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A blow-up criterion for compressible viscous heat-conductive flows

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Abstract

We study an initial boundary value problem for the three-dimensional Navier–Stokes equations of viscous heat-conductive fluids in a bounded smooth domain. We establish a blow-up criterion for the local strong solutions in terms of the temperature and the gradient of velocity only, similar to the Beale–Kato–Majda criterion for ideal incompressible flows.

Résumé

Nous étudions un problème de valeur limite initiale pour les équations de Navier–Stokes tridimensionnelles des fluides visqueux conducteurs de chaleur dans un domaine délimité lisse. Nous établissons un critère d'explosion pour les solutions fortes en termes de température et de gradient de vitesse seulement, semblable au critère de Beale–Kato–Majda pour les écoulements incompressibles idéaux.

MSC: 76N10; 35M10; 35Q30

Keywords: Blow-up criterion; Strong solutions; Compressible Navier–Stokes equations; Heat-conductive flows

1. Introduction

This paper is concerned with a blow-up criterion for the three-dimensional Navier–Stokes equations of a viscous heat-conductive gas which describe the conservation of mass, momentum and total energy, and can be written in the following form:

$$\partial_t \rho + \operatorname{div}(\rho u) = 0, \tag{1.1}$$

$$\partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) - \mu \Delta u - (\lambda + \mu)\nabla \operatorname{div} u + \nabla P = 0, \tag{1.2}$$

$$c_V\big(\partial_t(\rho\theta) + \operatorname{div}(\rho\theta)\big) - \kappa \Delta\theta + P \operatorname{div} u = \frac{\mu}{2}\left|\nabla u + \nabla u^t\right|^2 + \lambda(\operatorname{div} u)^2. \tag{1.3}$$

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Here we denote by ρ, θ and $u = (u_1, u_2, u_3)$ the density, temperature, and velocity, respectively. The physical constants μ, λ are the viscosity coefficients satisfying $\mu > 0$, $\lambda + 2\mu/3 \geqslant 0$, $c_V > 0$ and $\kappa > 0$ are the specific heat at constant volume and thermal conductivity coefficient, respectively. P is the pressure which is a known function of ρ and θ, and in the case of an ideal gas P has the following form

$$P = R\rho\theta, \tag{1.4}$$

where $R > 0$ is a generic gas constant.

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$ and exterior normal vector ν. We will consider an initial boundary value problem for (1.1)–(1.3) in $Q := (0, \infty) \times \Omega$ with initial and boundary conditions:

$$(\rho, u, \theta)|_{t=0} = (\rho_0, u_0, \theta_0) \quad \text{in } \Omega, \tag{1.5}$$

$$u = 0, \qquad \frac{\partial\theta}{\partial\nu} = 0 \quad \text{on } \partial\Omega. \tag{1.6}$$

In the last decades significant progress has been made in the study of global in time existence for the system (1.1)–(1.6). With the assumption that the initial data are sufficiently small, Matsumura and Nishida [14,15] first proved the global existence of smooth solutions to initial boundary value problems and the Cauchy problem for (1.1)–(1.3), and the existence of global weak solutions was shown by Hoff [7]. For large data, however, it is still an open question whether a global solution to (1.1)–(1.6) exists or not, except certain special cases, such as the spherically symmetric case in domains without the origin, see [10] for example. Recently, Feireisl [5,6] obtained the global existence of the so-called "variational solutions" to (1.1)–(1.3) in the case of real gases in the sense that the energy equation is replaced by an energy inequality. However, this result excludes the case of ideal gases unfortunately. We mention that in the isentropic case, the existence of global weak solutions of the multidimensional compressible Navier–Stokes equations was first shown by Lions [13], and his result was then improved and generalized in [4,11,12], and among others.

Xin [18], Rozanova [16] showed the non-existence of global smooth solutions when the initial density is compactly supported, or decreases to zero rapidly. Since the system (1.1)–(1.3) is a model of non-dilute fluids, these non-existence results are natural to expect when vacuum regions are present initially. Thus, it is very interesting to investigate whether a strong or smooth solution will still blow up in finite time, when there is no vacuum initially. Recently, Fan and Jiang [3] proved the following blow-up criteria for the local strong solutions to (1.1)–(1.6) in the case of two dimensions:

$$\lim_{T\to T^*}\left(\sup_{0\leqslant t\leqslant T}\left\{\|\rho\|_{L^\infty}, \left\|\rho^{-1}\right\|_{L^\infty}, \|\theta\|_{L^\infty}\right\}(t) + \int_0^T \left(\|\rho\|_{W^{1,q_0}} + \|\nabla\rho\|_{L^2}^4 + \|u\|_{L^{r,\infty}}^{\frac{2r}{r-2}}\right) dt\right) = \infty,$$

or,

$$\lim_{T\to T^*}\left(\sup_{0\leqslant t\leqslant T}\left\{\|\rho\|_{L^\infty}, \left\|\rho^{-1}\right\|_{L^\infty}, \|\theta\|_{L^\infty}\right\}(t) + \int_0^T \left(\|\rho\|_{W^{1,q_0}} + \|\nabla\rho\|_{L^2}^4\right) dt\right) = \infty,$$

provided $2\mu > \lambda$, where $T^* < \infty$ is the maximal time of existence of a strong solution (ρ, u), $q_0 > 3$ is a certain number, $3 < r \leqslant \infty$ with $2/s + 3/r = 1$, and $L^{r,\infty} \equiv L^{r,\infty}(\Omega)$ is the Lorentz space.

In the isentropic case, the result in [3] reduces to

$$\lim_{T\to T^*}\left(\sup_{0\leqslant t\leqslant T}\|\rho\|_{L^\infty} + \int_0^T \left(\|\rho\|_{W^{1,q_0}} + \|\nabla\rho\|_{L^2}^4\right)\right) = \infty, \tag{1.7}$$

provided $7\mu > 9\lambda$. Very recently, Huang and Xin [9] established the following blow-up criterion, similar to the Beale–Kato–Majda criterion for ideal incompressible flows [1], for the isentropic compressible Navier–Stokes equations:

$$\lim_{t\to T^*}\int_0^T \|\nabla u\|_{L^\infty}\, dt = \infty, \tag{1.8}$$

provided

$$7\mu > \lambda. \tag{1.9}$$

The aim of the current paper is to extend the result in [9] to the non-isentropic flows, that is, to establish a blow-up criterion similar to (1.8) for the non-isentropic Navier–Stokes equations. This is a nontrivial generalization, since one has to control the terms involved with the temperature by assuming additionally upper boundedness of the temperature, but not upper boundedness of derivatives of the temperature. Furthermore, it is not necessary to derive a positive lower bound of the temperature in our proof. These are exactly the new points of this paper.

At the same time, the current paper also generalizes the result in [3] in the sense that the restriction on the viscosity coefficients in (1.7) is relaxed and the vacuum is allowed in an open subset of Ω initially. Moreover, it is interesting to see that the a priori assumption (2.1) is more concise than the one in [3].

For the sake of generality, we will study the blow-up criterion for local strong solutions with initial vacuum, the existence of which is essentially obtained in [2]. The case that the initial density has a positive lower bound can be dealt with in the same manner (in fact, simpler) and the same result holds.

