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Asymptotic behavior of a Cahn–Hilliard–Navier–Stokes system in 2D

Ciprian G. Gal [a], Maurizio Grasselli [b,*]

[a] *Department of Mathematics, University of Missouri, Columbia, MO 65211, USA*
[b] *Dipartimento di Matematica "F. Brioschi", Politecnico di Milano, 20133 Milano, Italy*

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In memoriam Giovanni Prouse (1932–2008)

Abstract

We consider a model for the flow of a mixture of two homogeneous and incompressible fluids in a two-dimensional bounded domain. The model consists of a Navier–Stokes equation governing the fluid velocity coupled with a convective Cahn–Hilliard equation for the relative density of atoms of one of the fluids. Endowing the system with suitable boundary and initial conditions, we analyze the asymptotic behavior of its solutions. First, we prove that the initial and boundary value problem generates a strongly continuous semigroup on a suitable phase-space which possesses the global attractor $\mathcal{A}$. Then we establish the existence of an exponential attractors $\mathcal{E}$. Thus $\mathcal{A}$ has finite fractal dimension. This dimension is then estimated from above in terms of the physical parameters. Moreover, assuming the potential to be real analytic and in absence of volume forces, we demonstrate that each trajectory converges to a single equilibrium. We also obtain a convergence rate estimate in the phase-space metric.

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1. Introduction

It is widely accepted that the incompressible Navier–Stokes equation governs the complex motions of single-phase fluids such as air or water, while we are faced with the persistent and intriguing questions of recovering complex motions of binary fluid mixtures (see [52]). The turbulence issues for single-phase flows have been analyzed in many fundamental works (see, e.g., [14,24,25,45,47] and their references). On the other hand, the mathematical study of turbulent binary (or even multi-phase) mixture flows is only in its infancy. Thus, the present article may be viewed as a preliminary contribution to the analysis of the turbulence problem for multi-phase flows (cf. also [28]).

* Corresponding author.
E-mail addresses: galc@missouri.edu (C.G. Gal), maurizio.grasselli@polimi.it (M. Grasselli).

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The quenching of a system from a disordered phase into an ordered one produces a time-dependent growth process of ordered regions. The evolution of these regions is the subject of phase ordering dynamics, a relevant subject of investigation for a number of physical systems ranging from solid alloys to polymer blends, multi-phase fluids and nematic liquid crystals [5,7,13,40,36,46,49,53]. The first to address the problem were J.W. Cahn and J.E. Hilliard [16] who studied the spinodal decomposition of binary alloys (see also [15]). Similar phenomena occur in the phase separation of binary fluids, that is, fluids composed by either two phases of the same chemical species or phases of different composition. In this case, however, the phenomenology is much more complicated because of the interplay between the phase separation stage and the fluid dynamics.

The mathematical analysis of these phenomena is far from being well understood. For instance, the spinodal decomposition under shear consists of a two-stage evolution of a homogeneous initial mixture: a phase separation stage in which some macroscopic pattern appear, then a shear stage in which these patterns organize themselves into parallel layers (see, e.g., [50] for experimental snapshots). This model has to take into account the chemical interactions between the two phases at the interface, achieved using a Cahn–Hilliard approach, as well as the hydrodynamic properties of the mixture (e.g., in the shear case), for which Navier–Stokes equations with surface tension terms acting at the interface are needed. When the two fluids have the same constant density, the temperature differences are negligible and the diffusive interface between the two phases has a small but non-zero thickness, a well-known model is the so-called "Model H" (cf. [37], see [34] for a rigorous derivation). This is a system of equations where an incompressible Navier–Stokes equation for the (mean) velocity field $\boldsymbol{u} = (u_1, \dots, u_{\mathrm{N}})$, $\mathrm{N} = 2, 3$, is coupled with a convective Cahn–Hilliard equation for the order parameter ϕ which represents the relative concentration of one of the fluids (for the compressible case see [3] and its references). More precisely, the equations read as follows

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla p = \mathcal{K} \mu \nabla \phi + \boldsymbol{g}, \tag{1.1}$$

$$\nabla \cdot \boldsymbol{u} = 0, \tag{1.2}$$

$$\partial_t \phi + \boldsymbol{u} \cdot \nabla \phi - \varrho_0 \Delta \mu = 0, \tag{1.3}$$

$$\mu = -\varepsilon \Delta \phi + \alpha f(\phi), \tag{1.4}$$

in $\Omega \times (0, +\infty)$, where Ω is a bounded domain in $\mathbb{R}^{\mathrm{N}}$, $\mathrm{N} = 2, 3$, with smooth boundary Γ, $\boldsymbol{g}$ is an external time-independent volume force and we have assumed the density equal to one. We remind that an external nongradient force (e.g., a stirring force) can play a basic role in certain phenomena like coarsening (see [7]). The quantities ν, ϱ_0 and $\mathcal{K}$ are positive constants that correspond to the kinematic viscosity of fluid, mobility constant and capillarity (stress) coefficient, respectively. Here μ is the chemical potential of the binary mixture which is given by the variational derivative of the following free energy functional

$$\mathcal{F}(\phi) = \int_\Omega \left(\frac{\varepsilon}{2} |\nabla \phi|^2 + \alpha F(\phi) \right) dx,$$

where, e.g., $F(r) = \int_0^r f(\zeta)\, d\zeta$ is a suitable double-well potential. Here ε and α are two positive parameters describing the interactions between the two phases. In particular, ε is related to the thickness of the interface separating the two fluids. A typical example of potential F is of logarithmic type (see [16] and references therein). However, this potential is very often replaced by a polynomial approximation of the type $F(r) = \gamma_1 r^4 - \gamma_2 r^2$, γ_1 and γ_2 being positive constants. We also note that (1.1) can be replaced by

$$\partial_t \boldsymbol{u} + \boldsymbol{u} \cdot \nabla \boldsymbol{u} - \nu \Delta \boldsymbol{u} + \nabla \widetilde{p} = -\mathcal{K} \operatorname{div}(\nabla \phi \otimes \nabla \phi) + \boldsymbol{g}$$

with $\widetilde{p} = p - \kappa(\frac{\varepsilon}{2}|\nabla \phi|^2 + \alpha F(\phi))$ since

$$\kappa \mu \nabla \phi = \kappa \nabla \left(\frac{\varepsilon}{2} |\nabla \phi|^2 + \alpha F(\phi) \right) - \mathcal{K} \operatorname{div}(\nabla \phi \otimes \nabla \phi).$$

The stress tensor $\nabla \phi \otimes \nabla \phi$ is considered the main contribution modelling capillary forces due to surface tension at the interface between the two phases of the fluid.

Regarding possible boundary conditions for these models, we recall two cases considered in the literature: the mixing of two fluids in a driven cavity (see, e.g., [17] and the references therein) and the spinodal decomposition under shear in a channel (cf., for instance, [50]; see also [12]). In the first case, the boundary conditions for ϕ in (1.3) are the natural no-flux conditions

$$\partial_{\mathbf{n}}\phi = \partial_{\mathbf{n}}\Delta\phi = 0, \tag{1.5}$$

on $\Gamma \times (0, +\infty)$, where $\mathbf{n}$ is the outward normal to Γ. These conditions ensure the mass conservation. In fact, it is easy to check that (1.5) implies that

$$\partial_{\mathbf{n}}\mu = 0, \quad \text{on } \Gamma \times (0, +\infty),$$

which yields the conservation of the following quantity

$$\langle\phi(t)\rangle = \frac{1}{|\Omega|}\int_{\Omega} \phi(x,t)\,dx,$$

where $|\Omega|$ stands for the Lebesgue measure of Ω. More precisely, we get from (1.3) that $\langle\phi(t)\rangle = \langle\phi(0)\rangle$ for all $t \geqslant 0$. Concerning the boundary condition for $\boldsymbol{u}$, we will assume the Dirichlet (no-slip) boundary condition

$$\boldsymbol{u} = \mathbf{0}, \quad \text{on } \Gamma \times (0, +\infty). \tag{1.6}$$

Therefore we suppose that there is no relative motion at the fluid-solid interface. On the other hand, in the case of channel under shear, periodicity conditions may be imposed for ϕ, μ and $\boldsymbol{u}$, in the longitudinal direction. The periodicity conditions are natural because in the physical experiments the shear is obtained by putting the mixture between two rotating cylinders whose diameters are very close (Couette–Taylor flows), curvature effects are usually neglected because of the thickness of the domain (see, e.g., [12]). We could also consider these conditions here, but for the sake of exposition, we will focus our attention to (1.5)–(1.6) only. However, we remark that all the subsequent results concerning problem (1.1)–(1.4) can also be extended to the mentioned periodic boundary conditions on a rectangular domain Ω. Of course, system (1.1)–(1.5) is also subject to initial conditions, that is,

$$\boldsymbol{u}_{|t=0} = \boldsymbol{u}_0, \qquad \phi_{|t=0} = \phi_0, \quad \text{in } \Omega. \tag{1.7}$$

Problems like (1.1)–(1.7) have recently received lot of attention from the numerical viewpoint (see, e.g., [6,11,23, 39,42,44,48] and references therein). Well-posedness issues have been analyzed in [9] for a system where the Cahn–Hilliard equation has nonconstant mobility and the Navier–Stokes equation has non-matched viscosity $\nu = \nu(\phi)$ (see [10] for the nonhomogeneous case and [21,43] for non-Newtonian fluids). The concentration dependent mobility forces ϕ to take values within a bounded interval (say, $[-1, 1]$) and also logarithmic-type potentials can be handled (see [9]). In particular, the author has proven the existence and uniqueness of global weak and strong solutions in 2D as well as local asymptotic stability of suitable stationary solutions. The hard case of constant mobility, nonconstant viscosity and singular potentials has been analyzed in [2]. In this noteworthy paper, besides existence and uniqueness results, the regularity of solutions has been carefully examined and convergence to a single equilibrium has been established. The case $\Omega = \mathbb{R}^2$ with smooth potentials has also been considered and existence, uniqueness and stability of stationary solutions have been investigated [54]. A further interesting qualitative result is contained in [4, Appendix A]. There, the authors take $\mathcal{K} = \varepsilon$ and $\alpha = \varepsilon^{-1}$, and identify the limit as ε tends to 0 of system (1.1)–(1.4) endowed with suitable initial and boundary conditions. The resulting limiting system is a combination of the classical Navier–Stokes sharp interface model with a Mullins–Sekerka type problem (see [4] and references therein).

As far as the longtime behavior is concerned, existence of a global attractor for (1.1)–(1.4) has recently been proven in [1]. Here, we want to carry out a more detailed analysis of the same system endowed with (1.5)–(1.7) for $\mathrm{N} = 2$. The goals are similar to the ones of [28], where the 2D Navier–Stokes equation coupled with an Allen–Cahn equation has been examined. Both these systems have been then considered in a unified way in [29], where we have studied the longtime behavior in the 3D case, subject to a time-dependent external nongradient force using the trajectory approach [20]. Moreover, in [30], we have proved the instability of certain stationary solutions for systems (1.1)–(1.4) subject to periodic boundary conditions on elongated domains $\mathbf{T}_{\alpha_0} = (0, 2\pi/\alpha_0) \times (0, 2\pi)$ or $\mathbf{T}_{\alpha_0\beta_0} = (0, 2\pi/\alpha_0) \times (0, 2\pi) \times (0, 2\pi/\beta_0)$, α_0 and β_0 being small nondimensional parameters. In this case $\boldsymbol{g}$ is a suitable periodic external force (e.g., like the one in the Kolmogorov problem, see [38, Section 5] and its references). As a consequence, a lower bound for the Hausdorff dimension of the global attractor can be deduced. This bound shows that the coupling gives rise to additional instabilities and, thus, to novel and even more complex flow behavior (see [30] for details).

The plan of the paper goes as follows. In Section 2 we present and discuss the weak formulation of our problem. In Section 3, we prove that the problem generates a strongly continuous semigroup on a suitable phase-space. Moreover, we show that dynamical system possesses a global attractor and an exponential attractor. Section 4 is devoted to demonstrate an upper bound of the fractal dimension of the global attractor in terms of the most relevant physical parameters $\nu, \varepsilon, \mathcal{K}$ and α. Finally, in Section 5, assuming the potential F to be real analytic and no external nongradient forces ($\boldsymbol{g} = \boldsymbol{0}$), we prove that each trajectory converges to a single equilibrium with respect to the phase-space metric and find a convergence rate estimate.

2. Weak formulation

We begin by setting $\varrho_0 = 1$ for the sake of simplicity. Then we assume that $f \in C^2(\mathbb{R})$ and satisfies

$$\begin{cases} \liminf\limits_{|r|\to+\infty} f'(r) > 0, \\ \left|f''(r)\right| \leqslant c_f\left(1+|r|^{m-1}\right), \quad \forall r \in \mathbb{R}, \end{cases} \tag{2.1}$$

where c_f is some positive constant and $m \in [1, +\infty)$ is fixed, but otherwise arbitrary. It is immediate that (2.1) entails that

$$\left|f'(r)\right| \leqslant c_f\left(1+|r|^m\right), \qquad \left|f(r)\right| \leqslant c_f\left(1+|r|^{m+1}\right), \quad \forall r \in \mathbb{R}. \tag{2.2}$$

Note that the derivative f of the typical double-well potential F satisfies (2.1).

Let us describe the functional setup of Eqs. (1.1)–(1.4). From now on Ω denotes a two-dimensional bounded domain with C^2-boundary Γ. If X is real Hilbert space with inner product $(\cdot,\cdot)_X$, then we denote the induced norm by $|\cdot|_X$, while X^* will indicate its dual. Moreover, we indicate by $\mathbb{X}$ the space $X \times X$ endowed with the product structure. Let us consider the Hilbert spaces

$$\mathbb{H} := \overline{\left\{\boldsymbol{u} \in \mathbb{C}_c^\infty(\Omega)\colon \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\right\}}^{\mathbb{L}^2}, \qquad \mathbb{V} = \overline{\left\{\boldsymbol{u} \in \mathbb{C}_c^\infty(\Omega)\colon \operatorname{div} \boldsymbol{u} = 0 \text{ in } \Omega\right\}}^{\mathbb{H}_0^1}, \tag{2.3}$$

where $\mathbb{L}^2(\Omega, dx) = (L^2(\Omega, dx))^2$ and $\mathbb{H}_0^1(\Omega) = (H_0^1(\Omega))^2$. The space $\mathbb{H}$ is endowed with the scalar product and the norm of $\mathbb{L}^2(\Omega, dx)$ are denoted by $(\cdot,\cdot)$ and $|\cdot|$, respectively. The space $\mathbb{V}$ becomes is Hilbert with respect to the scalar product

$$\left((\boldsymbol{u}, \boldsymbol{v})\right) = \sum_{i=1}^{2} (\partial_{x_i}\boldsymbol{u}, \partial_{x_i}\boldsymbol{v}), \qquad \|\boldsymbol{u}\| = \left((\boldsymbol{u}, \boldsymbol{u})\right)^{1/2}.$$

We recall that the norm in $\mathbb{V}$ is equivalent to that induced by $\mathbb{H}_0^1(\Omega)$, due to Poincaré's inequality.

Let us indicate by A_0 the self-adjoint positive unbounded operators in $\mathbb{H}$ defined by

$$A_0\boldsymbol{u} = -\mathbb{P}\Delta\boldsymbol{u}, \quad \forall \boldsymbol{u} \in D(A_0) = \mathbb{H}^2(\Omega) \cap \mathbb{V},$$

where $\mathbb{P}$ is the Leray-Helmholtz projector in $\mathbb{L}^2(\Omega, dx)$ on $\mathbb{H}$. Observe that A_0^{-1} is a compact linear operator on $\mathbb{H}$ and $|A_0\cdot|$ is a norm on $D(A_0)$ that is equivalent to $\mathbb{H}^2$-norm.

Then we introduce the linear nonnegative unbounded operator on $L^2(\Omega)$

$$A_N\phi = -\Delta\phi, \quad \forall \phi \in D(A_N) = \left\{\phi \in H^2(\Omega)\colon \partial_{\mathbf{n}}\phi = 0, \text{ on } \Gamma\right\}$$

and we endow $D(A_N)$ with the norm $|A_N\cdot|_{L^2} + |\langle\cdot\rangle|$ which is equivalent to the H^2-norm. Also, we define the linear positive unbounded operator on the Hilbert space $L_0^2(\Omega)$ of the L^2-functions with null mean

$$B_N\phi = -\Delta\phi, \quad \forall \phi \in D(B_N) = D(A_N) \cap L_0^2(\Omega).$$

Observe that B_N^{-1} is a compact linear operator on $L_0^2(\Omega)$. More generally, we can define B_N^s for any $s \in \mathbb{R}$, noting that $|B_N^{s/2}\cdot|_{L^2}$, $s > 0$, is an equivalent to the canonical H^s-norm on $D(B_N^{s/2}) \subseteq H^s(\Omega) \cap L_0^2(\Omega)$. Note that $A_N \equiv B_N$ on $D(B_N)$. If ϕ is such that $\phi - \langle\phi\rangle \in D(B_N^{s/2})$ we have that $|B_N^{s/2}(\phi - \langle\phi\rangle)|_{L^2} + |\langle\phi\rangle|$ is equivalent to the H^s-norm. Moreover, we set $H^{-s}(\Omega) := (H^{-s}(\Omega))^*$ whenever $s < 0$.

In order to define the variational setting for the Navier–Stokes equations, we also need to introduce the bilinear operators B_0, B_1 (and their related trilinear forms b_0 and b_1) as well as the coupling mapping $\boldsymbol{R}_0$ which are defined, respectively, from $D(A_0) \times D(A_0)$ into $\mathbb{H}$, $D(A_0) \times D(A_N)$ into $L^2(\Omega)$ and $L^2(\Omega) \times (D(A_N) \cap H^3(\Omega))$ into $\mathbb{H}$. More precisely, we set

$$\big(B_0(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{w}\big) = \int_\Omega \big[(\boldsymbol{u}\cdot\nabla)\boldsymbol{v}\big]\cdot\boldsymbol{w}\,dx =: b_0(\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}), \quad \forall \boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in D(A_0),$$

$$\big(B_1(\boldsymbol{u},\phi),\psi\big)_{L^2} = \int_\Omega \big[(\boldsymbol{u}\cdot\nabla)\phi\big]\psi\,dx =: b_1(\boldsymbol{u},\phi,\psi), \quad \forall \boldsymbol{u} \in D(A_0),\ \forall \phi,\psi \in D(A_N),$$

$$\big(\boldsymbol{R}_0(\xi,\phi),\boldsymbol{w}\big) = \int_\Omega \xi[\nabla\phi\cdot\boldsymbol{w}]\,dx, \quad \forall \boldsymbol{w} \in D(A_0),\ \forall\phi \in D(A_N)\cap H^3(\Omega),\ \forall \xi \in L^2(\Omega).$$

Remark 2.1. The operators defined above enjoy continuity properties which depend on the space dimension (cf., e.g., [51, Chap. 9] or [55, Chap. 3]). In addition, note that $\boldsymbol{R}_0(\mu,\phi) = \mathbb{P}\mu\nabla\phi$.

We are now in a position to formulate problem (1.1)–(1.7) in a weak form. However, due to the mass conservation

$$\big\langle\phi(t)\big\rangle = \big\langle\phi(0)\big\rangle =: M_0, \quad \forall t \geqslant 0, \tag{2.4}$$

we need to put a constraint, namely, we have to take as phase-space the following

$$\mathbb{Y}_M = \mathbb{H} \times \big\{\phi \in H^1(\Omega):\ \big|\langle\phi\rangle\big| \leqslant M\big\},$$

where $M \geqslant 0$ is fixed. The space $\mathbb{Y}_M$ is a complete metric space with respect to the metric associated with the norm

$$\big\|(\boldsymbol{u},\phi)\big\|^2_{\mathbb{Y}_M} := \frac{1}{\mathcal{K}}|\boldsymbol{u}|^2 + \varepsilon\big(|\nabla\phi|^2 + \langle\phi\rangle^2\big). \tag{2.5}$$

Then our problem can be formulated as follows.

Problem P. For $\boldsymbol{g} \in \mathbb{V}^*$ and any given pair of initial data

$$(\boldsymbol{u}_0,\phi_0) \in \mathbb{Y}_M, \tag{2.6}$$

find a pair of functions

$$(\boldsymbol{u},\phi) \in C\big([0,+\infty);\mathbb{Y}_M\big) \cap L^2_{loc}\big([0,+\infty);\mathbb{V}\times\big(D(A_N)\cap H^3(\Omega)\big)\big) \tag{2.7}$$

such that

$$(\partial_t\boldsymbol{u},\partial_t\phi) \in L^2_{loc}\big([0,+\infty);\mathbb{V}^*\times H^{-1}(\Omega)\big), \tag{2.8}$$

which fulfills (1.7) and satisfies

$$\begin{cases} \partial_t\boldsymbol{u} + \nu A_0\boldsymbol{u} + B_0(\boldsymbol{u},\boldsymbol{u}) - \mathcal{K}\boldsymbol{R}_0(\varepsilon A_N\phi,\phi) = \boldsymbol{g}, & \text{in } \mathbb{V}^*,\ \text{a.e. in } (0,+\infty),\\ \mu = \varepsilon A_N\phi + \alpha f(\phi), & \text{a.e. in } \Omega\times(0,+\infty),\\ \partial_t\phi + A_N\mu + B_1(\boldsymbol{u},\phi) = 0, & \text{in } H^{-1},\ \text{a.e. in } (0,+\infty). \end{cases} \tag{2.9}$$

Remark 2.2. Note that the chemical potential does no longer appear in the first equation of (2.9). More precisely, $\mu\nabla\phi$ has been replaced by $\varepsilon A_N\phi\nabla\phi$ (cf. the right-hand side of Eq. (1.1)). This is justified since $f'(\phi)\nabla\phi$ is the gradient of $F(\phi)$ and can be incorporated into the pressure gradient. This remarks also holds when the volume force $\boldsymbol{g}$ is the gradient of some potential (e.g., gravity). In the sequel, for the sake of convenience, we will also replace μ in the last equation of (2.9) with $\overline{\mu} = \mu - \langle\mu\rangle$, that is,

$$\overline{\mu} = \varepsilon A_N\phi + \alpha f(\phi) - \alpha\big\langle f(\phi)\big\rangle, \quad \text{a.e. in } \Omega\times(0,+\infty).$$

Obviously, we have $\langle\overline{\mu}(t)\rangle = 0$ for all $t > 0$.

We finish this section by pointing out once more that other kind of boundary conditions can be handled with simple modifications of the phase-space. For instance, one can suppose that Ω is a rectangular domain and $\boldsymbol{u}$, its first spatial derivatives, p and ϕ are Ω-periodic or we can assume that $\boldsymbol{u}$ satisfies a free boundary condition (see, e.g., [55, Chap. III, Section 2]). In these cases all the subsequent results for $\mathbf{P}$ are still valid, provided that f satisfies suitable assumptions.

3. Global and exponential attractors

In this section, we first establish some uniform (in time) a priori estimates and prove the existence of a strongly continuous dissipative semigroup. Then, we show some smoothing properties of the solutions which allow us to demonstrate the existence of global and exponential attractors. All the estimates are obtained through formal arguments which can be justified within a suitable Faedo–Galerkin approximation scheme (see, e.g., [9]).

3.1. Uniform estimates on the solutions

Observe preliminarily that if $(\boldsymbol{u},\phi)$ is a smooth solution of $\mathbf{P}$, by taking the scalar product in $\mathbb{H}$ of Eq. (1.1) with $\boldsymbol{u}$, then integrating over Ω, and using Eqs. (1.3)–(1.4), we obtain the energy identity

$$\frac{d}{dt}\left[\frac{1}{2\mathcal{K}}\left|\boldsymbol{u}(t)\right|^2+\mathcal{F}\big(\phi(t)\big)\right]-\frac{1}{\mathcal{K}}\big(\boldsymbol{u}(t),\boldsymbol{g}\big)+\frac{\nu}{\mathcal{K}}\left\|\boldsymbol{u}(t)\right\|^2+\left|\nabla\mu(t)\right|^2_{L^2}=0. \tag{3.1}$$

It is also worth mentioning that (3.1) is a consequence of the orthogonality properties of the products below, which will be also employed in the sequel, namely,

$$\big(B_0(\boldsymbol{u},\boldsymbol{v}),\boldsymbol{v}\big)=0,\quad \forall \boldsymbol{u},\boldsymbol{v}\in\mathbb{V},\qquad \big(B_1(\boldsymbol{u},\phi),\phi\big)_{L^2}=0,\quad \forall \boldsymbol{u}\in\mathbb{V},\ \forall\phi\in H^1(\Omega). \tag{3.2}$$

By exploiting (3.1), we prove the following dissipative estimate.

