \leqslant c_{Li}2^{n+1+2s}\left(\theta^{-n-2-s}\varepsilon+\theta^s\right)\gamma^2, \tag{4.28}$$

where we have abbreviated $\widetilde{\xi} \equiv \xi - \gamma(D_nh)^+_{\theta\varrho/4}$. For the case $s = 2$ we note that (4.28) then holds for $(\xi_\varrho - \gamma(Dh)_{\theta\varrho/4})$ replaced by $\xi_{\theta\varrho/4}$, where $\xi_{\theta\varrho/4} = \frac{n+2}{(\theta\varrho/4)^2}\fint_{Q^+_{\theta\varrho/4}} u\cdot x_n\,dz$ is the vector in $\mathbb{R}^N$ minimizing the mapping $\xi \mapsto \fint_{Q^+_{\theta\varrho/4}} |u-\xi|^2\,dz$ (see (2.11) from Lemma 2.2). Recalling the definition of γ we therefore infer

$$\psi_2^+(0,\theta\varrho/4,\xi_{\theta\varrho/4}) = (\theta\varrho/4)^{-2} \fint_{Q^+_{\theta\varrho/4}} |u-\xi_{\theta\varrho/4}x_n|^2\,dz \leqslant c\left(\theta^{-n-4}\varepsilon+\theta^2\right)\left(E_{\mathrm{lat}}(0,\varrho,\xi_\varrho)+\delta^{-2}\varrho^{2\beta}\right),$$

where c is of the form $Lc(n, N, p, L\kappa_{M+1}/\nu, M, H(M), \|g_t\|_{L^{2,2-2\beta}(Q_1^+)})$. Similarly, for $s = p$ we can bound the integral on the left-hand side of (4.28) from below by the quantity $\fint_{Q^+_{\theta\varrho/4}} |u-\xi^{(p)}_{\theta\varrho/4}x_n|^p\,dz$, where $\xi^{(p)}_{\theta\varrho/4}$ denotes the unique vector in $\mathbb{R}^N$ minimizing the mapping $\xi \mapsto \fint_{Q^+_{\theta\varrho/4}} |u-\xi|^p\,dz$. Recalling once again the definition of γ we obtain in this case

$$(\theta\varrho/4)^{-p} \fint_{Q^+_{\theta\varrho/4}} \left|u-\xi^{(p)}_{\theta\varrho/4}x_n\right|^p dz \leqslant c\left(\theta^{-n-2-p}\varepsilon+\theta^p\right)\left(E_{\mathrm{lat}}(0,\varrho,\xi_\varrho)+\delta^{-2}\varrho^{2\beta}\right).$$

In the preceding estimate we want to replace $\xi^{(p)}_{\theta\varrho/4}$ by $\xi_{\theta\varrho/4}$. To this aim we use Lemma 2.2 to estimate for $x = (x', x_n) \in B^+_{\theta\varrho/4}$,

$$\left|\xi_{\theta\varrho/4}x_n-\xi^{(p)}_{\theta\varrho/4}x_n\right|^p \leqslant (n+2)^{\frac{p}{2}} \fint_{Q^+_{\theta\varrho/4}} \left|u-\xi^{(p)}_{\theta\varrho/4}x_n\right|^p dz.$$

Combining this with the second last estimate we deduce

$$\begin{aligned}\psi_p^+(0,\theta\varrho/4,\xi_{\theta\varrho/4}) &= \left(\frac{\theta\varrho}{4}\right)^{-p} \fint_{Q^+_{\theta\varrho/4}} \left|u-\xi_{\theta\varrho/4}x_n\right|^p dz \\ &\leqslant 2^p(n+2)^{\frac{p}{2}}\left(\frac{\theta\varrho}{4}\right)^{-p} \fint_{Q^+_{\theta\varrho/4}} \left|u-\xi^{(p)}_{\theta\varrho/4}x_n\right|^p dz \\ &\leqslant c_3\big(\theta^{-n-2-p}\varepsilon+\theta^p\big)\big(E_{\mathrm{lat}}(0,\varrho,\xi_\varrho)+\delta^{-2}\varrho^{2\beta}\big),\end{aligned}$$

where $c_3 = 2^p(n+2)^{\frac{p}{2}}c$. Combining the estimates for ψ_2^+ and ψ_p^+ and recalling that $\theta \leqslant 1$, $p \geqslant 2$ and $E = \psi_2^+ + \psi_p^+$ we arrive at

$$E_{\mathrm{lat}}(0,\theta\varrho/4,\xi_{\theta\varrho/4}) \leqslant c_3\big(\theta^{-n-2-p}\varepsilon+\theta^2\big)\big(E_{\mathrm{lat}}(0,\varrho,\xi_\varrho)+\delta^{-2}\varrho^{2\beta}\big).$$

Choosing $\varepsilon = \theta^{n+4+p}$ we obtain

$$E_{\mathrm{lat}}(0,\theta\varrho/4,\xi_{\theta\varrho/4}) \leqslant 2c_3\theta^2\big(E_{\mathrm{lat}}(0,\varrho,\xi_\varrho)+\delta^{-2}\varrho^{2\beta}\big). \tag{4.29}$$

Given α with $\beta < \alpha < 1$ we choose $0 < \theta \leqslant 1/2$ such that $2^{1+4\alpha}c_3\theta^2 \leqslant \theta^{2\alpha}$, so that

$$\theta = \theta\big(n, N, \nu, L, M, H(M), \kappa_{M+1}, \alpha, \|g_t\|_{L^{2,2-2\beta}(Q_1^+)}\big).$$

This also fixes ε and $\delta \in (0,1]$ depending on the same parameters. We now define $\vartheta := \frac{1}{4}\theta$. Then, (4.29) yields the assertion of the lemma. □

4.1.5. Iteration

Here we want to iterate the excess-decay estimate from Lemma 4.4.

Lemma 4.5. *Given $M > 1$ and $\alpha \in (\beta,1)$, there exist constants $\vartheta \in (0,\frac{1}{4}]$, $\mathcal{E}_{\mathrm{lat}} \in (0,1]$, $\varrho_{\mathrm{lat}} > 0$ and c_4 depending on $n, N, p, \nu, L, M, H(M), \kappa_{M+1}, \alpha, \beta$ and $\|g_t\|_{L^{2,2-2\beta}(Q_1^+)}$ such that the following holds: Suppose that*

(i) $|\xi_\varrho| \leqslant M$,

(ii) $\varrho \leqslant \varrho_{\mathrm{lat}}$,

(iii) $\widetilde{E}_{\mathrm{lat}}(\varrho) \leqslant \mathcal{E}_{\mathrm{lat}}$

are fulfilled on some parabolic half-cylinder $Q_\varrho^+(z_0)$ centered at $z_0 \in \Gamma_1$ with $Q_\varrho^+(z_0) \subset Q_1^+$. Then for every $j \in \mathbb{N}$ we have

$(\mathrm{I})_j$ $\widetilde{E}_{\mathrm{lat}}\big(\vartheta^j\varrho\big) \leqslant \vartheta^{2\alpha j}\widetilde{E}_{\mathrm{lat}}(\varrho) + c_4(\vartheta^j\varrho)^{2\beta}$,

$(\mathrm{II})_j$ $|\xi_{\vartheta^j\varrho}| \leqslant 2M$,

where we have abbreviated $\widetilde{E}_{\mathrm{lat}}(r) = \widetilde{E}_{\mathrm{lat}}(z_0, r, \xi_r)$. Furthermore, the limit

$$\Upsilon_{z_0} := \lim_{j\to\infty}(D_n u)^+_{z_0,\vartheta^j\varrho}$$

exists and the estimate

$$\fint_{Q_r^+(z_0)} |Du-\Upsilon_{z_0}\otimes e_n|^2 + |Du-\Upsilon_{z_0}\otimes e_n|^p\,dz \leqslant c_{\mathrm{lat}}\bigg[\bigg(\frac{r}{\varrho/2}\bigg)^{2\alpha}E_{\mathrm{lat}}(\varrho)+r^{2\beta}\bigg],$$

holds for $0 < r \leqslant \varrho/2$, where the constant c_{lat} depends on $n, N, p, \nu, L, M, H(M), \kappa_{M+1}, \alpha, \beta$ and $\|g_t\|_{L^{2,2-2\beta}(Q_1^+)}$.

Proof. In order to prove the assertions of the lemma we have to show that the smallness conditions (4.20)–(4.22) are also fulfilled on the cylinders $Q^+_{\vartheta^j\varrho}(z_0)$ provided they are fulfilled on $Q^+_{\vartheta^\ell\varrho}(z_0)$ for $\ell = 0, \dots, j-1$. Without loss of generality we once again assume that $z_0 = 0$. Given $M > 1$ and α with $\beta < \alpha < 1$ we determine the constants $\vartheta = \vartheta(2M)$, $\delta = \delta(2M)$ and $c_2 = c_2(2M)$ from Lemma 4.4, depending also on $n, N, p, \nu, L, M, H(M), \kappa_{M+1}$ and α, respectively. Then, there exists $\mathcal{E}_{\text{lat}} = \mathcal{E}_{\text{lat}}(M) > 0$, such that

$$\omega^2_{2M}(2\mathcal{E}_{\text{lat}}) + 2\mathcal{E}_{\text{lat}} \leqslant \frac{1}{2}\delta^2 \tag{4.30}$$

and

$$\mathcal{E}_{\text{lat}} \leqslant \frac{1}{4(n+2)} M^2 \vartheta^{n+4}\big(1-\vartheta^\alpha\big)^2. \tag{4.31}$$

Furthermore we choose $\varrho_{\text{lat}} = \varrho_{\text{lat}}(M) \in (0,1]$ such that with

$$c_4 \equiv c_4(M) \equiv \frac{c_2}{\vartheta^{2\beta} - \vartheta^{2\alpha}} \tag{4.32}$$

we have

$$c_4 \varrho_{\text{lat}}^{2\beta} \leqslant \min\left\{\delta^2, \mathcal{E}_{\text{lat}}, \frac{1}{4(n+2)} M^2 \vartheta^{n+4}\big(1-\vartheta^\beta\big)^2\right\}. \tag{4.33}$$

Thus, $\mathcal{E}_{\text{lat}}$, ϱ_{lat} and c_4 depend on $n, N, p, \nu, L, M, H(M), \kappa_{M+1}, \alpha$ and $\|g_t\|_{L^{2,2-2\beta}(Q_1^+)}$. We first note that $(\mathrm{I})_j$ combined with (ii), (iii) and (4.33) yields

$$(\mathrm{I}')_j \quad \widetilde{E}_{\text{lat}}\big(\vartheta^j\varrho\big) \leqslant 2\mathcal{E}_{\text{lat}}.$$

Now, suppose that the conditions (i)–(iii) are fulfilled on $Q^+_\varrho \subset Q^+$. Then, by induction we shall show $(\mathrm{I})_j$ and $(\mathrm{II})_j$ hold for $j \in \mathbb{N}$. We start with the case $j = 1$. From (iii), (4.30) and the monotonicity of ω_{M+1} we infer

$$\omega^2_{M+1}\big(\widetilde{E}_{\text{lat}}(\varrho)\big) + \widetilde{E}_{\text{lat}}(\varrho) \leqslant \omega^2_{2M}(2\mathcal{E}_{\text{lat}}) + 2\mathcal{E}_{\text{lat}} \leqslant \frac{1}{2}\delta^2.$$

Furthermore, (i) and (ii) guarantee that also the assumptions $\varrho \leqslant \varrho_{\text{lat}} \leqslant 1$ and $|\xi_\varrho| \leqslant M$ of Lemma 4.4 are fulfilled. Therefore, the application of the lemma ensures that $(\mathrm{I})_1$ holds. Using (i), Lemma 2.2, (iii) and (4.31) yields

$$\begin{aligned} |\xi_{\vartheta\varrho}| &\leqslant |\xi_\varrho| + |\xi_{\vartheta\varrho} - \xi_\varrho| \leqslant M + \left(\frac{n+2}{(\vartheta\varrho)^2} \fint_{Q^+_{\vartheta\varrho}} |u - \xi_\varrho x_n|^2\,dz\right)^{\frac{1}{2}} \leqslant M + \left(\frac{n+2}{\vartheta^{n+4}} \fint_{Q^+_\varrho} \left|\frac{u-\xi_\varrho x_n}{\varrho}\right|^2 dz\right)^{\frac{1}{2}} \\ &\leqslant M + \left(\frac{n+2}{\vartheta^{n+4}} \widetilde{E}_{\text{lat}}(\varrho)\right)^{\frac{1}{2}} \leqslant M + \frac{\sqrt{n+2}}{\sqrt{\vartheta^{n+4}}}\sqrt{\mathcal{E}_{\text{lat}}(M)} \leqslant 2M, \end{aligned}$$

so that $(\mathrm{II})_1$ holds. We now assume that $(\mathrm{I})_k$ and $(\mathrm{II})_k$ hold for $k = 1, \dots, j-1$, and therefore also $(I')_k)$ for those k. The validity of $(I')_k$, $(\mathrm{II})_k$ and (4.30) allows us to apply Lemma 4.4 with $\vartheta^k\varrho$ instead of ϱ and $2M$ instead of M for $k = 1, \dots, j-1$. This is possible since we have chosen $\vartheta = \vartheta(2M)$, $\delta = \delta(2M)$ and $c_2 = c_2(2M)$. Using Lemma 4.4 for $k = 1, \dots, j-1$ and (4.32), we find

$$\begin{aligned} \widetilde{E}_{\text{lat}}\big(\vartheta^j\varrho\big) &\leqslant \vartheta^{2\alpha j}\widetilde{E}_{\text{lat}}(\varrho) + c_2(2M)\big(\vartheta^{j-1}\varrho\big)^{2\beta} \sum_{i=0}^{j-1} \vartheta^{2(\alpha-\beta)i} \\ &\leqslant \vartheta^{2\alpha j}\widetilde{E}_{\text{lat}}(\varrho) + \frac{c_3(2M)}{\vartheta^{2\beta} - \vartheta^{2\alpha}}\big(\vartheta^j\varrho\big)^{2\beta} \\ &= \vartheta^{2\alpha j}\widetilde{E}_{\text{lat}}(\varrho) + c_4(M)\big(\vartheta^j\varrho\big)^{2\beta}, \end{aligned}$$

proving $(\mathrm{I})_j$. To show $(\mathrm{II})_j$ we use of Lemma 2.2, $(\mathrm{I})_k$ for $k = 1, \dots, j-1$, (4.33), (4.31), (ii) and (iii) to obtain

$$
\begin{aligned}
|\xi_{\vartheta^j\varrho}| &\leqslant |\xi_\varrho| + \sum_{i=1}^{j} |\xi_{\vartheta^i\varrho} - \xi_{\vartheta^{i-1}\varrho}| \\
&\leqslant M + \sum_{i=1}^{j} \biggl(\frac{n+2}{(\vartheta^i\varrho)^2} \fint_{Q^+_{\vartheta^i\varrho}} |u - \xi_{\vartheta^{i-1}\varrho} x_n|^2 \, dz \biggr)^{\frac12} \\
&\leqslant M + \frac{\sqrt{n+2}}{\sqrt{\vartheta^{n+4}}} \sum_{i=1}^{j} E_{\mathrm{lat}}\bigl(\vartheta^{i-1}\varrho\bigr)^{\frac12} \\
&\leqslant M + \frac{\sqrt{n+2}}{\sqrt{\vartheta^{n+4}}} \sum_{i=0}^{j-1} \bigl(\vartheta^{2\alpha i} \widetilde{E}_{\mathrm{lat}}(\varrho) + c_4(M)\bigl(\vartheta^i\varrho\bigr)^{2\beta}\bigr)^{\frac12} \\
&\leqslant M + \frac{\sqrt{n+2}}{\sqrt{\vartheta^{n+4}}} \biggl(\frac{\sqrt{\widetilde{E}_{\mathrm{lat}}(\varrho)}}{1-\vartheta^\alpha} + \frac{\sqrt{c_4(M)}\varrho^\beta}{1-\vartheta^\beta} \biggr) \\
&\leqslant M + \frac{M}{2} + \frac{M}{2} = 2M.
\end{aligned}
$$

This proves the second assertion of the lemma. The assertion about the limit Υ is proved by showing that $((D_n u)^+_{\vartheta^j\varrho/2})_{j\in\mathbb{N}}$ is a Cauchy sequence. Since $|\xi_{\vartheta^j\varrho}| \leqslant 2M$ we can apply Caccioppoli's inequality to infer for $s=2$, respectively $s=p$,

$$
\begin{aligned}
\phi_s^+\bigl(\vartheta^j\varrho/2, (D_n u)^+_{\vartheta^j\varrho/2}\bigr) &\leqslant \phi_s^+\bigl(\vartheta^j\varrho/2, \xi_{\vartheta^j\varrho}\bigr) \\
&\leqslant c_{Cac}(2M)\widetilde{E}_{\mathrm{lat}}\bigl(\vartheta^j\varrho\bigr) \\
&\leqslant c_{Cac}(2M)\bigl(\vartheta^{2\alpha j}\widetilde{E}_{\mathrm{lat}}(\varrho) + c_4(M)\bigl(\vartheta^j\varrho\bigr)^{2\beta}\bigr).
\end{aligned}
\tag{4.34}
$$