Before giving our main result, we state the following local existence of the strong solutions with initial vacuum, the proof of which can be found in [2].

Proposition 1.1 *(Local Existence). Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Suppose that the initial data ρ_0, u_0, θ_0 satisfy*

$$\begin{aligned} &\rho_0 \geqslant 0, \quad \rho_0 \in W^{1,q}(\Omega) \text{ for some } 3 < q \leqslant 6, \\ &u_0 \in H_0^1(\Omega) \cap H^2(\Omega), \qquad \inf_{x\in\Omega} \theta_0(x) > 0, \quad \theta_0 \in H^2(\Omega), \end{aligned} \tag{1.10}$$

and the compatibility conditions

$$\begin{aligned} &\mu\Delta u_0 + (\mu+\lambda)\nabla \operatorname{div} u_0 - R\nabla(\rho_0\theta_0) = \rho_0^{1/2} g_1, \\ &\kappa\Delta\theta_0 + \frac{\mu}{2}\left|\nabla u_0 + \nabla u_0^t\right|^2 + \lambda(\operatorname{div} u_0)^2 - R\rho_0\theta_0 \operatorname{div} u_0 = \rho_0^{1/2} g_2, \end{aligned} \tag{1.11}$$

for some $g_1, g_2 \in L^2(\Omega)$. Moreover, $\{x \in \Omega \mid \rho_0(x) = 0\}$ is an open subset of Ω. Then there exist a positive constant T_0 and a unique strong solution (ρ, θ, u) to (1.1)–(1.6)*, such that*

$$\begin{aligned} &\rho \geqslant 0, \quad \rho \in C\big([0,T_0]; W^{1,q}\big), \quad \rho_t \in C\big([0,T_0]; L^q\big), \\ &u \in C\big([0,T_0]; H_0^1 \cap H^2\big) \cap L^2\big(0,T_0; W^{2,q}\big), \\ &\sqrt{\rho} u_t \in L^\infty\big(0,T_0; L^2\big), \quad u_t \in L^2\big(0,T_0; H_0^1\big), \\ &\theta > 0, \quad \theta \in C\big([0,T_0]; H^2\big) \cap L^2\big(0,T_0; W^{2,q}\big), \\ &\sqrt{\rho}\theta_t \in L^\infty\big(0,T_0; L^2\big), \quad \theta_t \in L^2\big(0,T_0; H^1\big). \end{aligned} \tag{1.12}$$

We remark that in Proposition 1.1, $\theta > 0$ can be obtained when the initial temperature is bounded from below by a positive constant, although it is not discussed in [2]. In fact, it is not necessary to estimate the positive lower boundedness of $\theta(t,x)$ at $t = T_0$ in terms of $\inf_{x\in\Omega}\theta_0(x)$, since the boundedness of $\sup_{0\leqslant t\leqslant T_0}\int_\Omega \rho|\log\theta|\,dx$ in Lemma 2.1 below serves as a substitute condition for the extension of the local strong solution given in Proposition 1.1.

We also remark that u, θ and their weak derivatives u_t, θ_t are defined to be zero in the presence of vacuum, and also well defined in the usual sense away from vacuum. Thus by the regularities $u_t \in L^\infty(0,T_0;L^2)$ and $\theta_t \in L^\infty(0,T_0;L^2)$, $\|u_t(T_0)\|_{L^2(\Omega)}$ and $\|\theta_t(T_0)\|_{L^2(\Omega)}$, redefined if necessary, are finite, which leads to the validity of the compatibility conditions at $t = T_0$. One may refer to Remark 2 in [2] for the necessity of the compatibility conditions in (1.11).

Therefore, with the regularities in (1.12) and the new compatibility conditions at $t = T_0$, we are able to extend the solution to the time beyond T_0. Now, we are interested in the question what happens to the solution if we extend the solution repeatedly. One possible case is that the solution exists in $[0,\infty)$, while another case is that the solution will blow up in finite time in the sense of (1.12), that is, some of the regularities in (1.12) no longer hold.

Definition 1.1. $T^* \in (0, \infty)$ is called the maximal life time of existence of a strong solution to (1.1)–(1.6) in the regularity class (1.12) if for any $0 < T < T^*$, (ρ, u, θ) solves (1.1)–(1.6) in $[0, T] \times \Omega$ and satisfies (1.12) with $T_0 = T$, and moreover, (1.12) does not hold for $T_0 = T^*$.

Now, we are in a position to state the main theorem of this paper.

Theorem 1.1 *(Blow-up Criterion). Suppose that the assumptions in Proposition* 1.1 *are satisfied. Let* (ρ, u, θ) *be the strong solution obtained in Proposition* 1.1. *Then either this solution can be extended to* $[0, \infty)$, *or there exists a positive constant* $T^* < \infty$, *the maximal time of existence, such that the solution only exists in* $[0, T]$ *for every* $T < T^*$, *and*

$$\lim_{T \to T^*} \left(\|\theta\|_{L^\infty(0,T;L^\infty)} + \|\nabla u\|_{L^1(0,T;L^\infty)} \right) = \infty,$$

provided that the condition (1.9) *is satisfied.*

Remark 1.1.

(1) As aforementioned, the situation that $\inf_{x\in\Omega} \rho_0 > 0$ can be studied in the same manner (in fact, simpler) and the same result holds.
(2) Obviously, in the isentropic or isothermal case, Theorem 1.1 reduces to the result given in [9].
(3) It is interesting to see that, in comparison with the isentropic case in [9], the additional blow-up assumption for non-isentropic flows is made on θ only, but not on any derivative of θ.

We will prove Theorem 1.1 by contradiction in the next section. In fact, the proof of the theorem is based on a priori estimates under the assumption that $\|\theta\|_{L^\infty(0,T;L^\infty)} + \|\nabla u\|_{L^1(0,T;L^\infty)}$ is bounded independent of any $T \in [0, T^*)$. The a priori estimates are then sufficient for us to apply the local existence theorem repeatedly to extend a local solution beyond the maximal time of existence T^*, consequently, contradicting the maximality of T^*.

The key step in getting the a priori estimates is to bound $\|\nabla\rho\|_{L^\infty(0,T;L^2)}$, $\|u\|_{L^\infty(0,T;H_0^1)}$ and $\|u\|_{L^2(0,T;H^2)}$. This requires the assumption on the viscosity coefficients $7\mu > \lambda$, which also implies $\rho|u|^{3+\delta} \in L^\infty(0, T; L^1)$, other than the usual estimate $\sqrt{\rho}u \in L^\infty(0, T; L^2)$. Moreover, the boundedness of $\|u\|_{L^2(0,T;H^2)}$ relies heavily on $\|u\|_{L^2(0,T;L^2)}$ and $\|\nabla P\|_{L^2(0,T;L^2)}$ in view of the momentum equation (1.2). Note that these two terms cannot be bounded by a usual L^2-estimate as in the isentropic case (cf. [9]), since the viscous dissipation and thermal diffusion are involved in the evolution of the pressure. In the current paper we will circumvent this difficulty by estimating the equation for $\log\theta$ (cf. [3,5]). We also point out that due to presence of the temperature, the estimates on the temporal and higher-order spatial derivatives of the solution are much more involved than in the isentropic flow case, and depend essentially on bounds of $\|\theta\|_{L^2(0,T;H^1)}$.