Proposition 3.1. *Let $\boldsymbol{g}\in\mathbb{V}^*$ and $f\in C^2(\mathbb{R})$ satisfy* (2.1). *If $(\boldsymbol{u},\phi)$ is a solution to $\mathbf{P}$, then the following estimate holds*:

$$\begin{aligned}&\left\|\big(\boldsymbol{u}(t),\phi(t)\big)\right\|^2_{\mathbb{Y}_M}+\int_t^{t+1}\left(\frac{\nu}{\mathcal{K}}\left\|\boldsymbol{u}(s)\right\|^2+\left|\mu(s)\right|^2_{H^1}+\left|F\big(\phi(s)\big)\right|_{L^1}\right)ds\\&\quad+\int_t^{t+1}\left(\left\|\partial_t\boldsymbol{u}(s)\right\|^2_{\mathbb{V}^*}+\left|\phi(s)\right|^2_{H^3}+\left|\partial_t\phi(s)\right|^2_{H^{-1}}\right)ds\\&\quad\leqslant Q\left(\left\|\big(\boldsymbol{u}(0),\phi(0)\big)\right\|^2_{\mathbb{Y}_M}\right)e^{-\rho t}+C_0\big(\nu,\varepsilon,\alpha,\mathcal{K},M,\|\boldsymbol{g}\|_{\mathbb{V}^*}\big),\quad\forall t\geqslant 0,\end{aligned} \tag{3.3}$$

where the monotone non-decreasing function Q and the positive constants ρ and C_0 are independent of t and of the initial conditions.

Proof. We now introduce the functions $\overline{\phi}(t):=\phi(t)-M_0$ and $\overline{\mu}(t):=\mu(t)-\langle\mu(t)\rangle$ and note that $\langle\overline{\phi}(t)\rangle=0$, due to (2.4). Let us take the scalar product in $L^2(\Omega)$ of the second equation of (2.9) with $2\xi\overline{\phi}(t)$, $\xi>0$. We obtain

$$2\xi\big(\overline{\mu}(t),\overline{\phi}(t)\big)_{L^2}=2\xi\varepsilon\left|\nabla\overline{\phi}(t)\right|^2_{L^2}+2\alpha\xi\big(f\big(\phi(t)\big),\overline{\phi}(t)\big)_{L^2},$$

since $\langle\overline{\mu}(t)\rangle=0$. Then adding together the obtained relationship with (3.1), we get

$$\frac{d}{dt}E(t)+\kappa E(t)=\Lambda_1(t), \tag{3.4}$$

where $\kappa\in(0,\xi)$ and

$$E(t):=\left\|\big(\boldsymbol{u}(t),\overline{\phi}(t)\big)\right\|^2_{\mathbb{Y}_M}+2\alpha\big(F\big(\phi(t)\big),1\big)+c_E.$$

Here the constant $c_E=2\alpha C_F|\Omega|>0$, where C_F is taken large enough in order to ensure that E is nonnegative (note that F is bounded from below by a constant independent of ε and α). The function Λ_1 is given by

$$\Lambda_1(t) := -2\frac{\nu}{\mathcal{K}}\|\boldsymbol{u}(t)\|^2 + \frac{\kappa}{\mathcal{K}}|\boldsymbol{u}(t)|^2 - 2|\nabla\overline{\mu}(t)|^2_{L^2} - (2\xi-\kappa)\varepsilon|\nabla\overline{\phi}(t)|^2 + \frac{2}{\mathcal{K}}(\boldsymbol{u}(t),\boldsymbol{g})$$
$$+ 2\alpha\big[\kappa\big(F(\phi(t)) - f(\phi(t))\overline{\phi}(t),1\big)_{L^2} - (\xi-\kappa)\big(f(\phi(t))\overline{\phi}(t),1\big)_{L^2}\big]$$
$$+ 2\xi\big(\overline{\mu}(t),\overline{\phi}(t)\big)_{L^2} + \kappa c_E. \tag{3.5}$$

The Hölder, Friedrich and Young inequalities yield

$$2\xi(\overline{\mu},\overline{\phi})_{L^2} \leqslant 2\xi|\overline{\mu}|_{L^2}|\overline{\phi}|_{L^2} \leqslant 2\xi c_\Omega^{1/2}|\Omega|^{1/2}|\nabla\overline{\mu}|_{L^2}|\nabla\overline{\phi}|_{L^2}$$
$$\leqslant |\nabla\overline{\mu}|^2_{L^2} + \xi^2 c_\Omega|\Omega||\nabla\overline{\phi}|^2_{L^2}.$$

Moreover, owing to the first assumption of (2.1), we have

$$c_*|f(y)|(1+|y|) \leqslant 2f(y)(y-M_0) + c_{f,M_0}, \tag{3.6}$$
$$F(y) - f(y)(y-M_0) \leqslant c'_f(y-M_0)^2 + c''_{f,M_0}, \tag{3.7}$$

for any $y\in\mathbb{R}$. Here c_{f,M_0}, c_*, c'_f and c''_{f,M_0} are positive, sufficiently large constants that depend on f and M_0 only. From (3.5)–(3.7) and Poincaré's inequality (cf. [55, (3.17), p. 461]), it follows that

$$\Lambda_1(t) \leqslant -\frac{1}{\mathcal{K}}\big(\nu - \kappa c_\Omega|\Omega|\big)\|\boldsymbol{u}(t)\|^2 - |\nabla\overline{\mu}(t)|^2_{L^2} - \big[\xi\big(2-\xi c_\Omega|\Omega|\varepsilon^{-1}\big) - \kappa\big(1+2\alpha\varepsilon^{-1}c_\Omega c'_f|\Omega|\big)\big]\varepsilon|\nabla\overline{\phi}(t)|^2$$
$$- c_*\alpha(\xi-\kappa)\big(|f(\phi(t))|, 1+|\phi(t)|\big) + \frac{1}{\nu\mathcal{K}}\|\boldsymbol{g}\|^2_{\mathbb{V}^*} + c_1,$$

where c_Ω depends on the shape of Ω, but not on its size and $c_1>0$ depends on κ, c_f, M_0 and c''_f at most. Furthermore, performing a more careful computation of c_1, we get

$$c_1 = 2\kappa\alpha C_F|\Omega| + 2\alpha\kappa c''_{f,M_0}|\Omega| + c_{f,M_0}\alpha(\xi-\kappa)|\Omega|.$$

From now on, c_i stands for a positive constant which is independent on the initial data and on time.

Observe that it is possible to adjust $\xi = \varepsilon/(c_\Omega|\Omega|)$ and $\kappa\in(0,\xi)$ by letting

$$\kappa = \min\big\{\nu/\big(2c_\Omega|\Omega|\big), \varepsilon/\big(2c_\Omega|\Omega|\big), \xi/\big(1+2\alpha\varepsilon^{-1}c_\Omega c'_f|\Omega|\big)\big\},$$

in order to have

$$\frac{d}{dt}E(t) + \kappa E(t) + \kappa_1\left(\frac{\nu}{\mathcal{K}}\|\boldsymbol{u}(t)\|^2 + \varepsilon|\nabla\overline{\phi}(t)|^2\right) + |\nabla\overline{\mu}(t)|^2_{L^2} + \kappa_2\big(|f(\phi(t))|, 1+|\phi(t)|\big)_{L^2}$$
$$\leqslant \frac{1}{\nu\mathcal{K}}\|\boldsymbol{g}\|^2_{\mathbb{V}^*} + c_1.$$

Then, applying a suitable version of the Gronwall inequality (see, e.g., [32, Lemma 2.5]), we deduce that

$$E(t) + \int_t^{t+1}\left[\kappa_1\left(\frac{\nu}{\mathcal{K}}\|\boldsymbol{u}(s)\|^2 + \varepsilon|\nabla\overline{\phi}(s)|^2\right) + |\nabla\mu(s)|^2_{L^2}\right]ds + \kappa_2\int_t^{t+1}\big(|f(\phi(s))|, 1+|\phi(s)|\big)_{L^2}\,ds$$
$$\leqslant 2E(0)e^{-\kappa t} + 2\kappa^{-1}\left(\frac{1}{\nu\mathcal{K}}\|\boldsymbol{g}\|^2_{\mathbb{V}^*} + c_1\right), \quad \forall t\geqslant 0. \tag{3.8}$$

On the other hand, one can check that there exists a monotone non-decreasing function Q, independent of t and on the initial data, such that

$$\big\|\big(\boldsymbol{u}(t),\phi(t)\big)\big\|^2_{\mathbb{Y}_M} - \varepsilon\langle\phi(t)\rangle^2 \leqslant E(t) \leqslant Q\big(\big\|\big(\boldsymbol{u}(t),\phi(t)\big)\big\|^2_{\mathbb{Y}_M}\big). \tag{3.9}$$

Taking (3.9) into account and observing that assumption (2.1) also implies that

$$|F(y)| - c_{M_0} \leqslant |f(y)|(1+|y|),$$

for some positive constant c_{M_0} and all $y\in\mathbb{R}$, we obtain the following estimate:

$$\big\|\big(\boldsymbol{u}(t),\phi(t)\big)\big\|_{\mathbb{Y}_M}^2+\int_t^{t+1}\bigg(\frac{\nu}{\mathcal{K}}\big\|\boldsymbol{u}(s)\big\|^2+\big|\nabla\mu(s)\big|_{L^2}^2\bigg)\,ds+\int_t^{t+1}\big(\varepsilon\big|\nabla\phi(s)\big|^2+\big|F\big(\phi(s)\big)\big|_{L^1}\big)\,ds$$
$$\leqslant Q\big(\big\|\big(\boldsymbol{u}(t),\phi(t)\big)\big\|_{\mathbb{Y}_M}^2\big)e^{-\kappa t}+c_2, \tag{3.10}$$

where $c_2=2\kappa^{-1}(\nu^{-1}\mathcal{K}^{-1}\|\boldsymbol{g}\|_{\mathbb{V}^*}^2+c_1)+c_{M_0}+\varepsilon(M_0)^2$. It is left to prove the estimate for the remaining terms in (3.3). We proceed as follows. First, take the average over Ω of the second equation of (2.9) and notice that, due to (1.5) and assumption (2.1), we have

$$\langle\mu(t)\rangle^2=\alpha^2\big\langle f\big(\phi(t)\big)\big\rangle^2\leqslant\alpha^2c_f\big(1+\big|\phi(t)\big|_{L^{2m+2}}^{2m+2}\big)$$
$$\leqslant\alpha^2c_{f,m}\big[1+\varepsilon^{-m-1}\big(\varepsilon\big|\nabla\phi(t)\big|_{L^2}^2+\varepsilon\langle\phi(t)\rangle^2\big)^{m+1}\big].$$

Here we have used the injection $H^1(\Omega)\hookrightarrow L^{2m+2}(\Omega)$, $m\in[1,+\infty)$. Thus, we deduce from (3.10) the required estimate for the average of μ over Ω, that is,

$$\int_t^{t+1}\langle\mu(s)\rangle^2\,ds\leqslant Q\big(\big\|\big(\boldsymbol{u}(t),\phi(t)\big)\big\|_{\mathbb{Y}_M}^2\big)e^{-(m+1)\kappa t}+c_3,\quad\forall t\geqslant 0,$$

where $c_3=\alpha^2c_{f,m}[1+c_2^m\varepsilon^{-(m+1)}]$. Hence the above inequality together with the estimate for $|\nabla\mu|_{L^2}$ from (3.10), yields

$$\int_t^{t+1}\big|\mu(s)\big|_{H^1}^2\,ds\leqslant Q\big(\big\|\big(\boldsymbol{u}(t),\phi(t)\big)\big\|_{\mathbb{Y}_M}^2\big)e^{-\rho t}+c_4,\quad\forall t\geqslant 0, \tag{3.11}$$

for some positive constant ρ that depends only on κ and m, and where $c_4=c_2+c_3$. Furthermore, we observe that, from (3.10)–(3.11) and the injection $H^1(\Omega)\hookrightarrow L^\beta(\Omega)$, $\beta\in[1,+\infty)$, it follows that

$$\int_t^{t+1}\big|A_N\phi(s)\big|_{L^2}^2\,ds\leqslant\varepsilon^{-2}\int_t^{t+1}\big(\big|\mu(s)\big|_{L^2}^2+\alpha^2\big|f\big(\phi(s)\big)\big|_{L^2}^2\big)\,ds$$
$$\leqslant\varepsilon^{-2}\big(Q\big(\big\|\big(\boldsymbol{u}(t),\phi(t)\big)\big\|_{\mathbb{Y}_M}^2\big)e^{-\rho t}+c_6\big),$$

for all $t\geqslant 0$. Also, using a well-known regularity result, we obtain

$$\int_t^{t+1}\big|\phi(s)\big|_{H^3}^2\,ds\leqslant Q\big(\big\|\big(\boldsymbol{u}(t),\phi(t)\big)\big\|_{\mathbb{Y}_M}^2\big)e^{-\rho t}+c_7,\quad\forall t\geqslant 0. \tag{3.12}$$

In order to deduce an a priori bound on $\partial_t\phi$ in $L^2([t,t+1];H^{-1}(\Omega))$, we use the last two equations of (2.9). From (3.8), (3.11), and the fact that $\langle\partial_t\phi(t)\rangle=0$ for all $t\geqslant 0$, we have that

$$\int_t^{t+1}\big|\partial_t\phi(s)\big|_{H^{-1}}^2\,ds\leqslant\int_t^{t+1}\big(\big|A_N\mu(s)\big|_{H^{-1}}^2+\big|B_1\big(\boldsymbol{u}(s),\phi(s)\big)\big|_{H^{-1}}^2\big)\,ds$$
$$\leqslant\int_t^{t+1}\big(\big|\mu(s)\big|_{H^1}^2+c_\Omega\big\|\boldsymbol{u}(s)\big\|^2\big|\phi(s)\big|_{H^1}^2\big)\,ds$$
$$\leqslant Q\big(\big\|\big(\boldsymbol{u}(t),\phi(t)\big)\big\|_{\mathbb{Y}_M}^2\big)e^{-\rho t}+c_8,\quad\forall t\geqslant 0. \tag{3.13}$$

To get a uniform bound on $\partial_t\boldsymbol{u}$ in $L^2([t,t+1];\mathbb{V}^*)$, it is enough to observe that

$$\big\|B_0(\boldsymbol{u},\boldsymbol{u})\big\|_{\mathbb{V}^*}^2\leqslant c_\Omega|\boldsymbol{u}|^2\|\boldsymbol{u}\|_{\mathbb{V}}^2,\quad\forall\boldsymbol{u}\in\mathbb{V},$$

and $\nu A_0 \boldsymbol{u} \in L^2([t,t+1];\mathbb{V}^*)$. Besides, the following inequality holds (cf., e.g., [55]):

$$\left|\left(\boldsymbol{R}_0(\varepsilon A_N\phi,\phi),\boldsymbol{v}\right)\right| = \left|b_1(\boldsymbol{v},\phi,\varepsilon A_N\phi)\right| \leqslant c_9\|\boldsymbol{v}\||\phi|_{H^1}|A_N\phi|_{L^2}^{1/2}|\phi|_{H^3}^{1/2},$$

for all $\boldsymbol{v}\in\mathbb{V}$ and $\phi\in D(A_N)\cap H^3(\Omega)$. Therefore, we have

$$\left\|\boldsymbol{R}_0(\varepsilon A_N\phi,\phi)\right\|_{\mathbb{V}^*}^2 \leqslant c_{10}|\phi|_{H^1}^2|A_N\phi|_{L^2}|\phi|_{H^3}. \tag{3.14}$$

Hence, if $(\boldsymbol{u},\phi)$ satisfies (3.10) and (3.12), then $\boldsymbol{R}_0(\varepsilon A_N\phi,\phi)\in L^2([t,t+1];\mathbb{V}^*)$. Finally, from estimates (3.10) and (3.12)–(3.14), the integral control of $\partial_t\boldsymbol{u}$ in (3.3) is deduced by comparison from the first equation of (2.9). Summing up, we have completed the proof of (3.3). □

As a consequence, we also prove some bounds which will be useful to estimate the dimension of the global attractor in Section 4.

Proposition 3.2. *Let the assumptions of Proposition* 3.1 *hold. Then we have*

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^t\left\|\boldsymbol{u}(s)\right\|^2ds \leqslant \frac{\|\boldsymbol{g}\|_{\mathbb{V}^*}^2}{\nu^2}, \qquad \limsup_{t\to+\infty}\frac{1}{t}\int_0^t\left|\mu(s)\right|_{H^1}^2ds\leqslant\delta_2, \tag{3.15}$$

$$\limsup_{t\to+\infty}\left[\frac{1}{\mathcal{K}}\left|\boldsymbol{u}(t)\right|^2+\varepsilon\left(\left|\nabla\phi(t)\right|_{L^2}^2+\left\langle\phi(t)\right\rangle^2\right)\right]\leqslant\delta_1, \tag{3.16}$$

where

$$\delta_1 := \frac{2}{\nu\mathcal{K}\kappa_1}\|\boldsymbol{g}\|_{\mathbb{V}^*}^2+\frac{2c_1(M)}{\kappa_1}+\varepsilon M^2,$$
$$\delta_2 := (2\nu\mathcal{K})^{-1}\|\boldsymbol{g}\|_{\mathbb{V}^*}^2+\alpha^2c_f\left(1+\varepsilon^{-(m+1)}\delta_1^{m+1}\right),$$

with

$$\kappa_1=\min\left\{\nu/\left(2c_\Omega|\Omega|\right),\xi_1/2,\xi_1/\left(1+2c_f'\alpha\varepsilon^{-1}c_\Omega|\Omega|\right)\right\}, \quad \xi_1=\varepsilon/\left(c_\Omega|\Omega|\right)$$

and $c_1=c_1(M)$ *as in the proof of Proposition* 3.1. *In addition, we have*

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^t\left(\left|A_N\phi(s)\right|_{L^2}^2+\left\langle\phi(s)\right\rangle^2\right)ds\leqslant\delta_3, \tag{3.17}$$

where

$$\delta_3 := \varepsilon^{-2}\delta_2+\alpha^2\varepsilon^{-2}c_f+\alpha^2c_f''\varepsilon^{-(m+3)}\delta_1^{m+1}+M^2.$$

Proof. Integrating relation (3.1) over $(0,t)$ and employing the standard Hölder and Young inequalities, we get the energy inequality

$$\frac{1}{\mathcal{K}}\left|\boldsymbol{u}(t)\right|^2+2\mathcal{F}\left(\phi(t)\right)+\int_0^t\left(\frac{\nu}{\mathcal{K}}\left\|\boldsymbol{u}(s)\right\|^2+2\left|\nabla\mu(s)\right|_{L^2}^2\right)ds\leqslant\frac{1}{\mathcal{K}}\left|\boldsymbol{u}(0)\right|^2+2\mathcal{F}\left(\phi(0)\right)+\frac{\|\boldsymbol{g}\|_{\mathbb{V}^*}^2}{\nu\mathcal{K}}t, \quad \forall t\geqslant 0,$$

from which we deduce (3.16) and the first part of estimate (3.15). Moreover, we have

$$\limsup_{t\to+\infty}\frac{1}{t}\int_0^t\left|\nabla\mu(s)\right|_{L^2}^2ds\leqslant\frac{\|\boldsymbol{g}\|_{\mathbb{V}^*}^2}{2\nu\mathcal{K}}. \tag{3.18}$$

Using assumption (2.1) on the nonlinearity f, we readily see that

$$\int_0^t \langle\mu(s)\rangle^2\,ds = \alpha^2\int_0^t \langle f(\phi(s))\rangle^2\,ds$$
$$\leqslant \alpha^2 c_f\left(t+\varepsilon^{-(m+1)}\int_0^t \left(\varepsilon\left|\nabla\phi(s)\right|_{L^2}^2+\varepsilon\langle\phi(s)\rangle^2\right)^{m+1}ds\right).$$

Dividing both sides of the above inequality by t and employing estimate (3.16), the second part of estimate (3.15) is a straightforward consequence of (3.18).

Analogously, using the second equation of (2.9), we deduce

$$\int_0^t \left|A_N\phi(s)\right|_{L^2}^2 ds \leqslant \varepsilon^{-2}\int_0^t \left|\mu(s)\right|_{L^2}^2 ds + \alpha^2\varepsilon^{-2}c_f\int_0^t \left(1+\left|\phi(s)\right|_{L^{2m+2}}^{2m+2}\right)ds$$
$$\leqslant \varepsilon^{-2}\int_0^t \left|\mu(s)\right|_{L^2}^2 ds + \alpha^2\varepsilon^{-2}c_f t + \alpha^2\varepsilon^{-2}c_f''\varepsilon^{-(m+1)}\int_0^t \left(\varepsilon\left|\nabla\phi(s)\right|_{L^2}^2+\varepsilon\langle\phi(s)\rangle^2\right)^{m+1}ds,$$

for some positive constant c_f'' depending on c_f. Here we have also used the fact that $H^1(\Omega)\hookrightarrow L^{2m+2}(\Omega)$, for any arbitrary m. Dividing both sides of the above inequality by t and employing estimates (3.15)–(3.16) once again and the Hölder inequality, we infer (3.17). □

Proposition 3.1 is the basic ingredient to establish the existence of a solution to **P** by means of a Faedo–Galerkin approach (see, e.g., [9]). Instead, uniqueness of weak solutions and their time continuity are consequences of the following lemma.