We also used the minimizing property of $(D_n u)^+_{\vartheta^j\varrho/2}$ and $(\mathrm{I})_j$. Now, for $k > j$ we have

$$
\begin{aligned}
\bigl|(D_n u)^+_{\vartheta^j\varrho/2} - (D_n u)^+_{\vartheta^k\varrho/2}\bigr| &\leqslant \sum_{i=j+1}^{k} \bigl|(D_n u)^+_{\vartheta^i\varrho/2} - (D_n u)^+_{\vartheta^{i-1}\varrho/2}\bigr| \\
&\leqslant \vartheta^{-\frac{n+2}{s}} \sum_{i=j+1}^{k} \biggl(\fint_{Q^+_{\vartheta^i\varrho/2}(z_0)} \bigl|D_n u - (D_n u)^+_{\vartheta^i\varrho/2}\bigr|^s \, dz \biggr)^{\frac1s} \\
&\leqslant \vartheta^{-\frac{n+2}{s}} \sum_{i=j+1}^{k} \biggl(\fint_{Q^+_{\vartheta^i\varrho/2}(z_0)} \bigl|Du - (Du)^+_{\vartheta^i\varrho/2} \otimes e_n\bigr|^s \, dz \biggr)^{\frac1s} \\
&\leqslant \bigl(\vartheta^{-(n+2)} c_{Cac}(2M)\bigr)^{\frac1s} \sum_{i=j+1}^{k} \bigl(\vartheta^{2\alpha i}\widetilde{E}_{\mathrm{lat}}(\varrho) + c_4(M)\bigl(\vartheta^i\varrho\bigr)^{2\beta}\bigr)^{\frac1s} \\
&\leqslant \bigl(\vartheta^{-(n+2)} c_{Cac}(2M)\bigr)^{\frac1s} \biggl(\frac{\widetilde{E}_{\mathrm{lat}}(\varrho)^{\frac1s}}{1-\vartheta^\alpha}\vartheta^{2\alpha j} + \frac{(c_4(M)\varrho^{2\beta})^{\frac1s}}{1-\vartheta^\beta}\vartheta^{2\beta j} \biggr).
\end{aligned}
$$

This implies that $((D_n u)^+_{\vartheta^j\varrho/2})_{j\in\mathbb{N}}$ is a Cauchy sequence and therefore the limit $\Upsilon := \lim_{j\to\infty}(D_n u)^+_{\vartheta^j\varrho/2}$ exists. In the limit $k\to\infty$ we obtain from the preceding estimate

$$
\bigl|(D_n u)^+_{\vartheta^j\varrho/2} - \Upsilon\bigr|^s \leqslant c\bigl[\vartheta^{2\alpha j}\widetilde{E}_{\mathrm{lat}}(\varrho) + \bigl(\vartheta^j\varrho\bigr)^{2\beta}\bigr],
$$

where c depends on $n, N, \nu, L, M, H(M), \kappa_{M+1}, \alpha, \beta$ and $\|g_t\|_{L^{2,2-2\beta}(Q_1^+)}$. Combining this with (4.34) we get

$$\fint_{Q^+_{\vartheta^j\varrho/2}(z_0)} |Du-\Upsilon\otimes e_n|^s\,dz \leqslant 2^{s-1}\phi_s^+\big(\vartheta^j\varrho/2\big)+2^{s-1}\big|(D_nu)^+_{\vartheta^j\varrho/2}-\Upsilon\big|^s$$
$$\leqslant c(M)\big[\vartheta^{2\alpha j}\widetilde{E}_{\mathrm{lat}}(\varrho)+\big(\vartheta^j\varrho\big)^{2\beta}\big],$$

where $c(M)=c(n,N,\nu,L,M,H(M),\kappa_{M+1},\alpha,\beta,\|g_t\|_{L^{2,2-2\beta}(Q_1^+)})$. We now consider an arbitrary radius $0<r\leqslant\frac{\varrho}{2}$. Then, we find $k\in\mathbb{N}\cup\{0\}$ with $\vartheta^{k+1}\varrho/2<r\leqslant\vartheta^k\varrho/2$ and obtain from the previous estimate:

$$\fint_{Q_r^+(z_0)} |Du-\Upsilon\otimes e_n|^s\,dz \leqslant \vartheta^{-n-2}\fint_{Q^+_{\vartheta^k\varrho/2}(z_0)} |Du-\Upsilon\otimes e_n|^s\,dz$$
$$\leqslant \vartheta^{-n-2}c(M)\big[\vartheta^{2\alpha k}\widetilde{E}_{\mathrm{lat}}(\varrho)+\big(\vartheta^k\varrho\big)^{2\beta}\big]$$
$$\leqslant c(M)\bigg[\bigg(\frac{r}{\varrho/2}\bigg)^{2\alpha}E_{\mathrm{lat}}(\varrho)+(2r)^{2\beta}\bigg],$$

which finishes the proof of the lemma. □

In order to provide our characterization for regular boundary points we will have to combine Lemma 4.5 with an excess-decay estimate for interior points, stated below. In the interior situation we shall use a different excess functional, namely for $z_0\in Q_1^+$, $\varrho>0$ such that $Q_\varrho(z_0)\subset Q_1^+$ we define

$$E_{\mathrm{int}}(z_0,\varrho):=\fint_{Q_\varrho(z_0)}\bigg|\frac{u-\ell_{z_0,\varrho}}{\varrho}\bigg|^2+\bigg|\frac{u-\ell_{z_0,\varrho}}{\varrho}\bigg|^p\,dz, \tag{4.35}$$

where $\ell_{z_0,\varrho}$ denotes the unique affine map minimizing $\ell\mapsto\int_{Q_\varrho(z_0)}|u-\ell|^2\,dz$ and

$$\widetilde{E}_{\mathrm{int}}(z_0,\varrho)=E_{\mathrm{int}}(z_0,\varrho)+\varrho^{2\beta}.$$

Then, for interior points $z_0\in Q_1^+$ we have the following excess-decay estimate form [20, Lemma 4.8]. We here state it in a form which is convenient for our purpose (note that we can take the symmetric parabolic cylinders instead of the lower ones considered in [20]). Moreover, to be precise we have to take a non-homogeneous version of the lemma valid for systems involving a right-hand side g_t, since our model problem is of this form. This can be achieved by minor changes with the methods we have previously used (see for instance the proof of Lemmas 4.2, 4.3).

Proposition 4.6. *Given $M>1$ and $\alpha\in(\beta,1)$, there exist constants $\varrho_{\mathrm{int}}=\varrho_{\mathrm{int}}(M)\in(0,1)$ with $Q_{\varrho_{\mathrm{int}}}(z_0)\Subset Q_1$ and $\mathcal{E}_{\mathrm{int}}(M)\in(0,1)$ depending on $n,N,p,\nu,L,M,H(M),\kappa_{M+1},\alpha,\beta$ and $\|g_t\|_{L^{2,2-2\beta}(Q_1^+)}$ such that the following holds. Suppose that*

$$\big|(u)_{z_0;\varrho}\big|+|D\ell_{z_0;\varrho}|\leqslant M,\qquad \varrho\leqslant\varrho_{\mathrm{int}},\qquad \widetilde{E}_{\mathrm{int}}(z_0,\varrho)\leqslant\mathcal{E}_{\mathrm{int}}$$

are fulfilled for some parabolic cylinder $Q_\varrho(z_0)\subset Q_1^+$. Then it follows the existence of the limit

$$\mathfrak{X}_{z_0}\equiv\lim_{j\to\infty}(Du)_{z_0,\vartheta^j\varrho},$$

and moreover, for any $0<r\leqslant\varrho/2$ there holds the estimate

$$\fint_{Q_r(z_0)}|Du-\mathfrak{X}_{z_0}|^2\,dz\leqslant c_{\mathrm{int}}\bigg[\bigg(\frac{r}{\varrho/2}\bigg)^{2\alpha}E_{\mathrm{int}}(z_0,\varrho)+r^{2\beta}\bigg],$$

where c_{int} depends on $n,N,p,\nu,L,M,H(M),\kappa_{M+1},\alpha,\beta$ and $\|g_t\|_{L^{2,2-2\beta}(Q_1^+)}$.

4.1.6. Description of regular lateral boundary points

We are now in a position to prove our main result, i.e. the characterization of regular lateral boundary points as stated in Theorem 1.2 in the model situation $\Omega_T = Q_1^+$. Let us note that due to our considerations in Section 2.1, the assertion of Theorem 1.2 concerning lateral boundary points $z_0 \in \partial_{\mathrm{lat}}\Omega_T$ directly follows from Proposition 4.7. In order to prove the Hölder continuity of Du in the regular set $\partial_{\mathrm{lat}}\Omega_T \setminus \Sigma$ we use the integral characterization of Hölder continuous functions due to Campanato and Da Prato. To this aim we have to combine the excess-decay estimates in the interior and at the boundary.

Proposition 4.7. *Suppose that $u \in L^p(\Lambda_1; W^{1,p}(B^+; \mathbb{R}^N))$ with $u \equiv 0$ on Γ_1 is a weak solution of* (2.5) *where the structure conditions* (1.4)–(1.6) *are in force. Then, for any $z_0 \in \Gamma_1 \setminus \Sigma$ there exists a neighborhood U_{z_0} such that*

$$Du \in C^{\beta,\frac{\beta}{2}}\big(U_{z_0} \cap \big(Q_1^+ \cup \Gamma_1\big); \mathbb{R}^{Nn}\big),$$

where $\Sigma_{\mathrm{lat}} := \Sigma_{\mathrm{lat}}^1 \cup \Sigma_{\mathrm{lat}}^2$ and

$$\Sigma_{\mathrm{lat}}^1 = \bigg\{ z_0 \in \Gamma_1\colon \liminf_{\varrho \downarrow 0} \fint_{Q_\varrho^+(z_0)} \big|D_n u - (D_n u)_{z_0,\varrho}^+\big|^p \, dz > 0 \bigg\},$$

$$\Sigma_{\mathrm{lat}}^2 = \Big\{ z_0 \in \Gamma_1\colon \limsup_{\varrho \downarrow 0} \big|(D_n u)_{z_0,\varrho}^+\big| = \infty \Big\}.$$

Proof. For $\mathfrak{z} \in Q_1^+ \cup \Gamma_1$, $\varrho > 0$, we define

$$\phi_s^{(n)}(\mathfrak{z},\varrho) := \fint_{Q_\varrho^+(\mathfrak{z})} \big|D_n u - (D_n u)_{\mathfrak{z},\varrho}^+\big|^s \, dz, \quad s = 2, p,$$

while, for a fixed $M_0 > 1$ we define

$$M_1 := 2(1 + M_0)(n+2) \quad \text{and} \quad M_2 := 2^{2n+5}(2 + M_0)(n+3)c_{Cac}(0),$$

where $c_{Cac}(0)$ is the constant from Lemma 4.2 for the choice $M = 0$. Moreover, we recall from Proposition 4.6 respectively Lemma 4.5 the definition of $\varrho_{\mathrm{lat}}(M_1)$, $\mathcal{E}_{\mathrm{lat}}(M_1)$, $c_{\mathrm{lat}} = c_{\mathrm{lat}}(M_1)$ respectively $\varrho_{\mathrm{int}}(M_2)$, $\mathcal{E}_{\mathrm{int}}(M_2)$. Note that $\mathcal{E}_{\mathrm{lat}}(M_1) \leqslant 1$ by definition.

Now, for $z_0 \in \Gamma_1 \setminus (\Sigma_{\mathrm{lat}}^1 \cup \Sigma_{\mathrm{lat}}^2)$ we can find $M_0 > 1$ (depending on z_0) and $0 < \varrho \leqslant \min\{\varrho_{\mathrm{lat}}(M_1), \varrho_{\mathrm{int}}(M_2)\}$ with $Q_{2\varrho}^+(z_0) \subset Q_1^+$ such that $|(D_n u)_{z_0,\varrho}^+| < M_0$ and

$$\phi_2^{(n)}(z_0,\varrho) + \phi_p^{(n)}(z_0,\varrho) + \varrho^{2\beta} < \frac{1}{2c}\big(c_5 c_{\mathrm{lat}}(M_1)\big)^{-1} \min\big\{\mathcal{E}_{\mathrm{lat}}(M_1), \mathcal{E}_{\mathrm{int}}(M_2)\big\} \tag{4.36}$$

where $c = 2^p(n+2)^{\frac{p}{2}}$ and $c_5 = 2^{2n}3^{2p}(n+3)^p$. Using the Poincaré inequality from Lemma 2.3 we see that

$$\bigg(\varrho^{-p} \fint_{Q_\varrho^+(z_0)} |u|^p \, dz\bigg)^{\frac{1}{p}} \leqslant \bigg(\fint_{Q_\varrho^+(z_0)} |D_n u|^p \, dz\bigg)^{\frac{1}{p}} \leqslant \phi_p^{(n)}(z_0,\varrho)^{\frac{1}{p}} + \big|(D_n u)_{z_0,\varrho}^+\big| \leqslant 1 + M_0.$$

This leads us immediately to (cf. Lemma 2.2 for the definition of $\xi_{z_0,\varrho}$)

$$|\xi_{z_0,\varrho}| \leqslant (n+2)\varrho^{-1} \fint_{Q_\varrho^+(z_0)} |u| \, dz \leqslant (n+2)(1+M_0) \leqslant \frac{1}{2}M_1,$$

where we have used the definition of M_1. By an application of the Caccioppoli inequality from Lemma 4.2 with $\xi = 0$ we also have

$$
\begin{aligned}
\fint_{Q_\varrho^+(z_0)} |Du|\,dz &\leqslant \left[c_{Cac}(0) \fint_{Q_\varrho^+(z_0)} \varrho^{-2}|u|^2 + \varrho^{-p}|u|^p + \varrho^{2\beta}\,dz\right]^{\frac{1}{p}} \\
&\leqslant \left[2c_{Cac}(0) \fint_{Q_\varrho^+(z_0)} \varrho^{-p}|u|^p + 1\,dz\right]^{\frac{1}{p}} \\
&\leqslant 2c_{Cac}(0)(2+M_0) \\
&\leqslant \left(2^{2n+4}(n+3)\right)^{-1} M_2.
\end{aligned} \tag{4.37}
$$

Here we have used in the last line the particular choice of M_1. Moreover, from Lemma 2.2, the Poincaré inequality Lemma 2.3 and (4.36) we deduce

$$
\begin{aligned}
\widetilde{E}_{\mathrm{lat}}(z_0, \varrho, \xi_{z_0,\varrho}) &\leqslant 2^p (n+2)^{\frac{p}{2}}\, \widetilde{E}_{\mathrm{lat}}\big(z_0, \varrho, (D_n u)^+_{z_0,\varrho}\big) \\
&\leqslant 2^p (n+2)^{\frac{p}{2}} \big(\phi_2^{(n)}(z_0,\varrho) + \phi_p^{(n)}(z_0,\varrho) + \varrho^{2\beta}\big) \\
&< \frac{1}{2} \min\big\{\mathcal{E}_{\mathrm{lat}}(M_1), c_5^{-1}, c_6^{-1}\mathcal{E}_{\mathrm{int}}(M_1)\big\}.
\end{aligned}
$$

Since $\Gamma_1 \ni z \mapsto \xi_{z,\varrho}$ and $\Gamma_1 \ni z \mapsto \widetilde{E}_{\mathrm{lat}}(z, \varrho, \xi_{z,\varrho})$ are continuous with respect to the center z, there exists a radius $0 < R \leqslant \varrho/12$ such that

$$
|\xi_{z,\varrho}| < M_1, \tag{4.38}
$$

and

$$
\widetilde{E}_{\mathrm{lat}}(z, \varrho, \xi_{z,\varrho}) < \big(c_5 c_{\mathrm{lat}}(M_1)\big)^{-1} \min\big\{\mathcal{E}_{\mathrm{lat}}(M_1), \mathcal{E}_{\mathrm{int}}(M_2)\big\} \tag{4.39}
$$

for all $z \in \Gamma_R(z_0)$. Moreover, due to the choice $R \leqslant \varrho/12$ and the inclusion $Q_{2\varrho}^+(z_0) \subset Q_1^+$ we have $Q_\varrho^+(z) \subset Q_{2\varrho}^+(z_0) \subset Q_1^+$.

Now, given $\alpha \in (\beta, 1)$ and M_1 from above we choose ϑ in dependence of $n, N, p, \nu, L, M_1, H(M_1), \kappa_{M_1}, \alpha, \beta$ and $\|g_t\|_{L^{2,2-2\beta}(Q_1^+)}$ to be the constant from Lemma 4.5. Without loss of generality we can assume that the constant ϑ appearing Proposition 4.6 is equal to the one from Lemma 4.5. In the following we will show that for all $\mathfrak{z} \in Q_R^+(z_0) \cup \Gamma_R(z_0)$ the limit

$$
\mathfrak{X}_{\mathfrak{z}} := \lim_{j\to\infty} (Du)^+_{\mathfrak{z},\vartheta^j\varrho} \tag{4.40}
$$

exists and that

$$
\fint_{Q_r^+(\mathfrak{z})} |Du - \mathfrak{X}_{\mathfrak{z}}|^2\,dz \leqslant c\left(\frac{r}{\varrho}\right)^{2\beta}, \tag{4.41}
$$

for all $0 < r \leqslant \varrho/6$ and with a constant c depending on $n, N, p, \nu, L, M_1, M_2, H(M_1), H(M_2), \kappa_{M_1+1}, \kappa_{M_2+1}, \alpha, \beta$ and $\|g_t\|_{L^{2,2-2\beta}(Q_1^+)}$. For this we will distinguish between the cases $\mathfrak{z} \in \Gamma_R(z_0)$ and $\mathfrak{z} \in Q_R^+(z_0)$.

In the *first case* $\mathfrak{z} \in \Gamma_R(z_0)$, we see that by (4.38) and (4.39) the hypotheses of Lemma 4.5 are fulfilled so that the application of the lemma yields the existence of the limit $\Upsilon_{\mathfrak{z}} = \lim_{j\to\infty}(D_n u)^+_{\mathfrak{z},\vartheta^j\varrho}$. Moreover, for $0 < r \leqslant \varrho/2$ we have

$$
\begin{aligned}
\fint_{Q_r^+(\mathfrak{z})} |Du - \Upsilon_{\mathfrak{z}} \otimes e_n|^2 + |Du - \Upsilon_{\mathfrak{z}} \otimes e_n|^p\,dz &\leqslant c_{\mathrm{lat}}\left[\left(\frac{r}{\varrho/2}\right)^{2\alpha} E_{\mathrm{lat}}(\mathfrak{z}, \varrho, \xi_{\mathfrak{z},\varrho}) + r^{2\beta}\right] \\
&\leqslant c_{\mathrm{lat}}\left(\frac{r}{\varrho/2}\right)^{2\beta} \widetilde{E}_{\mathrm{lat}}(\mathfrak{z}, \varrho, \xi_{\mathfrak{z},\varrho}) \\
&\leqslant c\left(\frac{r}{\varrho/2}\right)^{2\beta},
\end{aligned} \tag{4.42}
$$

where c depends on $n, N, p, \nu, L, M_1, H(M_1), \kappa_{M_1+1}, \alpha, \beta$ and $\|g_t\|_{L^{2,2-2\beta}(Q_1^+)}$. Here we have used in the last line the bound (4.39) for $\widetilde{E}_{\mathrm{lat}}(\mathfrak{z}, \varrho, \xi_{\mathfrak{z},\varrho})$. The preceding estimate implies for the tangential directions $\sigma = 1, \ldots, n-1$ in particular that

$$\lim_{r\downarrow 0} \fint_{Q_r^+(\mathfrak{z})} |D_\sigma u|^2\, dz = 0.$$

Hence, (4.40) and (4.41) are valid with $\mathfrak{X}_{\mathfrak{z}} = \Upsilon_{\mathfrak{z}} \otimes e_n$.