Throughout this paper, we will use the following abbreviations:

$$L^p \equiv L^p(\Omega), \qquad H^m \equiv H^m(\Omega), \qquad H_0^m \equiv H_0^m(\Omega).$$

2. Proof of Theorem 1.1

Let $0 < T < T^*$ be arbitrary but fixed. Throughout this section we denote by C (or $C(X, \ldots)$ to emphasize the dependence of C on $X, \ldots$) a general positive constant which may depend continuously on T^*.

Let (ρ, u, θ) be a strong solution to the problem (1.1)–(1.6) in the function space given in (1.12) on the time interval $[0, T]$. Suppose that $T^* < \infty$. We will prove Theorem 1.1 by a contradiction argument. To this end, we suppose that for any $T < T^*$,

$$\|\theta\|_{L^\infty(0,T;L^\infty)} + \|\nabla u\|_{L^1(0,T;L^\infty)} \leqslant C < \infty, \tag{2.1}$$

we will deduce a contradiction to the maximality of T^*.

First, we show that the density ρ is non-negative and bounded from above due to the assumptions in (2.1). It is easy to see that the continuity equation (1.1) on the characteristic curve $\dot{\chi}(t) = u(\chi(t))$ can be written as

$$\frac{d}{dt}\rho\big(\chi(t),t\big) = -\rho\big(\chi(t),t\big)\operatorname{div}u\big(\chi(t),t\big).$$

Thus, by Gronwall's inequality and (2.1), one obtains that for any $x \in \bar{\Omega}$ and $t \in [0,T]$,

$$\begin{aligned}
0 &\leqslant \underline{\rho}\exp\left(-\int_0^T \|\operatorname{div}u\|_{L^\infty}\,dt\right) \leqslant \rho(x,t) \\
&\leqslant \bar{\rho}\exp\left(\int_0^T \|\operatorname{div}u\|_{L^\infty}\,dt\right) \leqslant C,
\end{aligned} \tag{2.2}$$

where $0 \leqslant \underline{\rho} \leqslant \rho_0 \leqslant \bar{\rho}$. Next, we show that θ is positive a.e. in $[0,T]\times\Omega$ (see also [5]). Let $H(\theta)(x,t) := c_V \min\{-\theta(x,t),0\}$. Clearly, $H'(\theta) \leqslant 0$ and $H''(\theta) = 0$. We multiply (1.3) by $H'(\theta)$ and integrate over Ω to obtain

$$\begin{aligned}
\int_\Omega \big(\rho\big(H(\theta)_t + u\cdot\nabla H(\theta)\big) + R\rho H(\theta)\operatorname{div}u\big)\,dx &= \int_\Omega H'(\theta)\left(\kappa\Delta\theta + \frac{\mu}{2}\big|\nabla u + \nabla u^t\big|^2 + \lambda(\operatorname{div}u)^2\right)dx \\
&\leqslant \kappa\int_{\partial\Omega} H'(\theta)\frac{\partial\theta}{\partial n}\,dS - \kappa\int_\Omega H''(\theta)|\nabla\theta|^2\,dx \leqslant 0.
\end{aligned}$$

By the continuity equation (1.1), we integrate by parts to get

$$\begin{aligned}
\frac{d}{dt}\int_\Omega \rho H(\theta)\,dx &\leqslant C\int_\Omega |\operatorname{div}u|\big|\rho H(\theta)\big|\,dx \\
&\leqslant -\|\operatorname{div}u\|_{L^\infty}\int_\Omega \rho H(\theta)\,dx.
\end{aligned}$$

Utilizing (2.1) and Gronwall's inequality, we have

$$\int_\Omega \rho H(\theta)\,dx \equiv 0, \quad \forall t \in [0,T],$$

since $\theta_0 \geqslant 0$. Thus $\theta \geqslant 0$ by the definition of $H(\theta)$ again. Observing that

$$\frac{\Delta\theta}{\theta} = \operatorname{div}\left(\frac{\nabla\theta}{\theta}\right) - \nabla\left(\frac{1}{\theta}\right)\cdot\nabla\theta = \operatorname{div}\left(\frac{\nabla\theta}{\theta}\right) + \frac{|\nabla\theta|^2}{\theta^2},$$

the function $s := \log\theta$ satisfies the equation:

$$\partial_t(\rho s) + \operatorname{div}(\rho s u) - \kappa\operatorname{div}\left(\frac{\nabla\theta}{\theta}\right) + R\rho\operatorname{div}u = \frac{1}{\theta}\left[\frac{\mu}{2}\big|\nabla u + \nabla u^T\big|^2 + \lambda(\operatorname{div}u)^2\right] + \frac{\kappa}{\theta^2}|\nabla\theta|^2.$$

Integrating the above equation over $(0,T)\times\Omega$ and applying (1.10), (2.1) and (2.4), we obtain

$$\begin{aligned}
&\int_0^T\int_\Omega \left(\frac{\alpha|\nabla u|^2}{\theta} + \frac{\kappa|\nabla\theta|^2}{\theta^2}\right)dx\,ds - \int_\Omega \rho\log\theta\,dx\bigg|_{t=T} \\
&\quad\leqslant C\|\rho\|_{L^\infty(0,T;L^\infty)}\|\operatorname{div}u\|_{L^1(0,T;L^\infty)} + \int_\Omega \rho_0\log\theta_0\,dx \leqslant C,
\end{aligned} \tag{2.3}$$

for some constant $\alpha > 0$. The second term on the left-hand side of (2.3) can be estimated as follows. Noting that ρ and θ are indeed continuous in $[0,T]\times\Omega$ by the Sobolev embedding theorem, we have

$$\int_{\Omega\cap\{\theta\geqslant 1\}} \rho\log\theta\,dx\bigg|_{t=T} \leqslant C\|\rho\|_{L^\infty(0,T;L^\infty)}\log\|\theta\|_{L^\infty(0,T;L^\infty)} \leqslant C.$$

Inserting the above inequality into (2.3), we obtain

$$\int_0^T\int_\Omega\left(\frac{\alpha|\nabla u|^2}{\theta}+\frac{\kappa|\nabla\theta|^2}{\theta^2}\right)dx\,ds+\int_{\Omega\cap\{0\leqslant\theta\leqslant 1\}}\rho|\log\theta|\,dx\bigg|_{t=T}\leqslant C.$$

It follows that

Lemma 2.1. *For any $T<T^*$, we have*

$$\sup_{0\leqslant t\leqslant T}\int_\Omega\rho(t)\big|\log\theta(t)\big|\,dx+\int_0^T\int_\Omega\left(|\nabla\log\theta|^2+|\nabla u|^2+|\nabla\theta|^2\right)dx\,dt\leqslant C. \tag{2.4}$$

With the help of the above lemma and the upper boundedness of ρ, we are able to deduce the positiveness of θ by the following auxiliary lemma from [6]:

Lemma 2.2. *(See [6].) Let Ω be a bounded domain in $\mathbb{R}^N$ and $\gamma>1$ be a constant. Given constants M and E_0 with $0<M<E_0$, there is a constant $C(E_0,M)$, such that for any non-negative function ρ satisfying*

$$M\leqslant\int_\Omega\rho\,dx,\ \int_\Omega\rho^\gamma\,dx\leqslant E_0$$

and any $v\in H^1(\Omega)$,

$$\|v\|_{L^2}^2\leqslant C\left[\|\nabla v\|_{L^2(\Omega)}^2+\left(\int_\Omega\rho|v|\,dx\right)^2\right].$$

Therefore, from (2.2) and Lemma 2.2, we get

$$\int_0^T\int_\Omega|\log\theta|^2\,dx\,dt\leqslant C. \tag{2.5}$$

Notice that $\theta\in C([0,T],H^2)$, which means that θ and thus $\log\theta$ is continuous in both space and time. It follows that $|\log\theta|<\infty$ everywhere. Moreover, the continuity of θ up to the initial time $t=0$ and the assumption that $\theta(\cdot,0)>0$ immediately imply that

$$\theta(x,t)>0,\quad\forall x\in\bar{\Omega},\ t\in[0,T]. \tag{2.6}$$

The following key lemma is due to Hoff [8] (see also [9]).

Lemma 2.3. *Let $7\mu>\lambda$. Then there is a small $\delta>0$, such that*

$$\sup_{0\leqslant t\leqslant T}\int_\Omega\rho(x,t)\big|u(x,t)\big|^{3+\delta}\,dx+\int_0^T\int_\Omega|u|^{1+\delta}|\nabla u|^2\,dx\,dt\leqslant C. \tag{2.7}$$

Proof. Denoting $q=3+\delta$ with $\delta>0$ to be determined below, after a straightforward calculation we derive from Eq. (1.2) that

$$\begin{aligned}&\rho\big[\big(|u|^q\big)_t+u\cdot\nabla\big(|u|^q\big)\big]+q|u|^{q-2}u\cdot\nabla P+q|u|^{q-2}\big[\mu|\nabla u|^2+(\mu+\lambda)(\operatorname{div}u)^2\big]\\&\quad=q|u|^{q-2}\left(\frac{1}{2}\mu\Delta\big(|u|^2\big)+(\mu+\lambda)\operatorname{div}(u\operatorname{div}u)\right).\end{aligned}$$

Using (1.1) and (1.5), we integrate the above identity over $(0,t)\times\Omega$ to get

$$\int_\Omega \rho|u|^q\,dx\Big|_0^t+\int_0^t\int_\Omega\big\{q|u|^{q-2}\big(\mu|\nabla u|^2+(\mu+\lambda)(\operatorname{div}u)^2+\mu(q-2)\big|\nabla|u|\big|^2\big)$$
$$+(\mu+\lambda)q(q-2)|u|^{q-3}u\cdot\nabla|u|\operatorname{div}u\big\}\,dx\,ds=\int_0^t\int_\Omega qR\rho\theta\operatorname{div}u|u|^{q-2}u\,dx\,ds. \tag{2.8}$$

Due to $7\mu>\lambda$, there exists a small $\delta>0$, such that for $q=3+\delta$,

$$4\mu(q-1)-(\mu+\lambda)(q-2)^2>0.$$

Hence, recalling the fact that $|\nabla|u||\leqslant|\nabla u|$, we find that the time- and spatial-integral term (the second term) on the left-hand side of (2.8) is bounded from below by

$$\left(\mu(q-1)-\frac{\mu+\lambda}{4}(q-2)^2\right)q|u|^{q-2}|\nabla u|^2\geqslant\frac{1}{C}|u|^{q-2}|\nabla u|^2. \tag{2.9}$$

Moreover, since the density ρ and the temperature θ are bounded, the right-hand side of (2.8) is less than

$$C\int_0^t\int_\Omega\rho|u|^{q-2}|\nabla u|\,dx\,ds\leqslant\epsilon\int_0^t\int_\Omega|u|^{q-2}|\nabla u|^2\,dx\,ds+C(\epsilon)\left(\int_0^t\int_\Omega\rho|u|^q\,dx\,ds\right)^{\frac{q-2}{q}}$$
$$\leqslant\epsilon\int_0^t\int_\Omega|u|^{q-2}|\nabla u|^2\,dx\,ds+\int_0^t\int_\Omega\rho|u|^q\,dx\,ds+C(\epsilon), \tag{2.10}$$

by Hölder's inequality and Young's inequality. Inserting (2.9) and (2.10) into (2.8), and choosing ϵ small enough, we obtain (2.7) by Gronwall's inequality. □

Now, we are ready to bound the first-order spatial derivatives of ρ and u, which are also necessary for estimating other quantities.

Lemma 2.4 *(Main Estimates). Under* (2.1)*, we have for any $T<T^*$ that*

$$\sup_{0\leqslant t\leqslant T}\big\|\nabla\rho(t)\big\|_{L^2}+\int_0^T\|\rho_t\|_{L^2}^2\,dt\leqslant C, \tag{2.11}$$

$$\sup_{0\leqslant t\leqslant T}\big\|u(t)\big\|_{H_0^1}^2+\int_0^T\int_\Omega\rho|u_t|^2\,dx\,dt\leqslant C, \tag{2.12}$$

$$\int_0^T\big\|u(t)\big\|_{H^2}^2\,dt\leqslant C. \tag{2.13}$$

Proof. We multiply Eq. (1.2) by u_t and then integrate over Ω. Using (1.1) and (1.5), we easily derive that

$$\frac{d}{dt}\int_\Omega\left(\frac{\mu}{2}|\nabla u|^2+\frac{\mu+\lambda}{2}(\operatorname{div}u)^2\right)dx+\frac12\int_\Omega\rho|u_t|^2\,dx$$
$$\leqslant\int_\Omega\rho|u\cdot\nabla u|^2\,dx-\int_\Omega\nabla P\cdot u_t\,dx, \tag{2.14}$$

where the first term on the right-hand side of (2.14) is estimated as follows, using (2.2), Lemma 2.3 and the interpolation inequality (cf. [17])

$$\begin{aligned}\int_\Omega \rho|u\cdot\nabla u|^2\,dx &\leqslant \int_\Omega \rho^{1/q}|u\cdot\nabla u|^2\,dx\\ &\leqslant \Big(\int_\Omega \rho|u|^q\,dx\Big)^{2/q}\|\nabla u\|^2_{L^{\frac{2q}{q-2}}}\\ &\leqslant \epsilon\|\nabla u\|^2_{H^1}+C(\epsilon)\|\nabla u\|^2_{L^2},\quad 0<\epsilon<1,\ q=3+\delta. \end{aligned}\tag{2.15}$$

Next, we rewrite the second integral on the right-hand side of (2.14), so that the time derivative of u is represented by spatial derivatives of u, that is,

$$\begin{aligned}\int_\Omega \nabla P\cdot u_t\,dx &= -\int_\Omega P\operatorname{div}u_t\,dx\\ &=\int_\Omega P_t\operatorname{div}u\,dx-\frac{d}{dt}\int_\Omega P\operatorname{div}u\,dx. \end{aligned}\tag{2.16}$$