Lemma 3.3. *Let the assumptions of Proposition* 3.1 *hold. Let* $(\boldsymbol{u}_i,\phi_i)$ *be the solution corresponding to the initial data* $(\boldsymbol{u}_i(0),\phi_i(0))\in\mathbb{Y}_M$, $i=1,2$. *Then, for any* $t\geqslant 0$, *the following estimate holds*:

$$\left\|\left((\boldsymbol{u}_1-\boldsymbol{u}_2)(t),(\phi_1-\phi_2)(t)\right)\right\|_{\mathbb{Y}_M}^2+\int_0^t\left[\nu\left\|(\boldsymbol{u}_1-\boldsymbol{u}_1)(s)\right\|^2+\varepsilon^2\left|(\phi_1-\phi_2)(s)\right|_{H^2}^2\right]ds$$
$$\leqslant Ce^{Lt}\left\|\left((\boldsymbol{u}_1-\boldsymbol{u}_2)(0),(\phi_1-\phi_2)(0)\right)\right\|_{\mathbb{Y}_M}^2, \tag{3.19}$$

where C *and* L *are positive constants depending only on the norms of the initial data in* $\mathbb{Y}_M$, *on* Ω *and on the parameters of the problem, but are both independent of time.*

Proof. Let us first set $\psi:=\phi_1-\phi_2$, $\boldsymbol{w}:=\boldsymbol{u}_1-\boldsymbol{u}_2$. Also, let us introduce the function $\overline{\mu}:=\widetilde{\mu}-\langle\widetilde{\mu}\rangle$, where

$$\widetilde{\mu}(t)=\varepsilon A_N\psi(t)-\alpha\left[f(\phi_2(t))-f(\phi_1(t))\right]$$

and note that $\langle\overline{\mu}(t)\rangle=0$ and $\langle\partial_t\psi(t)\rangle=0$, due (2.4). We also have

$$\langle\psi(t)\rangle=\langle\phi_1(0)\rangle-\langle\phi_2(0)\rangle=:M_{1,2}.$$

However, in general $\langle\phi_1(0)\rangle\neq\langle\phi_2(0)\rangle$. To this end, we introduce a new function $\overline{\psi}(t)=\psi(t)-M_{1,2}$ so that $\langle\overline{\psi}(t)\rangle=0$, by definition. Then we easily realize that $(\boldsymbol{w},\overline{\psi})$ solves the system

$$\begin{cases}\partial_t\boldsymbol{w}+\nu A_0\boldsymbol{w}=B_0(\boldsymbol{u}_2,\boldsymbol{u}_2)-B_0(\boldsymbol{u}_1,\boldsymbol{u}_1)-\mathcal{K}\boldsymbol{R}_0(\varepsilon A_N\phi_2,\phi_2)+\mathcal{K}\boldsymbol{R}_0(\varepsilon A_N\phi_1,\phi_1),\\ \overline{\mu}=\varepsilon A_N\overline{\psi}-\alpha\left[f(\phi_2)-f(\phi_1)\right]-\langle\widetilde{\mu}\rangle,\\ \partial_t\psi+A_N\overline{\mu}=B_1(\boldsymbol{u}_2,\phi_2)-B_1(\boldsymbol{u}_1,\phi_1),\end{cases}$$

which we rewrite, using the properties of the bilinear forms B_0, B_1 and $\boldsymbol{R}_0$, as

$$\begin{cases}\partial_t\boldsymbol{w}+\nu A_0\boldsymbol{w}=-\left(B_0(\boldsymbol{w},\boldsymbol{u}_1)+B_0(\boldsymbol{u}_2,\boldsymbol{w})\right)+\mathcal{K}\left(\boldsymbol{R}_0(\varepsilon A_N\phi_2,\overline{\psi})-\boldsymbol{R}_0(\varepsilon A_N\overline{\psi},\phi_1)\right),\\ \overline{\mu}=\varepsilon A_N\overline{\psi}-\alpha\left[f(\phi_2)-f(\phi_1)\right]-\langle\widetilde{\mu}\rangle,\\ \partial_t\overline{\psi}+A_N\overline{\mu}=-\left(B_1(\boldsymbol{w},\phi_1)+B_1(\boldsymbol{u}_2,\overline{\psi})\right).\end{cases}\tag{3.20}$$

Take $\boldsymbol{w}(t)$ as a test function in the first equation of (3.20). Then, take the duality coupling of the second and third equations of (3.20) with $A_N\overline{\mu}(t)+\varepsilon\zeta A_N\overline{\psi}(t)$ (with $\zeta>0$ sufficiently small to be selected in the sequel) and $\varepsilon A_N\overline{\psi}(t)$, respectively. On account of the orthogonality properties of b_0 and b_1, we add the resulting equations and we deduce the identity

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\mathcal{Y}_1(t)+\nu\|\boldsymbol{w}(t)\|^2+\big|\nabla\overline{\mu}(t)\big|_{L^2}^2+\varepsilon^2\zeta\big|A_N\overline{\psi}(t)\big|_{L^2}^2\\
&\quad=-b_0(\boldsymbol{w},\boldsymbol{u}_1,\boldsymbol{w})+\mathcal{K}\big(\boldsymbol{R}_0(\varepsilon A_N\phi_2,\overline{\psi}),\boldsymbol{w}\big)-\mathcal{K}\big(\boldsymbol{R}_0(\varepsilon A_N\overline{\psi},\phi_1),\boldsymbol{w}\big)\\
&\qquad-b_1(\boldsymbol{w},\phi_1,\varepsilon A_N\overline{\psi})-b_1(\boldsymbol{u}_2,\psi,\varepsilon A_N\overline{\psi})+\zeta(\overline{\mu},\varepsilon A_N\overline{\psi})_{L^2}\\
&\qquad+\alpha\zeta\varepsilon\big(f(\phi_1)-f(\phi_2),A_N\overline{\psi}\big)_{L^2}-\alpha\big(f(\phi_1)-f(\phi_2),A_N\overline{\mu}\big)_{L^2},
\end{aligned}\tag{3.21}$$

where

$$\mathcal{Y}_1(t):=\big|\boldsymbol{w}(t)\big|^2+\varepsilon\big|\nabla\overline{\psi}(t)\big|^2.$$

Before we proceed with estimating all the terms on the right-hand side of (3.21). From now on, throughout the paper, c will denote a generic positive constant (depending only on ν, ε, $\mathcal{K}$, α, Ω, M) which can take different values, sometimes even within the same line. This constant is independent of time and initial data. Using [51, Proposition 9.2, (9.26)–(9.27)] and suitable Young inequalities, we estimate the first, fourth and fifth terms on the right-hand side of (3.21), as follows:

$$\begin{aligned}
\big|b_0(\boldsymbol{w},\boldsymbol{u}_1,\boldsymbol{w})\big|&\leqslant c|\boldsymbol{w}|^{1/2}\|\boldsymbol{w}\|^{1/2}\|\boldsymbol{u}_1\|^{1/2}|\boldsymbol{w}|^{1/2}\|\boldsymbol{w}\|^{1/2}\\
&\leqslant\frac{\nu}{4}\|\boldsymbol{w}\|^2+c\|\boldsymbol{u}_1\|^2|\boldsymbol{w}|^2.
\end{aligned}\tag{3.22}$$

Similarly, we have

$$\begin{aligned}
\big|b_1(\boldsymbol{w},\phi_1,\varepsilon A_N\overline{\psi})\big|&\leqslant c|\boldsymbol{w}|^{1/2}\|\boldsymbol{w}\|^{1/2}|\phi_1|_{H^1}^{1/2}|\phi_1|_{H^2}^{1/2}|A_N\overline{\psi}|_{L^2}\\
&\leqslant c|\boldsymbol{w}|\|\boldsymbol{w}\||\phi_1|_{H^1}|\phi_1|_{H^2}+\frac{\varepsilon^2\zeta}{16}|A_N\overline{\psi}|_{L^2}^2\\
&\leqslant\frac{1}{4}\left(\nu\|\boldsymbol{w}\|^2+\frac{\varepsilon^2\zeta}{4}|A_N\overline{\psi}|_{L^2}^2\right)+c_\zeta|\phi_1|_{H^1}^2|\phi_1|_{H^2}^2|\boldsymbol{w}|^2
\end{aligned}\tag{3.23}$$

and

$$\begin{aligned}
\big|b_1(\boldsymbol{u}_2,\psi,\varepsilon A_N\overline{\psi})\big|&\leqslant c|\boldsymbol{u}_2|^{1/2}\|\boldsymbol{u}_2\|^{1/2}|\overline{\psi}|_{H^1}^{1/2}|A_N\overline{\psi}|_{L^2}^{1/2}|A_N\overline{\psi}|_{L^2}\\
&=c|\boldsymbol{u}_2|^{1/2}\|\boldsymbol{u}_2\|^{1/2}|\overline{\psi}|_{H^1}^{1/2}|A_N\overline{\psi}|_{L^2}^{3/2}\\
&\leqslant c|\boldsymbol{u}_2|^2\|\boldsymbol{u}_2\|^2\big(\varepsilon|\nabla\overline{\psi}|_{L^2}^2+\varepsilon\langle\overline{\psi}\rangle^2\big)+\frac{\varepsilon^2\zeta}{16}|A_N\overline{\psi}|_{L^2}^2,
\end{aligned}\tag{3.24}$$

where, in estimating (3.24), we have used the Young inequality with exponents 4 and 4/3. Regarding the last two terms in (3.21), employing the standard Hölder and Sobolev inequalities, we obtain

$$\begin{aligned}
\alpha\big|\big(f(\phi_1)-f(\phi_2),A_N\overline{\mu}\big)_{L^2}\big|&=\alpha\big|\big(\nabla\big(f(\phi_1)-f(\phi_2)\big),\nabla\overline{\mu}\big)_{L^2}\big|\\
&\leqslant Q\big(|\phi_1|_{H^1}+|\phi_2|_{H^1}\big)\big(|\phi_1|_{H^2}^2+|\phi_2|_{H^2}^2\big)\big(\varepsilon|\nabla\overline{\psi}|_{L^2}^2+M_{1,2}^2\big)+\frac{1}{2}|\nabla\overline{\mu}|_{L^2}^2
\end{aligned}\tag{3.25}$$

and

$$\alpha\zeta\varepsilon\big|\big(f(\phi_1)-f(\phi_2),A_N\overline{\psi}\big)_{L^2}\big|\leqslant Q_\zeta\big(|\phi_1|_{H^1}+|\phi_2|_{H^1}\big)\big(\varepsilon|\nabla\overline{\psi}|_{L^2}^2+M_{1,2}^2\big)+\frac{\varepsilon^2\zeta}{16}|A_N\overline{\psi}|_{L^2}^2,$$

for suitable monotone non-decreasing functions Q, Q_ζ independent of time, which clearly depend on ε and α. Besides, we have

$$\begin{aligned}
\mathcal{K}\big|\big(\boldsymbol{R}_0(\varepsilon A_N\phi_2,\overline{\psi}),\boldsymbol{w}\big)\big| &\leqslant c|\boldsymbol{w}|^{1/2}\|\boldsymbol{w}\|^{1/2}|\overline{\psi}|_{H^1}^{1/2}|A_N\overline{\psi}|_{L^2}^{1/2}|\phi_2|_{H^2}\\
&\leqslant c|\boldsymbol{w}|^{2/3}\|\boldsymbol{w}\|^{2/3}|\overline{\psi}|_{H^1}^{2/3}|\phi_2|_{H^2}^{4/3}+\frac{\varepsilon^2\zeta}{8}|A_N\overline{\psi}|_{L^2}^2\\
&\leqslant \frac{1}{4}\left(\nu\|\boldsymbol{w}\|^2+\frac{\varepsilon^2\zeta}{8}|A_N\overline{\psi}|_{L^2}^2\right)+c|\boldsymbol{w}||\overline{\psi}|_{H^1}|\phi_2|_{H^2}^2\\
&\leqslant \frac{1}{4}\left(\nu\|\boldsymbol{w}\|^2+\frac{\varepsilon^2\zeta}{8}|A_N\overline{\psi}|_{L^2}^2\right)+c\big(|\boldsymbol{w}|^2+\varepsilon|\nabla\overline{\psi}|_{L^2}^2\big)|\phi_2|_{H^2}^2
\end{aligned}\tag{3.26}$$

and

$$\begin{aligned}
\mathcal{K}\big|\big(\boldsymbol{R}_0(\varepsilon A_N\overline{\psi},\phi_1),\boldsymbol{w}\big)\big| &\leqslant c|\boldsymbol{w}|^{1/2}\|\boldsymbol{w}\|^{1/2}|\phi_1|_{H^1}^{1/2}|\phi_1|_{H^2}^{1/2}|A_N\overline{\psi}|_{L^2}\\
&\leqslant \frac{\nu}{8}\|\boldsymbol{w}\|^2+\frac{\varepsilon^2\zeta}{32}|A_N\overline{\psi}|_{L^2}^2+c|\boldsymbol{w}|^2|\phi_1|_{H^1}^2|\phi_1|_{H^2}^2\\
&\leqslant \frac{1}{4}\left(\frac{\nu}{2}\|\boldsymbol{w}\|^2+\frac{\varepsilon^2\zeta}{8}|A_N\overline{\psi}|_{L^2}^2\right)+c|\boldsymbol{w}|^2|\phi_1|_{H^1}^2|\phi_1|_{H^2}^2.
\end{aligned}\tag{3.27}$$

Finally, we estimate the remaining term in (3.21) as follows:

$$\zeta(\overline{\mu},\varepsilon A_N\overline{\psi})_{L^2}\leqslant\frac{\zeta C_\Omega}{2}|\nabla\overline{\mu}|_{L^2}^2+\frac{\varepsilon^2\zeta}{2}|A_N\overline{\psi}|_{L^2}^2.$$

Inserting the above estimates into the right-hand side of (3.21), we obtain

$$\frac{d}{dt}\mathcal{Y}_1(t)+\frac{\nu}{4}\big\|\boldsymbol{w}(t)\big\|^2+(1-\zeta C_\Omega)\big|\nabla\overline{\mu}(t)\big|_{L^2}^2+\frac{\varepsilon^2\zeta}{2}\big|A_N\overline{\psi}(t)\big|_{L^2}^2\leqslant\mathcal{J}_1(t)\mathcal{Y}(t)+M_{1,2}^2\mathcal{J}_2(t),\tag{3.28}$$

where

$$\begin{aligned}
\mathcal{J}_1(t):={}&c\big(\big\|\boldsymbol{u}_1(t)\big\|^2+\big(1+\big|\phi_1(t)\big|_{H^1}^2\big)\big|\phi_1(t)\big|_{H^2}^2+\big|\boldsymbol{u}_2(t)\big|^2\big\|\boldsymbol{u}_2(t)\big\|^2\big)\\
&+Q\big(\big|\phi_1(t)\big|_{H^1}+\big|\phi_2(t)\big|_{H^1}\big)\big(1+\big|\phi_1(t)\big|_{H^2}^2+\big|\phi_2(t)\big|_{H^2}^2\big)
\end{aligned}$$

and

$$\mathcal{J}_2(t):=Q\big(\big|\phi_1(t)\big|_{H^1}+\big|\phi_2(t)\big|_{H^1}\big)\big(1+\big|\phi_1(t)\big|_{H^2}^2+\big|\phi_2(t)\big|_{H^2}^2\big).$$

Obviously, (3.28) implies that

$$\mathcal{Y}_1(t)\leqslant\mathcal{J}_3(t)+\int_0^t\mathcal{J}_1(s)\mathcal{Y}_1(s)\,ds,$$

where

$$\mathcal{J}_3(t):=\mathcal{Y}_1(0)+M_{1,2}^2\int_0^t\mathcal{J}_2(s)\,ds.$$

From (3.3), it is readily seen that, for $i=1,2$,

$$\sup_{t\geqslant0}\int_t^{t+1}\mathcal{J}_i(s)\,ds\leqslant Q\big(\big\|\big(\boldsymbol{u}_1(0),\phi_1(0)\big)\big\|_{\mathbb{Y}_M}+\big\|\big(\boldsymbol{u}_2(0),\phi_2(0)\big)\big\|_{\mathbb{Y}_M}\big)+c.\tag{3.29}$$

Thus, exploiting a suitable version of the Gronwall inequality, and choosing ζ sufficiently small in (3.28), we deduce the following inequality:

$$\mathcal{Y}_1(t)+\int_0^t\big[\nu\big\|(\boldsymbol{u}_1-\boldsymbol{u}_1)(s)\big\|^2+\varepsilon^2\big|A_N\overline{\psi}(s)\big|_{L^2}^2\big]\,ds\leqslant\mathcal{J}_3(t)+\int_0^t\mathcal{J}_3(s)\mathcal{J}_1(s)\exp\left(\int_s^t\mathcal{J}_1(\tau)\,d\tau\right)ds,\quad\forall t\geqslant0.$$

Finally, employing both estimates of (3.29) and the obvious inequalities

$$\langle\psi(t)\rangle^2 = M_{1,2}^2 \leqslant c\big\|\big(\boldsymbol{w}(0), \psi(0)\big)\big\|_{\mathbb{Y}_M}^2, \qquad \mathcal{J}_3(t) \leqslant Ce^{Ct}\big\|\big(\boldsymbol{w}(0), \psi(0)\big)\big\|_{\mathbb{Y}_M}^2, \tag{3.30}$$

for all $t \geqslant 0$, the claim (3.19) follows immediately from (3.30). We recall that C is a positive constant that depends on the norm of the initial data in $\mathbb{Y}_M$, but is independent of t. The proof of lemma is now complete. □

Remark 3.4. Note that we can control the terms $\boldsymbol{R}_0(A_N\overline{\psi}, \phi_1)$ and $\boldsymbol{R}_0(\varepsilon A_N\phi_2, \overline{\psi})$ in the first equation of (3.20), thanks to the assumption $f \in C^2(\mathbb{R})$.

Thanks to Lemma 3.3 we can now state

Theorem 3.5. *Let the assumptions of Proposition* 3.1 *hold. Then* **P** *defines a* (*nonlinear*) *strongly continuous semigroup*

$$\mathcal{S}(t): \mathbb{Y}_M \to \mathbb{Y}_M, \tag{3.31}$$

by setting, for all $t \geqslant 0$,

$$\mathcal{S}(t)(\boldsymbol{u}_0, \phi_0) = \big(\boldsymbol{u}(t), \phi(t)\big), \tag{3.32}$$

where $(\boldsymbol{u}, \phi)$ is the unique solution to Problem **P**.

Besides, Proposition 3.1 yields

Proposition 3.6. *Let the assumptions of Proposition* 3.1 *hold. Then $\mathcal{S}(t)$ has a $\mathbb{Y}_M$-bounded absorbing set. For instance*

$$\mathbb{B} := \big\{(\boldsymbol{u}, \phi) \in \mathbb{Y}_M\colon \ \big\|(\boldsymbol{u}, \phi)\big\|_{\mathbb{Y}_M} \leqslant (C_0 + 1)^{1/2}\big\},$$

where C_0 is the positive constant in (3.3)*, is an absorbing set for $\mathcal{S}(t)$. This means that, for any bounded set $\mathcal{B}$ in $\mathbb{Y}_M$, there exists $t_0 = t_0(\mathcal{B}) > 0$ for which*

$$\sup_{(\boldsymbol{u}_0, \phi_0)\in\mathcal{B}} \big\|\mathcal{S}(t)(\boldsymbol{u}_0, \phi_0)\big\|_{\mathbb{Y}_M}^2 \leqslant C_0 + 1, \quad \forall t \geqslant t_0. \tag{3.33}$$

3.2. Existence of compact absorbing sets

In this subsection, we prove that our dynamical system has absorbing sets which are compact in the phase-space. These results will entail the existence of the global attractor (see next subsection).

Lemma 3.7. *Let $\boldsymbol{g} \in \mathbb{H}$ and $f \in C^2(\mathbb{R})$ satisfy* (2.1). *Then there is a positive constant C_1, only depending on the physical parameters, such that for any $\mathbb{Y}_M$-bounded set $\mathcal{B}$, there exists $t_1 = t_1(\mathcal{B}) > 0$ such that*

$$\sup_{(\boldsymbol{u}_0, \phi_0)\in\mathcal{B}} \big\|\mathcal{S}(t)(\boldsymbol{u}_0, \phi_0)\big\|_{\mathbb{V}\times H^2(\Omega)} \leqslant C_1, \quad \forall t \geqslant t_1. \tag{3.34}$$

Proof. The following estimates will be deduced by a formal argument as before. However, even in this case, they can be rigorously justified taking advantage once more of a standard approximation procedure (see [9]). We recall that c denotes a generic positive constant which is independent of time and of the initial data. This constant may vary even within the same line.

First we introduce the functions $\overline{\phi}(t) = \phi(t) - M_0$ and $\overline{\mu}(t) = \mu(t) - \langle\mu(t)\rangle$ (with $\overline{\mu}$ given by Remark 2.2) and observe that $\langle\overline{\phi}(t)\rangle = \langle\overline{\mu}(t)\rangle = 0$. Let us now take the inner product of the first equation of (2.9) in $\mathbb{H}$ with $2A_0\boldsymbol{u}(t)$ (recall that we can do that within a suitable Galerkin discretization scheme, see, e.g., [9]). Then, we take the inner product of both the second and third equations of (2.9) in $L^2(\Omega)$ with $2B_N^2\overline{\mu}(t) + 2\eta B_N^3\overline{\phi}(t)$ ($\eta > 0$ is a small parameter to be chosen later) and $2\varepsilon B_N^2\overline{\phi}(t)$, respectively. Adding up the resulting relationships, we obtain

$$\begin{aligned}
&\frac{d}{dt}\mathcal{Y}_2(t) + 2\nu\left|A_0\boldsymbol{u}(t)\right|^2 + 2\eta\varepsilon\left|B_N^2\overline{\phi}(t)\right|_{L^2}^2 + 2\left|B_N\overline{\mu}(t)\right|_{L^2}^2 \\
&\quad = 2\mathcal{K}\big(\boldsymbol{R}_0\big(\varepsilon A_N\phi(t), \phi(t)\big), A_0\boldsymbol{u}(t)\big) - 2\big(B_0\big(\boldsymbol{u}(t), \boldsymbol{u}(t)\big), A_0\boldsymbol{u}(t)\big) + \big(\boldsymbol{g}, A_0\boldsymbol{u}(t)\big) \\
&\qquad - 2\alpha\big(A_N\big(f\big(\phi(t)\big) - \big\langle f\big(\phi(t)\big)\big\rangle\big), B_N\overline{\mu}(t)\big)_{L^2} - 2\big(B_1\big(\boldsymbol{u}(t), \phi(t)\big), B_N^2\overline{\phi}(t)\big)_{L^2} \\
&\qquad + 2\eta\big(B_N\overline{\mu}(t), B_N^2\overline{\phi}(t)\big) + 2\alpha\eta\big(B_N\big(f\big(\phi(t)\big) - \big\langle f\big(\phi(t)\big)\big\rangle\big), B_N^2\overline{\phi}(t)\big)_{L^2},
\end{aligned} \tag{3.35}$$

where

$$\mathcal{Y}_2(t) := \left\|\boldsymbol{u}(t)\right\|^2 + \varepsilon\left|B_N\overline{\phi}(t)\right|_{L^2}^2, \quad \forall t \geqslant t_0. \tag{3.36}$$

We begin by estimating all the terms on the right-hand side of (3.35). Using the Agmon inequality in two dimensions and the Young inequality (with exponents $(4, 4/3)$ and $(3/2, 3)$, respectively), we obtain

$$\begin{aligned}
2\mathcal{K}\left|\big(\boldsymbol{R}_0(\varepsilon A_N\phi, \phi), A_0\boldsymbol{u}\big)\right| &= 2\mathcal{K}\left|\big(\boldsymbol{R}_0(\varepsilon B_N\overline{\phi}, \overline{\phi}), A_0\boldsymbol{u}\big)\right| \\
&\leqslant 2\mathcal{K}\varepsilon\left|\boldsymbol{R}_0(B_N\overline{\phi}, \overline{\phi})\right||A_0\boldsymbol{u}| \\
&\leqslant 2\mathcal{K}\varepsilon|B_N\overline{\phi}|_{L^2}|\nabla\overline{\phi}|_{L^\infty}|A_0\boldsymbol{u}| \\
&\leqslant c\varepsilon|B_N\overline{\phi}|_{L^2}|\nabla\overline{\phi}|_{L^2}^{1/2}\left|B_N^{3/2}\overline{\phi}\right|_{L^2}^{1/2}|A_0\boldsymbol{u}| \\
&\leqslant c|B_N\overline{\phi}|_{L^2}^{4/3}|\nabla\overline{\phi}|_{L^2}^{2/3}|A_0\boldsymbol{u}|^{4/3} + \frac{\eta\varepsilon}{2}\left|B_N^2\overline{\phi}\right|_{L^2}^2 \\
&\leqslant c|B_N\overline{\phi}|_{L^2}^4|\nabla\overline{\phi}|_{L^2}^2 + \frac{\nu}{2}|A_0\boldsymbol{u}|^2 + \frac{\eta\varepsilon}{2}\left|B_N^2\overline{\phi}\right|_{L^2}^2 \\
&= c|B_N\overline{\phi}|_{L^2}^2|\nabla\overline{\phi}|_{L^2}^2|B_N\overline{\phi}|_{L^2}^2 + \frac{\nu}{2}|A_0\boldsymbol{u}|^2 + \frac{\eta\varepsilon}{2}\left|B_N^2\overline{\phi}\right|_{L^2}^2.
\end{aligned} \tag{3.37}$$

By the continuity properties of B_0, we also get

$$\begin{aligned}
2\left|\big(B_0(\boldsymbol{u}, \boldsymbol{u}), A_0\boldsymbol{u}\big)\right| &\leqslant 2\left|B_0(\boldsymbol{u}, \boldsymbol{u})\right||A_0\boldsymbol{u}| \leqslant c|\boldsymbol{u}|^{1/2}\|\boldsymbol{u}\||A_0\boldsymbol{u}|^{1/2}|A_0\boldsymbol{u}| \\
&\leqslant c|\boldsymbol{u}|^2\|\boldsymbol{u}\|^2\|\boldsymbol{u}\|^2 + \frac{\nu}{2}|A_0\boldsymbol{u}|^2,
\end{aligned}$$

where we have employed the Young inequality with exponents $4/3$ and 4. Moreover, we have

$$\begin{aligned}
2\left|\big(B_1(\boldsymbol{u}, \phi), B_N^2\overline{\phi}\big)_{L^2}\right| &\leqslant 2\left|B_1(\boldsymbol{u}, \overline{\phi})\right|_{L^2}\left|B_N^2\overline{\phi}\right|_{L^2} \\
&\leqslant \frac{\eta\varepsilon}{2}\left|B_N^2\overline{\phi}\right|_{L^2}^2 + c|u|\|u\||\nabla\overline{\phi}|_{L^2}|B_N\overline{\phi}|_{L^2} \\
&\leqslant \frac{\eta\varepsilon}{2}\left|B_N^2\overline{\phi}\right|_{L^2}^2 + c\big(|u|^2\|u\|^2 + |\nabla\overline{\phi}|_{L^2}^2|B_N\overline{\phi}|_{L^2}^2\big)
\end{aligned}$$

and

$$2\eta\left|\big(B_N\overline{\mu}, B_N^2\overline{\phi}\big)_{L^2}\right| \leqslant \frac{\eta\varepsilon}{4}\left|B_N^2\overline{\phi}\right|_{L^2}^2 + 4\varepsilon^{-1}\eta|B_N\overline{\mu}|_{L^2}^2.$$