In *the second case* $\mathfrak{z} \in Q_R^+(z_0)$ we want to apply Proposition 4.6. Therefore we have to ensure that the hypotheses are fulfilled. From (4.35) we recall that $\ell_{\mathfrak{z},\mathfrak{x}_n}$ denotes the unique affine map minimizing $\ell \mapsto \int_{Q_{\mathfrak{x}_n}(\mathfrak{z})} |u-\ell|^2\, dz$ and by $\mathfrak{z}' = (\mathfrak{x}_1, \ldots, \mathfrak{x}_{n-1}, 0, \mathfrak{t})$ we denote the projection of $\mathfrak{z} = (\mathfrak{x}_1, \ldots, \mathfrak{x}_{n-1}, \mathfrak{x}_n, \mathfrak{t})$ onto Γ_1. Since $\mathfrak{z}' \in \Gamma_R(z_0)$ we can use the results from the first case with center $\mathfrak{z}'$ obtaining that the limit $\Upsilon_{\mathfrak{z}'} := \lim_{j\to\infty} (D_n u)^+_{\mathfrak{z}',\vartheta^j\varrho}$ exists and, moreover, that (4.42) holds with $\mathfrak{z}'$ instead of $\mathfrak{z}$. At this stage we recall that for $\mathfrak{z} \in Q_R^+(z_0)$ we have $Q_{\mathfrak{x}_n}(\mathfrak{z}) \subset Q^+_{2\mathfrak{x}_n}(\mathfrak{z}')$. Therefore, by the use of the minimizing property of $\ell_{\mathfrak{z},\mathfrak{x}_n}$ and Poincaré's inequality from Lemma 2.3 we find

$$\begin{aligned}
\mathfrak{x}_n^{-2} \fint_{Q_{\mathfrak{x}_n}(\mathfrak{z})} |u - \ell_{\mathfrak{z},\mathfrak{x}_n}|^2\, dz &\leqslant \mathfrak{x}_n^{-2} \fint_{Q_{\mathfrak{x}_n}(\mathfrak{z})} |u - x_n \Upsilon_{\mathfrak{z}'}|^2\, dz \\
&\leqslant 2^{n+1}\mathfrak{x}_n^{-2} \fint_{Q^+_{2\mathfrak{x}_n}(\mathfrak{z}')} |u - x_n \Upsilon_{\mathfrak{z}'}|^2\, dz \\
&\leqslant 2^{n+2} \fint_{Q^+_{2\mathfrak{x}_n}(\mathfrak{z}')} |D_n u - \Upsilon_{\mathfrak{z}'}|^2\, dz \\
&\leqslant 2^{n+2} \fint_{Q^+_{2\mathfrak{x}_n}(\mathfrak{z}')} |Du - \Upsilon_{\mathfrak{z}'} \otimes e_n|^2\, dz.
\end{aligned}$$

Similarly, replacing in the left-hand side of the preceding inequality the integrand by $|u - \ell^{(p)}_{\mathfrak{z},\mathfrak{x}_n}|^p$, where $\ell^{(p)}_{\mathfrak{z},\mathfrak{x}_n}$ is the unique affine map minimizing $\ell \mapsto \int_{Q_{\mathfrak{x}_n}(\mathfrak{z})} |u-\ell|^p\, dz$, we see that

$$\mathfrak{x}_n^{-p} \fint_{Q_{\mathfrak{x}_n}(\mathfrak{z})} \left|u - \ell^{(p)}_{\mathfrak{z},\mathfrak{x}_n}\right|^p dz \leqslant 2^{n+p} \fint_{Q^+_{2\mathfrak{x}_n}(\mathfrak{z}')} |Du - \Upsilon_{\mathfrak{z}'} \otimes e_n|^p\, dz.$$

In this estimate we want to replace $\ell^{(p)}_{\mathfrak{z},\mathfrak{x}_n}$ by $\ell_{\mathfrak{z},\mathfrak{x}_n}$, i.e. the affine map minimizing $\ell \mapsto \int_{Q_{\mathfrak{x}_n}(\mathfrak{z})} |u-\ell|^2\, dz$. From the proof of Lemma 5.1 in [20] (i.e. from (4-26)) we know that

$$\mathfrak{x}_n^{-p} \fint_{Q_{\mathfrak{x}_n}(\mathfrak{z})} |u - \ell_{\mathfrak{z},\mathfrak{x}_n}|^p\, dz \leqslant 3^p(n+2)^p \mathfrak{x}_n^{-p} \fint_{Q_{\mathfrak{x}_n}(\mathfrak{z})} \left|u - \ell^{(p)}_{\mathfrak{z},\mathfrak{x}_n}\right|^p dz. \tag{4.43}$$

Combining the previous estimates and using (4.42) with $(\mathfrak{z}', 2\mathfrak{x}_n, \varrho)$ instead of $(\mathfrak{z}, r, \varrho)$, the fact that $2\mathfrak{x}_n \leqslant 2R \leqslant \varrho/2$ and (4.39) we infer

$$\begin{aligned}
\widetilde{E}_{\mathrm{int}}(\mathfrak{z}, \mathfrak{x}_n) &= \mathfrak{x}_n^{-2} \fint_{Q_{\mathfrak{x}_n}(\mathfrak{z})} |u - \ell_{\mathfrak{z},\mathfrak{x}_n}|^2\, dz + \mathfrak{x}_n^{-p} \fint_{Q_{\mathfrak{x}_n}(\mathfrak{z})} |u - \ell_{\mathfrak{z},\mathfrak{x}_n}|^p\, dz + \mathfrak{x}_n^{2\beta} \\
&\leqslant c(n,p)\left(\fint_{Q^+_{2\mathfrak{x}_n}(\mathfrak{z}')} |Du - \Upsilon_{\mathfrak{z}'} \otimes e_n|^2 + |Du - \Upsilon_{\mathfrak{z}'} \otimes e_n|^p\, dz + (2\mathfrak{x}_n)^{2\beta}\right) \\
&\leqslant 2c(n,p)c_{\mathrm{lat}} \left(\frac{2\mathfrak{x}_n}{\varrho/2}\right)^{2\beta} \widetilde{E}_{\mathrm{lat}}(\mathfrak{z}', \varrho, \xi_{\mathfrak{z}',\varrho}) \leqslant \left(\frac{2\mathfrak{x}_n}{\varrho/2}\right)^{2\beta} \mathcal{E}_{\mathrm{int}}(M_2),
\end{aligned} \tag{4.44}$$

where we have abbreviated $c(n,p) = 3^p 2^{n+p}(n+2)^p$ and we note that $2c(n,p) \leqslant c_5$ due to the definition of c_5 in (4.36). Since $2\mathfrak{r}_n \leqslant 2R \leqslant \varrho/2$ this implies in particular that $\widetilde{E}_{\mathrm{int}}(\mathfrak{z},\mathfrak{r}_n) \leqslant \mathcal{E}_{\mathrm{int}}(M_2)$. Next, we will infer a bound for the mean-value of u and for $|D\ell_{\mathfrak{z},\mathfrak{r}_n}|$ on the cylinder $Q_{\mathfrak{r}_n}(\mathfrak{z})$. From Hölder's inequality, (4.42) and (4.37) we obtain

$$\begin{aligned} |\Upsilon_{\mathfrak{z}'}| &\leqslant \fint_{Q^+_{\varrho/2}(\mathfrak{z}')} |Du - \Upsilon_{\mathfrak{z}'} \otimes e_n|\,dz + \fint_{Q^+_{\varrho/2}(\mathfrak{z}')} |Du|\,dz \\ &\leqslant \big(c_{\mathrm{lat}} \widetilde{E}_{\mathrm{lat}}(\mathfrak{z}',\varrho,\xi_{\mathfrak{z}',\varrho})\big)^{\frac12} + 2^{n+2} \fint_{Q^+_{\varrho}(\mathfrak{z}')} |Du|\,dz \\ &\leqslant \big(c_{\mathrm{lat}} \widetilde{E}_{\mathrm{lat}}(\mathfrak{z}',\varrho,\xi_{\mathfrak{z}',\varrho})\big)^{\frac12} + \big(2^{n+2}(n+3)\big)^{-1} M_2. \end{aligned} \tag{4.45}$$

To proceed further we recall from (2-8) in [20] that

$$|D\ell_{\mathfrak{z},\mathfrak{r}_n}| = \frac{n+2}{\mathfrak{r}_n^2} \fint_{Q_{\mathfrak{r}_n}(\mathfrak{z})} u \otimes (x - \mathfrak{x})\,dz \leqslant \frac{n+2}{\mathfrak{r}_n} \fint_{Q_{\mathfrak{r}_n}(\mathfrak{z})} |u|\,dz.$$

Using the preceding estimate, Poincaré's inequality from Lemma 2.3, Hölder's inequality, (4.39), (4.42) and (4.45) we infer with $\tilde{c} = 2^{n+2}(n+3)$ (note also that $2\mathfrak{r}_n \leqslant \varrho/2 \leqslant 1$)

$$\begin{aligned} \big|(u)_{\mathfrak{z},\mathfrak{r}_n}\big| + |D\ell_{\mathfrak{z},\mathfrak{r}_n}| &\leqslant 2^{n+2}\frac{n+3}{2\mathfrak{r}_n} \fint_{Q^+_{2\mathfrak{r}_n}(\mathfrak{z}')} |u|\,dz \leqslant \tilde{c} \fint_{Q^+_{2\mathfrak{r}_n}(\mathfrak{z}')} |Du|\,dz \leqslant \tilde{c} \fint_{Q^+_{2\mathfrak{r}_n}(\mathfrak{z}')} |Du - \Upsilon_{\mathfrak{z}'} \otimes e_n|\,dz + \tilde{c}|\Upsilon_{\mathfrak{z}'}| \\ &\leqslant \tilde{c}\bigg(c_{\mathrm{lat}}\bigg(\frac{2\mathfrak{r}_n}{\varrho/2}\bigg)^{2\beta} \widetilde{E}_{\mathrm{lat}}(\mathfrak{z}',\varrho,\xi_{\mathfrak{z}',\varrho})\bigg)^{\frac12} + \tilde{c}\big(c_{\mathrm{lat}} \widetilde{E}_{\mathrm{lat}}(\mathfrak{z}',\varrho,\xi_{\mathfrak{z}',\varrho})\big)^{\frac12} + M_2 \\ &\leqslant 2\tilde{c}\big(c_{\mathrm{lat}} \widetilde{E}_{\mathrm{lat}}(\mathfrak{z}',\varrho,\xi_{\mathfrak{z}',\varrho})\big)^{\frac12} + M_2 \leqslant 2\tilde{c} c_5^{-\frac12} + M_2 \leqslant 1 + M_2. \end{aligned} \tag{4.46}$$

Recall that the constant c_5 was defined in (4.36) such that $2 \cdot 2^{2(n+2)}(n+3)^2 \leqslant c_5$ in (4.36). Hence, by (4.44) and (4.46) the hypotheses of Proposition 4.6 are satisfied. Therefore the Proposition can be applied with $(\mathfrak{z}, r, \mathfrak{r}_n, 1+M_2)$ instead of (z_0, r, ϱ, M) to conclude on the one hand that the limit in (4.40) exists and, on the other hand, that for any $0 < r \leqslant \mathfrak{r}_n/2$ there holds

$$\begin{aligned} \fint_{Q_r(\mathfrak{z})} |Du - \mathfrak{X}_{\mathfrak{z}}|^2\,dz &\leqslant c_{\mathrm{int}}\bigg[\bigg(\frac{r}{\mathfrak{r}_n/2}\bigg)^{2\alpha} E_{\mathrm{int}}(\mathfrak{z},\mathfrak{r}_n) + r^{2\beta}\bigg] \\ &\leqslant c_{\mathrm{int}}\bigg[\bigg(\frac{r}{\mathfrak{r}_n/2}\bigg)^{2\alpha}\bigg(\frac{2\mathfrak{r}_n}{\varrho/2}\bigg)^{2\beta} \mathcal{E}_{\mathrm{int}}(M_2) + r^{2\beta}\bigg] \leqslant c\bigg(\frac{r}{\varrho/2}\bigg)^{2\beta}, \end{aligned}$$

where c depends on $n, N, p, \nu, L, M_2, H(M_2), \kappa_{M_2+1}, \alpha$ and β. Here we have also used (4.44) in the second line and $\mathcal{E}_{\mathrm{int}}(M_2) \leqslant 1$ in the last line. In the remaining case $\mathfrak{r}_n/2 < r \leqslant \varrho/6$ we use (4.41) for $\mathfrak{z}'$ as well as the previous estimate to infer

$$\begin{aligned} \fint_{Q^+_r(\mathfrak{z})} |Du - \mathfrak{X}_{\mathfrak{z}}|^2\,dz &\leqslant 2 \fint_{Q^+_r(\mathfrak{z})} |Du - \mathfrak{X}_{\mathfrak{z}'}|^2\,dz + 4 \fint_{Q_{\mathfrak{r}_n/2}(\mathfrak{z})} |Du - \mathfrak{X}_{\mathfrak{z}'}|^2 + |Du - \mathfrak{X}_{\mathfrak{z}}|^2\,dz \\ &\leqslant c \fint_{Q^+_{3r}(\mathfrak{z}')} |Du - \mathfrak{X}_{\mathfrak{z}'}|^2\,dz + c \fint_{Q^+_{3\mathfrak{r}_n/2}(\mathfrak{z}')} |Du - \mathfrak{X}_{\mathfrak{z}'}|^2\,dz + c\bigg(\frac{\mathfrak{r}_n/2}{\varrho/2}\bigg)^{2\beta} \\ &\leqslant c\bigg(\frac{r}{\varrho/2}\bigg)^{2\beta}, \end{aligned}$$

where c depends only on $n, N, p, \nu, L, M_1, M_2, H(M_1), H(M_2), \kappa_{M_1+1}, \kappa_{M_2+1}, \alpha, \beta$ and $\|g_t\|_{L^{2,2-2\beta}(Q_1^+)}$. This completes the proof of (4.40) and (4.41).

Finally, we will prove that the Lebesgue representative $z \mapsto \mathfrak{X}_z$ of Du is Hölder continuous on $Q_R^+(z_0) \cup \Gamma_R(z_0)$. Given $z_1 = (x_1, t_1)$, $z_2 = (x_2, t_2) \in Q_R^+(z_0) \cup \Gamma_R(z_0)$, we put $r = \max\{|x_1 - x_2|, \sqrt{|t_1 - t_2|}\} \leqslant 2R \leqslant \varrho/6$ and $a \equiv \frac{z_1+z_2}{2}$. Then there holds

$$\begin{aligned}|\mathfrak{X}_{z_1} - \mathfrak{X}_{z_2}|^2 &= \fint_{Q_{r/2}^+(a)} |\mathfrak{X}_{z_1} - \mathfrak{X}_{z_2}|^2\,dz \leqslant 2 \fint_{Q_{r/2}^+(a)} |Du - \mathfrak{X}_{z_1}|^2\,dz + 2 \fint_{Q_{r/2}^+(a)} |Du - \mathfrak{X}_{z_2}|^2\,dz \\ &\leqslant 2^{n+3}\biggl[\fint_{Q_r^+(z_1)} |Du - \mathfrak{X}_{z_1}|^2\,dz + \fint_{Q_r^+(z_2)} |Du - \mathfrak{X}_{z_2}|^2\,dz\biggr].\end{aligned}$$

Using (4.41) for z_1 and z_2 we therefore infer

$$|\mathfrak{X}_{z_1} - \mathfrak{X}_{z_2}|^2 \leqslant c\left(\frac{r}{\varrho}\right)^{2\beta} \leqslant c\left(\frac{\mathrm{d}_{\mathcal{P}}(z_1, z_2)}{\varrho}\right)^{2\beta},$$

proving that the Lebesgue representative $z \mapsto \mathfrak{X}_z$ of Du is Hölder continuous with respect to the parabolic metric on $Q_R^+(z_0) \cup \Gamma_R(z_0)$ with Hölder exponent β. Here, the constant c depends only on $n, N, p, \nu, L, M_1, M_2, H(M_1)$, $H(M_2), \kappa_{M_1+1}, \kappa_{M_2+1}, \alpha, \beta$ and $\|g_t\|_{L^{2,2-2\beta}(Q_1^+)}$. This completes the proof of Proposition 4.7. □

This proves the assertion of Theorem 1.2 concerning the lateral boundary. We now turn our attention to initial boundary points.

4.2. Regular points on the initial boundary

In this section we prove the characterization of regular points z_0 lying on the initial boundary Ω_0. We shall always refer to the model situation (2.6) where the boundary values are equal to zero. The general result then follows by considering the map $v(x,t) = u(x,t) - g(x,0)$ as described in Section 2.1. Since the arguments are similar to the interior situation considered in [20], we shall only give an outline of the proof. Thereby we shall concentrate our attention on those arguments which are peculiar of the initial boundary situation.