Notice that by (1.1), (1.3) and (1.4), one gets

$$P_t+u\cdot\nabla P+\gamma P\operatorname{div}u=(\gamma-1)\kappa\Delta\theta+(\gamma-1)\Big(\frac{\mu}{2}\big|\nabla u+\nabla u^T\big|^2+\lambda(\operatorname{div}u)^2\Big),$$

thus, the first term on the right-hand side of (2.16) is bounded by

$$\begin{aligned}&C\Big|\int_\Omega \operatorname{div}u(\kappa\Delta\theta-u\cdot\nabla P)\,dx\Big|+C\Big|\int_\Omega\big(|\nabla u|^3+|\nabla u|^2\big)\,dx\Big|\\ &\quad\leqslant C\Big|\int_\Omega\big(|\nabla\operatorname{div}u||\nabla\theta|+\big|\operatorname{div}(u\operatorname{div}u)\big|\big)\,dx\Big|+C\Big|\int_\Omega\big(|\nabla u|^3+|\nabla u|^2\big)\,dx\Big|\\ &\quad\leqslant \epsilon\|u\|^2_{H^2}+C(\epsilon)\big(\|\nabla\theta\|^2_{L^2}+\big(1+\|\nabla u\|_{L^\infty}\big)\|\nabla u\|^2_{L^2}\big),\quad \forall 0<\epsilon<1, \end{aligned}\tag{2.17}$$

where we have also used Poincaré's inequality.

On the other hand, since u is a solution of the elliptic system

$$-\mu\Delta u-(\lambda+\mu)\nabla\operatorname{div}u=f,$$

where $f:=-\rho u_t-\rho u\cdot\nabla u-\nabla P$, it follows from the classical regularity theory and (2.15) that

$$\begin{aligned}\|u\|_{H^2}&\leqslant C\|f\|_{L^2}\\ &\leqslant C\big(\|\sqrt{\rho}u_t\|_{L^2}+\|\sqrt{\rho}u\cdot\nabla u\|_{L^2}+\|\nabla\rho\|_{L^2}+\|\nabla\theta\|_{L^2}\big)\\ &\leqslant C\big(\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla\rho\|_{L^2}+\|\nabla\theta\|_{L^2}+\|\nabla u\|_{L^2}\big)+\frac12\|u\|_{H^2}, \end{aligned}$$

whence,

$$\|u\|_{H^2}\leqslant C\big(\|\sqrt{\rho}u_t\|_{L^2}+\|\nabla u\|_{L^2}+\|\nabla\rho\|_{L^2}+\|\nabla\theta\|_{L^2}\big).\tag{2.18}$$

Substituting (2.15)–(2.18) into (2.14) and taking ϵ appropriately small, we conclude

$$\begin{aligned}&\frac{d}{dt}\int_\Omega\Big(\frac{\mu}{2}|\nabla u|^2+\frac{\lambda+\mu}{2}(\operatorname{div}u)^2-P\operatorname{div}u\Big)\,dx+\frac14\int_\Omega\rho u_t^2\,dx\\ &\quad\leqslant C\big[\big(1+\|\nabla u\|_{L^\infty}\big)\|\nabla u\|^2_{L^2}+\|\nabla\theta\|^2_{L^2}\big]+\|\nabla\rho\|^2_{L^2}. \end{aligned}\tag{2.19}$$

Clearly, it remains to estimate the L^2-norm of $\nabla\rho$. The calculations are routine, namely, we apply ∇ to Eq. (1.1), then multiply the resulting equation by $\nabla\rho$ and integrate over Ω to get

$$\begin{aligned}\frac{d}{dt}\int_\Omega |\nabla\rho|^2\,dx &\leqslant C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^2}^2 + C\|u\|_{H^2}\|\nabla\rho\|_{L^2}\\ &\leqslant C\big[\big(1+\|\nabla u\|_{L^\infty}\big)\|\nabla\rho\|_{L^2}^2 + \|\nabla u\|_{L^2}^2 + \|\nabla\theta\|_{L^2}^2\big] + \frac{1}{8}\|\sqrt{\rho}u_t\|_{L^2}^2,\end{aligned} \tag{2.20}$$

where we have also applied (2.18).

Moreover, observing that by (2.1) and (2.2), we have

$$\int_\Omega P\,\mathrm{div}\,u\,dx\bigg|_0^t \leqslant \frac{\mu}{4}\big\|\nabla u(t)\big\|_{L^2}^2 + C. \tag{2.21}$$

Adding (2.20) to (2.19), applying Gronwall's inequality, and employing (2.21), (2.1) and (2.3), we obtain

$$\sup_{t\in[0,T]}\int_\Omega \big(|\nabla u|^2 + |\nabla\rho|^2\big)(x,t)\,dx + \int_0^T\int_\Omega \rho u_t^2\,dx\,dt \leqslant C. \tag{2.22}$$

Thus, (2.13) follows from (2.18), (2.22) and (2.3) immediately. Finally, from (1.1), (2.2), Sobolev's inequality and (2.13), we have

$$\begin{aligned}\int_0^T \|\rho_t\|_{L^2}^2\,dt &\leqslant C\int_0^T \big(\|\rho\|_{L^\infty}\|\nabla u\|_{L^2}^2 + \|u\|_{L^\infty}^2\|\nabla\rho\|_{L^2}^2\big)\,dt\\ &\leqslant C + C\int_0^T \|u\|_{H^2}^2\,dt \leqslant C.\end{aligned}$$

This completes the proof. □

Next, we will exploit the a priori estimates obtained so far to derive bounds on higher derivatives.

Lemma 2.5. *Let*

$$\Phi(t) := 1 + \bigg(\int_0^t \big\|\theta_t(s)\big\|_{H^1}^2\,ds\bigg)^{1/2}.$$

Then for any $T < T^*$, we have

$$\sup_{0\leqslant t\leqslant T}\big\|\theta(t)\big\|_{H^1}^2 + \int_0^T\int_\Omega \rho\theta_t^2\,dx\,dt \leqslant C\Phi(T), \tag{2.23}$$

$$\sup_{0\leqslant t\leqslant T}\big\|u(t)\big\|_{H^2}^2 + \int_0^T \big\|\theta(t)\big\|_{H^2}^2\,dt \leqslant C\Phi(T), \tag{2.24}$$

$$\sup_{0\leqslant t\leqslant T}\big\|\sqrt{\rho(t)}u_t(t)\big\|_{L^2}^2 + \int_0^T \big\|u_t(t)\big\|_{H_0^1}^2\,dt \leqslant C\Phi(T). \tag{2.25}$$