Then, using the Hölder, Young and Sobolev inequalities, we obtain (cf. also (2.1))

$$\begin{aligned}
&2\alpha\left|\big(B_N\big(f(\phi) - \langle f(\phi)\rangle\big), B_N\overline{\mu}\big)_{L^2}\right| \\
&\quad \leqslant \alpha\eta|B_N\overline{\mu}|_{L^2}^2 + c\left|B_N\big(f(\phi) - \langle f(\phi)\rangle\big)\right|_{L^2}^2 \\
&\quad \leqslant \alpha\eta|B_N\overline{\mu}|_{L^2}^2 + c\big(\left|f''(\phi)|\nabla\overline{\phi}|^2\right|_{L^2}^2 + \left|f'(\phi)B_N\overline{\phi}\right|_{L^2}^2\big) \\
&\quad \leqslant \alpha\eta|B_N\overline{\mu}|_{L^2}^2 + Q\big(|\phi|_{H^1}\big)|B_N\overline{\phi}|_{L^2}^2\big(1 + |B_N\overline{\phi}|_{L^2}^2\big) + \frac{\eta\varepsilon}{4}\left|B_N^{3/2}\overline{\phi}\right|_{L^2}^2,
\end{aligned} \tag{3.38}$$

for some monotone non-decreasing function Q, which is independent of time and of initial data. Finally, arguing exactly as in (3.38), we also have that

$$2\alpha\eta\left|\big(B_N\big(f(\phi) - \langle f(\phi)\rangle\big), B_N^2\overline{\phi}\big)_{L^2}\right| \leqslant \frac{\eta\varepsilon}{4}\left|B_N^2\overline{\phi}\right|_{L^2}^2 + Q\big(|\phi|_{H^1}\big)|B_N\overline{\phi}|_{L^2}^2\big(1 + |B_N\overline{\phi}|_{L^2}^2\big). \tag{3.39}$$

Collecting now all estimates (3.37)–(3.39), using them to estimate the right-hand side of (3.35) and observing that (3.33) also holds, after standard transformations, we obtain that

$$\frac{d}{dt}\mathcal{Y}_2(t)+\nu\left|A_0\boldsymbol{u}(t)\right|^2+\frac{\eta\varepsilon}{4}\left|B_N^2\overline{\phi}(t)\right|_{L^2}^2+\left[2-\left(4\varepsilon^{-1}+\alpha\right)\eta\right]\left|B_N\overline{\mu}(t)\right|_{L^2}^2$$
$$\leqslant \mathcal{G}_1(t)\mathcal{Y}(t)+\mathcal{G}_2(t), \quad \forall t\geqslant t_0, \tag{3.40}$$

where

$$\mathcal{G}_1(t):=c\left|\boldsymbol{u}(t)\right|^2\left\|\boldsymbol{u}(t)\right\|^2+c\left|\nabla\overline{\phi}(t)\right|_{L^2}^2\left|B_N\overline{\phi}(t)\right|_{L^2}^2+Q\left(\left|\phi(t)\right|_{H^1}\right)\left(1+\left|B_N\overline{\phi}(t)\right|_{L^2}^2\right),$$
$$\mathcal{G}_2(t):=\nu^{-1}|\boldsymbol{g}|^2+c\left(1+\left|u(t)\right|^2+\left|\nabla\overline{\phi}(t)\right|_{L^2}^2\right).$$

Choosing $\eta:=1/(4\varepsilon^{-1}+\alpha)$ in (3.40), recalling that $\overline{\phi}(t)=\phi(t)-M_0$ and $\overline{\mu}(t)=\mu(t)-\langle\mu(t)\rangle$, and exploiting (3.3), it is easy to check that there exist positive constants a_i, $i=1,2,3$ (independent of time and initial data) such that

$$\sup_{t\geqslant t_0}\int_t^{t+1}\mathcal{Y}_2(s)\,ds\leqslant\alpha_1,\qquad \sup_{t\geqslant t_0}\int_t^{t+1}\mathcal{G}_1(s)\,ds\leqslant\alpha_2,\qquad \sup_{t\geqslant t_0}\int_t^{t+1}\mathcal{G}_2(s)\,ds\leqslant\alpha_3. \tag{3.41}$$

From (3.40)–(3.41), owing to the uniform Gronwall lemma (see, e.g., [55, Chap. III, Lemma 1.1]), we conclude that

$$\mathcal{Y}_2(t+1)\leqslant(\alpha_3+\alpha_2)e^{\alpha_1}, \quad \forall t\geqslant t_0,$$

which entails, for all $t\geqslant t_1=t_0+1$,

$$\left\|\boldsymbol{u}(t)\right\|^2+\varepsilon\left|B_N\overline{\phi}(t)\right|_{L^2}^2+\int_t^{t+1}\nu\left|A_0\boldsymbol{u}(s)\right|^2ds+\int_t^{t+1}\left(\left|B_N\overline{\mu}(s)\right|_{L^2}^2+\varepsilon\left|B_N^2\overline{\phi}(t)\right|_{L^2}^2\right)ds\leqslant c. \tag{3.42}$$

The claim (3.34) follows from (3.33) and (3.42). The proof is finished. □

Remark 3.8. Observe that, thanks to (3.34) and to the embedding $H^2(\Omega)\hookrightarrow L^\infty(\Omega)$, we have

$$\left|\phi(t)\right|_{L^\infty}\leqslant c, \quad \forall t\geqslant t_1. \tag{3.43}$$

We can also prove (see [26] or [51, Chap. 12] for the Navier–Stokes equations)

Proposition 3.9. *Let the assumptions of Lemma* 3.7 *hold. Assume Γ is of class C^4. Then there exists a $\mathbb{H}^2(\Omega)\times H^4(\Omega)$-bounded absorbing set, for the semigroup $\mathcal{S}(t)$. More precisely, there exist a time $t_2\geqslant t_1$ and a positive constant C_2 such that*

$$\left|A_0\boldsymbol{u}(t)\right|^2+\left|\phi(t)\right|_{H^4}^2+\left|\mu(t)\right|_{D(A_N)}^2\leqslant C_2, \quad \forall t\geqslant t_2. \tag{3.44}$$

Proof. First, observe that, from (3.3) and (3.42), we also have that

$$\sup_{t\geqslant t_1}\int_t^{t+1}\left(\left|\mu(s)\right|_{H^2}^2+\varepsilon\left|\phi(t)\right|_{H^4}^2\right)ds\leqslant c, \tag{3.45}$$

which yields, using (3.42) once more and (3.43), and arguing exactly as in [51, Proposition 12.4, (12.9)–(12.10)],

$$\sup_{t\geqslant t_1}\left[\left|\mu(t)\right|_{L^2}^2+\int_t^{t+1}\left|\partial_t\boldsymbol{u}(s)\right|^2ds\right]\leqslant c. \tag{3.46}$$

From (2.9), we have that

$$\partial_t\phi(t)=-B_N\left(\mu(t)-\left\langle\mu(t)\right\rangle\right)-B_1\left(\boldsymbol{u}(t),\phi(t)\right), \quad \text{a.e. in } \Omega\times(t_1,+\infty). \tag{3.47}$$

Using known properties of the bilinear form B_1 (see, e.g., [51, p. 243]), the Hölder and Ladyzhenskaya inequalities, we have

$$\begin{aligned}\left|B_1(\boldsymbol{u},\phi)\right|_{L^2}^2 &\leqslant \sum_{i=1}^{2}\left(|\boldsymbol{u}_i|_{L^4}^2|\partial_{x_i}\phi|_{L^4}^2\right)\\ &\leqslant c_\Omega\sum_{i=1}^{2}\left(|\boldsymbol{u}_i|_{L^2}|\boldsymbol{u}_i|_{H^1}|\partial_{x_i}\phi|_{L^2}|\partial_{x_i}\phi|_{H^1}\right)\\ &\leqslant c_\Omega|\boldsymbol{u}|\|\boldsymbol{u}\||\phi|_{H^1}|\phi|_{H^2},\end{aligned}\tag{3.48}$$

where $c_\Omega > 0$ depends only on Ω. Then, using estimates (3.34), (3.45), (3.48), and recalling that $\langle\partial_t\phi(t)\rangle = 0$, it is not difficult to realize that

$$\sup_{t\geqslant t_1}\left|B_N^{-1}\partial_t\phi(t)\right|_{L^2}^2+\sup_{t\geqslant t_1}\int_t^{t+1}\left|\partial_t\phi(s)\right|_{L^2}^2\,ds\leqslant c.\tag{3.49}$$

To prove (3.44), we need to differentiate all the equations of (2.9) with respect to time. Taking the inner products of the resulting equations in $\mathbb{H}$ and $L^2(\Omega)$ with $2\partial_t\boldsymbol{u}(t)$, $2B_N\partial_t\phi(t)$ and $2\partial_t\phi(t)$, respectively, and adding the resulting relations, after standard transformations (i.e., orthogonality properties of the trilinear forms b_0, b_1 and the fact that $\langle\partial_t\phi(t)\rangle = 0$), we infer that

$$\frac{d}{dt}\mathcal{Y}_3(t)+2\nu\left\|\partial_t\boldsymbol{u}(t)\right\|^2+2\varepsilon\left|B_N\partial_t\phi(t)\right|_{L^2}^2=\Lambda_1(t),\quad \forall t\geqslant t_1,\tag{3.50}$$

where

$$\mathcal{Y}_3(t):=\left|\partial_t\boldsymbol{u}(t)\right|^2+\left|\partial_t\phi(t)\right|_{L^2}^2$$

and

$$\begin{aligned}\Lambda_1(t):=&-2b_0\big(\partial_t\boldsymbol{u}(t),\boldsymbol{u}(t),\partial_t\boldsymbol{u}(t)\big)-2b_1\big(\partial_t\boldsymbol{u}(t),\phi(t),\partial_t\phi(t)\big)\\ &+2\mathcal{K}\big(\boldsymbol{R}_0\big(\varepsilon B_N\partial_t\phi(t),\phi(t)\big),\partial_t\boldsymbol{u}(t)\big)+2\mathcal{K}\big(\boldsymbol{R}_0\big(\varepsilon A_N\phi(t),\partial_t\phi(t)\big),\partial_t\boldsymbol{u}(t)\big)\\ &-2\alpha\big(f'\big(\phi(t)\big)\partial_t\phi(t),B_N\partial_t\phi(t)\big)_{L^2}.\end{aligned}\tag{3.51}$$

Using the continuity properties of the trilinear forms b_0, b_1, we estimate the first two terms in $\Lambda_1(t)$, as follows:

$$\begin{aligned}&2\left|b_0(\partial_t\boldsymbol{u},\boldsymbol{u},\partial_t\boldsymbol{u})+b_1(\partial_t\boldsymbol{u},\phi,\partial_t\phi)\right|\\ &\quad\leqslant c|\partial_t\boldsymbol{u}|\|\boldsymbol{u}\|^{1/2}\|\partial_t\boldsymbol{u}\|+c|\partial_t\boldsymbol{u}|^{1/2}\|\partial_t\boldsymbol{u}\|^{1/2}|\phi|_{H^1}^{1/2}|B_N\partial_t\phi|_{L^2}^{1/2}|\partial_t\phi|_{L^2}\end{aligned}\tag{3.52}$$

and by applying Young's inequality repeatedly, we get

$$\begin{aligned}&2\left|b_0(\partial_t\boldsymbol{u},\boldsymbol{u},\partial_t\boldsymbol{u})+b_1(\partial_t\boldsymbol{u},\phi,\partial_t\phi)\right|\\ &\quad\leqslant\left(\frac{\nu}{4}\|\partial_t\boldsymbol{u}\|^2+c|\partial_t\boldsymbol{u}|^2\|\boldsymbol{u}\|\right)+\left(\frac{\nu}{4}\|\partial_t\boldsymbol{u}\|^2+\frac{\varepsilon}{4}|B_N\partial_t\phi|_{L^2}^2+c|\partial_t\boldsymbol{u}||\phi|_{H^1}|\partial_t\phi|_{L^2}^2\right)\\ &\quad\leqslant\frac{\nu}{2}\|\partial_t\boldsymbol{u}\|^2+\frac{\varepsilon}{4}|B_N\partial_t\phi|_{L^2}^2+c|\partial_t\boldsymbol{u}|^2\|\boldsymbol{u}\|+c|\partial_t\boldsymbol{u}||\phi|_{H^1}|\partial_t\phi|_{L^2}^2.\end{aligned}$$

Analogously, using the generalized Hölder and Agmon inequalities, we obtain

$$\begin{aligned}2\mathcal{K}\big(\boldsymbol{R}_0(\varepsilon B_N\partial_t\phi,\phi),\partial_t\boldsymbol{u}\big)&=2\mathcal{K}\varepsilon b_1(\partial_t\boldsymbol{u},\phi,B_N\partial_t\phi)\\ &\leqslant c|\partial_t\boldsymbol{u}||\nabla\phi|_{L^\infty}|B_N\partial_t\phi|_{L^2}\\ &\leqslant c|\partial_t\boldsymbol{u}|\big(|\nabla\phi|_{L^2}^{1/2}|\phi|_{H^3}^{1/2}\big)|B_N\partial_t\phi|_{L^2}\\ &\leqslant\frac{\varepsilon}{4}|B_N\partial_t\phi|_{L^2}^2+c|\partial_t\boldsymbol{u}|^2|\phi|_{H^1}|\phi|_{H^3}.\end{aligned}$$

Here and in the sequel of this proof, $Q(\cdot)$ stands for some continuous, positive and monotone non-decreasing function independent of time and initial data. Arguing now as in the proof of Lemma 3.7 (see (3.37)), we easily get

$$2\mathcal{K}\big(\boldsymbol{R}_0(\varepsilon B_N\phi,\partial_t\phi),\partial_t\boldsymbol{u}\big)-2\alpha\big(f'(\phi)\partial_t\phi,B_N\partial_t\phi\big)_{L^2}$$
$$\leqslant \frac{\varepsilon}{4}|\nabla\partial_t\phi|_{L^2}^2+\frac{\nu}{2}\|\partial_t\boldsymbol{u}\|^2+c|\partial_t\boldsymbol{u}|_{L^2}^2|\phi|_{H^2}^2|\phi|_{H^3}^2+Q\big(|\phi|_{H^1}\big)|\partial_t\phi|_{L^2}^2+\frac{\varepsilon}{4}|B_N\partial_t\phi|_{L^2}^2. \tag{3.53}$$

Recalling (3.51) and inserting all estimates (3.52)–(3.53) into the right-hand side of (3.50), we get

$$\frac{d}{dt}\mathcal{Y}_3(t)+\nu\big\|\partial_t\boldsymbol{u}(t)\big\|^2+\varepsilon\big|B_N\partial_t\phi(t)\big|_{L^2}^2\leqslant \Lambda_2(t)\mathcal{Y}_3(t),\quad \forall t\geqslant t_1, \tag{3.54}$$

where

$$\Lambda_2:=c\big(|\partial_t\boldsymbol{u}||\phi|_{H^1}+\big|\phi(t)\big|_{H^1}|\phi|_{H^3}+|\phi|_{H^2}^2|\phi|_{H^3}^2\big)+Q\big(|\phi|_{H^1}\big)+c\|\boldsymbol{u}\|.$$

Applying the uniform Gronwall inequality once more (see, e.g., [55, Chap. III, Lemma 1.1]), and using estimates (3.34), (3.43), (3.45), (3.46), (3.49), we can find a time $t_2\geqslant t_1$ such that

$$\big|\partial_t\boldsymbol{u}(t)\big|^2+\big|\partial_t\phi(t)\big|_{L^2}^2\leqslant c,\quad \forall t\geqslant t_2. \tag{3.55}$$

Finally, using estimates (3.34) and (3.55), we infer from (3.46)–(3.48), that

$$\big|\mu(t)\big|_{D(A_N)}\leqslant c,\quad \forall t\geqslant t_2. \tag{3.56}$$

Rewriting now the first two equations of system (2.9) into the following form

$$A_0\boldsymbol{u}=-\nu^{-1}\big(\partial_t\boldsymbol{u}+B_0(\boldsymbol{u},\boldsymbol{u})-\mathcal{K}\boldsymbol{R}_0(\varepsilon A_N\phi,\phi)-\boldsymbol{g}\big),$$
$$A_N\phi=\varepsilon^{-1}\big(\mu-\alpha f(\phi)\big),$$

and exploiting the above estimates together with (3.55)–(3.56), recalling (2.1) and the regularity of Γ, we deduce (3.44). This finishes the proof. □

3.3. Global and exponential attractors

We are now in a position to prove the following.

Theorem 3.10. *Let the assumptions of Lemma 3.7 hold. The dynamical system $(\mathbb{Y}_M,\mathcal{S}(t))$ possesses a connected global attractor $\mathcal{A}_M$ which is bounded in $\mathbb{V}\times H^2(\Omega)$. Moreover, if Γ is of class C^4, then $\mathcal{A}_M$ is bounded in $\mathbb{H}^2(\Omega)\times H^4(\Omega)$.*

Proof. Proposition 3.6, Lemma 3.7 and Proposition 3.9 imply that the dynamical system $(\mathbb{Y}_M,\mathcal{S}(t))$ has a bounded absorbing set and a compact absorbing set which is contained in $\mathbb{V}\times D(A_N)$ or $D(A_0)\times(D(A_N)\cap H^4(\Omega))$, according to the smoothness of Γ. Therefore, recalling that $\mathcal{S}(t)$ is also a Lipschitz continuous semigroup (cf. Lemma 3.3), the proof follows from a well-known result (see, e.g., [51, Theorem 10.5]). □

Remark 3.11. Let $h>1$ and assume $\boldsymbol{g}\in\mathbb{H}^{h-1}(\Omega)$ is divergence free and $f\in C^{h+1}(\mathbb{R})$ satisfies (2.1). Then, arguing as in Proposition 3.9, we can prove that any functional invariant set for the semigroup $\mathcal{S}(t)$ is in fact bounded in $\mathbb{H}^{h+1}(\Omega)\times H^{h+3}(\Omega)$, provided that Γ is smooth enough (e.g., of class C^{h+3}).

The second main result of this subsection is concerned with the existence of exponential attractors.

Theorem 3.12. *Let Γ be of class C^4, $\boldsymbol{g}\in\mathbb{H}$ and $f\in C^3(\mathbb{R})$ satisfy* (2.1). *Then $\mathcal{S}(t)$ possesses an exponential attractor $\mathcal{E}_M\subset\mathbb{Y}_M$ which is bounded in $\mathbb{H}^2(\Omega)\times H^4(\Omega)$. Thus, by definition, we have that*:

(i) *$\mathcal{E}_M$ is compact and semi-invariant with respect $\mathcal{S}(t)$, i.e.,*

$$\mathcal{S}(t)(\mathcal{E}_M)\subset\mathcal{E}_M,\quad \forall t\geqslant 0.$$

(ii) *The fractal dimension* $\dim_F(\mathcal{E}_M,\mathbb{Y}_M)$ *of $\mathcal{E}_M$ is finite.*

(iii) *$\mathcal{E}_M$ attracts exponentially fast any bounded subset B of $\mathbb{Y}_M$, that is, there exist a positive non-decreasing function Q and a constant $\rho > 0$ such that*

$$dist_{\mathbb{Y}_M}\big(\mathcal{S}(t)B, \mathcal{E}_M\big) \leqslant Q\big(\|B\|_{\mathbb{Y}_M}\big)e^{-\rho t}, \quad \forall t \geqslant 0.$$

Here $dist_{\mathbb{Y}_M}$ denotes the Hausdorff semi-distance between sets in $\mathbb{Y}_M$ and $\|B\|_{\mathbb{Y}_M}$ stands for the size of B in $\mathbb{Y}_M$. Both Q and ρ can be explicitly calculated.

Remark 3.13. Theorem 3.12 entails that $\mathcal{A}_M$ has finite fractal dimension. In the next section, this dimension will be estimated from above in terms of ν, ε, $\mathcal{K}$, M and α. In addition, it is worth observing that, due to the boundedness of $\mathcal{E}$ in $\mathbb{H}^2(\Omega) \times H^4(\Omega)$, then, through interpolation, one can prove that (ii) and (iii) hold with respect to the $\mathbb{V} \times H^3(\Omega)$-metric.

The proof of Theorem 3.12 is based on a fundamental result on discrete semigroups (see [22]), which is reported here below for the reader's convenience.

Theorem 3.14. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two Banach spaces such that $\mathcal{X}_2$ is compactly embedded in $\mathcal{X}_1$. Let X_0 be a bounded subset of $\mathcal{X}_2$ and consider a nonlinear map $\Sigma : X_0 \to X_0$ satisfying the smoothing property*

$$\big\|\Sigma(x_1) - \Sigma(x_2)\big\|_{\mathcal{X}_2} \leqslant d\|x_1 - x_2\|_{\mathcal{X}_1}, \tag{3.57}$$

for all $x_1, x_2 \in X_0$, where $d > 0$ depends on X_0. Then the discrete dynamical system (X_0, Σ^n) possesses a discrete exponential attractor $\mathcal{E}_M^* \subset \mathcal{X}_2$, that is, a compact set in $\mathcal{X}_1$ with finite fractal dimension such that

$$\Sigma\big(\mathcal{E}_M^*\big) \subset \mathcal{E}_M^*, \tag{3.58}$$

$$dist_{\mathcal{X}_1}\big(\Sigma^n(X_0), \mathcal{E}_M^*\big) \leqslant d_X e^{-\rho_* n}, \quad n \in \mathbb{N}, \tag{3.59}$$

where d_X and ρ_* are positive constants independent of n, with the former depending on X_0.

The validity of the smoothing property as well as the extension of the discrete case to the continuous one are consequences of the following lemmas.

Lemma 3.15. *Let the assumptions of Theorem* 3.12 *be satisfied. Indicate by $(\boldsymbol{u}_i, \phi_i)$ the solution to* **P** *which corresponds to the initial data $(\boldsymbol{u}_i(0), \phi_i(0)) \in \mathbb{Y}_M$, $i = 1, 2$. Then the following estimate holds*:

$$\big\|(\boldsymbol{u}_1 - \boldsymbol{u}_1)(t)\big\|^2 + \varepsilon\big|(\phi_1 - \phi_2)(t)\big|_{H^2}^2 \leqslant C_3\frac{\bar{t}+1}{\bar{t}}e^{C_4 t}\big(\big\|\big((\boldsymbol{u}_1 - \boldsymbol{u}_2)(0), (\phi_1 - \phi_2)(0)\big)\big\|_{\mathbb{Y}_M}^2\big), \quad \forall t > t_2, \tag{3.60}$$

where $\bar{t} := t - t_2$, while C_3 and C_4 are positive constants which only depend on the norms of the initial data in $\mathbb{Y}_M$, on Ω and on the other structural parameters of the problem.