4.2.1. A-caloric approximation

Lemma 4.8. *Given $\varepsilon > 0$, $0 < \nu \leqslant L$ and $p \geqslant 2$ there exists a positive function $\delta = \delta(n, p, \nu, L, \varepsilon) \in (0, 1]$ with the following property. Whenever A is a bilinear form on $\mathbb{R}^{Nn}$ which is strongly elliptic with ellipticity constant $\nu > 0$ and upper bound L, i.e.*

$$\nu|w|^2 \leqslant \langle Aw, w\rangle \quad \textit{and} \quad \langle Aw, \widetilde{w}\rangle \leqslant L|w||\widetilde{w}|$$

holds whenever $w, \widetilde{w} \in \mathbb{R}^{Nn}$ and $u \in L^p(0, \varrho^2; W^{1,p}(B_\varrho; \mathbb{R}^N))$ with $u \equiv 0$ on B_ϱ at the initial time $t = 0$ (in the usual L^2-sense) and

$$\fint_{Q_\varrho^0} \left|\frac{u}{\varrho}\right|^2 + |Du|^2\,dz + \gamma^{p-2}\fint_{Q_\varrho^0} \left|\frac{u}{\varrho}\right|^p + |Du|^p\,dz \leqslant 1,$$

where $0 < \gamma \leqslant 1$, is approximately A-caloric in the sense that

$$\left|\fint_{Q_\varrho^0} u \cdot \varphi_t - \langle ADu, D\varphi\rangle\,dz\right| \leqslant \delta \sup_{Q_\varrho^0} |D\varphi|, \quad \textit{for every } \varphi \in C_0^\infty\bigl(Q_\varrho^0; \mathbb{R}^N\bigr),$$

then there exists an A-caloric map $h \in L^p(0, (\varrho/2)^2; W^{1,p}(B_{\varrho/2}; \mathbb{R}^N))$, i.e.

$$\int_{Q_{\varrho/2}^0} h \cdot \varphi_t - \langle ADh, D\varphi\rangle\,dz = 0 \quad \textit{for every } \varphi \in C_0^\infty\bigl(Q_{\varrho/2}^0; \mathbb{R}^N\bigr),$$

with $h \equiv 0$ on $B_{\varrho/2}$ in the L^2-sense satisfying

$$\fint_{Q_\varrho^0} \left|\frac{h}{\varrho}\right|^2 + |Dh|^2 \, dz + \gamma^{p-2} \fint_{Q_\varrho^0} \left|\frac{h}{\varrho}\right|^p + |Dh|^p \, dz \leqslant 2 \cdot 2^{n+2+2p}$$

and

$$\fint_{Q_{\varrho/2}^0} \left|\frac{u-h}{\varrho/2}\right|^2 + \gamma^{p-2}\left|\frac{u-h}{\varrho/2}\right|^p dz \leqslant \varepsilon.$$

Proof. The proof goes as the one for Lemma 4.1, with a few modifications we are going to describe. After reducing to the case $Q_\varrho^0 \equiv Q_1^0$ via the usual scaling we proceed by contradiction: we get the existence of $\varepsilon > 0$ and sequences $(A_j)_{j\in\mathbb{N}}$ of bilinear forms on $\mathbb{R}^{Nn}$ with uniform ellipticity constant $\nu > 0$ and upper bound L, $(v_j)_{j\in\mathbb{N}}$ with $v_j \in L^p(0,1; W^{1,p}(B_1;\mathbb{R}^N))$ satisfying $v_j(\cdot,0) \equiv 0$ on B_1 and $\gamma_j \in (0,1]$ such that

$$\fint_{Q_1^0} |v_j|^2 + |Dv_j|^2 \, dz + \gamma_j^{p-2} \fint_{Q_1^0} |Dv_j|^p + |Dv_j|^p \, dz \leqslant 1 \tag{4.47}$$

and

$$\left|\int_{Q_1^0} v_j \cdot \varphi_t - \langle A_j Dv_j, D\varphi\rangle \, dz\right| \leqslant \frac{1}{j} \sup_{Q_1^0} |D\varphi| \quad \text{for every } \varphi \in C_0^\infty\big(Q_1^0;\mathbb{R}^N\big), \tag{4.48}$$

but

$$\fint_{Q_{1/2}^0} 4|v_j - h|^2 + 2^p \gamma_j^{p-2} |v_j - h|^p \, dz > \varepsilon \tag{4.49}$$

for all A_j-caloric maps h on $Q_{1/2}^0$ with $h(\cdot,0) \equiv 0$ on $B_{1/2}$ and

$$\fint_{Q_{1/2}^0} |h|^2 + |Dh|^2 \, dz + \gamma^{p-2} \fint_{Q_{1/2}^0} |h|^p + |Dh|^p \, dz \leqslant 2 \cdot 2^{n+2+2p}. \tag{4.50}$$

We define $\tilde v_j = \gamma_j^{\frac{p-2}{p}} v_j$ as in (4.5), and proceed as thereafter, up to proving the following strong convergence:

$$\begin{cases} v_j \to v & \text{strongly in } L^2\big(Q_1^0;\mathbb{R}^N\big), \\ \tilde v_j \to \tilde v & \text{strongly in } L^p\big(Q_1^0;\mathbb{R}^N\big). \end{cases} \tag{4.51}$$

Note that at this stage, and in contrast the case of a Dirichlet condition at the lateral boundary, we cannot immediately conclude by a trace theorem that $v \equiv 0$ on B_1. This can be derived after establishing equicontinuity of v_j with respect to t in $W^{-\ell,2}(B_1,\mathbb{R}^N)$ for some $\ell \in \mathbb{N}$. More precisely, by the way, we have proved the continuity estimate

$$\gamma_j^{\frac{s-2}{s}} \big\|v_j(\cdot,\tau_2) - v_j(\cdot,\tau_1)\big\|_{W^{-\ell,2}(B_1,\mathbb{R}^N)} \leqslant \tilde c\left(|A_j|(\tau_2-\tau_1)^{\frac12} + \frac{1}{j}\right), \tag{4.52}$$

which holds whenever $\tau_1, \tau_2 \in (0,1)$, and $j \in \mathbb{N}$. Notice that the previous inequality tells us that the family of Banach space valued maps $v_j : (0,1) \to W^{-\ell,2}(B_1,\mathbb{R}^N)$ is equi-uniformly continuous in $(0,1)$. Therefore we first observe that they can be extended, again in an equicontinuous way, as maps defined in $[0,1]$, i.e. they do have an initial trace $v_j(\cdot,0) \in W^{-\ell,2}(B_1,\mathbb{R}^N)$ in the sense that $\|v_j(\cdot,\tau) - v_j(\cdot,0)\|_{W^{-\ell,s}(B_1,\mathbb{R}^N)} \to 0$ when $\tau \to 0$; we call this a "weak trace". Moreover, the fact that $v_j \to v$ in $L^2(Q_1^0,\mathbb{R}^N)$ ensures that there exists $\tau \in (0,1)$ such that $v_j(\cdot,\tau) \to v(\cdot,\tau)$ in $L^2(B_1,\mathbb{R}^N)$ and therefore also in $W^{-\ell,2}(B_1,\mathbb{R}^N)$. Together with the last inequality we deduce that v_j is bounded in $C^0([0,1];W^{-\ell,2}(B_1,\mathbb{R}^N))$. This allows us to apply Ascoli–Arzelá's theorem to conclude that, up to extracting a non-relabeled subsequence, we may assume that $v_j\to v$ in $C^0([0,1],W^{-\ell,s}(B_1,\mathbb{R}^N))$. Therefore, by uniform convergence we have that $v_j(0)\to v(0)$ in $W^{-\ell,s}(B_1,\mathbb{R}^N)$. On the other hand we have that $v_j(\cdot,0)\equiv 0$ in the strong L^2-sense by assumption, and therefore also the weak trace of v_j at the initial time is zero, since a strong trace is also a weak trace – this follows from $\|\cdot\|_{W^{-\ell,2}(B_1,\mathbb{R}^N)}\leqslant\|\cdot\|_{L^2(B_1,\mathbb{R}^N)}$. We deduce at once that the weak trace of v is zero. Now we know that v is A-caloric, and therefore it has a strong trace, i.e. $\|v(\cdot,\tau)-v(\cdot,0)\|_{L^2(B_1)}\to 0$; this follows from the fact that $v\in C^0([0,1];L^2(B_1))$; again, as a strong trace is also a weak trace we finally conclude with $\|v(\cdot,\tau)\|_{L^2(B_1,\mathbb{R}^N)}\to 0$, that is v has zero trace at the initial time in the sense of (1.2). With this information we conclude the proof: we can define $w_j\in L^2(\Lambda^0_{(3/4)^2};W^{1,2}(B^+_{3/4},\mathbb{R}^N))$ as the unique solution of the following Cauchy–Dirichlet problem:

$$\begin{cases}\displaystyle\int_{Q^0_{3/4}} w_j\cdot\varphi_t-\langle A_jDw_j,D\varphi\rangle\,dz=0 \quad \text{for every } \varphi\in C_0^\infty\big(Q^0_{3/4},\mathbb{R}^N\big),\\ w_j=v \quad \text{on } \partial_{\mathcal{P}}Q^0_{3/4}.\end{cases}$$

Since now we have defined the A_j-caloric map w_j with $w_j(\cdot,0)=v(\cdot,0)=0$ on $B_{3/4}$, the rest of the proof follows as in the lateral boundary case Lemma 4.8, using the corresponding results from Section 3.2, instead of those from Section 3.1. □

4.2.2. Caccioppoli inequality

We now state the Caccioppoli inequality on initial cylinders of the form $Q^0_\varrho(z_0)$ where $z_0=(x_0,0)$ touching the initial boundary Ω_0. Since the proof is essentially the same as the one from Lemma 4.2 we shall only outline the changes that have to be made.

Lemma 4.9. *Suppose that $u\in L^p(0,T;W^{1,p}(\Omega;\mathbb{R}^N))$ is a weak solution of the non-linear parabolic system* (2.6) *with $u(\cdot,0)=0$ on Ω, where the structure conditions* (1.4)–(1.6) *are in force. Then, for any $z_0=(x_0,0)\in\Omega_0$ and $\varrho\in(0,1)$ such that $Q^0_\varrho(z_0)\subset\Omega_T$ there holds*

$$\fint_{Q^0_{\varrho/2}(z_0)}|Du|^2+|Du|^p\,dz\leqslant c_{Cac}\bigg(\fint_{Q^0_\varrho(z_0)}\bigg|\frac{u}{\varrho}\bigg|^2+\bigg|\frac{u}{\varrho}\bigg|^p\,dz+\varrho^{2\beta}\bigg),$$

where $c_{Cac}=(1+\|g_t\|^2_{L^{2,2-2\beta}(\Omega_T)})c(n,p,L/\nu,K(1),\kappa_1)$.

Sketch of the proof. Here, we let $0<\varepsilon<\varrho^2/2$ and choose the test-function $\varphi_\varepsilon(x,t)=\eta(x)\zeta_\varepsilon(t)u(x,t)$, where η is as in the proof of Lemma 4.2 and $\zeta_\varepsilon\in W_0^{1,\infty}((0,\varrho^2))$ such that $\zeta_\varepsilon\equiv 1$ on $[\varepsilon,\varrho^2-\varepsilon]$, $\zeta_\varepsilon(t)=t/\varepsilon$ on $(0,\varepsilon)$ and $\zeta_\varepsilon(t)=(\varrho^2-t)/\varepsilon$ on $(\varrho^2-\varepsilon,\varrho^2)$. Testing the parabolic system (2.6) formally with φ_ε we arrive at the analogue of (4.17) with $\xi=0$ and $\mathfrak{X}=0$. Now, the estimates for the terms I and II are similar to the ones in the proof of Lemma 4.2. Indeed, the only difference is the estimation of the term $IV=IV_\varepsilon$ which now depends on ε. Here, we can exploit the initial condition on u to show that

$$\begin{aligned}IV_\varepsilon&=\int_{Q^0_\varrho}u\cdot\partial_t\varphi_\varepsilon\,dz\\&=\int_{Q^0_\varrho}|u|^2\eta^2\partial_t\zeta_\varepsilon\,dz+\frac12\int_{Q^0_\varrho}\partial_t|u|^2\eta^2\zeta_\varepsilon\,dz\end{aligned}$$

$$= \frac{1}{2} \int\limits_{Q_\varrho^0} |u|^2 \eta^2 \partial_t \zeta_\varepsilon \, dz$$

$$= \frac{1}{2\varepsilon} \int\limits_0^\varepsilon \int\limits_{B_\varrho} |u|^2 \eta^2 \, dx \, dt - \frac{1}{2\varepsilon} \int\limits_{\varrho^2-\varepsilon}^{\varrho^2} \int\limits_{B_\varrho} |u|^2 \eta^2 \, dx \, dt.$$

Due to our initial condition on u we have

$$\lim_{\varepsilon \downarrow 0} \frac{1}{2\varepsilon} \int\limits_0^\varepsilon \int\limits_{B_R} |u|^2 \eta^2 \, dx \, dt = 0.$$

With this information we can conclude the proof as in Lemma 4.2. $\square$

4.2.3. Linearization

For $s \geqslant 1$, $z_0 = (x_0, 0) \in \Omega_0$ and a parabolic cylinder $Q_\varrho^0(z_0) \subset \Omega_T$ we define the excess functionals by

$$\phi_s^0(z_0, \varrho) := \fint\limits_{Q_\varrho^0(z_0)} |Du|^s \, dz, \qquad \psi_s^0(z_0, \varrho) := \fint\limits_{Q_\varrho^0(z_0)} \left| \frac{u}{\varrho} \right|^s dz,$$

and we shall often abbreviate

$$\phi_s^0 = \phi_s^0(z_0, \varrho) \quad \text{and} \quad \psi_s^0 = \psi_s^0(z_0, \varrho).$$

We now state the linearization lemma for cylinders touching the initial boundary, i.e. the analogue of Lemma 4.3. Here, we are allowed to apply the linearization lemma for interior cylinders from [20, Lemma 4.4] also on the cylinder $Q_\varrho^0(z_0)$ where $z_0 = (x_0, t_0)$, and with the choice $\ell = 0$ and $M = 0$, because the test-function φ is assumed to have compact support in $Q_\varrho^0(z_0)$. We only have to take into account the fact that we are dealing with inhomogeneous systems and that we did impose a slightly weaker regularity assumption on the vector field a, compared to [20], namely we did not assume any regularity in t on the vector field a in (1.6). Therefore we "freeze" the coefficients in $a((0,t), 0, 0)$ instead of $a((0,0), 0, 0)$ as we did in (4.18) in the lateral boundary situation. Taking into account this slight change in the proof we come up with $\omega_1(\phi_p^0 + \varrho^p)$ on the right-hand side rather than $\omega_1(\phi_p^0)$. The linearization lemma at the initial time boundary situation then reads as follows:

Lemma 4.10. *Suppose that $u \in L^p(0, T; W^{1,p}(\Omega; \mathbb{R}^N))$ is a weak solution of* (2.6) *satisfying $u(\cdot, 0) = 0$ on Ω, where the structure conditions* (1.4)–(1.6) *are in force. Then we have*

$$\left| \fint\limits_{Q_\varrho^0(z_0)} u \cdot \varphi_t - \langle \partial_w a(z_0, 0, 0) Du, D\varphi \rangle \, dz \right|$$

$$\leqslant c_{Eu} \left[\omega_1\big(\phi_p^0 + \varrho^p\big) \sqrt{\phi_2^0} + \phi_p^0 + \psi_2^0 + \varrho^\beta \big(\phi_p^0\big)^{1-\frac{1}{p}} \big(1 + \big(\psi_p^0\big)^{\frac{\beta}{p}}\big) + \varrho^\beta \right] \sup_{Q_\varrho^0(z_0)} |D\varphi|,$$

for any $\varphi \in C_0^\infty(Q_\varrho^0(z_0); \mathbb{R}^N)$, $z_0 = (x_0, 0) \in \Omega_0$ and $Q_\varrho^0(z_0) \subset \Omega_T$. The constant c_{Eu} depends on is of the form $c_{Eu} = L(1 + \|g_t\|^2_{L^{2,2-2\beta}(\Omega_T)}) c(n, p, K(1), \kappa_1)$.

4.2.4. A decay estimate at the initial boundary

At the initial boundary we want to approximate our solution u by an affine map constructed from the initial values, i.e. the affine map $\ell : \mathbb{R}^n \to \mathbb{R}^N$ minimizing $\ell \mapsto \int_{B_\varrho(x_0)} |g_0 - \ell|^2 \, dx$. Since we have transformed the problem to the model situation where $g_0 = 0$ we shall take $\ell = 0$. Therefore, we define the following excess functional

$$E_{\text{ini}} \equiv E_{\text{ini}}(z_0, \varrho) := \fint\limits_{Q_\varrho^0(z_0)} \left| \frac{u}{\varrho} \right|^2 + \left| \frac{u}{\varrho} \right|^p dz$$

for $z_0 = (x_0, 0) \in \Omega_0$ and

$$\widetilde{E}_{\text{ini}} \equiv \widetilde{E}_{\text{ini}}(z_0, \varrho) := E_{\text{ini}}(z_0, \varrho) + \varrho^{2\beta}.$$

Then, we can show the following excess-decay estimate.

Lemma 4.11. *Given $\alpha \in (\beta, 1)$, there exist constants $\vartheta \in (0, \frac{1}{4}]$, $\mathcal{E}_{\text{ini}} \in (0, 1]$, $\varrho_{\text{ini}} > 0$ and c_4 depending on $n, N, p, \nu, L, K(1), \kappa_1, \alpha, \beta, \|g_t\|^2_{L^{2,2-2\beta}(\Omega_T)}$ such that the following holds. Suppose that $u \in L^p(0, T; W^{1,p}(\Omega; \mathbb{R}^N))$ is a weak solution of* (2.6) *satisfying $u(\cdot, 0) = 0$ on Ω, where the structure conditions* (1.4)–(1.6) *are in force and suppose that*

(i) $\varrho \leqslant \varrho_{\text{ini}}$,
(ii) $\widetilde{E}_{\text{ini}}(\varrho) \leqslant \mathcal{E}_{\text{ini}}$

are fulfilled on some cylinder $Q^0_\varrho(z_0)$ with $z_0 = (x_0, 0) \in \Omega_0$ and $B_\varrho(x_0) \Subset \Omega$. Then for any $0 < r \leqslant \varrho/2$ we have

$$\fint_{Q^0_r(z_0)} \left|\frac{u}{r}\right|^2 + \left|\frac{u}{r}\right|^p dz + \fint_{Q^0_r(z_0)} |Du|^2 + |Du|^p \, dz \leqslant c_{\text{ini}} \left[\left(\frac{r}{\varrho}\right)^{2\alpha} E_{\text{ini}}(\varrho) + r^{2\beta}\right],$$

where the constant c_{ini} depends on $n, N, p, \nu, L, K(1), \kappa_1, \alpha, \beta, \|g_t\|^2_{L^{2,2-2\beta}(\Omega_T)}$. In particular this estimate implies

$$\mathfrak{X}_{z_0} := \lim_{j\to\infty} (Du)^0_{z_0, \vartheta^j \varrho} = 0.$$

Sketch of the proof. Since the proof is very much similar to the one from Lemma 4.5 for the lateral boundary situation, we shall only give a brief outline here. From the linearization Lemma 4.10 and the Caccioppoli inequality 4.9 we infer that u satisfies approximately a linear system with constant coefficients $\partial_w a(z_0, 0, 0)$. This allows us to apply the A-caloric approximation Lemma 4.8 to find a suitable A-caloric map h on $Q^0_{\varrho/2}(z_0)$ which is close to u in $L^p(Q^0_{\varrho/2}(z_0), \mathbb{R}^N)$ and with $h(\cdot, 0) = 0$ on $B_{\varrho/2}(x_0)$.