Proof. Multiplying (1.3) by θ_t in $L^2(\Omega)$, we make use of (2.4), (2.11), (2.12) and (2.18) to infer

$$\begin{aligned}
&\frac{k}{2}\frac{d}{dt}\int_\Omega |\nabla\theta|^2\,dx + c_V\int_\Omega \rho\theta_t^2\,dx\\
&\quad= -c_V\int_\Omega \rho(u\cdot\nabla)\theta\theta_t\,dx - \int_\Omega P\operatorname{div}u\theta_t\,dx + \int_\Omega\left(\frac{\mu}{2}\left|\nabla u+\nabla u^T\right|^2+\lambda(\operatorname{div}u)^2\right)\theta_t\,dx\\
&\quad\leqslant C\|u\|_{L^\infty}\|\nabla\theta\|_{L^2}\|\sqrt{\rho}\theta_t\|_{L^2} + C\|\nabla u\|_{L^2}\big(\|\sqrt{\rho}\theta_t\|_{L^2} + \|\nabla u\|_{L^3}\|\theta_t\|_{H^1}\big)\\
&\quad\leqslant C\|u\|_{H^2}\|\nabla\theta\|_{L^2}\|\sqrt{\rho}\theta_t\|_{L^2} + C\|\sqrt{\rho}\theta_t\|_{L^2} + C\|u\|_{H^2}^{1/2}\|\theta_t\|_{H^1}\\
&\quad\leqslant \epsilon\|\sqrt{\rho}\theta_t\|_{L^2}^2 + C(\epsilon)\big(1+\|u\|_{H^2}^2\|\nabla\theta\|_{L^2}^2 + \|u\|_{H^2}^{1/2}\|\theta_t\|_{H^1}\big),
\end{aligned}$$

for any $0<\epsilon<1$. Taking ϵ appropriately small, we integrate the above inequality over $[0,t]$ and apply Gronwall's inequality to obtain (2.23) by (2.13).

Now, taking ∂_t to Eq. (1.2), multiplying then the resulting equation by u_t in $L^2(\Omega)$, integrating by parts, and employing (1.1) and (2.11), we find that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho u_t^2\,dx + \int_\Omega \big(\mu|\nabla u_t|^2 + (\lambda+\mu)(\operatorname{div}u_t)^2\big)\,dx\\
&\quad= \int_\Omega P_t\operatorname{div}u_t\,dx - \int_\Omega \rho u\cdot\nabla\big[(u_t+u\cdot\nabla u)u_t\big]\,dx - \int_\Omega \rho u_t\cdot\nabla u\cdot u_t\,dx\\
&\quad:= I_1+I_2+I_3.
\end{aligned}\tag{2.26}$$

Observing that $P_t = R\rho_t\theta + R\rho\theta_t$, we have

$$|I_1| \leqslant \epsilon\|\nabla u_t\|_{L^2}^2 + C(\epsilon)\big(\|\rho_t\|_{L^2}^2 + \|\sqrt{\rho}\theta_t\|_{L^2}^2\big),\tag{2.27}$$

$$\begin{aligned}
|I_2| &\leqslant \int_\Omega \rho|u||u_t||\nabla u_t|\,dx + \int_\Omega \rho|u||\nabla u|^2|u_t|\,dx + \int_\Omega \rho|u|^2\left|\nabla^2 u\right||u_t|\,dx + \int_\Omega \rho|u||\nabla u||\nabla u_t|\,dx\\
&:= I_{21}+I_{22}+I_{23}+I_{24},
\end{aligned}\tag{2.28}$$

where each term on the right-hand side of (2.28) can be estimated as follows, using (2.12), the interpolation inequality and Sobolev's imbedding theorem

$$\begin{aligned}
|I_{21}| &\leqslant C\|u\|_{H^1}\|\nabla u_t\|_{L^2}\|\sqrt{\rho}u_t\|_{L^3}\\
&\leqslant C\|\nabla u_t\|_{L^2}\|\sqrt{\rho}u_t\|_{L^3}\\
&\leqslant \epsilon\|\nabla u_t\|_{L^2}^2 + C\epsilon^{-1}\|\sqrt{\rho}u_t\|_{L^6}\|\sqrt{\rho}u_t\|_{L^2}\\
&\leqslant \epsilon\|u_t\|_{H^1}^2 + \epsilon\|u_t\|_{L^6}^2 + C\epsilon^{-3}\|\sqrt{\rho}u_t\|_{L^2}^2\\
&\leqslant C\epsilon\|u_t\|_{H^1}^2 + C\epsilon^{-3}\|\sqrt{\rho}u_t\|_{L^2}^2,
\end{aligned}\tag{2.29}$$

$$\begin{aligned}
|I_{22}| &\leqslant C\|u\|_{L^6}\|\nabla u\|_{L^2}\|\nabla u\|_{L^6}\|u_t\|_{L^6}\\
&\leqslant C\|u\|_{H^1}^2\|u\|_{H^2}\|u_t\|_{H^1}\\
&\leqslant \epsilon\|u_t\|_{H^1}^2 + C\epsilon^{-1}\|u\|_{H^2}^2.
\end{aligned}\tag{2.30}$$

Similarly,

$$|I_{23}| \leqslant C\|u\|_{H^1}^2\left\|\nabla^2 u\right\|_{L^2}\|u_t\|_{H^1} \leqslant \epsilon\|u_t\|_{H^1}^2 + C\epsilon^{-1}\|u\|_{H^2}^2,\tag{2.31}$$

and

$$|I_{24}| \leqslant C\|u\|_{H^1}^2\|\nabla u\|_{H^1}\|\nabla u_t\|_{L^2} \leqslant \epsilon\|u_t\|_{H^1}^2 + C\epsilon^{-1}\|u\|_{H^2}^2.\tag{2.32}$$

Again, we apply the interpolation inequality and Sobolev's imbedding theorem to get

$$\begin{aligned}|I_3| &\leqslant C\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^4}^2\\ &\leqslant C\|\nabla u\|_{L^2}\|\sqrt{\rho}u_t\|_{L^6}^{3/2}\|\sqrt{\rho}u_t\|_{L^2}^{1/2}\\ &\leqslant C\|u_t\|_{H^1}^{3/2}\|\sqrt{\rho}u_t\|_{L^2}^{1/2}\\ &\leqslant \epsilon\|u_t\|_{H^1}^2 + C\epsilon^{-1}\|\sqrt{\rho}u_t\|_{L^2}^2.\end{aligned} \tag{2.33}$$

Substituting (2.27)–(2.33) into (2.26), and taking ϵ suitably small, we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_\Omega \rho u_t^2\,dx + \int_\Omega \big(\mu|\nabla u_t|^2 + (\lambda+\mu)(\operatorname{div} u_t)^2\big)\,dx\\ &\quad\leqslant C\big(\|\sqrt{\rho}u_t\|_{L^2}^2 + \|u\|_{H^2}^2 + \|\rho_t\|_{L^2}^2\big) + C_1\|\sqrt{\rho}\theta_t\|_{L^2}^2.\end{aligned}$$

Applying Gronwall's inequality to the above inequality and using (2.23), one obtains (2.25). Moreover, (2.24) follows from Eq. (1.3) and the inequality (2.18), together with the estimates obtained so far. This completes the proof. □