Proof. Let us again set $\psi := \phi_1 - \phi_2$, $\boldsymbol{w} := \boldsymbol{u}_1 - \boldsymbol{u}_2$ and $\overline{\psi} = \psi - M_{1,2}$, where $M_{1,2}$ is as in the proof of Lemma 3.3. Recall that $(\boldsymbol{w}, \overline{\psi})$ solves system (3.20) and that each solution $(\boldsymbol{w}(t), \overline{\psi}(t))$ satisfies (3.19) for every $t \geqslant t_2$ (t_2 is as in the proof of Proposition 3.9). We are now ready to verify estimate (3.60). We take the inner product of the first equation of (3.20) with $A_0\boldsymbol{w}(t)$ in $\mathbb{H}$. Then, take the inner product in $L^2(\Omega)$ of the second and third equations of (3.20) with $B_N^2\overline{\mu}(t) + \varepsilon\zeta B_N^2\overline{\psi}(t)$ (with $\zeta > 0$ sufficiently small to be selected in the sequel) and $\varepsilon B_N^2\overline{\psi}(t)$, respectively. Adding the resulting equations, we deduce that

$$\frac{1}{2}\frac{d}{dt}\mathcal{Y}_4(t) + \nu\big|A_0\boldsymbol{w}(t)\big|^2 + \varepsilon^2\zeta\big|B_N^{3/2}\overline{\psi}(t)\big|_{L^2}^2 + \big|B_N\overline{\mu}(t)\big|_{L^2}^2 = \Lambda_3(t), \tag{3.61}$$

for all $t \geqslant t_2$, where $\mathcal{Y}_4(t) := \|\boldsymbol{w}(t)\|^2 + \varepsilon|B_N\overline{\psi}(t)|_{L^2}^2$ and

$$\begin{aligned}\Lambda_3 := {} & -b_0(\boldsymbol{w}, \boldsymbol{u}_1, A_0\boldsymbol{w}) - b_0(\boldsymbol{u}_2, \boldsymbol{w}, A_0\boldsymbol{w}) + \mathcal{K}\big(\boldsymbol{R}_0(\varepsilon B_N\phi_1, \overline{\psi}), A_0\boldsymbol{w}\big) \\ & + \mathcal{K}\big(\boldsymbol{R}_0(\varepsilon B_N\overline{\psi}, \phi_2), A_0\boldsymbol{w}\big) - b_1\big(\boldsymbol{w}, \phi_1, \varepsilon B_N^2\overline{\psi}\big) - b_1\big(\boldsymbol{u}_2, \overline{\psi}, \varepsilon B_N^2\overline{\psi}\big) \\ & + \alpha\big(f(\phi_1) - f(\phi_2), B_N^2\overline{\mu}\big)_{L^2} + \varepsilon\zeta(B_N\overline{\psi}, B_N\overline{\mu})_{L^2} - \varepsilon\zeta\alpha\big(f(\phi_1) - f(\phi_2), B_N^2\overline{\psi}\big)_{L^2}.\end{aligned}$$

Before we begin estimating Λ_3, it is worth recalling that $(\boldsymbol{u}_i, \phi_i)$ satisfies (3.34), (3.43)–(3.44), (3.46)–(3.49) and (3.56). In particular, we have that

$$\left|A_0\boldsymbol{u}_i(t)\right|^2 + \left|\phi_i(t)\right|_{H^4}^2 + \left|\mu_i(t)\right|_{H^2}^2 \leqslant c, \quad \forall t \geqslant t_2,\ i = 1, 2. \tag{3.62}$$

Using the continuity properties of b_0 and suitable Young inequalities, we control the first two terms in Λ_3, as follows:

$$\begin{aligned}
&\left|b_0(\boldsymbol{w}, \boldsymbol{u}_1, A_0\boldsymbol{w}) + b_0(\boldsymbol{u}_2, \boldsymbol{w}, A_0\boldsymbol{w})\right| \\
&\quad\leqslant \left(\frac{\nu}{4}|A_0\boldsymbol{w}|^2 + c_\Omega|\boldsymbol{w}|\|\boldsymbol{w}\|\|\boldsymbol{u}_1\||A_0\boldsymbol{u}_1|\right) + \left(\frac{\nu}{4}|A_0\boldsymbol{w}|^2 + c_\Omega|\boldsymbol{u}_2|^2\|\boldsymbol{u}_2\|^2\|\boldsymbol{w}\|^2\right) \\
&\quad\leqslant \frac{\nu}{2}|A_0\boldsymbol{w}|^2 + c\|\boldsymbol{w}\|^2.
\end{aligned} \tag{3.63}$$

Here, we have employed (3.62). Similarly, we obtain

$$\begin{aligned}
\left|b_1\left(\boldsymbol{w}, \phi_1, \varepsilon B_N^2\overline{\psi}\right)\right| &\leqslant c\left|B_N^{1/2}B_1(\boldsymbol{w}, \phi_1)\right|_{L^2}\left|B_N^{3/2}\overline{\psi}\right|_{L^2} \\
&\leqslant c|\boldsymbol{w}|^{1/2}|A_0\boldsymbol{w}|^{1/2}|\phi_1|_{H^1}^{1/2}|\phi_1|_{H^2}^{1/2}\left|B_N^{3/2}\overline{\psi}\right|_{L^2} + c|\boldsymbol{w}|^{1/2}\|\boldsymbol{w}\|^{1/2}|\phi_1|_{H^2}^{1/2}|\phi_1|_{H^3}^{1/2}\left|B_N^{3/2}\overline{\psi}\right|_{L^2} \\
&\leqslant \frac{\varepsilon^2\zeta}{2}\left|B_N^{3/2}\overline{\psi}\right|_{L^2}^2 + c_\zeta|\boldsymbol{w}||A_0\boldsymbol{w}||\phi_1|_{H^1}|\phi_1|_{H^2} + c_\zeta|\boldsymbol{w}|\|\boldsymbol{w}\||\phi_1|_{H^2}|\phi_1|_{H^3} \\
&\leqslant \frac{\varepsilon^2\zeta}{2}\left|B_N^{3/2}\overline{\psi}\right|_{L^2}^2 + \frac{\nu}{4}|A_0\boldsymbol{w}|^2 + c\|\boldsymbol{w}\|^2
\end{aligned} \tag{3.64}$$

and

$$\begin{aligned}
\left|b_1\left(\boldsymbol{u}_2, \overline{\psi}, \varepsilon B_N^2\overline{\psi}\right)\right| &\leqslant c\left|B_N^{1/2}B_1(\boldsymbol{u}_2, \overline{\psi})\right|_{L^2}\left|B_N^{3/2}\overline{\psi}\right|_{L^2} \\
&\leqslant c|\boldsymbol{u}_2|^{1/2}|A_0\boldsymbol{u}_2|^{1/2}|\overline{\psi}|_{H^1}^{1/2}|B_N\overline{\psi}|_{L^2}^{1/2}\left|B_N^{3/2}\overline{\psi}\right|_{L^2} + c|\boldsymbol{u}_2|^{1/2}|\nabla\boldsymbol{u}_2|^{1/2}|B_N\overline{\psi}|_{L^2}^{1/2}\left|B_N^{3/2}\overline{\psi}\right|_{L^2}^{3/2} \\
&\leqslant \frac{\varepsilon^2\zeta}{4}\left|B_N^{3/2}\overline{\psi}\right|_{L^2}^2 + c_\zeta|\boldsymbol{u}_2||A_0\boldsymbol{u}_2||\overline{\psi}|_{H^1}|B_N\overline{\psi}|_{L^2} + c_\zeta|\boldsymbol{u}_2|^2|\nabla\boldsymbol{u}_2|^2|B_N\overline{\psi}|_{L^2}^2.
\end{aligned} \tag{3.65}$$

Moreover, we have that

$$\begin{aligned}
\varepsilon\zeta\alpha\left|\left(f(\phi_1) - f(\phi_2), B_N^2\overline{\psi}\right)_{L^2}\right| &= \alpha\varepsilon\zeta\left|\left(\nabla\left(f(\phi_1) - f(\phi_2)\right), \nabla B_N\overline{\psi}\right)_{L^2}\right| \\
&\leqslant c_\zeta\left|f(\phi_1) - f(\phi_2)\right|_{H^1}^2 + \frac{\varepsilon^2\zeta}{4}\left|B_N^{3/2}\overline{\psi}\right|_{L^2}^2 \\
&\leqslant c_\zeta\left(|\overline{\psi}|_{H^1}^2 + M_{1,2}^2\right) + \frac{\varepsilon^2\zeta}{4}\left|B_N^{3/2}\overline{\psi}\right|_{L^2}^2.
\end{aligned} \tag{3.66}$$

Analogously to (3.66), we deduce

$$\begin{aligned}
\alpha\left(f(\phi_1) - f(\phi_2), B_N^2\overline{\mu}\right)_{L^2} &= \alpha\left(A_N\left(f(\phi_1) - f(\phi_2)\right), B_N\overline{\mu}\right)_{L^2} \\
&\leqslant \frac{\alpha^2\zeta}{2}|B_N\overline{\mu}|_{L^2}^2 + c\left|A_N\left(f(\phi_1) - f(\phi_2)\right)\right|_{L^2}^2 \\
&\leqslant \frac{\alpha^2\zeta}{2}|B_N\overline{\mu}|_{L^2}^2 + c\left(|B_N\overline{\psi}|_{L^2}^2 + M_{1,2}^2\right),
\end{aligned}$$

where we have exploited the fact that $f \in C^3(\mathbb{R})$ and used the bound (3.62), repeatedly. Let us now consider the remaining terms of Λ_3. First, Young's inequality yields

$$\varepsilon\zeta(B_N\overline{\psi}, B_N\overline{\mu})_{L^2} \leqslant \frac{\zeta}{2}|B_N\overline{\mu}|_{L^2}^2 + c|B_N\overline{\psi}|_{L^2}^2. \tag{3.67}$$

Then, exploiting the generalized Hölder and Young inequalities combined with some interpolation inequalities, and arguing as in the proof of Lemma 3.7 (see (3.37)), we get

$$\begin{aligned}
&\mathcal{K}\big|\big(\boldsymbol{R}_0(\varepsilon A_N\phi_1,\overline{\psi}),A_0\boldsymbol{w}\big)+\big(\boldsymbol{R}_0(\varepsilon B_N\overline{\psi},\phi_2),A_0\boldsymbol{w}\big)\big|\\
&\quad\leqslant \varepsilon\mathcal{K}\big(\big|\boldsymbol{R}_0(A_N\phi_1,\overline{\psi})\big|+\big|\boldsymbol{R}_0(B_N\overline{\psi},\phi_2)\big|\big)|A_0\boldsymbol{w}|\\
&\quad\leqslant c|\phi_1|_{H^2}^2|B_N\overline{\psi}|_{L^2}^2|\overline{\psi}|_{H^1}^2+\frac{\nu}{8}|A_0\boldsymbol{w}|^2+\frac{\varepsilon^2\zeta}{4}\big|B_N^{3/2}\overline{\psi}\big|_{L^2}^2+c|B_N\overline{\psi}|_{L^2}|\phi_2|_{W^{1,\infty}}|A_0\boldsymbol{w}|\\
&\quad\leqslant c|B_N\overline{\psi}|_{L^2}^2+\frac{\nu}{4}|A_0\boldsymbol{w}|^2+\frac{\varepsilon^2\zeta}{4}\big|B_N^{3/2}\overline{\psi}\big|_{L^2}^2+c|B_N\overline{\psi}|_{L^2}^2|\phi_2|_{W^{1,\infty}}^2\\
&\quad\leqslant \frac{\nu}{4}|A_0\boldsymbol{w}|^2+\frac{\varepsilon^2\zeta}{4}\big|B_N^{3/2}\overline{\psi}\big|_{L^2}^2+c|B_N\overline{\psi}|_{L^2}^2+c|B_N\overline{\psi}|_{L^2}^2|\phi_2|_{H^1}|\phi_2|_{H^3}\\
&\quad\leqslant \frac{\nu}{4}|A_0\boldsymbol{w}|^2+\frac{\varepsilon^2\zeta}{4}\big|B_N^{3/2}\overline{\psi}\big|_{L^2}^2+c|B_N\overline{\psi}|_{L^2}^2. \qquad (3.68)
\end{aligned}$$

Consequently, collecting all the above estimates and choosing $\zeta > 0$ sufficiently small, from (3.61), we deduce the following differential inequality:

$$\frac{d}{dt}\mathcal{Y}_4(t)\leqslant c\big(\mathcal{Y}_4(t)+M_{1,2}^2\big),\quad \forall t>t_2. \qquad (3.69)$$

Multiplying now both sides of this inequality by $\bar{t}=t-t_2$ and integrating the resulting relation over (t_2,t), we get

$$\bar{t}\mathcal{Y}_4(t)\leqslant c\int_{t_2}^{t}(s-t_2+1)\mathcal{Y}_4(s)\,ds+\frac{c}{2}M_{1,2}^2\bar{t}^2,\quad \forall t>t_2, \qquad (3.70)$$

which entails (3.60), thanks to Lemma 3.3 and the inequality $M_{1,2}^2\leqslant c\|(\boldsymbol{w}(0),\psi(0))\|_{\mathbb{Y}_M}^2$. The proof is complete. □

The second lemma is concerned with the time regularity of the semigroup $\mathcal{S}(t)$. The proof is standard and is left to the reader (just recall (3.55)).

Lemma 3.16. *Let the assumptions of Theorem* 3.12 *be satisfied. Then, for any bounded set $\mathcal{B}\subset\mathbb{Y}_M$ there is a positive constant c and a time $t^*=t^*(\mathcal{B})>0$ such that*

$$\big\|\mathcal{S}(t)(\boldsymbol{u}_0,\phi_0)-\mathcal{S}(\widetilde{t})(\boldsymbol{u}_0,\phi_0)\big\|\leqslant c\big(|t-\widetilde{t}|^{1/2}+|t-\widetilde{t}|^{1/4}\big), \qquad (3.71)$$

for all $t,\widetilde{t}\in[t^*,+\infty)$ and any $(\boldsymbol{u}_0,\phi_0)\in\mathcal{B}\subset\mathbb{Y}_M$.

Proof of Theorem 3.12. Using Lemma 3.3, Proposition 3.9 and (3.60), we can find a bounded subset X_0 of $D(A_0)\times(D(A_N)\cap H^4(\Omega))$ and $t^\sharp>0$ such that, setting $\Sigma=S(t^\sharp)$, the mapping $\Sigma: X_0\to X_0$ enjoys the smoothing property (3.57). Therefore Theorem 3.14 applies to Σ and there exists a compact set $\mathcal{E}_M^*\in X_0$ with finite fractal dimension (with respect to the metric topology of $\mathbb{Y}_M$) that satisfies (3.58) and (3.59). Hence, setting

$$\mathcal{E}_M=\bigcup_{t\in[t^\sharp,2t^\sharp]}\mathcal{S}(t)\mathcal{E}_M^*,$$

we deduce that (i) and (iii) are fulfilled, while (ii) is a consequence of (3.19) and (3.71). □

Remark 3.17. Thanks to some results concerning second-order differential operators with variable coefficients (see, e.g., [2,1]), it should be possible to extend the main results of this section to the case of concentration dependent viscosities $\nu=\nu(\phi)\in C^2(\mathbb{R},[\nu_0,\nu_1])$, for some $\nu_1>\nu_0>0$. In this case, the operator $-\nu\Delta\boldsymbol{u}$ is replaced by $-\operatorname{div}(\nu(\phi)D\boldsymbol{u})$, where $D\boldsymbol{u}$ is the rate-of-strain tensor (see, for instance, [29]).

3.4. The modified problem and its semigroup

In the next two sections, we aim to estimate in terms of the physical parameters the dimension of the global attractor and to study the convergence of a given solution of Problem **P** to a single equilibrium. In order to do that, it is more convenient to concentrate our attention on $\mathcal{S}(t)$ restricted to the phase-space

$$\mathbb{Y}_{M_0}:=\mathbb{H}\times\big\{\phi\in H^1(\Omega):\ \langle\phi\rangle=M_0\big\},$$

where M_0 is fixed. In this case, **P** can be rewritten into an equivalent form. More precisely, we set, as in the proof of Proposition 3.1, $\overline{\phi}(t) = \phi(t) - M_0$ and $\overline{f}(\phi) = f(\overline{\phi} + M_0)$. Then we observe that **P** rewritten for $(\boldsymbol{u}, \overline{\phi})$ reads

Problem $\mathbf{P_0}$. For $\boldsymbol{g} \in \mathbb{V}^*$ and any given pair of initial data $(\boldsymbol{u}_0, \phi_0) \in \mathbb{Y}_{M_0}$, find a pair of functions $(\boldsymbol{u}, \overline{\phi})$ satisfying (2.7)–(2.8) and

$$\begin{cases} \partial_t \boldsymbol{u} + \nu A_0 \boldsymbol{u} + B_0(\boldsymbol{u}, \boldsymbol{u}) - \mathcal{K}\boldsymbol{R}_0(\varepsilon A_N \overline{\phi}, \overline{\phi}) = \boldsymbol{g}, \quad \text{in } \mathbb{V}^*, \text{ a.e. in } (0, +\infty), \\ \overline{\mu} = \varepsilon A_N \overline{\phi} + \alpha \overline{f}(\phi), \quad \text{a.e. in } \Omega \times (0, +\infty), \\ \partial_t \phi + A_N \overline{\mu} + B_1(\boldsymbol{u}, \overline{\phi}) = 0, \quad \text{in } H^{-1}, \text{ a.e. in } (0, +\infty), \end{cases} \tag{3.72}$$

and the initial conditions

$$\boldsymbol{u}_{|t=0} = \boldsymbol{u}_0, \qquad \overline{\phi}_{t=0} = \phi_0 - M_0.$$

It is clear that $\overline{f}(r) = f(r + M_0)$ also satisfies (2.1). Thus, all the a priori estimates and the results of the previous sections still hold for the solutions of Problem $\mathbf{P}_0$.

We can then define the solving semigroup associated with Problem $\mathbf{P}_0$, namely,

$$\overline{\mathcal{S}}(t) : \mathbb{Y}_0 \to \mathbb{Y}_0, \quad \overline{\mathcal{S}}(t)(\boldsymbol{u}_0, \overline{\phi}_0) = \big(\boldsymbol{u}(t), \overline{\phi}(t)\big), \tag{3.73}$$

where $(\boldsymbol{u}, \overline{\phi})$ is the unique solution of (3.72) with initial data $(\boldsymbol{u}_0, \overline{\phi}_0) \in \mathbb{Y}_0$ and

$$\mathbb{Y}_0 := \mathbb{H} \times \big(H^1(\Omega) \cap L^2_0(\Omega)\big),$$

which is a Hilbert space with norm

$$\big\|(\boldsymbol{u}, \overline{\phi})\big\|^2_{\mathbb{Y}_0} := \frac{1}{\mathcal{K}} |\boldsymbol{u}|^2 + \varepsilon |\nabla \overline{\phi}|^2_{L^2}. \tag{3.74}$$

Of course, $\overline{\mathcal{S}}(t)(\boldsymbol{u}_0, \phi_0 - M_0) = (\boldsymbol{u}(t), \phi(t) - M_0)$, where $(\boldsymbol{u}(t), \phi(t)) = \mathcal{S}(t)(\boldsymbol{u}_0, \phi_0)$ is the unique solution to Problem **P** with initial data $(\boldsymbol{u}_0, \phi_0) \in \mathbb{Y}_{M_0}$.

4. The fractal dimension of the global attractor

In this section, we let the assumptions of Lemma 3.7 hold. Then we consider the dynamical system $(\mathbb{Y}_0, \overline{\mathcal{S}}(t))$ which possesses the global attractor $\overline{\mathcal{A}} \subset \mathbb{Y}_0$. In the sequel, for the sake of exposition, we will drop the bars from $\overline{\phi}$, $\overline{\mu}$ and $\overline{f}$.

Our goal is to estimate in terms of the physical parameters the fractal dimension of $\overline{\mathcal{A}}$. We begin by reviewing a few results taken from [14]. Recalling Theorem 3.5, we consider a solution $(\boldsymbol{u}, \phi)$ to $\mathbf{P}_0$ and we write the first variation equations with given initial values $\xi = (\xi_1, \xi_2) \in \mathbb{Y}_0$, namely,

$$\begin{cases} \partial_t \boldsymbol{U} + \nu A_0 \boldsymbol{U} + \mathbf{B}_0(\boldsymbol{U}) - \mathcal{R}_0(\Phi) = 0, \\ \Psi = \varepsilon A_N \Phi + \alpha f'(\phi + M_0)\Phi, \\ \partial_t \Phi + A_N \Psi + \mathbf{B}_1(\Phi) = 0, \\ \boldsymbol{U}(0) = \xi_1, \qquad \Phi(0) = \xi_2, \end{cases} \tag{4.1}$$

where

$$\mathbf{B}_0(\boldsymbol{U}) := B_0(\boldsymbol{u}, \boldsymbol{U}) + B_0(\boldsymbol{U}, \boldsymbol{u}), \qquad \mathbf{B}_1(\Phi) := B_1(\boldsymbol{u}, \Phi) + B_1(\boldsymbol{U}, \phi), \tag{4.2}$$

$$\mathcal{R}_0(\Phi) := \mathcal{K}\boldsymbol{R}_0(\varepsilon A_N \phi, \Phi) + \mathcal{K}\boldsymbol{R}_0(\varepsilon A_N \Phi, \phi). \tag{4.3}$$

Then, we recall the following (adapted) definition of Fréchet differentiability for $\overline{\mathcal{S}}(t)$.

Definition 4.1. Let $\mathbb{X} \subset \mathbb{Y}_0$ be a bounded functional invariant set for $\overline{\mathcal{S}}(t)$ and let $\Xi_i := (\boldsymbol{u}_i, \phi_i) \in \mathbb{X}$, $i = 0, 1$. We say that the mapping $\Xi \mapsto \overline{\mathcal{S}}(t)\Xi$ is differentiable on $\mathbb{X}$ if for any $\Xi_0 \in \mathbb{Y}_0$, there exists an operator $\mathbf{L}(t, \Xi_0) \in \mathcal{L}(\mathbb{Y}_0)$ such that

$$\sup_{\substack{\Xi_0, \Xi_1 \in \mathbb{X} \\ 0 < \|\Xi_0 - \Xi_1\|_{\mathbb{Y}_0} \leqslant \sigma}} \frac{\|\overline{\mathcal{S}}(t)\Xi_1 - \overline{\mathcal{S}}(t)\Xi_0 - \mathbf{L}(t, \Xi_0) \cdot (\Xi_1 - \Xi_0)\|_{\mathbb{Y}_0}}{\|(\Xi_1 - \Xi_0)\|_{\mathbb{Y}_0}} \to 0, \tag{4.4}$$

as $\sigma \to 0$.

Using the techniques developed in Section 3 and known regularity results for parabolic equations (see, e.g., [55, Theorem II.3.4]), we can rigorously prove the following result (the details are left to the reader since they are very similar to the ones in [55, Chap. 6, Section 8]).

Proposition 4.2. *For any fixed $T > 0$, problem* (4.1)–(4.3) *possesses a unique solution*

$$(\boldsymbol{U}, \Phi) \in C\big([0,T]; \mathbb{Y}_0\big) \cap L^2\big([0,T]; \mathbb{V} \times \big(D(A_N) \cap H^3(\Omega)\big)\big). \tag{4.5}$$

Furthermore, for every $t > 0$, the function $\Xi_0 \mapsto \overline{S}(t)\Xi_0$ is Fréchet differentiable in $\mathbb{Y}_0$ at Ξ_0 with differential

$$\mathbf{L}(t, \Xi_0) : \xi = (\xi_1, \xi_2) \in \mathbb{Y}_0 \mapsto (\boldsymbol{U}, \Phi) \in \mathbb{Y}_0, \tag{4.6}$$

where $(\boldsymbol{U}, \Phi)$ is the unique solution to (4.1)–(4.3).

For $\mathbf{L} \in \mathrm{Lin}(\mathbb{Y}_0)$ and $j \in \mathbb{N}$, we denote by $\omega_j(\mathbf{L})$ the norm of the exterior product $\bigwedge^j \mathbf{L}$ in $\bigwedge^j \mathbb{Y}_0$, thus if $\mathbf{L} = \mathbf{L}(t, \Xi_0)$ and $\Upsilon_j := (\boldsymbol{U}_j, \Phi_j)$, we have

$$\omega_j\big(\mathbf{L}(t,\Xi_0)\big) = \sup_{\substack{\xi^1,\dots,\xi^j \in \mathbb{Y}_0 \\ \|\xi^i\|_{\mathbb{Y}_0} \leqslant 1,\, i=1,\dots,j}} \big|\Upsilon_1(t) \wedge \dots \wedge \Upsilon_j(t)\big|_{\bigwedge^j \mathbb{Y}_0}, \tag{4.7}$$

where $\Upsilon_1, \dots, \Upsilon_j$, are j solutions of (4.1)–(4.3) corresponding respectively to given initial values $\xi^1 = (\xi_1^1, \xi_2^1), \dots, \xi^j = (\xi_1^j, \xi_2^j)$. We then set

$$\omega_j(t) = \sup_{\Xi_0 \in \mathbb{X}} \omega_j\big(\mathbf{L}(t, \Xi_0)\big)$$

and observe that these numbers are subexponential with respect to t (see, e.g., [55, Chap. 5]). As a consequence, the limit

$$\lim_{t \to +\infty} \omega_j(t)^{1/t} = \Pi_j$$

exists for every j. The uniform Lyapunov numbers λ_j for $\mathbb{X}$ are then defined by the formula $\lambda_1 = \Pi_1$, $\lambda_j = \Pi_j/\Pi_{j-1}$, $j \geqslant 2$. The uniform Lyapunov exponents are the numbers $\pi_j = \log \lambda_j$, $j \geqslant 1$.

We now recall the following basic result (see [14]).

Theorem 4.3. *Let $\mathbb{X}$ be a compact functional invariant set for the semigroup $\overline{S}(t)$. If for some integer $n \geqslant 1$,*

$$\pi_1 + \dots + \pi_n < 0, \tag{4.8}$$

then

$$d_H(\mathbb{X}) \leqslant n \quad \text{and} \quad d_F(\mathbb{X}) \leqslant n \left\{ \max_{1 \leqslant j \leqslant n-1} 1 + \frac{\pi_1 + \dots + \pi_j}{|\pi_1 + \dots + \pi_j|} \right\} \leqslant 2n, \tag{4.9}$$

where $d_H(\mathbb{X})$ and $d_F(\mathbb{X})$ are the Hausdorff and fractal dimension of $\mathbb{X}$, respectively, with respect to the $\mathbb{Y}_0$-metric.