Next, we use the excess-decay estimate from Theorem 3.5 for the A-caloric map h and the fact that h is close to u in L^p to infer an estimate for the excess functional $\widetilde{E}_{\text{ini}}(\vartheta\varrho, z_0)$ on a smaller cylinder $Q^0_{\vartheta\varrho}(z_0)$ with some $\vartheta \in (0, \frac{1}{4})$ and under certain smallness assumptions. Finally, we iterate this estimate to get an excess-decay estimate for u on cylinders of the type $Q^0_{\vartheta^j\varrho}(z_0)$, $j \in \mathbb{N}$. From this we deduce an excess-decay estimate for u on arbitrary cylinders $Q^0_r(z_0)$ with $r \leqslant \varrho$. Finally, with the help of the Caccioppoli inequality from Lemma 4.2 we also infer the asserted excess-decay estimate for Du which completes the proof of the lemma. □

4.2.5. Description of regular initial boundary points

As usual, we prove the Hölder continuity of Du on the regular set $\Omega_0 \setminus \Sigma$ by the integral characterization of Hölder continuous functions of Campanato and Da Prato. Therefore we have to combine the excess-decay estimates for cylinders touching the initial boundary from above with the one for cylinders lying in the interior of Ω_T from Proposition 4.6.

Proposition 4.12. *Suppose that $u \in L^p(0, T; W^{1,p}(\Omega; \mathbb{R}^N))$ with $u(\cdot, 0) = 0$ on Ω is a weak solution of* (2.6) *where the structure conditions* (1.4)–(1.6) *are in force. Then, for any $z_0 = (x_0, 0) \in \Omega_0 \setminus \Sigma_{\text{ini}}$ there exists a neighborhood U_{z_0} such that*

$$Du \in C^{\beta, \frac{\beta}{2}}\big(U_{z_0} \cap (\Omega_T \cup \Omega_0); \mathbb{R}^{Nn}\big),$$

where

$$\Sigma_{\text{ini}} := \left\{ z_0 \in \Omega_0 \colon \liminf_{\varrho \downarrow 0} \varrho^{-p} \fint_{Q^0_\varrho(z_0)} |u|^p \, dz > 0 \right\}.$$

Proof. In the following we denote by where ϱ_{ini}, $\mathcal{E}_{\mathrm{ini}}$, c_{ini}, $\varrho_{\mathrm{int}}(1)$ and $\mathcal{E}_{\mathrm{int}}(1)$ are the constants from Lemma 4.11, respectively Proposition 4.6 for the choice $M=1$ Now, for $z_0 \in \Omega_0 \setminus \Sigma_{\mathrm{ini}}$ we can find $0<\varrho \leqslant \min\{\varrho_{\mathrm{ini}}, \varrho_{\mathrm{int}}(1)\}$ with $Q^0_{2\varrho}(z_0) \subset \Omega_T$ such that

$$\widetilde{E}_{\mathrm{ini}}(z_0,\varrho) < \frac{1}{2}(c_5 c_{\mathrm{ini}})^{-1} \min\{\mathcal{E}_{\mathrm{ini}}, \mathcal{E}_{\mathrm{int}}(1)\}, \tag{4.53}$$

where we define $c_5 = 2^{2n}3^{2p}(n+3)^p$. Since $\Omega_0 \ni z \mapsto \widetilde{E}_{\mathrm{ini}}(z,\varrho)$ is continuous with respect to the center z, there exists a radius $0<R\leqslant \varrho/12$ such that

$$\widetilde{E}_{\mathrm{ini}}(z,\varrho) < (c_5 c_{\mathrm{ini}})^{-1} \min\{\mathcal{E}_{\mathrm{ini}}, \mathcal{E}_{\mathrm{int}}(1)\} \tag{4.54}$$

for all $z=(x,0) \in D_R(z_0)$. Moreover, due to the choice $R \leqslant \varrho/12$ and the inclusion $Q^0_{2\varrho}(z_0) \subset \Omega_T$ we have $Q^0_\varrho(z) \subset Q^0_{2\varrho}(z_0) \subset \Omega_T$.

Now, given $\alpha \in (\beta,1)$ we choose ϑ in dependence of $n, N, p, \nu, L, K(1), \kappa_1, \alpha, \beta$ to be the constant from Lemma 4.11. Without loss of generality we can assume that the constant ϑ appearing in Proposition 4.6 (with the choice $M=1$) is equal to the one from Lemma 4.11. In the following we will show that for all $\mathfrak{z} \in Q^0_R(z_0) \cup D_R(z_0)$ the limit

$$\mathfrak{X}_{\mathfrak{z}} := \lim_{j\to\infty} (Du)^0_{\mathfrak{z},\vartheta^j\varrho} \tag{4.55}$$

exists and that

$$\fint_{Q^0_r(\mathfrak{z})} |Du - \mathfrak{X}_{\mathfrak{z}}|^2\,dz \leqslant c\left(\frac{r}{\varrho}\right)^{2\beta}, \tag{4.56}$$

for all $0<r\leqslant \varrho/6$ and with a constant c depending on $n, N, p, \nu, L, K(1), \kappa_1, \alpha, \beta$. For this we will distinguish between the cases $\mathfrak{z} \in D_R(z_0)$ and $\mathfrak{z} \in Q^0_R(z_0)$. In the first case, i.e. *the case* $\mathfrak{z} \in D_R(z_0)$, we see that by (4.54) the hypotheses of Lemma 4.11 are fulfilled (note that $c_5 \geqslant 1$) so that the application of the lemma yields that

$$\mathfrak{X}_{\mathfrak{z}} = \lim_{j\to\infty} (Du)^0_{\mathfrak{z},\vartheta^j\varrho} = 0.$$

Moreover, for $0<r\leqslant\varrho/2$ we have

$$\begin{aligned}
\fint_{Q^0_r(\mathfrak{z})} \left|\frac{u}{r}\right|^2 + \left|\frac{u}{r}\right|^p + |Du|^2 + |Du|^p\,dz &\leqslant c_{\mathrm{ini}}\left[\left(\frac{r}{\varrho}\right)^{2\alpha} E_{\mathrm{ini}}(\mathfrak{z},\varrho) + r^{2\beta}\right] \\
&\leqslant c_{\mathrm{ini}}\left(\frac{r}{\varrho}\right)^{2\beta} \widetilde{E}_{\mathrm{ini}}(\mathfrak{z},\varrho) \\
&\leqslant c\left(\frac{r}{\varrho}\right)^{2\beta},
\end{aligned} \tag{4.57}$$

where c depends on $n, N, p, \nu, L, K(1), \kappa_1, \alpha, \beta, \|g_t\|^2_{L^{2,2-2\beta}(\Omega_T)}$. Here we have used in the last line the bound (4.54) for $\widetilde{E}_{\mathrm{ini}}(\mathfrak{z},\varrho)$. Hence, (4.55) and (4.56) are valid with $\mathfrak{X}_{\mathfrak{z}}=0$.

In *the case* $\mathfrak{z} \in Q^0_R(z_0)$ we want to apply Proposition 4.6. Therefore we first have to ensure that the hypotheses are satisfied. By $\mathfrak{z}' = (\mathfrak{x},0)$ we denote the projection of $\mathfrak{z}=(\mathfrak{x},\mathfrak{t})$ onto Ω_0. From (4.35) we recall that $\ell_{\mathfrak{z},\sqrt{\mathfrak{t}}}$ denotes the unique affine map minimizing $\ell \mapsto \int_{Q_{\sqrt{\mathfrak{t}}}(\mathfrak{z})} |u-\ell|^2\,dz$. Therefore, by the use of the minimizing property of $\ell_{\mathfrak{z},\sqrt{\mathfrak{t}}}$ we find

$$\fint_{Q_{\sqrt{\mathfrak{t}}}(\mathfrak{z})} \left|\frac{u-\ell_{\mathfrak{z},\sqrt{\mathfrak{t}}}}{\sqrt{\mathfrak{t}}}\right|^2 dz \leqslant \fint_{Q_{\sqrt{\mathfrak{t}}}(\mathfrak{z})} \left|\frac{u}{\sqrt{\mathfrak{t}}}\right|^2 dz.$$

Furthermore, we recall that $\ell^{(p)}_{\mathfrak{z},\sqrt{\mathfrak{t}}}$ denotes the unique affine map minimizing the functional $\ell \mapsto \int_{Q_{\sqrt{\mathfrak{t}}}(\mathfrak{z})} |u-\ell|^p\,dz$. Using (4.43) and the minimizing property of $\ell^{(p)}_{\mathfrak{z},\mathfrak{x}_n}$ we also find

$$\fint_{Q_{\sqrt{\mathfrak{t}}}(\mathfrak{z})} \left|\frac{u-\ell_{\mathfrak{z},\sqrt{\mathfrak{t}}}}{\sqrt{\mathfrak{t}}}\right|^p dz \leqslant 3^p(n+2)^p \fint_{Q_{\sqrt{\mathfrak{t}}}(\mathfrak{z})} \left|\frac{u}{\sqrt{\mathfrak{t}}}\right|^p dz.$$

At this stage we recall that $Q_{\sqrt{\mathfrak{t}}}(\mathfrak{z}) \subset Q^0_{\sqrt{2\mathfrak{t}}}(\mathfrak{z}')$ which allows us to enlarge the domain of integration from $Q_{\sqrt{\mathfrak{t}}}(\mathfrak{z})$ to $Q^0_{\sqrt{2\mathfrak{t}}}(\mathfrak{z}')$. Now, since $\mathfrak{z}' \in D_R(z_0)$ we can use the results from the first case with center $\mathfrak{z}'$ obtaining that the limit $\mathfrak{X}_{\mathfrak{z}'} := \lim_{j\to\infty}(Du)^0_{\mathfrak{z}',\vartheta^j\varrho} = 0$ exists and, moreover, that (4.57) holds with $\mathfrak{z}'$ instead of $\mathfrak{z}$. Combining the previous estimates and using (4.57) with $(\mathfrak{z}', \sqrt{2\mathfrak{t}}, \varrho)$ instead of $(\mathfrak{z}, r, \varrho)$, the fact that $\sqrt{2\mathfrak{t}} \leqslant 2R \leqslant \varrho/2$ and (4.54) we infer

$$\begin{aligned}
\widetilde{E}_{\mathrm{int}}(\mathfrak{z},\sqrt{\mathfrak{t}}) &\leqslant 3^p(n+2)^p \fint_{Q_{\sqrt{\mathfrak{t}}}(\mathfrak{z})} \left|\frac{u}{\sqrt{\mathfrak{t}}}\right|^2 + \left|\frac{u}{\sqrt{\mathfrak{t}}}\right|^p dz + \sqrt{\mathfrak{t}}^{2\beta} \\
&\leqslant 2^{n+p}3^p(n+2)^p\left(\fint_{Q^0_{\sqrt{2\mathfrak{t}}}(\mathfrak{z}')} \left|\frac{u}{\sqrt{2\mathfrak{t}}}\right|^2 + \left|\frac{u}{\sqrt{2\mathfrak{t}}}\right|^p dz + \sqrt{2\mathfrak{t}}^{2\beta}\right) \\
&\leqslant 2\cdot 3^p 2^{n+p}(n+2)^p c_{\mathrm{ini}} \left(\frac{\sqrt{2\mathfrak{t}}}{\varrho}\right)^{2\beta} \widetilde{E}_{\mathrm{ini}}(\mathfrak{z}',\varrho) \\
&\leqslant \left(\frac{\sqrt{2\mathfrak{t}}}{\varrho}\right)^{2\beta} \mathcal{E}_{\mathrm{int}}(1),
\end{aligned} \tag{4.58}$$

where we have abbreviated $\widetilde{E}_{\mathrm{int}}(\mathfrak{z},\sqrt{\mathfrak{t}}) = \widetilde{E}_{\mathrm{int}}(\mathfrak{z},\sqrt{\mathfrak{t}},\ell_{\mathfrak{z},\sqrt{\mathfrak{t}}})$. Note that the constant c_5 was defined in (4.53) such that $2\cdot 3^p 2^{n+p}(n+2)^p \leqslant c_5$. Since $\sqrt{2\mathfrak{t}} \leqslant 2R \leqslant \varrho/2$ this implies in particular that $\widetilde{E}_{\mathrm{int}}(\mathfrak{z},\sqrt{\mathfrak{t}}) \leqslant \mathcal{E}_{\mathrm{int}}(1)$. Next, we will infer a bound for the mean-value of u and for $|D\ell_{\mathfrak{z},\sqrt{\mathfrak{t}}}|$ on the cylinder $Q_{\sqrt{\mathfrak{t}}}(\mathfrak{z})$. Starting as in (4.46) and then using Hölder's inequality, (4.57) and (4.54) we infer with $\tilde{c} = 2^{n+2}(n+3)$ (note also that $\mathcal{E}_{\mathrm{ini}} \leqslant 1$ and $\sqrt{2\mathfrak{t}} \leqslant 1$),

$$\left|(u)_{\mathfrak{z},\sqrt{\mathfrak{t}}}\right| + |D\ell_{\mathfrak{z},\sqrt{\mathfrak{t}}}| \leqslant 2^{n+1}\frac{n+3}{\sqrt{2\mathfrak{t}}} \fint_{Q^0_{\sqrt{2\mathfrak{t}}}(\mathfrak{z}')} |u|\,dz \leqslant \tilde{c}\left(c_{\mathrm{ini}}\left(\frac{\sqrt{2\mathfrak{t}}}{\varrho}\right)^{2\beta} \widetilde{E}_{\mathrm{ini}}(\mathfrak{z}',\varrho)\right)^{\frac12} \leqslant \tilde{c}c_5^{-\frac12} \leqslant 1.$$

Recall that c_5 was defined such that $\tilde{c}^2 \leqslant c_5$ in (4.53). Hence, by the preceding estimate and (4.46) the hypotheses of Proposition 4.6 are satisfied. Therefore the Proposition can be applied with $(\mathfrak{z}, r, \sqrt{\mathfrak{t}}, 1)$ instead of (z_0, r, ϱ, M) to conclude on the one hand that the limit in (4.55) exists and, on the other hand, that for any $0 < r \leqslant \sqrt{\mathfrak{t}}/2$ there holds

$$\begin{aligned}
\fint_{Q_r(\mathfrak{z})} |Du - \mathfrak{X}_{\mathfrak{z}}|^2\,dz &\leqslant c_{\mathrm{int}}\left[\left(\frac{r}{\sqrt{\mathfrak{t}}/2}\right)^{2\alpha} E_{\mathrm{int}}(\mathfrak{z},\sqrt{\mathfrak{t}}) + r^{2\beta}\right] \\
&\leqslant c_{\mathrm{int}}\left[\left(\frac{r}{\sqrt{\mathfrak{t}}/2}\right)^{2\alpha}\left(\frac{\sqrt{2\mathfrak{t}}}{\varrho}\right)^{2\beta} \mathcal{E}_{\mathrm{int}}(1) + r^{2\beta}\right] \\
&\leqslant c\left(\frac{r}{\varrho}\right)^{2\beta},
\end{aligned}$$

where c depends on $n, N, p, \nu, L, K(1), \kappa_1, \alpha, \beta$ and $\|g_t\|^2_{L^{2,2-2\beta}(\Omega_T)}$. Here we have also used (4.58) in the second line and $\mathcal{E}_{\mathrm{int}}(1) \leqslant 1$ in the last line. In the remaining case $\sqrt{\mathfrak{t}}/2 < r \leqslant \varrho/6$ we use the previous estimate as well as (4.56) for $\mathfrak{z}'$ (note that $\mathfrak{X}_{\mathfrak{z}'} = 0$) to infer

$$\begin{aligned}
\fint_{Q_r^0(\mathfrak{z})} |Du-\mathfrak{X}_{\mathfrak{z}}|^2\,dz &\leqslant 2\fint_{Q_r^0(\mathfrak{z})} |Du|^2\,dz + 4\fint_{Q_{\sqrt{\mathfrak{t}}/2}(\mathfrak{z})} |Du|^2+|Du-\mathfrak{X}_{\mathfrak{z}}|^2\,dz \\
&\leqslant c\fint_{Q_{3r}^0(\mathfrak{z}')} |Du|^2\,dz + c\fint_{Q^0_{3\sqrt{\mathfrak{t}}/2}(\mathfrak{z}')} |Du|^2\,dz + c\left(\frac{\sqrt{\mathfrak{t}}/2}{\varrho}\right)^{2\beta} \\
&\leqslant c\left(\frac{r}{\varrho}\right)^{2\beta},
\end{aligned}$$

where c depends only on $n, N, p, \nu, L, K(1), \kappa_1, \alpha, \beta, \|g_t\|^2_{L^{2,2-2\beta}(\Omega_T)}$. This completes the proof of (4.55) and (4.56). Now, we can proceed completely similar to the proof of Proposition 4.7 to show that the Lebesgue representative $z\mapsto \mathfrak{X}_z$ of Du is Hölder continuous on $Q_R^0(z_0)\cup D_R(z_0)$ which completes the proof of Proposition 4.12. □

4.2.6. Poincaré type inequality, at last

The characterization of regular initial boundary points we have proved so far is not the one stated in Theorem 1.2. Therefore, we still have to show that $\Sigma_{\mathrm{ini}}\subset(\Sigma^1\cup\Sigma^2)\cap\Omega_0$. This is a consequence of the following Poincaré type inequality, after which, the proof of Theorem 1.2 concerning the initial boundary is complete.