Next, we derive bounds for θ_t to close the desired energy estimates. We have

Lemma 2.6. *For any $T < T^*$, we have*

$$\sup_{0\leqslant t\leqslant T}\int_\Omega \rho(x,t)\theta_t^2(x,t)\,dx + \int_0^T \big\|\theta_t(t)\big\|_{H^1}^2\,dt \leqslant C, \tag{2.34}$$

$$\sup_{0\leqslant t\leqslant T}\big\|\theta(t)\big\|_{H^2}^2 \leqslant C. \tag{2.35}$$

Proof. Taking ∂_t on both sides of Eq. (1.3), then multiplying the resulting equation by θ_t in $L^2(\Omega)$, we obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega \rho\theta_t^2\,dx + \kappa\int_\Omega |\nabla\theta_t|^2\,dx &= \int_\Omega R\rho\theta_t^2\operatorname{div} u\,dx + \int_\Omega R\rho_t\theta\operatorname{div} u\theta_t\,dx + \int_\Omega R\rho\theta\operatorname{div} u_t\theta_t\,dx\\ &\quad+ \int_\Omega\big[\mu\big(\nabla u + \nabla u^t\big):\big(\nabla u_t + \nabla u_t^t\big) + 2\lambda\operatorname{div} u\operatorname{div} u_t\big]\theta_t\,dx\\ &\quad- \int_\Omega \rho_t u\cdot\nabla\theta\theta_t\,dx - \int_\Omega \rho u_t\cdot\nabla\theta\theta_t\,dx - \int_\Omega \rho_t\theta_t^2\,dx := \sum_{i=1}^7 J_i.\end{aligned} \tag{2.36}$$

We have to estimate each term on the right-hand side of (2.36). First, from (1.1), Lemma 2.4, and Sobolev's imbedding theorem, we easily get

$$|J_1| \leqslant C\|\theta_t\|_{H^1}\|\sqrt{\rho}\theta_t\|_{L^2}\|\operatorname{div} u\|_{H^1} \leqslant \epsilon\|\theta_t\|_{H^1}^2 + C\epsilon^{-1}\|u\|_{H^2}^2\|\sqrt{\rho}\theta_t\|_{L^2}^2, \tag{2.37}$$

$$\begin{aligned}|J_2| &\leqslant \left|\int_\Omega R(\rho\operatorname{div} u + \nabla\rho\cdot u)\theta\operatorname{div} u\theta_t\,dx\right|\\ &\leqslant C\|\sqrt{\rho}\theta_t\|_{L^2}\|\nabla u\|_{L^4}^2 + C\|\nabla\rho\|_{L^2}\|u\|_{H^1}\|\operatorname{div} u\|_{H^1}\|\theta_t\|_{H^1}\\ &\leqslant C\epsilon^{-1}\big(\|\nabla u\|_{H^1}^2\|\sqrt{\rho}\theta_t\|_{L^2}^2 + \|u\|_{H^2}^2\big) + \epsilon\|\theta_t\|_{H^1}^2,\end{aligned} \tag{2.38}$$

and

$$|J_3| \leqslant C\|\sqrt{\rho}\theta_t\|_{L^2}\|\operatorname{div} u_t\|_{L^2} \leqslant \epsilon\|u_t\|_{H^1}^2 + C\epsilon^{-1}\|\sqrt{\rho}\theta_t\|_{L^2}^2. \tag{2.39}$$

Next, we calculate the crucial terms J_4 and J_5. To bound J_4, observing that

$$|J_4| \leqslant C\|\nabla u\|_{L^3}\|\nabla u_t\|_{L^2}\|\theta_t\|_{H^1} \leqslant C\|u\|_{H^2}^{1/2}\|\theta_t\|_{H^1}\|u_t\|_{H^1},$$

we make use of (2.18) and Lemma 2.5 to deduce that

$$\begin{aligned}\int_0^t |J_4|\,ds &\leqslant C\Big(\sup_{0\leqslant s\leqslant T}\big\|u(s)\big\|_{H^2}\Big)^{1/2}\|u_t\|_{L^2(0,t;H^1)}\|\theta_t\|_{L^2(0,t;H^1)}\\ &\leqslant C\big(1+\|\theta_t\|_{L^2(0,t;H^1)}^{1/4}\big)\big(1+\|\theta_t\|_{L^2(0,t;H^1)}^{1/2}\big)\|\theta_t\|_{L^2(0,t;H^1)}\\ &\leqslant C\big(1+\|\theta_t\|_{L^2(0,t;H^1)}^{7/4}\big)\\ &\leqslant C\epsilon^{-1}+\epsilon\|\theta_t\|_{L^2(0,t;H^1)}^2. \end{aligned}\tag{2.40}$$

Recalling that $\rho_t=-\rho\operatorname{div}u-\nabla\rho\cdot u$ and $\|u\|_{L^\infty}\leqslant C\|u\|_{W^{1,4}}\leqslant C\|u\|_{H^2}^{3/4}\|u\|_{H^1}^{1/4}$, we find that

$$\begin{aligned}|J_5| &\leqslant C\int_\Omega \big(\rho|\operatorname{div}u|+|u||\nabla\rho|\big)|u||\nabla\theta||\theta_t|\,dx\\ &\leqslant C\big(\|\sqrt{\rho}\theta_t\|_{L^2}\|\operatorname{div}u\|_{H^1}+\|\theta_t\|_{H^1}\|\nabla\rho\|_{L^2}\|u\|_{L^\infty}\big)\|u\|_{H^1}\|\nabla\theta\|_{H^1}\\ &\leqslant \epsilon\|\theta\|_{H^2}^2+C\epsilon^{-1}\big(\|u\|_{H^2}^2\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|\theta_t\|_{H^1}\|u\|_{H^2}^{3/4}\|\theta\|_{H^2}\big)\\ &\leqslant \epsilon\big(\|\theta\|_{H^2}^2+\|\theta_t\|_{H^1}^2\big)+C\epsilon^{-1}\big(\|u\|_{H^2}^2\|\sqrt{\rho}\theta_t\|_{L^2}^2+\|u\|_{H^2}^{3/2}\|\theta\|_{H^2}^2\big),\end{aligned}$$

which, together with Lemma 2.5 and Young's inequality, yields

$$\begin{aligned}\int_0^t |J_5|\,ds &\leqslant C+\epsilon\|\theta_t\|_{L^2(0,t;H^1)}^2+C\epsilon^{-1}\int_0^t \|u\|_{H^2}^2\|\sqrt{\rho}\theta_t\|_{L^2}^2\,ds\\ &\quad +C\big(1+\|\theta_t\|_{L^2(0,t;H^1)}^{3/4}\big)\big(\epsilon^{-1}+\epsilon\|\theta_t\|_{L^2(0,t;H^1)}\big)\\ &\leqslant C\epsilon\|\theta_t\|_{L^2(0,t;H^1)}^2+C(\epsilon)\left(1+\int_0^t\|u\|_{H^2}^2\|\sqrt{\rho}\theta_t\|_{L^2}^2\,ds\right).\end{aligned}\tag{2.41}$$