It is well known that the estimation of the Lyapunov numbers depends on the following inequality (cf. [14,55])

$$\omega_n(t) \leqslant \sup_{\Xi_0 \in \mathbb{X}} \exp\left(-\int_0^t \inf_{\xi^j \in \mathbb{Y}_0} Tr\, \mathcal{M}\big(\boldsymbol{u}(s), \phi(s)\big) \cdot Q_n(s)\, ds \right) \tag{4.10}$$

and $\Pi_n \leqslant \exp(-q_n)$, where

$$q_n = \lim_{t \to \infty} \sup \left\{ \inf_{\Xi_0 \in \mathbb{X}} \frac{1}{t} \int_0^t \inf_{\xi^j \in \mathbb{Y}_0} Tr\, \mathcal{M}\big(\boldsymbol{u}(s), \phi(s)\big) \cdot Q_n(s)\, ds \right\}. \tag{4.11}$$

Here $\mathcal{M}(\boldsymbol{u}, \phi) = \mathcal{M}(\boldsymbol{u}(s), \phi(s))$ is the linear mapping

$$\begin{pmatrix} \boldsymbol{U} \\ \Phi \end{pmatrix} \mapsto \begin{pmatrix} \nu A_0 \boldsymbol{U} + \mathcal{B}_0(\boldsymbol{U}) - \mathcal{R}_0(\Phi) \\ \varepsilon B_N^2 \Phi + \alpha A_N(f'(\phi + M_0)\Phi) + \mathcal{B}_1(\Phi) \end{pmatrix} \tag{4.12}$$

and $Q_n(s) = Q_n(s, \xi^1, \dots, \xi^n)$ is the projection in $\mathbb{Y}_0$ onto $span\{\Upsilon_1(s), \dots, \Upsilon_n(s)\}$. Furthermore, we infer from (4.9) that if $q_n > 0$ for some n, then the Hausdorff dimension of $\mathbb{X}$ is less or equal than n and its fractal dimension is bounded by $n(1 + \max_{1\leqslant k\leqslant n}(-q_k/q_n))$.

To this end, we proceed as follows. Let $\{\Psi_j(s)\} = \{(\boldsymbol{v}_j(s), \psi_j(s))\}$ be an orthonormal basis of $\mathbb{Y}_0$ with $\Psi_j(s) \in \mathbb{V} \times D(B_N^{3/2})$, for any j and almost any s, such that $\Psi_1(s), \dots, \Psi_n(s)$ spans $Q_n\mathbb{Y}_0$. Then the family

$$\big(\mathcal{K}^{-1/2}\boldsymbol{v}_j(s), \varepsilon^{1/2}\psi_j(s)\big)$$

is orthonormal in $\mathbb{H} \times (H^1(\Omega) \cap L_0^2(\Omega))$ with respect to $(\cdot,\cdot) + (\nabla\cdot, \nabla\cdot)$. Consequently, the families

$$\big\{\mathcal{K}^{-1/2}\boldsymbol{v}_j(s)\big\}_{j=1,\dots,n} \quad \text{and} \quad \big\{\varepsilon^{1/2}\psi_j(s)\big\}_{j=1,\dots,n}$$

are suborthonormal in $(\mathbb{H}, (\cdot,\cdot))$ and in $(H^1(\Omega) \cap L_0^2(\Omega), (\nabla\cdot, \nabla\cdot))$, respectively (cf. [31,38]). Before proceeding with calculating the trace of the linearized operator $\mathcal{M}$ on $Q_n\mathbb{Y}_0$, we report below two basic inequalities which will be helpful in the sequel. Following [38, Corollary 2.1, Theorems 2.3 and 3.1], generalized versions of the Lieb–Thirring inequalities can be applied to the families above. More precisely, there exists a positive constant C_ℓ, which is independent of n, such that

$$\int_\Omega \big[\rho_0(x)\big]^2\,dx \leqslant C_\ell \sum_{j=1}^n \int_\Omega \big|\nabla \boldsymbol{v}_j(x)\big|^2\,dx, \tag{4.13}$$

$$\int_\Omega \big[a_{p,q}(x)\big]^{1+q}\,dx \leqslant C_\ell \sum_{j=1}^n \int_\Omega \big|B_N^{q/2+i}\psi_j(x)\big|^2\,dx, \tag{4.14}$$

where

$$\rho_0(x) := \sum_{j=1}^n \mathcal{K}^{-1/2}\big|\boldsymbol{v}_j(x)\big|^2, \qquad a_{p,q}(x) := \sum_{j=1}^n \varepsilon^{q/(1+q)}\big|B_N^p\psi_j(x)\big|^2,$$

with the following choices of indices: $(p,q) \in \{(0,1),(0,2),(0,3)\}$ if $i = 0$, $(p,q) \in \{(1/2,1),(1/2,2)\}$ if $i = 1/2$ and $(p,q) \in \{(1,1)\}$ if $i = 1$. Moreover, the constant C_ℓ does not increase when passing from a suborthonormal family to an orthonormal one (cf. [38, Sections 3 and 4]). In the rest of this section, all the positive constants indicated with c_i, c_i', c_i'', c_i''', $i \in \mathbb{N}$, are independent of time, ν, ε, $\mathcal{K}$, α, n and M_0.

We now state a result on the behavior of the eigenvalues for the operator $(\boldsymbol{v}, \psi) \mapsto (A_0\boldsymbol{v}, B_N^2\psi)$, so that we can estimate the first two terms on the right-hand side of (4.22) (see below).

Lemma 4.4. *Let $\{\Psi_j\} = \{(\boldsymbol{v}_j, \psi_j)\}$, $1 \leqslant j \leqslant n$, be a finite family of $\mathbb{V} \times D(B_N^{3/2})$, which is orthonormal in $\mathbb{Y}_0$. We have*:

$$\sum_{j=1}^n \bigg(\frac{\nu}{\mathcal{K}}\|\boldsymbol{v}_j\|^2 + \varepsilon^2\big|B_N^{3/2}\psi_j\big|_{L^2}^2\bigg) \geqslant c_0\frac{n^2}{|\Omega|}\bigg(\frac{\nu\varepsilon}{\nu+\varepsilon}\bigg) - c_0'|\Omega|\bigg(\frac{\nu\varepsilon}{\nu+\varepsilon}\bigg), \tag{4.15}$$

where the constants c_0, c_0' depend on the shape of Ω, but are independent of the size of Ω, ν, ε, $\mathcal{K}$, α, n, M_0 and of Ψ_j.

Proof. The proof of (4.15) is based on a slight modification of inequalities (4.13)–(4.14). To this end, set

$$(\widetilde{\boldsymbol{v}}_j, \widetilde{\psi}_j) := \big(\mathcal{K}^{-1/2}\boldsymbol{v}_j, \varepsilon^{1/2}\psi_j\big), \quad 1 \leqslant j \leqslant n, \tag{4.16}$$

and note that this family is orthonormal in $\mathbb{H} \times (H^1(\Omega) \cap L_0^2(\Omega))$ with respect to $(\cdot,\cdot) + (\nabla\cdot, \nabla\cdot)$. Consequently, the families

$$\big\{\widetilde{\boldsymbol{v}}_j(s)\big\}_{j=1,\dots,n} \quad \text{and} \quad \big\{\widetilde{\psi}_j(s)\big\}_{j=1,\dots,n} \tag{4.17}$$

are suborthonormal in $\mathbb{H}$ and $H^1(\Omega) \cap L_0^2(\Omega)$, respectively, and the following Lieb–Thirring inequalities hold (see [38] again):

$$\int_\Omega \big[\widetilde{\rho}_0(x)\big]^2\,dx \leqslant C_\ell \sum_{i=1}^n \int_\Omega \big|\nabla \widetilde{\boldsymbol{v}}_j(x)\big|^2\,dx, \tag{4.18}$$

$$\int_\Omega \big[\widetilde{\rho}_1(x)\big]^3\,dx \leqslant C_\ell \sum_{i=1}^n \int_\Omega \big|B_N^{3/2}\widetilde{\psi}_j(x)\big|^2\,dx, \tag{4.19}$$

where

$$\widetilde{\rho}_0(x) := \sum_{j=1}^n \big|\widetilde{\boldsymbol{v}}_j(x)\big|^2, \qquad \widetilde{\rho}_1(x) := \sum_{j=1}^n \big|\nabla \widetilde{\psi}_j(x)\big|^2.$$

Also, the constant C_ℓ in (4.18)–(4.19) depends only on the shape of Ω (but not on its size) and does not increase. Let $\widetilde{\rho}(x) := \widetilde{\rho}_0(x) + \widetilde{\rho}_1(x)$. By the Hölder inequality,

$$n = \int_\Omega \widetilde{\rho}(x)\,dx \leqslant |\Omega|^{1/2}\bigg(\int_\Omega \big[\widetilde{\rho}(x)\big]^2\,dx\bigg)^{1/2},$$

which easily yields, on account of (4.18)–(4.19), that

$$\begin{aligned} n^2 &\leqslant c_1|\Omega|\bigg(\int_\Omega \big[\widetilde{\rho}_0(x)\big]^2\,dx + \int_\Omega \big[\widetilde{\rho}_1(x)\big]^2\,dx\bigg) \\ &\leqslant c_1'|\Omega|\bigg[\int_\Omega \big[\widetilde{\rho}_0(x)\big]^2\,dx + \bigg(|\Omega| + \int_\Omega \big[\widetilde{\rho}_1(x)\big]^3\,dx\bigg)\bigg] \\ &\leqslant c_1''|\Omega| \sum_{j=1}^n \big(\|\widetilde{\boldsymbol{v}}_j\|^2 + \big|B_N^{3/2}\widetilde{\psi}_j\big|_{L^2}^2\big) + c_1'''|\Omega|^2. \end{aligned} \tag{4.20}$$

By rewriting (4.20) in terms of (4.16), we deduce

$$\begin{aligned} n^2 &\leqslant c_1''|\Omega| \sum_{j=1}^n \bigg(\frac{1}{\mathcal{K}}\|\boldsymbol{v}_j\|^2 + \varepsilon\big|B_N^{3/2}\psi_j\big|_{L^2}^2\bigg) + c_1'''|\Omega|^2 \\ &\leqslant c_1''|\Omega|\big(\nu^{-1} + \varepsilon^{-1}\big) \sum_{j=1}^n \bigg(\frac{\nu}{\mathcal{K}}\|\boldsymbol{v}_j\|^2 + \varepsilon^2\big|B_N^{3/2}\psi_j\big|_{L^2}^2\bigg) + c_1'''|\Omega|^2. \end{aligned} \tag{4.21}$$

Thus, (4.15) is a straightforward consequence of (4.21). The proof is finished. □

We can now calculate $Tr\,\mathcal{M}(\boldsymbol{u}(s), \phi(s)) \cdot Q_n(s)$. Omitting the s-dependence, by (3.2), (4.2)–(4.4) and (4.10), we have

$$\begin{aligned} Tr\,\mathcal{M}(\boldsymbol{u}, \phi) \cdot Q_n &= \sum_{j=1}^n \big(\mathcal{M}(\boldsymbol{u}, \phi)\Psi_j, \Psi_j\big)_{\mathbb{Y}_0} \\ &= \sum_{j=1}^n \bigg\{\frac{\nu}{\mathcal{K}}\|\boldsymbol{v}_j\|^2 + \varepsilon^2\big|B_N^{3/2}\psi_j\big|_{L^2}^2 + \frac{1}{\mathcal{K}}\big(B_0(\boldsymbol{v}_j, \boldsymbol{u}), \boldsymbol{v}_j\big) + \varepsilon\big(B_1(\boldsymbol{u}, \psi_j), B_N\psi_j\big)_{L^2} \\ &\quad - \big(\boldsymbol{R}_0(\varepsilon B_N\phi, \psi_j), \boldsymbol{v}_j\big) - \big(\boldsymbol{R}_0(\varepsilon B_N\psi_j, \phi), \boldsymbol{v}_j\big) \\ &\quad + \varepsilon\big(B_1(\boldsymbol{v}_j, \phi), B_N\psi_j\big)_{L^2} + \alpha\varepsilon\big(\nabla\big(f'(\phi + M_0)\psi_j\big), \nabla B_N\psi_j\big)_{L^2}\bigg\}. \end{aligned} \tag{4.22}$$

We start by estimating from above the third term on the right-hand side of (4.22). Thanks to the pointwise Schwarz inequality

$$\big|\big(\big((\boldsymbol{v}_j \cdot \nabla) \cdot \boldsymbol{u}\big)\boldsymbol{v}_j\big)(x)\big| \leqslant \big|\nabla \boldsymbol{u}(x)\big|\big|\boldsymbol{v}_j(x)\big|^2, \tag{4.23}$$

so that

$$\mathcal{K}^{-1}\left|\sum_{j=1}^{n}\big(B_0(\boldsymbol{v}_j,\boldsymbol{u}),\boldsymbol{v}_j\big)\right| \leqslant \int_\Omega \frac{|\nabla \boldsymbol{u}(x)|}{\sqrt{\mathcal{K}}}\rho_0(x)\,dx \leqslant \frac{\|\boldsymbol{u}\|}{\sqrt{\mathcal{K}}}|\rho_0|_{L^2}. \tag{4.24}$$

Using now (4.13), we can bound (4.24) by

$$c_1\frac{\|\boldsymbol{u}\|}{\sqrt{\mathcal{K}}}\left(\sum_{j=1}^{n}\|\boldsymbol{v}_j\|^2\right)^{1/2} \leqslant \frac{\nu}{8\mathcal{K}}\sum_{j=1}^{n}\|\boldsymbol{v}_j\|^2+\frac{c_1'}{\nu}\|\boldsymbol{u}\|^2. \tag{4.25}$$

Note that, employing a similar pointwise Schwarz inequality as in (4.23), we have that

$$\begin{aligned}
&\left|\sum_{j=1}^{n}\varepsilon\big(B_1(\boldsymbol{u},\psi_j),B_N\psi_j\big)_{L^2}\right| \\
&\quad\leqslant \varepsilon^{-1/6}\int_\Omega \big|\boldsymbol{u}(x)\big|a_{1,1}^{1/2}(x)a_{1/2,2}^{1/2}(x)\,dx \\
&\quad\leqslant \varepsilon^{-1/6}\left(\int_\Omega \big|\boldsymbol{u}(x)\big|^{4/3}a_{1/2,2}^{2/3}(x)\,dx\right)^{3/4}\left(\int_\Omega a_{1,1}^2(x)\,dx\right)^{1/4} \\
&\quad\leqslant \frac{\varepsilon^2}{16}\sum_{j=1}^{n}\big|B_N^{3/2}\psi_j\big|_{L^2}^2+\frac{c_2}{\varepsilon^{8/9}}\int_\Omega \big|\boldsymbol{u}(x)\big|^{4/3}a_{1/2,2}^{2/3}(x)\,dx \\
&\quad\leqslant \frac{\varepsilon^2}{16}\sum_{j=1}^{n}\big|B_N^{3/2}\psi_j\big|_{L^2}^2+\frac{c_2}{\varepsilon^{8/9}}\left(\int_\Omega \big|\boldsymbol{u}(x)\big|^{12/7}\,dx\right)^{7/9}\left(\int_\Omega a_{1/2,2}^3(x)\,dx\right)^{2/9}.
\end{aligned} \tag{4.26}$$

Since $a_{1/2,2}$ satisfies (4.14), the last expression in (4.26) can be estimated by

$$\frac{\varepsilon^2}{16}\sum_{j=1}^{n}\big|B_N^{3/2}\psi_j\big|_{L^2}^2+\left(\frac{\varepsilon^2}{16}\sum_{j=1}^{n}\big|B_N^{3/2}\psi_j\big|_{L^2}^2+\frac{c_2'}{\varepsilon^{12/7}}|\boldsymbol{u}|_{\mathbb{L}^{12/7}}^{12/7}\right) \leqslant \frac{\varepsilon^2}{8}\sum_{j=1}^{n}\big|B_N^{3/2}\psi_j\big|_{L^2}^2+\frac{c_2''}{\varepsilon^{12/7}}\big(|\Omega|+|\boldsymbol{u}|^2\big).$$

Thus, we deduce that

$$\left|\sum_{j=1}^{n}\varepsilon\big(B_1(\boldsymbol{u},\psi_j),B_N\psi_j\big)_{L^2}\right| \leqslant \frac{\varepsilon^2}{8}\sum_{j=1}^{n}\big|B_N^{3/2}\psi_j\big|_{L^2}^2+\frac{c_2'''}{\varepsilon^{12/7}}\big(|\Omega|+|\boldsymbol{u}|^2\big). \tag{4.27}$$

Next, we estimate

$$\sum_{j=1}^{n}\varepsilon\big(B_1(\boldsymbol{v}_j,\phi),B_N\psi_j\big)_{L^2}=\sum_{j=1}^{n}\int_\Omega \varepsilon\big(\boldsymbol{v}_j(x)\cdot\nabla\phi(x)\big)B_N\psi_j(x)\,dx.$$

Using the Cauchy–Schwarz inequality and (4.14) again, the right-hand side is bounded by

$$\begin{aligned}
\varepsilon^{3/4}\mathcal{K}^{1/4}\int_\Omega \big|\nabla\phi(x)\big|\rho_0^{1/2}(x)a_{1,1}^{1/2}(x)\,dx &\leqslant \varepsilon^{3/4}\mathcal{K}^{1/4}\left(\int_\Omega \rho_0^2(x)\,dx\right)^{1/4}\left(\int_\Omega \big|\nabla\phi(x)\big|^{4/3}a_{1,1}^{2/3}(x)\,dx\right)^{3/4} \\
&\leqslant \frac{\nu}{8\varepsilon\mathcal{K}}\sum_{j=1}^{n}\|\boldsymbol{v}_j\|^2+c_3\varepsilon\nu^3\mathcal{K}^{-8/9}\int_\Omega \big|\nabla\phi(x)\big|^{4/3}a_{1,1}^{2/3}(x)\,dx.
\end{aligned} \tag{4.28}$$

We estimate the last term on the right-hand side of (4.28), using (4.14), as follows:

$$c_3\varepsilon\nu^3\mathcal{K}^{-8/9}\int_\Omega \left|\nabla\phi(x)\right|^{4/3}a_{1,1}^{2/3}(x)\,dx \leqslant c_3\varepsilon\nu^3\mathcal{K}^{-8/9}\left(\int_\Omega \left|\nabla\phi(x)\right|^2 dx\right)^{2/3}\left(\int_\Omega a_{1,1}^2(x)\,dx\right)^{1/3}$$
$$\leqslant \frac{\varepsilon^2}{16}\sum_{j=1}^n \left|B_N^{3/2}\psi_j\right|_{L^2}^2 + c_3'\nu^{9/2}\varepsilon^{1/2}\mathcal{K}^{-4/3}|\nabla\phi|_{L^2}^2. \tag{4.29}$$

Then, from (4.28) and (4.29), we readily see that

$$\left|\sum_{j=1}^n \varepsilon\big(B_1(\boldsymbol{v}_j,\phi), A_\gamma\psi_j\big)_{L^2}\right| \leqslant \frac{\varepsilon^2}{16}\sum_{j=1}^n \left|B_N^{3/2}\psi_j\right|_{L^2}^2 + \frac{\nu}{8\mathcal{K}}\sum_{j=1}^n \|\boldsymbol{v}_j\|^2 + c_3'\nu^{9/2}\varepsilon^{1/2}\mathcal{K}^{-4/3}|\nabla\phi|_{L^2}^2. \tag{4.30}$$

We now estimate the fifth term on the right-hand side of (4.22),

$$\begin{aligned}\left|\sum_{j=1}^n \big(\boldsymbol{R}_0(\varepsilon B_N\phi,\psi_j),\boldsymbol{v}_j\big)_{L^2}\right| &\leqslant \varepsilon^{2/3}\mathcal{K}^{1/4}\int_\Omega \left|B_N\phi(x)\right|\rho_0^{1/2}(x)a_{1/2,2}^{1/2}(x)\,dx\\
&\leqslant \varepsilon^{2/3}\mathcal{K}^{1/4}\left(\int_\Omega \rho_0^{3/5}(x)\left|B_N\phi(x)\right|^{6/5}dx\right)^{5/6}\left(\int_\Omega a_{1/2}^3(x)\,dx\right)^{1/6}\\
&\leqslant c_4\varepsilon^{2/5}\mathcal{K}^{3/10}\int_\Omega \rho_0^{3/5}(x)\left|B_N\phi(x)\right|^{6/5}dx + \frac{\varepsilon^2}{16}\sum_{j=1}^n \left|B_N^{3/2}\psi_j\right|_{L^2}^2\\
&\leqslant c_4\varepsilon^{2/5}\mathcal{K}^{3/10}\left(\int_\Omega \rho_0^2(x)\,dx\right)^{3/10}\left(\int_\Omega \left|B_N\phi(x)\right|^{12/7}dx\right)^{7/10}\\
&\quad + \frac{\varepsilon^2}{16}\sum_{j=1}^n \left|B_N^{3/2}\psi_j\right|_{L^2}^2\\
&\leqslant \frac{c_4'\varepsilon^{4/7}\mathcal{K}^{6/7}}{\nu^{3/7}}|B_N\phi|_{L^{12/7}}^{12/7} + \frac{\nu}{8\mathcal{K}}\sum_{j=1}^n \|\boldsymbol{v}_j\|^2 + \frac{\varepsilon^2}{16}\sum_{j=1}^n \left|B_N^{3/2}\psi_j\right|_{L^2}^2.\end{aligned} \tag{4.31}$$

Thus, we deduce

$$\left|\sum_{j=1}^n \big(\boldsymbol{R}_0(\varepsilon B_N\phi,\psi_j),\boldsymbol{v}_j\big)_{L^2}\right| \leqslant \frac{\nu}{8\mathcal{K}}\sum_{j=1}^n \|\boldsymbol{v}_j\|^2 + \frac{\varepsilon^2}{16}\sum_{j=1}^n \left|B_N^{3/2}\psi_j\right|_{L^2}^2 + \frac{c_4''\varepsilon^{4/7}\mathcal{K}^{6/7}}{\nu^{3/7}}\big(|\Omega| + |B_N\phi|_{L^2}^2\big). \tag{4.32}$$

Furthermore, we have from assumption (2.1) and the Cauchy–Schwarz inequality, that

$$\begin{aligned}&\left|\sum_{j=1}^n \alpha\varepsilon\big(\nabla\big(f'(\phi+M_0)\psi_j\big),\nabla B_N\psi_j\big)_{L^2}\right|\\
&\quad\leqslant \alpha\varepsilon c_f\sum_{j=1}^n\int_\Omega \left|\psi_j(x)\right|\left|\nabla\phi(x)\right|\left|B_N^{3/2}\psi_j(x)\right|\big(1+\left|\phi(x)+M_0\right|^{m-1}\big)\,dx\\
&\qquad + \alpha\varepsilon c_f\sum_{j=1}^n\int_\Omega \left|\nabla\psi_j(x)\right|\left|B_N^{3/2}\psi_j(x)\right|\big(1+\left|\phi(x)+M_0\right|^m\big)\,dx\\
&\quad\leqslant \alpha\varepsilon^{5/8}c_{f,m}\int_\Omega a_{0,3}^{1/2}(x)a_{3/2,0}^{1/2}(x)\left|\nabla\phi(x)\right|\big(1+M_0^{m-1}+\left|\phi(x)\right|^{m-1}\big)\,dx\\
&\qquad + \alpha\varepsilon^{2/3}c_{f,m}\int_\Omega a_{1/2,2}^{1/2}(x)a_{3/2,0}^{1/2}(x)\big(1+M_0^m+\left|\phi(x)\right|^m\big)\,dx\\
&\quad =: J_1 + J_2.\end{aligned} \tag{4.33}$$