Lemma 4.13. *Let $M>0$ and suppose that $u\in L^p(0,T;W^{1,p}(\Omega;\mathbb{R}^N))$ is a weak solution of* (2.6) *satisfying $u(\cdot,0)=0$ on Ω, and that the structure conditions* (1.4)–(1.6) *are in force. Moreover, let $z_0=(x_0,0)\in\Omega_0$ and $\varrho\leqslant 1$ such that $B_\varrho(x_0)\subset\Omega$. If $|(Du)^0_{z_0,\varrho}|\leqslant M$ then there exists a constant $c=c(n,N,L,p,M,K(1),\kappa_{M+1},\|g_t\|_{L^{2,2-2\beta}(Q_1^+)})$ such that*

$$\fint_{Q_\varrho^0(z_0)} |u|^p\,dz \leqslant c\varrho^p\big(\big(1+\Phi_p^{\frac{p-2}{p}}\big)\Phi_p+\varrho^{p\beta}\big),$$

holds with $\Phi_p\equiv\fint_{Q_\varrho^0(z_0)}|Du-(Du)^0_{z_0,\varrho}|^p\,dz$.

Proof. For notational convenience we omit the reference point and write B_ϱ and Q_ϱ rather than $B_\varrho(x_0)$ and $Q_\varrho(z_0)$. As usual when proving a Poincaré type inequality for solutions of parabolic systems we will avoid the use of time derivatives of u by proving suitable estimates for differences in time of the weighted means introduced below. Let $\eta\in C_0^\infty(B_\varrho)$ be a non-negative weight-function satisfying $0\leqslant\eta\leqslant c_\eta$, $|D\eta|\leqslant c_\eta/\varrho$ and $\fint_{B_\varrho}\eta\,dx=1$. We define the weighted mean of $u(\cdot,t)$ on B_ϱ for a.e. $t\in(0,T)$ by $(u)_\eta(t)=\fint_{B_\varrho}u(\cdot,t)\eta\,dx$ and prove in

Step 1. For $k=0,1$ and a.e. $t,\tau\in(0,\varrho^2)$ there holds

$$\big|\big(D^ku\big)_\eta(t)-\big(D^ku\big)_\eta(\tau)\big|^p\leqslant c\varrho^{(1-k)p}\bigg(\big(1+\Phi_p^{\frac{p-2}{p}}\big)\Phi_p+\fint_{Q_\varrho^0}|u|^{p\beta}\,dz+\varrho^{p\beta}\bigg), \tag{4.59}$$

with $c=c(N,L,M,K(1),\kappa_{M+1},\|g_t\|_{L^{2,2-2\beta}(Q_1^+)})$.

For any $i\in\{1,\dots,N\}$ we take $\varphi:\mathbb{R}^{n+1}\to\mathbb{R}^N$ with $\varphi_i=\eta$ and $\varphi_j=0$ for $j\neq i$ as test-function in the Steklov formulation (2.10) of the system (2.6) and obtain for the weighted means of $[u_i]_h$ (note that $[u]_h=([u_1]_h,\dots,[u_N]_h)$) for a.e. $t,\tau\in(0,\varrho^2)$,

$$\big([u_i]_h\big)_\eta(t)-\big([u_i]_h\big)_\eta(\tau)=\int_\tau^t\frac{\partial([u_i]_h)_\eta}{\partial s}\,ds=\int_\tau^t\fint_{B_\varrho}\big\langle\big[a_i(\cdot,u,Du)\big]_h,D\eta\big\rangle+[g_t]_h\eta\,dx\,ds.$$

Passing to the limit $h\downarrow 0$, enlarging the domain of integration if necessary and noting that

$$\int_\tau^t\fint_{B_\varrho}\big\langle a_i\big((0,t),0,(Du)^0_\varrho\big),D\eta\big\rangle\,dx\,ds=0,$$

we find

$$\begin{aligned}\big|(u_i)_\eta(t)-(u_i)_\eta(\tau)\big| &\leqslant \varrho^2\|D\eta\|_\infty \fint_{Q^0_\varrho}\big|a\big((x,t),u,Du\big)-a\big((0,t),0,Du\big)\big|\,d(x,t)\\ &\quad+\varrho^2\|D\eta\|_\infty\fint_{Q^0_\varrho}\big|a\big((0,t),0,Du\big)-a\big((0,t),0,(Du)^0_\varrho\big)\big|\,d(x,t)+\varrho^2\|\eta\|_\infty\fint_{Q^0_\varrho}|g_t|\eta\,dz\\ &=:\varrho^2\|D\eta\|_\infty(I_1+I_2)+\varrho^2\|\eta\|_\infty I_3,\end{aligned}$$

with the obvious meaning of I_1–I_3. We now in turn infer bounds for the terms I_1, I_2 and I_3. To estimate I_1 we use (1.6) and the assumption $|(Du)^0_\varrho|\leqslant M$. Then, we exploit the properties of θ, namely, for the term involving $1+M^{p-1}$ we use (2.2) with $u_0=0$ and for the term involving $|Du-(Du)^0_\varrho|$ we use that $\theta\leqslant 1$. Proceeding this way we infer

$$\begin{aligned}|I_1| &\leqslant 2^{p-2}L\fint_{Q^0_\varrho}\theta\big(|u|,\varrho+|u|\big)\big(\big(1+M^{p-1}\big)+\big|Du-(Du)^0_\varrho\big|^{p-1}\big)\,dz\\ &\leqslant 2^{p-2}L\fint_{Q^0_\varrho}\big(1+M^{p-1}\big)K(1)\big(\varrho^\beta+|u|^\beta\big)+\big|Du-(Du)^0_\varrho\big|^{p-1}\,dz.\end{aligned}$$

To estimate I_2 we decompose $Q^0_\varrho=S_1\cup S_2$, where

$$S_1=\big\{z\in Q^0_\varrho\colon\ \big|Du-(Du)^0_\varrho\big|\leqslant 1\big\},\qquad S_2=\big\{z\in Q^0_\varrho\colon\ \big|Du-(Du)^0_\varrho\big|>1\big\}$$

and rewrite I_2 as follows

$$I_2=\frac{1}{|Q^0_\varrho|}\int_{S_1}(\ldots)\,dz+\frac{1}{|Q^0_\varrho|}\int_{S_2}(\ldots)\,dz=\frac{1}{|Q^0_\varrho|}(I_{2,1}+I_{2,2}),$$

with the obvious labeling of $I_{2,1}$ and $I_{2,2}$. For $I_{2,1}$ we use (1.5) and note that $|(Du)^0_\varrho+s(Du-(Du)^0_\varrho)|\leqslant M+1$ on S_1 to obtain

$$\begin{aligned}I_{2,1}&=\int_{S_1}\left|\int_0^1\partial_w a\big((0,t),0,(Du)^0_\varrho+s\big(Du-(Du)^0_\varrho\big)\big)\big(Du-(Du)^0_\varrho\big)\,ds\right|dz\\ &\leqslant L\kappa_{M+1}\int_{S_1}\big|Du-(Du)^0_\varrho\big|\,dz.\end{aligned}$$

For $I_{2,2}$ we use the growth condition (1.3) instead of (1.5), the assumption $|(Du)^0_\varrho|\leqslant M$ and the fact that $|Du-(Du)^0_\varrho|>1$ on S_2 to obtain

$$I_{2,2}\leqslant L\int_{S_2}\big(2+M^{p-1}+|Du|^{p-1}\big)\,dz\leqslant 3L\big(1+M^{p-1}\big)\int_{S_2}\big|Du-(Du)^0_\varrho\big|^{p-1}\,dz.$$

Finally, for I_3 we obtain

$$|I_3|\leqslant c(n)\varrho^{\beta-1}\left(\varrho^{2-2\beta-(n+2)}\int_{Q^0_\varrho}|g_t|^2\,dz\right)^{\frac12}\leqslant c(n)\varrho^{\beta-1}\|g_t\|_{L^{2,2-2\beta}(Q^0_1)}.$$

Combining the previous estimates for I_1, I_2 and I_3 and summing over $i=1,\ldots,N$ and applying Hölder's inequality we arrive at

$$\big|(u)_\eta(t)-(u)_\eta(\tau)\big| \leqslant c\varrho^2\|D\eta\|_\infty\bigg(\big(\Phi_p^{\frac{p-2}{p}}+1\big)\Phi_p^{\frac{1}{p}}+\fint_{Q_\varrho^0}|u|^\beta\,dz+\varrho^\beta\bigg)+c\varrho^{\beta+1}\|\eta\|_\infty\|g_t\|_{L^{2,2-2\beta}(Q_1^0)},$$

where $c=c(N,L,M,K(1),\kappa_{M+1})$. Taking this to the power p, applying Hölder's inequality and noting that $\|\eta\|_\infty\leqslant c_\eta$ and $\|D\eta\|_\infty\leqslant c_\eta/\varrho$ we infer (4.59) for the case $k=0$. We get an analogous estimate for the weighted means of Du by taking $D^\alpha\eta$ instead of η as test-function in (2.10), where $\alpha=1,\dots,n$. Indeed using integration by parts we find that $(D^\alpha u)_\eta(t)=-(u)_{D^\alpha\eta}(t)$. Replacing η with $D^\alpha\eta$ in the previous estimate and summing over $\alpha=1,\dots,n$ we obtain

$$\big|(Du)_\eta(t)-(Du)_\eta(\tau)\big| \leqslant c\varrho^2\|D^2\eta\|_\infty\bigg(\big(\Phi_p^{\frac{p-2}{p}}+1\big)\Phi_p^{\frac{1}{p}}+\fint_{Q_\varrho^0}|u|^\beta\,dz+\varrho^\beta\bigg)+c\varrho^{\beta+1}\|D\eta\|_\infty\|g_t\|_{L^{2,2-2\beta}(Q_1^0)},$$

where $c=c(n,m,N,L,M,K(1),\kappa_{M+1})$. Noting that $\|D\eta\|_\infty\leqslant c/\varrho$ and $\|D^2\eta\|_\infty\leqslant c/\varrho^2$, we infer (4.59) also in the case $k=1$.

Step 2. Proof of the Poincaré type inequality. We fix $h\in(0,\varrho^2)$. Exploiting the weighted means of u we decompose

$$\fint_{Q_\varrho^0}|u|^p\,dz \leqslant 3^{p-1}\Bigg[\fint_{Q_\varrho^0}\big|u-(u)_\eta\big|^p\,dz+\bigg|\fint_0^{\varrho^2}(u)_\eta(t)\,dt-\fint_0^h(u)_\eta(\tau)\,d\tau\bigg|^p+\bigg|\fint_0^h(u)_\eta\,d\tau\bigg|^p\Bigg]$$
$$=: 3^{p-1}\big(I+II^{(h)}+III^{(h)}\big). \tag{4.60}$$

For the *estimate of $II^{(h)}$* we use (4.59) with $k=0$ and Hölder's inequality to infer

$$II^{(h)}\leqslant \sup_{t,\tau\in(0,\varrho^2)}\big|(u)_\eta(t)-(u)_\eta(\tau)\big|^p\leqslant c\varrho^p\bigg(\big(1+\Phi_p^{p-2}\big)\Phi_p+\fint_{Q_\varrho^0}|u|^{p\beta}\,dz+\varrho^{p\beta}\bigg).$$

Note that the previous bound is independent of h. Passing to the limit $h\downarrow 0$ in $III^{(h)}$ and exploiting our initial condition on u we find that $\lim_{h\downarrow 0}III^{(h)}=0$. Therefore it remains to *estimate I*. Here, we apply Poincaré's inequality slicewise to $(u-(u)_\eta)(\cdot,t)$ and obtain for $\lambda\in(0,\varrho^2)$ that

$$I\leqslant c\varrho^p\fint_{Q_\varrho^0}|Du|^p\,dx$$
$$\leqslant c\varrho^p\Bigg[\fint_{Q_\varrho^0}\big|Du-(Du)_\varrho^0\big|^p+\big|(Du)_\varrho^0-(Du)_\eta\big|^p\,dz+\bigg|\fint_0^{\varrho^2}(Du)_\eta(t)\,dt-\fint_0^\lambda(Du)_\eta(\tau)\,d\tau\bigg|^p+\bigg|\fint_0^\lambda(Du)_\eta\,d\tau\bigg|^p\Bigg]$$
$$=: c(n,p)\varrho^p\big(I_1+I_2+I_3^{(\lambda)}+I_4^{(\lambda)}\big).$$

We now in turn infer estimates for the terms I_2, $I_3^{(\lambda)}$ and $I_4^{(\lambda)}$. We start with I_2. Recalling that $\fint_{B_\varrho}\eta\,dx=1$ and $0\leqslant\eta\leqslant c_\eta$ we rewrite and estimate

$$I_2=\fint_0^{\varrho^2}\bigg|\fint_{B_\varrho}\big((Du)_\varrho^0-Du(y,\tau)\big)\eta(y)\,dy\bigg|^p\,d\tau\leqslant c_\eta\fint_{Q_\varrho^0}\big|Du-(Du)_\varrho^0\big|^p\,dz.$$

For the estimate of $I_3^{(\lambda)}$ we use the bound (4.59) with $k=1$ and Hölder's inequality to infer

$$I_3^{(\lambda)}\leqslant\sup_{t,\tau\in(0,\varrho^2)}\big|(Du)_\eta(t)-(Du)_\eta(\tau)\big|^p\leqslant c\bigg(\big(1+\Phi_p^{p-2}\big)\Phi_p+\fint_{Q_\varrho^0}|u|^{p\beta}\,dz+\varrho^{p\beta}\bigg).$$

Note that the preceding bound is independent of λ. In $I_4^{(\lambda)}$ we want to exploit the initial condition on u. Therefore we first integrate by parts and then pass to the limit $\lambda \downarrow 0$

$$I_4^{(\lambda)} = \left| \fint_0^\lambda \fint_{B_\varrho} u(x,\tau) \otimes D\eta(x)\,dx\,d\tau \right|^p \leqslant c\varrho^{-1} \fint_0^\lambda \fint_{B_\varrho} |u|\,dx\,d\tau \leqslant c\varrho^{-1} \left(\frac{1}{\lambda} \int_0^\lambda \fint_{B_\varrho} |u|^2\,dx\,d\tau \right)^{\frac{1}{2}} \to 0.$$

Therefore, collecting terms we conclude with the following bound for I:

$$I \leqslant c\varrho^p \left(\left(1 + \Phi_p^{p-2}\right)\Phi_p + \fint_{Q_\varrho^0} |u|^{p\beta} dz + \varrho^{p\beta} \right).$$

Combining the previous estimates for I, $II^{(h)}$ and $III^{(h)}$ we obtain from (4.60):

$$\fint_{Q_\varrho^0} |u|^p\,dz \leqslant c\varrho^p \left(\left(1+\Phi_p^{p-2}\right)\Phi_p + \fint_{Q_\varrho^0} |u|^{p\beta}\,dz + \varrho^{p\beta} \right),$$

where $c = c(n, N, L, M, K(1), \kappa_{M+1}, \|g_t\|_{L^{2,2-2\beta}(Q_1^+)})$. By Young's inequality and since $\varrho \leqslant 1$ we have $\varrho^p |u|^{p\beta} \leqslant \frac{1}{2c}|u|^p + c\varrho^{\frac{p}{1-\beta}} \leqslant \frac{1}{2c}|u|^p + c\varrho^{p\beta}$. Therefore, we can absorb the term involving u on the left-hand side in the standard way and end up the asserted Poincaré type inequality. □

This finishes the proof of the characterization of regular initial boundary points. Finally, we consider the remaining configuration, namely points lying on the edge $\partial\Omega \times \{0\}$.

4.3. Regular points on the edge

Here we will prove the characterization of regular edge-points in the model situation (2.7) on Q_1^* which was explained in Section 2.1. The statement of the main Theorem 1.2 concerning edge-points is therefore equivalent with Proposition 4.18 which deals with edge-points $z_0 = (x_0', 0, 0) \in \Gamma_1 \cap D_1$, with $x_0' \in \mathbb{R}^{n-1}$ of Q_1^*. Note that in the edge-situation we can decide whether we proceed similar to the lateral or initial boundary. We will choose the second one, since it seems slightly more convenient.

4.3.1. A-caloric approximation

As in the preceding sections we shall need a version of the lemma of A-caloric approximation for the edge-situation. Since the arguments are clear by now, we only state the result. Indeed, the only difference in the present situation is that we have to identify the zero trace of the limit function v found in (4.51) on both parts of the boundary, i.e. on Γ_1^0 and D_1^+. The first one follows from the trace theorem (as in the proof of Lemma 4.1) whereas the second one follows by the argument after (4.52) in the proof of Lemma 4.8.

Lemma 4.14. *Given $\varepsilon > 0$, $0 < \nu \leqslant L$ and $p \geqslant 2$ there exists a positive function $\delta = \delta(n, p, \nu, L, \varepsilon) \in (0, 1]$ with the following property: whenever A is a bilinear form on $\mathbb{R}^{Nn}$ which is strongly elliptic with ellipticity constant $\nu > 0$ and upper bound L, i.e.*

$$\nu|w|^2 \leqslant \langle Aw, w\rangle \quad \textit{and} \quad \langle Aw, \widetilde{w}\rangle \leqslant L|w||\widetilde{w}|$$

holds whenever $w, \widetilde{w} \in \mathbb{R}^{Nn}$ and $u \in L^p(\Lambda^0_{\varrho^2}(t_0); W^{1,p}(B_\varrho^+(x_0), \mathbb{R}^N))$ with $u \equiv 0$ on $\Gamma_\varrho^0(z_0) \cup D_\varrho^+(z_0)$ with $z_0 \in \Gamma_1 \cap D_1$ and

$$\fint_{Q_\varrho^*(z_0)} |Du|^2 + \gamma^{p-2}|Du|^p\,dz \leqslant 1,$$

where $0 < \gamma \leqslant 1$, is approximately A-caloric in the sense that

$$\bigg| \fint_{Q_\varrho^*(z_0)} u \cdot \varphi_t - \langle ADu, D\varphi \rangle \, dz \bigg| \leqslant \delta \sup_{Q_\varrho^*(z_0)} |D\varphi| \quad \text{for every } \varphi \in C_0^\infty\big(Q_\varrho^*(z_0); \mathbb{R}^N\big),$$

then there exists an A-caloric map $h \in L^p(\Lambda^0_{(\varrho/2)^2}(t_0); W^{1,p}(B^+_{\varrho/2}(x_0); \mathbb{R}^N))$, i.e.

$$\int_{Q_{\varrho/2}^*(z_0)} h \cdot \varphi_t - \langle ADh, D\varphi \rangle \, dz = 0 \qquad \text{for every } \varphi \in C_0^\infty\big(Q_{\varrho/2}^*(z_0); \mathbb{R}^N\big),$$

with $h \equiv 0$ on $\Gamma^0_{\varrho/2}(z_0) \cup D^+_{\varrho/2}(z_0)$ satisfying

$$\fint_{Q_{\varrho/2}^*(z_0)} |Dh|^2 + \gamma^{p-2} |Dh|^p \, dz \leqslant 2 \cdot 2^{n+2}$$

and

$$\fint_{Q_{\varrho/2}^*(z_0)} \left| \frac{u-h}{\varrho/2} \right|^2 + \gamma^{p-2} \left| \frac{u-h}{\varrho/2} \right|^p dz \leqslant \varepsilon.$$

4.3.2. Caccioppoli inequality

Since the proof of the Caccioppoli inequality in the edge-situation is performed by a combination of the arguments for the lateral and the initial boundary situation we shall omit it and only state the result.

Lemma 4.15. *Suppose that $u \in L^p(\Lambda^0_1; W^{1,p}(B^+_1; \mathbb{R}^N))$ is a weak solution of the non-linear parabolic system* (2.7) *with $u = 0$ on $\Gamma^0_1 \cup D^+_1$, where the structure conditions* (1.4)–(1.6) *are in force. Then, for any $z_0 \in \Gamma_1 \cap D_1$ and $\varrho \in (0, 1)$ such that $Q_\varrho(z_0) \subset Q_1$ there holds*

$$\fint_{Q_{\varrho/2}^*(z_0)} |Du|^2 + |Du|^p \, dz \leqslant c_{Cac} \bigg(\fint_{Q_\varrho^*(z_0)} \left| \frac{u}{\varrho} \right|^2 + \left| \frac{u}{\varrho} \right|^p dz + \varrho^{2\beta} \bigg),$$

where $c_{Cac} = (1 + \|g_t\|^2_{L^{2,2-2\beta}(Q_1^*)}) c(n, p, L/\nu, M, H(M), \kappa_{M+1})$.

4.3.3. Linearization

Here we state a version of the linearization lemma which is applicable in the edge. Since the proof is completely similar to the one of Lemma 4.3 for the initial boundary situation we shall omit it and only state the result. Indeed, since the test-function has compact support in the domain of integration we do not reach the boundary and therefore the proof can be completely adopted.

For $s \geqslant 1$, $z_0 \in \Gamma_1 \cap D_1$ and a parabolic cylinder $Q_\varrho(z_0) \subset Q_1$ we define the edge-point excess functionals by

$$\phi_s^*(z_0, \varrho) := \fint_{Q_\varrho^*(z_0)} |Du|^s \, dz, \qquad \psi_s^*(z_0, \varrho) := \fint_{Q_\varrho^*(z_0)} \left| \frac{u}{\varrho} \right|^s dz,$$

and we shall often abbreviate

$$\phi_s^* = \phi_s^*(z_0, \varrho) \quad \text{and} \quad \psi_s^* = \psi_s^*(z_0, \varrho).$$

Lemma 4.16. *Suppose that $u \in L^p(\Lambda^0_1; W^{1,p}(B^+_1; \mathbb{R}^N))$ is a weak solution of* (2.7) *satisfying $u = 0$ on $\Gamma^0_1 \cup D^+_1$, where the structure conditions* (1.4)–(1.6) *are in force. Then we have*

$$\bigg| \fint_{Q_\varrho^*(z_0)} u \cdot \varphi_t - \big\langle \partial_w a(z_0, 0, 0) Du, D\varphi \big\rangle \, dz \bigg|$$
$$\leqslant c_{Eu} \Big[\omega_1\big(\phi_p^* + \varrho^p\big) \sqrt{\phi_2^*} + \phi_p^* + \psi_2^* + \varrho^\beta \big(\phi_p^*\big)^{1-\frac{1}{p}} \big(1 + \big(\psi_p^*\big)^{\frac{\beta}{p}}\big) + \varrho^\beta \Big] \sup_{Q_\varrho^*(z_0)} |D\varphi|$$

for any $\varphi \in C_0^\infty(Q_\varrho^*(z_0); \mathbb{R}^N)$, $z_0 \in \Gamma_1 \cap D_1$ *and* $Q_\varrho(z_0) \subset Q_1$. *The constant* c_{Eu} *is of the form*

$$c_{Eu} = L\big(1 + \|g_t\|_{L^{2,2-2\beta}(Q_1^*)}\big)c\big(n, p, M, K(1), \kappa_{M+1}\big).$$

4.3.4. A decay estimate at the edge

Having all the prerequisites at hand, we can now use Lemmas 4.14–4.16 to prove an excess-decay estimate valid for edge-points $z_0 \in \Gamma_1 \cap D_1$. Since the first two steps of the proof, i.e. the application of the A-caloric approximation lemma and the iteration are completely similar to the proof of Lemma 4.11 for the initial boundary situation we shall omit the proof and only state the result, i.e. the analogue of Lemma 4.11. We first define the following edge-point excess functional

$$E_{\mathrm{ed}} \equiv E_{\mathrm{ed}}(z_0, \varrho) := \fint_{Q_\varrho^*(z_0)} \left|\frac{u}{\varrho}\right|^2 + \left|\frac{u}{\varrho}\right|^p dz$$

for $z_0 \in \Gamma_1 \cap D_1$ and $\widetilde{E}_{\mathrm{ed}} = \widetilde{E}_{\mathrm{ed}}(z_0, \varrho) = E_{\mathrm{ed}}(z_0, \varrho) + \varrho^{2\beta}$.

Lemma 4.17. *Given* $\alpha \in (\beta, 1)$, *there exist constants* $\vartheta \in (0, \frac{1}{4}]$, $\mathcal{E}_{\mathrm{ed}} \in (0, 1]$, $\varrho_{\mathrm{ed}} > 0$ *and* c_4 *depending on* $n, N, p, \nu, L, K(1), \kappa_1, \alpha, \beta, \|g_t\|_{L^{2,2-2\beta}(Q_1^*)}$ *such that the following holds. Suppose that* $u \in L^p(\Lambda_1^0; W^{1,p}(B_1^+; \mathbb{R}^N))$ *is a weak solution of* (2.7) *satisfying* $u(\cdot, 0) = 0$ *on* $\Gamma_1^0 \cup D_1^+$, *where the structure conditions* (1.4)–(1.6) *are in force and suppose that*

(i) $\varrho \leqslant \varrho_{\mathrm{ed}}$,

(ii) $\widetilde{E}_{\mathrm{ed}}(\varrho) \leqslant \mathcal{E}_{\mathrm{ed}}$

are fulfilled on some cylinder $Q_\varrho^*(z_0)$ *with* $z_0 \in \Gamma_1 \cap D_1$ *and* $Q_\varrho(z_0) \subset Q_1$. *Then for any* $0 < r \leqslant \varrho/2$ *we have*

$$\fint_{Q_r^*(z_0)} \left|\frac{u}{r}\right|^2 + \left|\frac{u}{r}\right|^p dz + \fint_{Q_r^*(z_0)} |Du|^2 + |Du|^p\, dz \leqslant c_{\mathrm{ed}}\left[\left(\frac{r}{\varrho}\right)^{2\alpha} E_{\mathrm{ed}}(\varrho) + r^{2\beta}\right],$$

where the constant c_{ed} *depends on* $n, N, p, \nu, L, K(1), \kappa_1, \alpha, \beta, \|g_t\|_{L^{2,2-2\beta}(Q_1^*)}$. *In particular this estimate implies*

$$\mathfrak{X}_{z_0} := \lim_{j\to\infty} (Du)^*_{z_0, \vartheta^j\varrho} = 0.$$

4.3.5. Description of regular edge-points

Now we come to the most interesting part concerning the proof of the characterization of regular edge-points. Indeed, when considering the neighborhood of an edge-point we have to take into account all four possible configurations, namely the edge and interior situation and the lateral and initial boundary situation. This is performed in the following

Proposition 4.18. *Suppose that* $u \in L^p(\Lambda_1^0; W^{1,p}(B_1^+; \mathbb{R}^N))$ *with* $u(\cdot, 0) = 0$ *on* $\Gamma_1^0 \cup D_1^+$ *is a weak solution of* (2.7) *where the structure conditions* (1.4)–(1.6) *are in force. Then, for any* $z_0 \in (\Gamma_1 \cap D_1) \setminus \Sigma_{\mathrm{ed}}$ *there exists a neighborhood* U_{z_0} *such that*

$$Du \in C^{\beta, \frac{\beta}{2}}\big(U_{z_0} \cap \overline{Q_1^*}; \mathbb{R}^{Nn}\big),$$

where

$$\Sigma_{\mathrm{ed}} := \left\{z_0 \in \Gamma_1 \cap D_1\colon \liminf_{\varrho\downarrow 0} \varrho^{-p} \fint_{Q_\varrho^*(z_0)} |u|^p\, dz > 0\right\}.$$

Proof. First of all we recall the definition of the constants $\varrho_{\mathrm{ed}}, \mathcal{E}_{\mathrm{ed}}, c_{\mathrm{ed}}, \varrho_{\mathrm{ini}}, \mathcal{E}_{\mathrm{ini}}, c_{\mathrm{ini}}$, $\varrho_{\mathrm{lat}} = \varrho_{\mathrm{lat}}(1)$, $\mathcal{E}_{\mathrm{lat}} = \mathcal{E}_{\mathrm{lat}}(1)$, $c_{\mathrm{lat}} = c_{\mathrm{lat}}(1)$, $\varrho_{\mathrm{int}} = \varrho_{\mathrm{int}}(1)$, $\mathcal{E}_{\mathrm{int}} = \mathcal{E}_{\mathrm{int}}(1)$ and $c_{\mathrm{int}} = c_{\mathrm{int}}(1)$ from Lemmas 4.17, 4.11, 4.5 and 4.6, respectively for the choice $M = 1$. For $z_0 \in (\Gamma_1 \cap D_1) \setminus \Sigma_{\mathrm{ed}}$ we can find $0 < \varrho \leqslant \min\{\varrho_{\mathrm{ed}}, \varrho_{\mathrm{ini}}, \varrho_{\mathrm{lat}}, \varrho_{\mathrm{int}}\}$ with $Q_{2\varrho}^*(z_0) \subset Q_1^*$ such that

$$\widetilde{E}_{\mathrm{ed}}(z_0,\varrho) < \frac{1}{2}\big(c_5 \max\{c_{\mathrm{ed}}, c_{\mathrm{ini}}, c_{\mathrm{lat}}, c_{\mathrm{int}}\}\big)^{-2} \min\{\mathcal{E}_{\mathrm{ed}}, \mathcal{E}_{\mathrm{ini}}, \mathcal{E}_{\mathrm{lat}}, \mathcal{E}_{\mathrm{int}}\} \tag{4.61}$$

where we have defined $c_5 = 2^{2n}3^{2p}(n+3)^p$. Since $\Gamma_1 \cap D_1 \ni z \mapsto \widetilde{E}_{\mathrm{ed}}(z,\varrho)$ is continuous with respect to the center z, there exists a radius $0 < R \leqslant \varrho/12$ such that

$$\widetilde{E}_{\mathrm{ed}}(z,\varrho) < \big(c_5 \max\{c_{\mathrm{ed}}, c_{\mathrm{ini}}, c_{\mathrm{lat}}, c_{\mathrm{int}}\}\big)^{-2} \min\{\mathcal{E}_{\mathrm{ed}}, \mathcal{E}_{\mathrm{ini}}, \mathcal{E}_{\mathrm{lat}}, \mathcal{E}_{\mathrm{int}}\} \tag{4.62}$$

for all $z \in \Gamma_R(z_0) \cap D_R(z_0)$. Moreover, due to the choice $R \leqslant \varrho/12$ and the inclusion $Q^*_{2\varrho}(z_0) \subset Q^*_1$ we have $Q^*_\varrho(z) \subset Q^*_{2\varrho}(z_0) \subset Q^*_1$.

Now, given $\alpha \in (\beta, 1)$ we choose ϑ in dependence of $n, N, p, \nu, L, K(1), \kappa_1, \alpha, \beta$ and $\|g_t\|_{L^{2,2-2\beta}(Q^*_1)}$ to be the constant from Lemma 4.11. Without loss of generality we can assume that the constant ϑ appearing in Proposition 4.6 (with the choice $M = 1$) is equal to the ones from Lemmas 4.5, 4.11 and 4.17. In the following we will show that for all $\mathfrak{z} \in Q^*_R(z_0) \cup \Gamma^0_R(z_0) \cup D^+_R(z_0) \cup (\Gamma_R(z_0) \cap D_R(z_0))$ the limit

$$\mathfrak{X}_{\mathfrak{z}} := \lim_{j\to\infty} (Du)^*_{\mathfrak{z},\vartheta^j\varrho} \tag{4.63}$$

exists and that

$$\fint_{Q^*_r(\mathfrak{z})} |Du - \mathfrak{X}_{\mathfrak{z}}|^2\,dz \leqslant c\left(\frac{r}{\varrho}\right)^{2\beta}, \tag{4.64}$$

for all $0 < r \leqslant \varrho/6$ and with a constant c depending on $n, N, p, \nu, L, K(1), \kappa_1, \alpha, \beta$ and $\|g_t\|_{L^{2,2-2\beta}(Q^*_1)}$. For this we will distinguish between the cases $\mathfrak{z} \in \Gamma_R(z_0) \cap D_R(z_0)$, $\mathfrak{z} \in D^+_R(z_0)$, $\mathfrak{z} \in \Gamma^0_R(z_0)$ and $\mathfrak{z} \in Q^*_R(z_0)$.

In the first case, i.e. *the case* $\mathfrak{z} \in \Gamma_R(z_0) \cap D_R(z_0)$, we see that by (4.62) the hypotheses of Lemma 4.17 are fulfilled so that the application of the lemma yields that

$$\mathfrak{X}_{\mathfrak{z}} = \lim_{j\to\infty} (Du)^*_{\mathfrak{z},\vartheta^j\varrho} = 0.$$

Moreover, for $0 < r \leqslant \varrho/2$ we have

$$\begin{aligned}
\fint_{Q^*_r(\mathfrak{z})} \left|\frac{u}{r}\right|^2 + \left|\frac{u}{r}\right|^p + |Du|^2 + |Du|^p\,dz &\leqslant c_{\mathrm{ed}}\left[\left(\frac{r}{\varrho}\right)^{2\alpha} E_{\mathrm{ed}}(\mathfrak{z},\varrho) + r^{2\beta}\right] \\
&\leqslant c_{\mathrm{ed}}\left(\frac{r}{\varrho}\right)^{2\beta} \widetilde{E}_{\mathrm{ed}}(\mathfrak{z},\varrho) \\
&\leqslant c\left(\frac{r}{\varrho}\right)^{2\beta},
\end{aligned} \tag{4.65}$$

where c depends on $n, N, p, \nu, L, K(1), \kappa_1, \alpha, \beta, \|g_t\|_{L^{2,2-2\beta}(Q^*_1)}$. Here we have used in the last line the bound (4.62) for $\widetilde{E}_{\mathrm{ed}}(\mathfrak{z},\varrho)$. Hence, (4.63) and (4.64) are valid with $\mathfrak{X}_{\mathfrak{z}} = 0$.

In the second case, i.e. *the case* $\mathfrak{z} \in D^+_R(z_0)$, we want to apply Lemma 4.11 and therefore first have to ensure that the hypothesis are satisfied. By $\mathfrak{z}' = (\mathfrak{x}_1, \dots, \mathfrak{x}_{n-1}, 0, 0)$ we denote the projection of $\mathfrak{z} = (\mathfrak{x}_1, \dots, \mathfrak{x}_{n-1}, \mathfrak{x}_n, 0)$ onto $\Gamma_1 \cap D_1$. At this stage we recall that $Q_{\mathfrak{x}_n}(\mathfrak{z}) \subset Q^*_{2\mathfrak{x}_n}(\mathfrak{z}')$ which allows us to enlarge the domain of integration from $Q_{\mathfrak{x}_n}(\mathfrak{z})$ to $Q^*_{2\mathfrak{x}_n}(\mathfrak{z}')$. Now, since $\mathfrak{z}' \in \Gamma_R(z_0) \cap D_R(z_0)$ we can use the results from the first case with center $\mathfrak{z}'$ obtaining that the limit $\mathfrak{X}_{\mathfrak{z}'} := \lim_{j\to\infty} (Du)^*_{\mathfrak{z}',\vartheta^j\varrho} = 0$ exists and, moreover, that (4.65) holds with $\mathfrak{z}'$ instead of $\mathfrak{z}$. Therefore, using (4.65) with $(\mathfrak{z}', 2\mathfrak{x}_n, \varrho)$ instead of $(\mathfrak{z}, r, \varrho)$, the fact that $2\mathfrak{x}_n \leqslant 2R \leqslant \varrho/2$ and (4.62) we infer

$$\begin{aligned}
\widetilde{E}_{\mathrm{ini}}(\mathfrak{z},\mathfrak{x}_n) &\leqslant 2^{n+2+p}\left(\fint_{Q^*_{2\mathfrak{x}_n}(\mathfrak{z}')} \left|\frac{u}{2\mathfrak{x}_n}\right|^2 + \left|\frac{u}{2\mathfrak{x}_n}\right|^p dz + (2\mathfrak{x}_n)^{2\beta}\right) \\
&\leqslant 2^{n+2+p} c_{\mathrm{ed}}\left(\frac{2\mathfrak{x}_n}{\varrho/2}\right)^{2\beta} \widetilde{E}_{\mathrm{ed}}(\mathfrak{z}',\varrho) \\
&\leqslant \left(\frac{2\mathfrak{x}_n}{\varrho/2}\right)^{2\beta} \mathcal{E}_{\mathrm{ini}}.
\end{aligned} \tag{4.66}$$