We can estimate the term J_2 as follows:

$$\begin{aligned} J_2 &\leqslant c_5\alpha\varepsilon^{2/3}\biggl(\int_\Omega a_{1/2,2}(x)\bigl(1+M_0^{2m}+\bigl|\phi(x)\bigr|^{2m}\bigr)\,dx\biggr)^{1/2}\biggl(\int_\Omega a_{3/2,0}(x)\,dx\biggr)^{1/2} \\ &\leqslant \frac{\varepsilon^2}{16}\sum_{j=1}^n\bigl|B_N^{3/2}\psi_j\bigr|_{L^2}^2 + c_5'\alpha^2\varepsilon^{-2/3}\int_\Omega a_{1/2,2}(x)\bigl(1+M_0^{2m}+\bigl|\phi(x)\bigr|^{2m}\bigr)\,dx \\ &\leqslant \frac{\varepsilon^2}{16}\sum_{j=1}^n\bigl|B_N^{3/2}\psi_j\bigr|_{L^2}^2 + c_5''\alpha^2\varepsilon^{-2/3}\biggl(\int_\Omega a_{1/2,2}^3(x)\,dx\biggr)^{1/3}\biggl(\int_\Omega\bigl(1+M_0^{2m}+\bigl|\phi(x)\bigr|^{2m}\bigr)^{3/2}\,dx\biggr)^{2/3} \\ &\leqslant \frac{\varepsilon^2}{8}\sum_{j=1}^n\bigl|B_N^{3/2}\psi_j\bigr|_{L^2}^2 + c_5'''\alpha^3\varepsilon^{-2}\int_\Omega\bigl(1+M_0^{3m}+\bigl|\phi(x)\bigr|^{3m}\bigr)\,dx, \end{aligned} \tag{4.34}$$

where c_5''' is a suitable constant that depends on c_5, but is independent of Ω. Similarly, we have

$$\begin{aligned} J_1 &\leqslant c_6\alpha\varepsilon^{5/8}\biggl(\int_\Omega a_{0,3}(x)\bigl|\nabla\phi(x)\bigr|^2\bigl(1+M_0^{2(m-1)}+\bigl|\phi(x)\bigr|^{2(m-1)}\bigr)\,dx\biggr)^{1/2}\biggl(\int_\Omega a_{3/2,0}(x)\,dx\biggr)^{1/2} \\ &\leqslant \frac{\varepsilon^2}{16}\sum_{j=1}^n\bigl|B_N^{3/2}\psi_j\bigr|_{L^2}^2 + \alpha c_6'\varepsilon^{-3/4}\int_\Omega a_{0,3}(x)\bigl|\nabla\phi(x)\bigr|^2\bigl(1+M_0^{2(m-1)}+\bigl|\phi(x)\bigr|^{2(m-1)}\bigr)\,dx \\ &\leqslant \frac{\varepsilon^2}{16}\sum_{j=1}^n\bigl|B_N^{3/2}\psi_j\bigr|_{L^2}^2 \\ &\quad + \alpha c_6'\varepsilon^{-3/4}\biggl(\int_\Omega a_0^4(x)\,dx\biggr)^{1/4}\biggl(\int_\Omega\bigl|\nabla\phi(x)\bigr|^{8/3}\bigl(1+M_0^{8(m-1)/3}+\bigl|\phi(x)\bigr|^{8(m-1)/3}\bigr)\,dx\biggr)^{3/4}. \end{aligned} \tag{4.35}$$

Since $a_{0,3}$ satisfies (4.14), we can bound the last expression in (4.35) by

$$\begin{aligned} &\alpha c_6'\varepsilon^{-3/4}\biggl(\int_\Omega a_0^4(x)\,dx\biggr)^{1/4}\biggl(\int_\Omega\bigl|\nabla\phi(x)\bigr|^{8/3}\bigl(1+M_0^{8(m-1)/3}+\bigl|\phi(x)\bigr|^{8(m-1)/3}\bigr)\,dx\biggr)^{3/4} \\ &\quad\leqslant \frac{\varepsilon^2}{16}\sum_{j=1}^n\bigl|B_N^{3/2}\psi_j\bigr|_{L^2}^2 + c_6''\alpha^{4/3}\varepsilon^{-5/3}\biggl(\int_\Omega\bigl|\nabla\phi(x)\bigr|^{8/3}\bigl(1+M_0^{8(m-1)/3}+\bigl|\phi(x)\bigr|^{8(m-1)/3}\bigr)\,dx\biggr) \\ &\quad\leqslant \frac{\varepsilon^2}{16}\sum_{j=1}^n\bigl|B_N^{3/2}\psi_j\bigr|_{L^2}^2 + c_6'''\alpha^{4/3}\varepsilon^{-5/3}|\nabla\phi|_{L^2}^{4/3}\bigl(1+|B_N\phi|_{L^2}^2\bigr)\bigl(|\nabla\phi|_{L^2}^2+\langle\phi\rangle^2\bigr)^{4(m-1)/3}. \end{aligned}$$

Combining the above estimates, from (4.33) and the continuous embedding $H^1(\Omega)\hookrightarrow L^s(\Omega)$, $s\in[1,+\infty)$, we readily deduce that

$$\begin{aligned} &\biggl|\sum_{j=1}^n\alpha\varepsilon\bigl(\nabla\bigl(f'(\phi+M_0)\psi_j\bigr),\nabla B_N\psi_j\bigr)_{L^2}\biggr| \\ &\quad\leqslant \frac{\varepsilon^2}{4}\sum_{j=1}^n\bigl|B_N^{3/2}\psi_j\bigr|_{L^2}^2 + c_5'''\alpha^3\varepsilon^{-2}\bigl[|\Omega|+\bigl(|\nabla\phi|_{L^2}^2+M_0^2\bigr)^{3m/2}\bigr] \\ &\qquad + c_6'''\alpha^{4/3}\varepsilon^{-5/3}|\nabla\phi|_{L^2}^{4/3}\bigl(1+|B_N\phi|_{L^2}^2\bigr)\bigl(|\nabla\phi|_{L^2}^2+M_0^2\bigr)^{4(m-1)/3}. \end{aligned} \tag{4.36}$$

We now treat the seventh term on the right-hand side of (4.22), namely,

$$\sum_{j=1}^{n}\big(\boldsymbol{R}_0(\varepsilon B_N\psi_j,\phi),\boldsymbol{v}_j\big)_{L^2}=\varepsilon\sum_{j=1}^{n}\int_{\Omega}(\boldsymbol{v}_j\cdot\nabla\phi)(x)B_N\psi_j(x)\,dx=:J_3.$$

We have

$$\begin{aligned}|J_3|&\leqslant\varepsilon\int_{\Omega}\big|\nabla\phi(x)\big|\sum_{j=1}^{n}\big|\boldsymbol{v}_j(x)\big|\big|B_N\psi_j(x)\big|\,dx\\&\leqslant\varepsilon^{1/2}\mathcal{K}^{1/4}\int_{\Omega}\big|\nabla\phi(x)\big|\big[\rho_0(x)\big]^{1/2}\big[a_{1,1}(x)\big]^{1/2}\,dx\\&\leqslant\varepsilon^{1/2}\mathcal{K}^{1/4}\bigg(\int_{\Omega}\big|\nabla\phi(x)\big|^{4/3}\rho_0^{2/3}(x)\,dx\bigg)^{3/4}\bigg(\int_{\Omega}a_{1,1}^2(x)\,dx\bigg)^{1/4}\\&\leqslant\frac{\varepsilon^2}{8}\sum_{j=1}^{n}\big|B_N^{3/2}\psi_j\big|_{L^2}^2+c_7\mathcal{K}^{1/3}\int_{\Omega}\big|\nabla\phi(x)\big|^{4/3}\rho_0^{2/3}(x)\,dx\\&\leqslant\frac{\varepsilon^2}{8}\sum_{j=1}^{n}\big|B_N^{3/2}\psi_j\big|_{L^2}^2+c_7\mathcal{K}^{1/3}\bigg(\int_{\Omega}\big|\nabla\phi(x)\big|^2\,dx\bigg)^{2/3}\bigg(\int_{\Omega}\rho_0^2(x)\,dx\bigg)^{1/3}.\end{aligned}\tag{4.37}$$

Thus, by a standard interpolation inequality applied to the second term on the right-hand side of (4.37), we get

$$\bigg|\sum_{j=1}^{n}\big(\boldsymbol{R}_0(\varepsilon B_N\psi_j,\phi),\boldsymbol{v}_j\big)_{L^2}\bigg|\leqslant\frac{\varepsilon^2}{8}\sum_{j=1}^{n}\big|B_N^{3/2}\psi_j\big|_{L^2}^2+\frac{\nu}{8\mathcal{K}}\sum_{j=1}^{n}\|\boldsymbol{v}_j\|^2+c_7'\mathcal{K}\nu^{-1/2}|\nabla\phi|_{L^2}^2.\tag{4.38}$$

We are now ready to estimate $Tr\,\mathcal{M}(\boldsymbol{u},\phi)\cdot Q_n$. Recalling (4.22) and collecting inequalities (4.24)–(4.27), (4.30), (4.32), (4.36) and (4.38), after simple computations, we find that

$$Tr\,\mathcal{M}(\boldsymbol{u},\phi)\cdot Q_n\geqslant\frac{\nu}{2\mathcal{K}}\sum_{j=1}^{n}\|\boldsymbol{v}_j\|^2+\frac{5\varepsilon^2}{8}\sum_{j=1}^{n}\big|B_N^{3/2}\psi_j\big|_{L^2}^2-c_8\mathcal{Z}_{\varepsilon,\nu}(\boldsymbol{u},\phi),$$

where we have set

$$\begin{aligned}\mathcal{Z}_{\varepsilon,\nu}(\boldsymbol{u},\phi)&:=\nu^{-1}\|\boldsymbol{u}\|^2+\varepsilon^{-12/7}\big(|\Omega|+|\boldsymbol{u}|^2\big)+\big(\nu^{9/2}\varepsilon^{1/2}\mathcal{K}^{-4/3}+\mathcal{K}\nu^{-1/2}\big)|\nabla\phi|_{L^2}^2\\&\quad+\nu^{-3/7}\varepsilon^{4/7}\mathcal{K}^{6/7}\big(|\Omega|+|B_N\phi|_{L^2}^2\big)+\alpha^{4/3}\varepsilon^{-5/3}|\nabla\phi|_{L^2}^{4/3}\big(1+|B_N\phi|_{L^2}^2\big)\big(|\nabla\phi|_{L^2}^2+M_0^2\big)^{4(m-1)/3}\\&\quad+\alpha^3\varepsilon^{-2}\big[|\Omega|+\big(|\nabla\phi|_{L^2}^2+M_0^2\big)^{3m/2}\big].\end{aligned}\tag{4.39}$$

Finally, on account of (4.15), we get

$$Tr\,\mathcal{M}(\boldsymbol{u},\phi)\cdot Q_n\geqslant c_9\frac{n^2}{|\Omega|}\bigg(\frac{\nu\varepsilon}{\nu+\varepsilon}\bigg)-c_9'|\Omega|\bigg(\frac{\nu\varepsilon}{\nu+\varepsilon}\bigg)-c_8\mathcal{Z}_{\varepsilon,\nu}(\boldsymbol{u},\phi).\tag{4.40}$$

Since the right-hand side of (4.40) does not depend on $\xi^1,\dots,\xi^n\in\mathbb{Y}_0$, we integrate with respect to the hidden variable s and we find that

$$\begin{aligned}&\frac{1}{t}\int_0^t\inf_{\xi^j\in\mathbb{Y}_0}Tr\,\mathcal{M}\big(\boldsymbol{u}(s),\phi(s)\big)\cdot Q_n(s)\,ds\\&\quad\geqslant c_9\frac{n^2}{|\Omega|}\bigg(\frac{\nu\varepsilon}{\nu+\varepsilon}\bigg)-c_9'|\Omega|\bigg(\frac{\nu\varepsilon}{\nu+\varepsilon}\bigg)-\frac{c_8}{t}\int_0^t\mathcal{Z}_{\varepsilon,\nu}\big(\boldsymbol{u}(s),\phi(s)\big)\,ds.\end{aligned}\tag{4.41}$$

On the other hand, due to Proposition 3.2 (cf. (3.16)–(3.17)), it is easy to see that

$$\begin{aligned}\frac{1}{t}\int_0^t \mathcal{Z}_{\varepsilon,\nu}\big(\boldsymbol{u}(s),\phi(s)\big)\,ds &\leqslant \nu^{-3}\|\boldsymbol{g}\|^2_{\mathbb{V}^*}+\varepsilon^{-12/7}|\Omega|\big(1+\nu^{-2}\|\boldsymbol{g}\|^2_{\mathbb{V}^*}\big)\\ &\quad+\varepsilon^{-1}\big(\nu^{9/2}\varepsilon^{1/2}\mathcal{K}^{-4/3}+\mathcal{K}\nu^{-1/2}\big)\delta_1+\nu^{-3/7}\varepsilon^{4/7}\mathcal{K}^{6/7}\big(|\Omega|+\delta_3\big)\\ &\quad+\alpha^{4/3}\varepsilon^{-(4m+1)/3}\delta_1^{2(2m-1)/3}(1+\delta_3)+\alpha^3\varepsilon^{-2}\big(|\Omega|+\big(\varepsilon^{-1}\delta_1\big)^{3m/2}\big)\\ &=: \delta_4. \end{aligned} \tag{4.42}$$

Recalling the definition of q_n from (4.11), we infer from (4.41)–(4.42) that

$$q_n \geqslant c_9\frac{n^2}{|\Omega|}\left(\frac{\nu\varepsilon}{\nu+\varepsilon}\right)-c_9'|\Omega|\left(\frac{\nu\varepsilon}{\nu+\varepsilon}\right)-c_8\delta_4=\delta_5 n^2-\delta_6 =: \ell(n), \tag{4.43}$$

where

$$\delta_5 := \frac{c_9}{|\Omega|}\left(\frac{\nu\varepsilon}{\nu+\varepsilon}\right), \qquad \delta_6 := c_9'|\Omega|\left(\frac{\nu\varepsilon}{\nu+\varepsilon}\right)+c_8\delta_4.$$

In conclusion, thanks to (4.43), Theorem 4.3 yields

Theorem 4.5. *We consider the dynamical system* $(\mathbb{Y}_0,\overline{\mathcal{S}}(t))$ *associated with Problem* $\mathbf{P}_0$. *Let* $n^*=(\delta_6/\delta_5)^{1/2}$ *and let* $\bar{n}$ *be the first integer such that*

$$\bar{n}\geqslant n^* > \bar{n}-1. \tag{4.44}$$

Then, the corresponding global attractor $\overline{\mathcal{A}}$ *defined by Theorem* 3.10 *has a Hausdorff dimension less than or equal to* $\bar{n}$ *and a fractal dimension less than or equal to* $2\bar{n}$.

Remark 4.6. Actually, we can refer to [19, Corollary 3.1] (see also [8,18]) to deduce that

$$d_F(\overline{\mathcal{A}})\leqslant \bar{n}. \tag{4.45}$$

Indeed, $\ell''(y)>0$ for all $y>0$, so that ℓ is convex. In addition, making use of more refined Lieb–Thirring type inequalities a smaller $\bar{n}$ can possibly be found (cf. [19,38]).

Remark 4.7. Estimate (4.45) gives information about the complexity of a two-phase flow. Although chaotic behavior can be measured and observed for Navier–Stokes equations for single-phase flows (even in two dimensions), the coupling with a convective Cahn–Hilliard equation gives rise to novel and possibly even more complex flow behavior. Indeed, estimate (4.44) yields a number that depends on the kinematic viscosity ν of the fluid, as well as on the capillarity coefficient $\mathcal{K}$ and on the fluid–fluid interface parameter ε, which are as small as ν in many experiments, and on α which is of order ε^{-1}. The dynamics restricted to the global attractor is described by a finite number of parameters, but our estimate indicates that this number might be larger than the one obtained for single-phase flows. Indeed, this is confirmed by a lower estimate recently obtained by analyzing a Kolmogorov-type problem (see [30]).

5. Convergence to equilibria

In this section, we analyze the convergence of given trajectories to stationary states in absence of external forces, i.e., $\boldsymbol{g}=\mathbf{0}$. In particular, we prove that each trajectory converges to a single equilibrium, provided that f is real analytic. A convergence result of this kind is also proven in [2] for a similar system with singular potential, but no convergence rate estimate is provided.

Let us begin with the following straightforward proposition.

Proposition 5.1. *Let the assumptions of Proposition* 3.1 *hold. Then the semigroup* $\mathcal{S}(t)$ *has a* (*strict*) *global Lyapunov functional defined by the free energy, namely,*

$$\mathcal{L}(\boldsymbol{u}_0,\phi_0)=\frac{1}{2}\left[\varepsilon|\nabla\phi_0|^2_{L^2}+\frac{1}{\mathcal{K}}|\boldsymbol{u}_0|^2\right]+\alpha\int_\Omega F(\phi_0)\,dx, \quad \forall(\boldsymbol{u}_0,\phi_0)\in\mathbb{Y}_M.$$

In particular, we have, for all $t > 0$,

$$\frac{d}{dt}\mathcal{L}\big(\boldsymbol{u}(t), \phi(t)\big) = -\frac{\nu}{\mathcal{K}}\big\|\boldsymbol{u}(t)\big\|^2 - \big|\nabla\mu(t)\big|_{L^2}^2. \tag{5.1}$$

Let us now examine more closely the set of equilibria. The stationary problem corresponding to **P** is

$$\begin{cases} \boldsymbol{v} = \mathbf{0}, & \text{in } \Omega, \\ -\Delta\big(-\varepsilon\Delta\psi + \alpha f(\psi)\big) = 0, & \text{in } \Omega, \\ \partial_{\mathbf{n}}\psi = \partial_{\mathbf{n}}\Delta\psi = 0, & \text{on } \Gamma, \\ \langle\psi\rangle = M_0. \end{cases} \tag{5.2}$$

This can be seen from the next two standard results, whose proofs are similar to [28, Section 5] and are left to the reader.

Lemma 5.2. *Let the assumptions of Proposition* 3.1 *hold and let f be real analytic. Suppose that $(\boldsymbol{v}, \psi)$ such that $\boldsymbol{v} \in D(A_0)$ and $\psi, A_N\psi \in D(A_N)$, satisfies* (5.2)*. Then $(\boldsymbol{v}, \psi)$ is a critical point of the functional $\mathcal{L}$ over $\mathbb{Y}_M$. Conversely, if $(\boldsymbol{v}, \psi) \in \mathbb{Y}_M$ is a critical point of $\mathcal{L}$ and Γ is C^∞, then $(\boldsymbol{v}, \psi) \in C^\infty(\overline{\Omega}) \times C^\infty(\overline{\Omega})$ and it is a classical solution to problem* (5.2)*.*

Lemma 5.3. *Let the assumptions of Proposition* 3.1 *hold and let f be real analytic. The functional $\mathcal{L}$ has at least one minimizer $(\mathbf{0}, \psi) \in \mathbb{Y}_M$ whose smoothness depends on Γ, that is,*

$$\mathcal{L}(\mathbf{0}, \psi) = \inf_{(\boldsymbol{u},\phi)\in\mathbb{Y}_M} \mathcal{L}(\boldsymbol{u}, \phi). \tag{5.3}$$

In other words, problem (5.2) *admits at least one* (*possibly*) *classical solution.*

Remark 5.4. When $\boldsymbol{g}$ is small enough (e.g., $\boldsymbol{g} = \mathbf{0}$), it is well known that the global attractor of the 2D Navier–Stokes equation reduces to a single steady state which is globally asymptotically stable (see, e.g., [20, Chap. II]). In presence of a two-phase flow, the global attractor $\mathcal{A}_M$ given by Theorem 3.10 is much richer in structure. Indeed, if $\boldsymbol{g} = \mathbf{0}$, thanks to (5.1), we know that $(\mathbb{Y}_M, \mathcal{S}(t))$ is a gradient system (see, e.g., [55]) so that $\mathcal{A}_M$ coincides with the unstable manifold of the set of the stationary points $(\mathbf{0}, \psi)$ (see (5.2)). However, this set can be a continuum (cf., for instance, [35]). Moreover, in addition to the equilibria, $\mathcal{A}_M$ also contains heteroclinic orbits connecting different equilibria.

We now report some standard implications of the fact that $(\mathbb{Y}_M, \mathcal{S}(t))$ is a gradient system with precompact trajectories (see, e.g., [35]).

Lemma 5.5. *Let the assumptions of Proposition* 3.1 *hold. Then, for any $(\boldsymbol{u}_0, \phi_0) \in \mathbb{Y}_M$, the set $\omega(\boldsymbol{u}_0, \phi_0)$ is a nonempty compact connected subset of $\mathbb{Y}_M$. Furthermore, we have*:

(i) *$\omega(\boldsymbol{u}_0, \phi_0)$ is fully invariant for $\mathcal{S}(t)$;*
(ii) *$\mathcal{L}$ is constant on $\omega(\boldsymbol{u}_0, \phi_0)$;*
(iii) *$dist_{\mathbb{Y}_M}(\mathcal{S}(t)(\boldsymbol{u}_0, \phi_0), \omega(\boldsymbol{u}_0, \phi_0)) \to 0$ as $t \to +\infty$;*
(iv) *$\omega(\boldsymbol{u}_0, \phi_0)$ consists of equilibria only.*

While global and exponential attractors represent the maximal level of complexity that can be observed in a dynamical system, they do not provide, in general, information on the asymptotic behavior of single trajectories. The result below is concerned with the convergence of a trajectory to a single equilibrium, which shows, in a strong form, their asymptotic stability. This constitutes the main result of this section.

Theorem 5.6. *Let the assumptions of Proposition* 3.9 *hold. Suppose, in addition, that f is real analytic. For any given initial datum $(\boldsymbol{u}_0, \phi_0) \in \mathbb{Y}_M$, the corresponding solution $(\boldsymbol{u}(t), \phi(t)) = \mathcal{S}(t)(\boldsymbol{u}_0, \phi_0)$ to* **P** *converges to a single equilibrium $(\mathbf{0}, \psi)$ in the topology of $\mathbb{V} \times D(A_N)$, that is,*

$$\lim_{t\to+\infty}\big(\big\|\boldsymbol{u}(t)\big\| + \big|\phi(t) - \psi\big|_{H^2}\big) = 0. \tag{5.4}$$

Moreover, there exist $C \geqslant 0$ and $\xi \in (0, 1/2)$ depending on $(\mathbf{0}, \psi)$ such that

$$\left|\boldsymbol{u}(t)\right| + \left|\phi(t) - \psi\right|_{H^1} \leqslant C(1+t)^{-\xi/(1-2\xi)}, \quad \forall t \geqslant 0. \tag{5.5}$$

Remark 5.7. It is also worth noting that, by using the smoothing property of the solutions, the convergence result (5.4) as well as the convergence rate estimate (5.5) can be demonstrated with respect to higher-order norms, provided that Γ is smooth enough.

To prove Theorem 5.6, we can assume, without loss of generality, that the solution $(\boldsymbol{u}(t), \phi(t))$ to Problem **P** satisfies the condition $\langle\phi\rangle = 0$ (that is, $M_0 = 0$), since it suffices to replace the solution $(\boldsymbol{u}, \phi)$ by $(\boldsymbol{u}, \phi - M_0)$ and to note that $(\boldsymbol{u}(t), \phi(t) - M_0)$ satisfies the system of Eqs. (3.72) with initial data $(\boldsymbol{u}_0, \phi_0 - M_0)$. Therefore, we replace $F(s)$ by $F(s + M_0)$ in the functional $\mathcal{L} : \mathbb{Y}_0 \to \mathbb{R}$, respectively. The question of whether $(\boldsymbol{u}(t), \phi(t))$ converges as $t \to +\infty$ is not affected by this normalization.

We next state a result which is crucial for the proof of Theorem 5.6. The version of the Łojasiewicz–Simon inequality we need is given by

Lemma 5.8. *Let $(\mathbf{0}, \psi) \in \mathbb{Y}_0$ satisfy (5.2), that is, $(\mathbf{0}, \psi)$ is a critical point of $\mathcal{L}$. Assume that f is real analytic. There exist constants $\zeta \in (0, 1/2)$ and $C_L > 0$, $\zeta > 0$ depending on $(\mathbf{0}, \psi)$ such that, for any $(\boldsymbol{u}, \phi) \in \mathbb{Y}_0$, if*

$$\left\|(\boldsymbol{u}, \phi) - (\mathbf{0}, \psi)\right\|_{\mathbb{Y}_0} \leqslant \zeta,$$

denoting by $\mathcal{L}'$ the Fréchet derivative of $\mathcal{L}$, we have

$$C_L \left\|\mathcal{L}'(\boldsymbol{u}, \phi)\right\|_{\mathbb{Y}_0^*} \geqslant \left|\mathcal{L}(\boldsymbol{u}, \phi) - \mathcal{L}(\mathbf{0}, \psi)\right|^{1-\zeta}. \tag{5.6}$$

Remark 5.9. The proof of Lemma 5.8 can be achieved arguing as in [41] (see also [27,33]).