Note that the last inequality holds since $2^{n+2+p} \leqslant c_5$ which was defined in (4.61). Since $2\mathfrak{x}_n \leqslant 2R \leqslant \varrho/2$ this implies in particular that $\widetilde{E}_{\mathrm{ini}}(\mathfrak{z},\mathfrak{x}_n) \leqslant \mathcal{E}_{\mathrm{ini}}$. Therefore, the hypotheses of Lemma 4.11 are satisfied and the application with $(\mathfrak{z}, r, \mathfrak{x}_n)$ instead of (z_0, r, ϱ) allows us to conclude on the one hand that the limit in (4.63) exists and, on the other hand, that for any $0 < r \leqslant \mathfrak{x}_n/2$ there holds

$$
\begin{aligned}
\fint_{Q_r^0(\mathfrak{z})} |Du - \mathfrak{X}_{\mathfrak{z}}|^2\,dz &\leqslant c_{\mathrm{ini}}\biggl[\biggl(\frac{r}{\mathfrak{x}_n/2}\biggr)^{2\alpha} E_{\mathrm{ini}}(\mathfrak{z},\mathfrak{x}_n) + r^{2\beta}\biggr] \\
&\leqslant c_{\mathrm{ini}}\biggl[\biggl(\frac{r}{\mathfrak{x}_n/2}\biggr)^{2\alpha}\biggl(\frac{2\mathfrak{x}_n}{\varrho}\biggr)^{2\beta} \mathcal{E}_{\mathrm{ini}} + r^{2\beta}\biggr] \\
&\leqslant c\biggl(\frac{r}{\varrho}\biggr)^{2\beta},
\end{aligned}
$$

where c depends on $n, N, p, \nu, L, K(1), \kappa_1, \alpha, \beta$ and $\|g_t\|_{L^{2,2-2\beta}(Q_1^*)}$. Here we have also used (4.66) and $\mathcal{E}_{\mathrm{ini}} \leqslant 1$ in the last line. In the remaining case $\mathfrak{x}_n/2 < r \leqslant \varrho/6$ we use the previous estimate as well as (4.64) for $\mathfrak{z}'$ (note that $\mathfrak{X}_{\mathfrak{z}'} = 0$) to infer

$$
\begin{aligned}
\fint_{Q_r^*(\mathfrak{z})} |Du - \mathfrak{X}_{\mathfrak{z}}|^2\,dz &\leqslant 2\fint_{Q_r^*(\mathfrak{z})} |Du|^2\,dz + 4\fint_{Q^0_{\mathfrak{x}_n/2}(\mathfrak{z})} |Du|^2 + |Du - \mathfrak{X}_{\mathfrak{z}}|^2\,dz \\
&\leqslant c\fint_{Q_{3r}^*(\mathfrak{z}')} |Du|^2\,dz + c\fint_{Q^*_{3\mathfrak{x}_n/2}(\mathfrak{z}')} |Du|^2\,dz + c\biggl(\frac{\mathfrak{x}_n/2}{\varrho}\biggr)^{2\beta} \\
&\leqslant c\biggl(\frac{r}{\varrho}\biggr)^{2\beta},
\end{aligned}
$$

where c depends only on $n, N, p, \nu, L, K(1), \kappa_1, \alpha, \beta$ and $\|g_t\|_{L^{2,2-2\beta}(Q_1^*)}$. This completes the proof of (4.63) and (4.64) in the case $\mathfrak{z} \in D_R^+(z_0)$.

In *the case* $\mathfrak{z} \in \Gamma_R^0(z_0)$ we want to apply Lemma 4.5. As in the last case we first will ensure that the hypotheses of the lemma are satisfied. By $\mathfrak{z}' = (\mathfrak{x}_1, \dots, \mathfrak{x}_{n-1}, 0, 0)$ we again denote the projection of $\mathfrak{z} = (\mathfrak{x}_1, \dots, \mathfrak{x}_{n-1}, 0, \mathfrak{t})$ onto $\Gamma_1 \cap D_1$. From (4.19) we recall that $\xi_{\mathfrak{z},\sqrt{\mathfrak{t}}} = \frac{n+2}{\mathfrak{t}} \fint_{Q^+_{\sqrt{\mathfrak{t}}}(\mathfrak{z})} u x_n\,dz$ denotes the vector minimizing $\xi \mapsto \int_{Q^+_{\sqrt{\mathfrak{t}}}(\mathfrak{z})} |u - \xi x_n|^2\,dz$. By the use of the minimizing property of $\xi_{\mathfrak{z},\sqrt{\mathfrak{t}}}$ we therefore find

$$
\fint_{Q^+_{\sqrt{\mathfrak{t}}}(\mathfrak{z})} \biggl|\frac{u - \xi_{\mathfrak{z},\sqrt{\mathfrak{t}}} x_n}{\sqrt{\mathfrak{t}}}\biggr|^2 dz \leqslant \fint_{Q^+_{\sqrt{\mathfrak{t}}}(\mathfrak{z})} \biggl|\frac{u}{\sqrt{\mathfrak{t}}}\biggr|^2 dz.
$$

Furthermore, we recall that $\xi^{(p)}_{\mathfrak{z},\sqrt{\mathfrak{t}}}$ denotes the vector minimizing $\xi \mapsto \int_{Q^+_{\sqrt{\mathfrak{t}}}(\mathfrak{z})} |u - \xi x_n|^p\,dz$. By the use of Lemma 2.2 and the minimizing property of $\xi^{(p)}_{\mathfrak{z},\mathfrak{x}_n}$ we therefore find

$$
\fint_{Q^+_{\sqrt{\mathfrak{t}}}(\mathfrak{z})} \biggl|\frac{u - \xi_{\mathfrak{z},\sqrt{\mathfrak{t}}} x_n}{\sqrt{\mathfrak{t}}}\biggr|^p dz \leqslant 2^{p-1}\biggl(\fint_{Q^+_{\sqrt{\mathfrak{t}}}(\mathfrak{z})} \biggl|\frac{u}{\sqrt{\mathfrak{t}}}\biggr|^p dz + \bigl|\xi^{(p)}_{\mathfrak{z},\mathfrak{x}_n} - \xi_{\mathfrak{z},\sqrt{\mathfrak{t}}}\bigr|^p\biggr) \leqslant 2^p(n+2)^p \fint_{Q_{\sqrt{\mathfrak{t}}}(\mathfrak{z})} \biggl|\frac{u}{\sqrt{\mathfrak{t}}}\biggr|^p dz.
$$

At this stage we recall that $Q^+_{\sqrt{\mathfrak{t}}}(\mathfrak{z}) \subset Q^*_{\sqrt{2\mathfrak{t}}}(\mathfrak{z}')$ which allows us to enlarge the domain of integration from $Q^+_{\sqrt{\mathfrak{t}}}(\mathfrak{z})$ to $Q^*_{\sqrt{2\mathfrak{t}}}(\mathfrak{z}')$. Since $\mathfrak{z}' \in \Gamma_R(z_0) \cap D_R(z_0)$ we can use the results from the first case with center $\mathfrak{z}'$ obtaining that the limit $\mathfrak{X}_{\mathfrak{z}'} := \lim_{j\to\infty} (Du)^*_{\mathfrak{z}',\vartheta^j\varrho} = 0$ exists and, moreover, that (4.65) holds with $\mathfrak{z}'$ instead of $\mathfrak{z}$. Combining the previous estimates and using (4.65) with $(\mathfrak{z}', \sqrt{2\mathfrak{t}}, \varrho)$ instead of $(\mathfrak{z}, r, \varrho)$, the fact that $\sqrt{2\mathfrak{t}} \leqslant 2R \leqslant \varrho/2$ and (4.62) we infer

$$
\begin{aligned}
\widetilde{E}_{\mathrm{lat}}(\mathfrak{z},\sqrt{\mathfrak{t}}) &\leqslant 2^p(n+2)^p \fint_{Q^+_{\sqrt{\mathfrak{t}}}(\mathfrak{z})} \left|\frac{u}{\sqrt{\mathfrak{t}}}\right|^2 + \left|\frac{u}{\sqrt{\mathfrak{t}}}\right|^p dz + \sqrt{\mathfrak{t}}^{2\beta} \\
&\leqslant 2^{n+2p}(n+2)^p\Bigg(\fint_{Q^*_{\sqrt{2\mathfrak{t}}}(\mathfrak{z}')} \left|\frac{u}{\sqrt{2\mathfrak{t}}}\right|^2 + \left|\frac{u}{\sqrt{2\mathfrak{t}}}\right|^p dz + \sqrt{2\mathfrak{t}}^{2\beta}\Bigg) \\
&\leqslant 2^{n+2p}(n+2)^p c_{\mathrm{ed}}\left(\frac{\sqrt{2\mathfrak{t}}}{\varrho}\right)^{2\beta} \widetilde{E}_{\mathrm{ed}}(\mathfrak{z}',\varrho) \\
&\leqslant \left(\frac{\sqrt{2\mathfrak{t}}}{\varrho}\right)^{2\beta} \mathcal{E}_{\mathrm{lat}},
\end{aligned}
\tag{4.67}
$$

where we have abbreviated $\widetilde{E}_{\mathrm{lat}}(\mathfrak{z},\sqrt{\mathfrak{t}}) = \widetilde{E}_{\mathrm{lat}}(\mathfrak{z},\sqrt{\mathfrak{t}},\xi_{\mathfrak{z},\sqrt{\mathfrak{t}}})$. Note that the last inequality holds since $2^{n+2p}(n+2)^p \leqslant c_5$ which was defined in (4.53). Since $\sqrt{2\mathfrak{t}} \leqslant 2R \leqslant \varrho/2$ this implies in particular that $\widetilde{E}_{\mathrm{lat}}(\mathfrak{z},\sqrt{\mathfrak{t}}) \leqslant \mathcal{E}_{\mathrm{lat}}$. Next, we will infer a bound for $|\xi_{\mathfrak{z},\sqrt{\mathfrak{t}}}|$. Enlarging the domain of integration from $Q^+_{\sqrt{\mathfrak{t}}}(\mathfrak{z})$ to $Q^*_{\sqrt{2\mathfrak{t}}}(\mathfrak{z}')$, using Hölder's inequality, (4.65) and (4.62) we infer with $\tilde{c} = 2^{n+1}(n+2)$ (note also that $\mathcal{E}_{\mathrm{lat}} \leqslant 1$ and $\sqrt{2\mathfrak{t}} \leqslant 1$),

$$
|\xi_{\mathfrak{z},\sqrt{\mathfrak{t}}}| \leqslant 2^{n+1}\frac{n+2}{\sqrt{2\mathfrak{t}}} \fint_{Q^*_{\sqrt{2\mathfrak{t}}}(\mathfrak{z}')} |u|\,dz \leqslant \tilde{c}\left(c_{\mathrm{ed}}\left(\frac{\sqrt{2\mathfrak{t}}}{\varrho}\right)^{2\beta} \widetilde{E}_{\mathrm{ed}}(\mathfrak{z}',\varrho)\right)^{\frac{1}{2}} \leqslant \tilde{c}c_5^{-\frac{1}{2}} \leqslant 1.
$$

Here, in the last line we have also used the fact that $\tilde{c}^2 \leqslant c_5$ by the choice of c_5 in (4.61). Hence, by the preceding estimate and (4.67) the hypotheses of Lemma 4.5 are satisfied. Therefore the lemma can be applied with $(\mathfrak{z}, r, \sqrt{\mathfrak{t}}, 1)$ instead of (z_0, r, ϱ, M) to conclude on the one hand that the limit in (4.63) exists and, on the other hand, that for any $0 < r \leqslant \sqrt{\mathfrak{t}}/2$ there holds

$$
\begin{aligned}
\fint_{Q^+_r(\mathfrak{z})} |Du - \mathfrak{X}_{\mathfrak{z}}|^2\,dz &\leqslant c_{\mathrm{lat}}\left[\left(\frac{r}{\sqrt{\mathfrak{t}}/2}\right)^{2\alpha} E_{\mathrm{lat}}(\mathfrak{z},\sqrt{\mathfrak{t}}) + r^{2\beta}\right] \\
&\leqslant c_{\mathrm{lat}}\left[\left(\frac{r}{\sqrt{\mathfrak{t}}/2}\right)^{2\alpha}\left(\frac{\sqrt{2\mathfrak{t}}}{\varrho/2}\right)^{2\beta} \mathcal{E}_{\mathrm{lat}} + r^{2\beta}\right] \\
&\leqslant c\left(\frac{r}{\varrho}\right)^{2\beta},
\end{aligned}
$$

where c depends on $n, N, p, \nu, L, K(1), \kappa_1, \alpha, \beta$ and $\|g_t\|_{L^{2,2-2\beta}(Q_1^*)}$. Here we have also used (4.67) and $\mathcal{E}_{\mathrm{ed}} \leqslant 1$ in the last line. In the remaining case $\sqrt{\mathfrak{t}}/2 < r \leqslant \varrho/6$ we use the previous estimate as well as (4.64) for $\mathfrak{z}'$ (note that $\mathfrak{X}_{\mathfrak{z}'} = 0$) to infer

$$
\begin{aligned}
\fint_{Q^*_r(\mathfrak{z})} |Du - \mathfrak{X}_{\mathfrak{z}}|^2\,dz &\leqslant 2\fint_{Q^*_r(\mathfrak{z})} |Du|^2\,dz + 4\fint_{Q^+_{\sqrt{\mathfrak{t}}/2}(\mathfrak{z})} |Du|^2 + |Du - \mathfrak{X}_{\mathfrak{z}}|^2\,dz \\
&\leqslant c\fint_{Q^*_{3r}(\mathfrak{z}')} |Du|^2\,dz + c\fint_{Q^*_{3\sqrt{\mathfrak{t}}/2}(\mathfrak{z}')} |Du|^2\,dz + c\left(\frac{\sqrt{\mathfrak{t}}/2}{\varrho}\right)^{2\beta} \\
&\leqslant c\left(\frac{r}{\varrho}\right)^{2\beta},
\end{aligned}
$$

where c depends only on $n, N, p, \nu, L, K(1), \kappa_1, \alpha, \beta$ and $\|g_t\|_{L^{2,2-2\beta}(Q_1^*)}$. This completes the proof of (4.63) and (4.64) in the case $\mathfrak{z} \in \Gamma_R(z_0)$.

Finally, we come to the remaining *case* $\mathfrak{z} \in Q^*_R(z_0)$. Here we can revert to the proofs of Proposition 4.7 and 4.12. To this aim we write $\mathfrak{z} = (\mathfrak{x}', \mathfrak{x}_n, \mathfrak{t})$ and distinguish the cases whether $\sqrt{\mathfrak{t}} \leqslant \mathfrak{x}_n$ or $\mathfrak{x}_n < \sqrt{\mathfrak{t}}$. In the first case, i.e. when

$\sqrt{\mathfrak{t}} \leqslant \mathfrak{x}_n$, then we denote by $\mathfrak{z}' = (\mathfrak{x}', \mathfrak{x}_n, 0)$ the projection of $\mathfrak{z}$ onto D_1^+. Then, we have $\mathfrak{z} \in Q^0_{\sqrt{\mathfrak{t}}}(\mathfrak{z}')$ and $\mathfrak{z}' \in D_R^+(z_0)$. Moreover, combining the second last estimate of (4.66) with (4.62) and taking also into account the definition of c_5, we obtain

$$\widetilde{E}_{\mathrm{ini}}(\mathfrak{z}', \mathfrak{x}_n) \leqslant (c_5 c_{\mathrm{ini}})^{-1} \min\{\mathcal{E}_{\mathrm{ini}}, \mathcal{E}_{\mathrm{int}}\}.$$

Therefore, (4.54) from the proof of Lemma 4.11 is satisfied for $(\mathfrak{z}', \mathfrak{x}_n)$ instead of (z_0, ϱ). This allows us to apply the arguments from the proof of Lemma 4.11 and conclude (4.63) and (4.64).

In the second case, i.e. when $\mathfrak{x}_n < \sqrt{\mathfrak{t}}$ we denote by $\mathfrak{z}' = (\mathfrak{x}', 0, \mathfrak{t})$ the projection of $\mathfrak{z}$ onto Γ_1^0. Now, we have $\mathfrak{z} \in Q^+_{\mathfrak{x}_n}(\mathfrak{z}')$ and $\mathfrak{z}' \in \Gamma_R^0(z_0)$. As before, we combine the second last estimate of (4.67) and (4.62) and recall the definition of c_5 to find that

$$\widetilde{E}_{\mathrm{lat}}(\mathfrak{z}', \sqrt{\mathfrak{t}}) \leqslant (c_5 c_{\mathrm{lat}})^{-1} \min\{\mathcal{E}_{\mathrm{lat}}, \mathcal{E}_{\mathrm{int}}\}.$$

Moreover due to (4.68) we also know that $|\xi_{\mathfrak{z}', \sqrt{\mathfrak{t}}}| \leqslant 1$. Therefore, (4.38) and (4.39) from the proof of Proposition 4.7 are satisfied for $(\mathfrak{z}', \sqrt{\mathfrak{t}}, 1, 1)$ instead of (z_0, ϱ, M_1, M_2). This allows us to apply the arguments from there to deduce (4.63) and (4.64) also in this case. Since now we have treated all the possible cases, this finally finishes the proof of the lemma. □

At this stage the same comment we made at the beginning of Section 4.2.6 in the initial boundary situation applies. More precisely, in Lemma 4.17 we indeed proved that the set of singular edge-points is contained in Σ_{ed}. But this is not the characterization we stated in Theorem 1.2. Therefore, it remains to show that $\Sigma_{\mathrm{ed}} \subset (\Sigma^1 \cup \Sigma^2) \cap \Omega_0$. But this follows from a version of the Poincaré type inequality in Proposition 4.12 for the edge-point situation. Indeed, by a different choice of the weight-function η in the proof of Lemma 4.13 such that $\operatorname{spt} \eta$ is now contained in B_ϱ^+ instead of B_ϱ the proof can be adopted line by line. Finally, let us note that it could also be slightly simplified in some points since in the edge-point situation we are allowed to apply the Poincaré inequality from Lemma 2.3. For the sake of brevity we shall not repeat the proof here. Therefore, using the edge-point version of Lemma 4.13 we now have completed the proof of Theorem 1.2 concerning the remaining edge-point situation.

As explained above the proof of Theorem 1.2 is now complete; as mentioned at the beginning of the paper, in a forthcoming sequel [6] we shall provide estimates ensuring that the boundary regularity criterium found applies at almost every boundary point.

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