Proof of Theorem 5.6. We first observe that, if there is $t^\natural \geqslant 0$ such that

$$\mathcal{L}\big(\boldsymbol{u}\big(t^\natural\big), \phi\big(t^\natural\big)\big) = \mathcal{L}_\infty,$$

then, for all $t \geqslant t^\natural$, $\mathcal{L}(\boldsymbol{u}(t), \phi(t)) = \mathcal{L}_\infty$, that is,

$$\phi(t) = \psi, \qquad \boldsymbol{u}(t) = \mathbf{0}, \quad \forall t \geqslant t^\natural.$$

In this case, there is nothing to prove. Therefore, without loss of generality, suppose now that, for all $t \geqslant t_0 \geqslant 0$, we have $\mathcal{L}(\boldsymbol{u}(t), \phi(t)) > \mathcal{L}_\infty$. We observe that, by Lemmas 5.5 and 5.8, the functional $\mathcal{L}$ satisfies the Łojasiewicz–Simon inequality (5.6) near every $(\mathbf{0}, \psi) \in \omega(\boldsymbol{u}_0, \phi_0)$. Since $\omega(\boldsymbol{u}_0, \phi_0)$ is compact in $\mathbb{Y}_0$, we can cover it by the union of finitely many balls $\mathcal{B}_j$ with centers $(\mathbf{0}, \psi^j)$ and radii r_j, where each radius is such that (5.6) holds in $\mathcal{B}_j$. Since $\mathcal{L} = \mathcal{L}_\infty$ on $\omega(\boldsymbol{u}_0, \phi_0)$, it follows from Lemma 5.8 that there exist uniform constants $\xi \in (0, 1/2)$, $C_L > 0$ (depending on $(\mathbf{0}, \psi)$) and a neighborhood $\mathcal{V}$ of $\omega(\boldsymbol{u}_0, \phi_0)$ such that

$$C_L \left\|\mathcal{L}'(\boldsymbol{u}, \phi)\right\|_{\mathbb{Y}_0^*} \geqslant \left|\mathcal{L}(\boldsymbol{u}, \phi) - \mathcal{L}_\infty\right|^{1-\xi}, \quad \forall (\boldsymbol{u}, \phi) \in \mathcal{V}. \tag{5.7}$$

Recalling property (iii) of Lemma 5.5, we can find a time $t_1 > 0$ such that $(\boldsymbol{u}(t), \phi(t))$ belongs to $\mathcal{V}$, for all $t \geqslant t_1$. Set now $t_2 \geqslant \max\{t_0, t_1\}$ so that Proposition 3.9 holds. Recalling (5.1), we obtain, for every $t \geqslant t_2$,

$$\begin{aligned} -\frac{d}{dt}\big(\mathcal{L}\big(\boldsymbol{u}(t), \phi(t)\big) - \mathcal{L}_\infty\big)^\xi &= \xi\left(-\frac{d}{dt}\mathcal{L}\big(\boldsymbol{u}(t), \phi(t)\big)\right)\big(\mathcal{L}\big(\boldsymbol{u}(t), \phi(t)\big) - \mathcal{L}_\infty\big)^{\xi-1} \\ &\geqslant \frac{\xi}{C_L}\frac{(\nu/\mathcal{K})\|\boldsymbol{u}(t)\|^2 + |\nabla\mu(t)|_{L^2}^2}{\|\mathcal{L}'(\boldsymbol{u}(t), \phi(t))\|_{\mathbb{Y}_0^*}}. \end{aligned} \tag{5.8}$$

Using now Green's formula on Ω, since $k \in \{\phi \in H^1(\Omega): \langle\phi\rangle = 0\}$, we obtain

$$\big\langle\mathcal{L}'(\boldsymbol{u}, \phi), (\boldsymbol{h}, k)\big\rangle_{\mathbb{Y}_0^*, \mathbb{Y}_0} = \int_\Omega \big(-\varepsilon\Delta\phi + \alpha f(\phi) - \langle\mu\rangle\big)k\, dx + \int_\Omega \boldsymbol{u}\cdot\boldsymbol{h}\frac{dx}{\mathcal{K}}, \tag{5.9}$$

where $\mu = -\varepsilon\Delta\phi + \alpha f(\phi)$. Hence, by using the Cauchy–Schwarz inequality and Poincaré's inequality, we obtain

$$\begin{aligned}\|\mathcal{L}'(\boldsymbol{u},\phi)\|_{\mathbb{Y}_0^*} &= \sup_{\|(\boldsymbol{h},k)\|_{\mathbb{Y}_0}\leqslant 1} \langle \mathcal{L}'(\boldsymbol{u},\phi),(\boldsymbol{h},k)\rangle_{\mathbb{Y}_0^*,\mathbb{Y}_0} \\ &\leqslant C_*\big(|\mu-\langle\mu\rangle|_{L^2}+\sqrt{\nu/\mathcal{K}}\|\boldsymbol{u}\|\big) \\ &\leqslant C_*\big(|\nabla\mu|_{L^2}+\sqrt{\nu/\mathcal{K}}\|\boldsymbol{u}\|\big),\end{aligned} \tag{5.10}$$

where C_* depends on ν and Ω, but is independent of time and initial data. Inserting now estimate (5.10) into estimate (5.8), we deduce

$$-\frac{d}{dt}\big(\mathcal{L}\big(\boldsymbol{u}(t),\phi(t)\big)-\mathcal{L}_\infty\big)^\xi \geqslant C\big(|\nabla\mu(t)|_{L^2}+\sqrt{\nu/\mathcal{K}}\|\boldsymbol{u}(t)\|\big). \tag{5.11}$$

Here C is some positive constant depending on C_*, C_L and ξ. By integrating this inequality on $[t_2,+\infty)$, and using the fact that $\mathcal{L}(\boldsymbol{u}(t),\phi(t)) \to \mathcal{L}_\infty$ as t goes to $+\infty$, we also infer that

$$\nabla\mu \in L^1\big([t_2,+\infty);\mathbb{L}^2(\Omega)\big), \qquad \boldsymbol{u} \in L^1\big([t_2,+\infty);\mathbb{V}\big). \tag{5.12}$$

Consequently, since $|B_1(\boldsymbol{u},\phi)|_{H^{-1}} \leqslant c\|\boldsymbol{u}\||\nabla\phi|_{L^2}$, we also deduce, on account of (5.12) and the last equation of (2.9), that

$$\partial_t\phi \in L^1\big([t_2,+\infty);H^{-1}(\Omega)\big). \tag{5.13}$$

Furthermore, setting $\mathbb{W} := \mathbb{V}\cap\mathbb{H}^2(\Omega)$, the following bounds are also consequences of [51, Proposition 9.2, (9.25)–(9.26)] and standard Sobolev embeddings:

$$\begin{aligned}&\|B_0(\boldsymbol{u},\boldsymbol{u})\|_{\mathbb{V}^*} \leqslant c|\boldsymbol{u}|\|\boldsymbol{u}\|, \\ &\|\boldsymbol{R}_0(\varepsilon A_N\phi,\phi)\|_{\mathbb{W}^*} = \|\boldsymbol{R}_0\big(\mu-\langle\mu\rangle,\phi\big)\|_{\mathbb{W}^*} \leqslant c|\nabla\phi|_{L^2}|\mu-\langle\mu\rangle|_{L^2}.\end{aligned} \tag{5.14}$$

Consequently, employing these inequalities, on account of (5.12), Poincaré's inequality and the first equation of (2.9), we also deduce that

$$\partial_t\boldsymbol{u} \in L^1\big([t_2,+\infty);\mathbb{W}^*\big). \tag{5.15}$$

We now recall that, due to Lemma 3.7, there exists an increasing unbounded sequence $\{t_k\}$ and an element $(\boldsymbol{0},\psi) \in \omega(\boldsymbol{u}_0,\phi_0)$ such that $(\boldsymbol{u}(t_l),\phi(t_l)) \to (\boldsymbol{0},\psi)$ in $\mathbb{H}\times H^1(\Omega)$ as l goes to $+\infty$. This fact combined with the above L^1-integrability (5.13)–(5.15) imply that $(\boldsymbol{u}(t),\phi(t)) \to (\boldsymbol{0},\psi)$ in $\mathbb{W}^*\times H^{-1}(\Omega)$ as t goes to $+\infty$ and in $\mathbb{V}\times D(A_N)$ as well, thanks to Proposition 3.9. Hence $\omega(\boldsymbol{u}_0,\phi_0) = \{(\boldsymbol{0},\psi)\}$ and (5.4) holds.

It remains to prove (5.5). From now on C will stand for a generic positive constant which depends on the initial data, on the equilibrium $(\boldsymbol{0},\psi)$ and on the parameters of the problem, but it is independent of time. For $t \geqslant t_2$, it follows from (5.7) and (5.8) that

$$\frac{d}{dt}\big(\mathcal{L}\big(\boldsymbol{u}(t),\phi(t)\big)-\mathcal{L}_\infty\big)^\xi + C\big(\mathcal{L}\big(\boldsymbol{u}(t),\phi(t)\big)-\mathcal{L}_\infty\big)^{1-\xi} \leqslant 0. \tag{5.16}$$

Then, we deduce that

$$\mathcal{L}\big(\boldsymbol{u}(t),\phi(t)\big)-\mathcal{L}_\infty \leqslant C(1+t)^{-1/(1-2\xi)}, \quad \forall t\geqslant t_2. \tag{5.17}$$

Thus, integrating (5.11) on $[t,+\infty)$, thanks to estimate (5.17), we get

$$\int_t^{+\infty}\big(|\nabla\mu(s)|_{L^2}+\sqrt{\nu/\mathcal{K}}\|\boldsymbol{u}(s)\|\big)\,ds \leqslant C(1+t)^{-\xi/(1-2\xi)}, \quad \forall t\geqslant t_2. \tag{5.18}$$

By properly adjusting the constant C in (5.18), from (5.13)–(5.15) we also infer

$$|\phi(t)-\psi|_{H^{-1}} \leqslant C(1+t)^{-\xi/(1-2\xi)}, \quad \forall t\geqslant t_2, \tag{5.19}$$

$$\|\boldsymbol{u}(t)\|_{\mathbb{W}^*} \leqslant C(1+t)^{-\xi/(1-2\xi)}, \quad \forall t\geqslant t_2. \tag{5.20}$$

Taking advantage of the above (lower order) convergence estimates we can prove the higher-order ones. For this purpose, let us set $\varphi := \phi - \psi$, $\boldsymbol{w} := \boldsymbol{u} - \boldsymbol{0}$ and notice that these functions solve the following equations

$$\begin{cases} \partial_t \boldsymbol{w} + \nu A_0 \boldsymbol{w} + B_0(\boldsymbol{w}, \boldsymbol{w}) = \mathcal{K}\boldsymbol{R}_0(\varepsilon A_N \varphi, \phi) + \mathcal{K}\boldsymbol{R}_0(\varepsilon A_N \psi, \varphi), \\ \widehat{\mu} = \varepsilon A_N \varphi + \alpha\big(f(\phi) - f(\psi)\big), \\ \partial_t \varphi + A_N \widehat{\mu} + B_1(\boldsymbol{w}, \varphi) + B_1(\boldsymbol{w}, \psi) = 0. \end{cases} \tag{5.21}$$

We multiply the first equation of (5.21) by $2A_0^{-1}\boldsymbol{w}(t)$ and the second and third ones each one by $2A_N\varphi(t)$ and $2\varphi(t)$, respectively. Integrating the obtained relations over Ω, and then adding the results, we obtain, after obvious manipulations,

$$\begin{aligned} &\frac{d}{dt}\big[\|\boldsymbol{w}(t)\|_{\mathbb{V}^*}^2 + |\varphi(t)|_{L^2}^2\big] + 2\nu|\boldsymbol{w}(t)|^2 + 2\varepsilon|A_N\varphi(t)|_{L^2}^2 \\ &\quad = -2b_0\big(\boldsymbol{w}(t), \boldsymbol{w}(t), A_0^{-1}\boldsymbol{w}(t)\big) + 2\mathcal{K}\big(\boldsymbol{R}_0\big(\varepsilon A_N\psi(t), \varphi(t)\big), A_0^{-1}\boldsymbol{w}(t)\big)_{L^2} \\ &\qquad + 2\mathcal{K}\big(\boldsymbol{R}_0\big(\varepsilon A_N\varphi(t), \phi(t)\big), A_0^{-1}\boldsymbol{w}(t)\big)_{L^2} - 2b_1\big(\boldsymbol{w}(t), \psi(t), \varphi(t)\big) - 2\alpha\big(f\big(\phi(t)\big) - f\big(\psi(t)\big), A_N\varphi(t)\big)_{L^2} \\ &\quad =: \Lambda_4(t). \end{aligned} \tag{5.22}$$

Setting now

$$\mathcal{Y}_5(t) := \|\boldsymbol{w}(t)\|_{\mathbb{V}^*}^2 + |\varphi(t)|_{L^2}^2,$$

we can rewrite the above energy equality as follows:

$$\frac{d}{dt}\mathcal{Y}_5(t) + \kappa\mathcal{Y}_5(t) + (2\nu - \kappa c_\Omega)|\boldsymbol{w}(t)|^2 + (2\varepsilon - \kappa c_\Omega)|A_N\varphi(t)|_{L^2}^2 = \Lambda_4(t),$$

provided that $\kappa \in (0, \max\{\nu, \varepsilon\})$ is sufficiently small.

Observe now that

$$\begin{aligned} |\Lambda_4| &\leqslant 2\big|b_0\big(\boldsymbol{w}, \boldsymbol{w}, A_0^{-1}\boldsymbol{w}\big)\big| + 2\mathcal{K}\big|\big(\boldsymbol{R}_0(\varepsilon A_N\psi, \varphi), A_0^{-1}\boldsymbol{w}\big)_{L^2}\big| + 2\mathcal{K}\big|\big(\boldsymbol{R}_0(\varepsilon A_N\varphi, \phi), A_0^{-1}\boldsymbol{w}\big)_{L^2}\big| \\ &\quad + 2\big|b_1(\boldsymbol{w}, \psi, \varphi)\big| + 2\alpha\big|\big(f(\phi) - f(\psi), A_N\varphi\big)_{L^2}\big|. \end{aligned} \tag{5.23}$$

Using Agmon's inequality, standard interpolation, Young's inequality and Proposition 3.9, we have

$$\begin{aligned} 2b_0\big(\boldsymbol{w}, \boldsymbol{w}, A_0^{-1}\boldsymbol{w}\big) &\leqslant c|\boldsymbol{w}|_{L^\infty}\|\boldsymbol{w}\|\big|A_0^{-1}\boldsymbol{w}\big|_{L^2} \\ &\leqslant C|\boldsymbol{w}|^{1/2}|\boldsymbol{w}|_{\mathbb{W}}^{1/2}\|\boldsymbol{w}\|\|\boldsymbol{w}\|_{\mathbb{V}^*} \\ &\leqslant C|\boldsymbol{w}||\boldsymbol{w}|_{\mathbb{W}}\|\boldsymbol{w}\|_{\mathbb{V}^*} \\ &\leqslant C|\boldsymbol{w}|^{3/2}|\boldsymbol{w}|_{\mathbb{W}}\|\boldsymbol{w}\|_{\mathbb{W}^*}^{1/2} \leqslant \kappa c_\Omega|\boldsymbol{w}|^2 + C\|\boldsymbol{w}\|_{\mathbb{W}^*}^2. \end{aligned} \tag{5.24}$$

Using again standard interpolation inequalities, Proposition 3.9 and Poincaré's inequality, noting that $\langle\varphi(t)\rangle = 0$, we deduce

$$\begin{aligned} 2b_1(\boldsymbol{w}, \psi, \varphi) &\leqslant c|\boldsymbol{w}||\psi|_{H^1}|\varphi|_{L^2}^{1/2}|A_N\varphi|_{L^2}^{1/2} \\ &\leqslant C|\boldsymbol{w}|^{4/3}|\varphi|_{L^2}^{2/3} + \kappa c_\Omega|A_N\varphi|_{L^2}^2 \\ &\leqslant C|\boldsymbol{w}|^{4/3}|\varphi|_{H^{-1}}^{1/3}|\nabla\varphi|_{L^2}^{1/3} + \kappa c_\Omega|A_N\varphi|_{L^2}^2 \\ &\leqslant C|\boldsymbol{w}|^{8/5}|\varphi|_{H^{-1}}^{2/5} + 2\kappa c_\Omega|A_N\varphi|_{L^2}^2 \\ &\leqslant C|\varphi|_{H^{-1}}^2 + \kappa c_\Omega|\boldsymbol{w}|^2 + 2\kappa c_\Omega|A_N\varphi|_{L^2}^2. \end{aligned} \tag{5.25}$$

Arguing similarly, we get

$$\begin{aligned} 2\mathcal{K}\big(\boldsymbol{R}_0(\varepsilon A_N\psi, \varphi), A_0^{-1}\boldsymbol{w}\big)_{L^2} &\leqslant c_\Omega\big|A_0^{-1}\boldsymbol{w}\big|^{1/2}\big\|A_0^{-1}\boldsymbol{w}\big\|^{1/2}|\nabla\varphi|_{L^2}|A_N\psi|_{L^2}^{1/2}|A_N\psi|_{H^1}^{1/2} \\ &\leqslant C\|\boldsymbol{w}\|_{\mathbb{W}^*}^{1/2}\|\boldsymbol{w}\|_{\mathbb{V}^*}^{1/2}|A_N\varphi|_{L^2} \\ &\leqslant C\|\boldsymbol{w}\|_{\mathbb{W}^*}^{1/2}\big(|\boldsymbol{w}|^{1/4}\|\boldsymbol{w}\|_{\mathbb{W}^*}^{1/4}\big)|A_N\varphi|_{L^2} \end{aligned}$$

$$\begin{aligned}&\leqslant \kappa c_\Omega |A_N\varphi|_{L^2}^2 + C|\boldsymbol{w}|^{1/2}\|\boldsymbol{w}\|_{\mathbb{W}^*}^{3/2}\\&\leqslant \kappa c_\Omega |A_N\varphi|_{L^2}^2 + \kappa c_\Omega|\boldsymbol{w}|^2 + C\|\boldsymbol{w}\|_{\mathbb{W}^*}^2 \end{aligned}\tag{5.26}$$

and

$$\begin{aligned}2\mathcal{K}\big(\boldsymbol{R}_0(\varepsilon A_N\varphi,\phi), A_0^{-1}\boldsymbol{w}\big)_{L^2} &\leqslant c_\Omega\big|A_0^{-1}\boldsymbol{w}\big|^{1/2}\big\|A_0^{-1}\boldsymbol{w}\big\|^{1/2}|\phi|_{H^1}^{1/2}|A_N\phi|_{L^2}^{1/2}|A_N\varphi|_{L^2}\\&\leqslant C\|\boldsymbol{w}\|_{\mathbb{W}^*}^{1/2}\|\boldsymbol{w}\|_{\mathbb{V}^*}^{1/2}|A_N\varphi|_{L^2}\\&\leqslant C\|\boldsymbol{w}\|_{\mathbb{W}^*}^{1/2}\big(|\boldsymbol{w}|^{1/4}\|\boldsymbol{w}\|_{\mathbb{W}^*}^{1/4}\big)|A_N\varphi|_{L^2}\\&\leqslant \kappa c_\Omega|A_N\varphi|_{L^2}^2 + C\|\boldsymbol{w}\|_{\mathbb{W}^*}^{3/2}|\boldsymbol{w}|^{1/2}\\&\leqslant \kappa c_\Omega|A_N\varphi|_{L^2}^2 + \kappa c_\Omega|\boldsymbol{w}|^2 + C\|\boldsymbol{w}\|_{\mathbb{W}^*}^2.\end{aligned}$$

Besides, we obtain

$$\begin{aligned}2\alpha\big|\big(f(\phi)-f(\psi),A_N\varphi\big)_{L^2}\big| &\leqslant C|\varphi|_{L^2}|A_N\varphi|_{L^2} \leqslant C|\varphi|_{H^{-1}}^{1/2}|\nabla\varphi|_{L^2}^{1/2}|A_N\varphi|_{L^2}\\&\leqslant \kappa c_\Omega|A_N\varphi|_{L^2}^2 + C|\varphi|_{H^{-1}}^2.\end{aligned}$$

Thus, combining all the above estimates, from (5.23), we deduce that

$$|\Lambda_4| \leqslant 5\kappa c_\Omega|A_N\varphi|_{L^2}^2 + 4\kappa c_\Omega|\boldsymbol{w}|^2 + C\big(|\varphi|_{H^{-1}}^2 + \|\boldsymbol{w}\|_{\mathbb{W}^*}^2\big).\tag{5.27}$$

Combining (5.27) with (5.22), then using (5.27) and (5.18)–(5.19), it is possible to find $\kappa > 0$ and $\kappa' > 0$ such that, for all $t \geqslant t_2$,

$$\frac{d}{dt}\mathcal{Y}_5(t) + \kappa\mathcal{Y}_5(t) + \kappa'\big(\big|\boldsymbol{w}(t)\big|^2 + \big|A_N\varphi(t)\big|_{L^2}^2\big) \leqslant C(1+t)^{-2\xi/(1-2\xi)}.\tag{5.28}$$

Consequently, from (5.28), we deduce that

$$\begin{aligned}\mathcal{Y}_5(t) &\leqslant \mathcal{Y}_5(t_2)e^{\kappa(t_2-t)} + Ce^{-\kappa t}\int_{t_2}^{t} e^{\kappa\tau}(1+\tau)^{-2\xi/(1-2\xi)}\,d\tau\\&\leqslant Ce^{-\kappa t} + Ce^{-\kappa t}\left(\int_0^{t/2} e^{\kappa\tau}(1+\tau)^{-2\xi/(1-2\xi)}\,d\tau + \int_{t/2}^{t} e^{\kappa\tau}(1+\tau)^{-2\xi/(1-2\xi)}\,d\tau\right)\\&\leqslant Ce^{-\kappa t} + Ce^{-\kappa t}\left(e^{(\kappa/2)t}\int_0^{t/2}(1+\tau)^{-2\xi/(1-2\xi)}\,d\tau + C(1+t)^{-2\xi/(1-2\xi)}e^{\kappa t}\right)\\&\leqslant C(1+t)^{-2\xi/(1-2\xi)}, \quad \forall t \geqslant t_2,\end{aligned}\tag{5.29}$$

which implies that, for any $t \geqslant t_2$,

$$\int_{t_2}^{t}\big(\big|\boldsymbol{w}(s)\big|^2 + \big|A_N\varphi(s)\big|_{L^2}^2\big)\,ds \leqslant C(1+t)^{-2\xi/(1-2\xi)},$$

$$\big\|\boldsymbol{w}(t)\big\|_{\mathbb{V}^*} + \big|\varphi(t)\big|_{L^2} \leqslant C(1+t)^{-\xi/(1-2\xi)}.$$

In order to deduce (5.5), we now multiply again the first equation of (5.21) by $2\boldsymbol{w}(t)$, and the remaining two each one by $2B_N^2\varphi(t)$ and $2B_N\varphi(t)$, respectively. Then we integrate the obtained relations over Ω. Adding these energy equalities, we obtain the following energy equality (recall that, in the present case, $B_N\varphi = A_N\varphi$)

$$\begin{aligned}&\frac{d}{dt}\mathcal{Y}_6(t) + 2\nu\big\|\boldsymbol{w}(t)\big\|^2 + 2\varepsilon\big|B_N^{3/2}\varphi(t)\big|_{L^2}^2\\&\quad= 2\mathcal{K}\big(\boldsymbol{R}_0\big(\varepsilon B_N\psi(t),\varphi(t)\big),\boldsymbol{w}(t)\big)_{L^2} - 2b_1\big(\boldsymbol{w}(t),\varphi(t),B_N\varphi(t)\big) + 2\mathcal{K}\big(\boldsymbol{R}_0\big(\varepsilon B_N\varphi(t),\phi(t)\big),\boldsymbol{w}(t)\big)_{L^2}\\&\qquad - 2b_1\big(\boldsymbol{w}(t),\psi(t),B_N\varphi(t)\big) - 2\alpha\big(\nabla\big(f\big(\phi(t)\big)-f\big(\psi(t)\big)\big),\nabla B_N\varphi(t)\big)_{L^2} =: \Lambda_5(t),\end{aligned}\tag{5.30}$$

where

$$\mathcal{Y}_6(t) := \left|\boldsymbol{w}(t)\right|^2 + \left|\nabla\varphi(t)\right|_{L^2}^2.$$

After repeated manipulations and computations, similar to (5.22)–(5.28), it is not difficult to show that

$$\left|\mathcal{Y}_6(t)\right| \leqslant C\left(\left\|\boldsymbol{w}(t)\right\|_{\mathbb{V}^*}^2 + \left|\varphi(t)\right|_{L^2}^2\right) + c\kappa\left(\left\|\boldsymbol{w}(t)\right\|^2 + \left|B_N^{3/2}\varphi(t)\right|_{L^2}^2\right),$$

provided that $\kappa > 0$ is sufficiently small. Finally, we have that $\mathcal{Y}_6$ satisfies an energy inequality analogous to (5.28) and to argue exactly as in (5.29), in order to obtain the conclusion of our theorem. The rigorous details are left to the reader, the argument being the same as the one leading to (5.29). □

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