On the other hand, we integrate by parts and apply Lemma 2.5 to get

$$\begin{aligned}\int_0^t |J_6|\,ds &\leqslant \int_0^t\int_\Omega \big(\theta\big(|\nabla\rho||\theta_t|+\rho|\nabla\theta_t|\big)|u_t|+\rho\theta|\theta_t||\operatorname{div}u_t|\big)\,dx\,ds\\ &\leqslant C\int_0^t\big[\big(1+\|\nabla\rho\|_{L^2}\big)\|\theta_t\|_{H^1}\|u_t\|_{H^1}+\|\theta_t\|_{H^1}\|\operatorname{div}u_t\|_{L^2}\big]\,ds\\ &\leqslant \epsilon\|\theta_t\|_{L^2(0,t;H^1)}^2+C\epsilon^{-1}\|u_t\|_{L^2(0,t;H^1)}^2\\ &\leqslant C(\epsilon)+C\epsilon\|\theta_t\|_{L^2(0,t;H^1)}^2.\end{aligned}\tag{2.42}$$

Recalling that $\rho_t=-\rho\operatorname{div}u-\nabla\rho\cdot u$, we have in the same manner that

$$\begin{aligned}|J_7| &\leqslant \left|\int_\Omega \big(\rho\operatorname{div}u\theta_t^2-\rho\operatorname{div}\big(\theta_t^2u\big)\big)\,dx\right|\\ &\leqslant C\int_\Omega\big(\rho|\operatorname{div}u||\theta_t|^2+\rho\big(|\theta_t|^2|\operatorname{div}u|+|\theta_t||\nabla\theta_t||u|\big)\big)\,dx\\ &\leqslant \|\sqrt{\rho}\theta_t\|_{L^2}\big(\|\operatorname{div}u\|_{H^1}\|\theta_t\|_{H^1}+\|u\|_{H^1}\|\nabla\theta_t\|_{L^2}\big)\\ &\leqslant \epsilon\|\theta_t\|_{H^1}^2+C\epsilon^{-1}\|u\|_{H^2}^2\|\sqrt{\rho}\theta_t\|_{L^2}^2.\end{aligned}\tag{2.43}$$

Finally, we integrate (2.36) and utilize (2.37)–(2.43) with ϵ sufficiently small to conclude

$$\left\|\sqrt{\rho}(t)\theta_t(t)\right\|_{L^2}^2+\|\theta_t\|_{L^2(0,T;H^1)}^2\leqslant C+C\int_0^t\left(1+\|u\|_{H^2}^2\right)\left\|\sqrt{\rho}(s)\theta_t(s)\right\|_{L^2}^2\,ds,\quad 0\leqslant t\leqslant T,$$

which, by applying Gronwall's inequality, implies (2.34). As a consequence of (2.34), we see that the left-hand sides of (2.23)–(2.25) are all bounded by a positive constant. Moreover, from the energy equation (1.3) and the energy estimates obtained so far, we easily obtain (2.35). □

Finally, in the next lemma we show the additional L^q bounds of the solution.

Lemma 2.7. *Let q be the same as in Theorem* 1.1. *Then,*

$$\sup_{0\leqslant t\leqslant T}\left(\left\|\rho_t(t)\right\|_{L^q}+\left\|\rho(t)\right\|_{W^{1,q}}\right)\leqslant C, \tag{2.44}$$

$$\int_0^T\left(\left\|u(t)\right\|_{W^{2,q}}^2+\left\|\theta(t)\right\|_{W^{2,q}}^2\right)dt\leqslant C. \tag{2.45}$$

Proof. Differentiating (1.1) with respect to x_j and multiplying the resulting equation by $|\partial_j\rho|^{q-2}\partial_j\rho$ in $L^2(\Omega)$, one deduces that

$$\begin{aligned}\frac{d}{dt}\int_\Omega|\nabla\rho|^q\,dx&\leqslant C\int_\Omega\left(|\nabla u||\nabla\rho|^q+|\rho||\nabla\rho|^{q-1}|\nabla^2u|\right)dx\\&\leqslant C\|\nabla u\|_{L^\infty}\|\nabla\rho\|_{L^q}^q+C\left\|\nabla^2u\right\|_{L^q}\|\nabla\rho\|_{L^q}^{q-1},\end{aligned}$$

which gives

$$\begin{aligned}\sup_{0\leqslant t\leqslant T}\left\|\nabla\rho(t)\right\|_{L^q}&\leqslant C\exp\left(\int_0^t\left\|\nabla u(s)\right\|_{L^\infty}ds\right)\left(\|\nabla\rho_0\|_{L^q}+\int_0^t\left\|\nabla^2u(s)\right\|_{L^q}ds\right)\\&\leqslant C(\sqrt{T})\epsilon^{-1}+\epsilon\left\|\nabla^2u\right\|_{L^2(0,t;L^q)},\end{aligned}\tag{2.46}$$

by Gronwall's inequality. Using the regularity theory of elliptic equations again, we have

$$\begin{aligned}\left\|u(t)\right\|_{W^{2,q}}&\leqslant C\left(\|u_t\|_{L^q}+\|u\cdot\nabla u\|_{L^q}+\|\nabla\rho\|_{L^q}+\|\nabla\theta\|_{L^q}\right)\\&\leqslant C\left(\|\nabla u_t\|_{L^2}+\|u\|_{L^\infty}\|\nabla u\|_{L^q}+\|\nabla\rho\|_{L^q}+\|\theta\|_{H^2}\right)\\&\leqslant C\left(\|\nabla u_t\|_{L^2}+\|u\|_{H^2}^2+\|\nabla\rho\|_{L^q}+\|\theta\|_{H^2}\right).\end{aligned}$$

If we integrate the above inequality over $(0,T)$ and make use of (2.24), (2.25) and (2.46), we obtain

$$\int_0^T\left\|u(t)\right\|_{W^{2,q}}^2\,dt\leqslant C, \tag{2.47}$$

and thus, from (2.46) one gets

$$\sup_{0\leqslant t\leqslant T}\left\|\rho(t)\right\|_{W^{1,q}}\leqslant C.$$

Since $\rho_t=-u\nabla\rho-\rho\,\mathrm{div}\,u$, we also have

$$\left\|\rho_t(t)\right\|_{L^q}\leqslant\|u\|_{L^\infty}\|\nabla\rho\|_{L^q}+\|\rho\|_{L^\infty}\|\mathrm{div}\,u\|_{L^q}\leqslant C.$$

Then the boundedness of θ in $L^2(0,T;W^{2,q})$ follows from (1.3), (2.47) and the above inequality. The proof is finished. □

By virtue of Lemmas 2.1–2.7, we obtain the bounds of the norms of (ρ, u, θ) in $[0, T] \times \Omega$ in the sense of (1.12) for any $T < T^*$. These bounds depend only on Ω, the initial data, and continuously on T^* (in fact, the bounds depend on T^* either polynomially or exponentially!). Thus, we can take $(\rho, u, \theta, \rho_t, \sqrt{\rho}u_t, \sqrt{\rho}\theta_t)|_{t=T}$, redefined if necessary, as the initial data at $t = T$ and apply Proposition 1.1 to extend the solution to $t = T + T_1$. Note that the bound for $\int_\Omega \rho|\log\theta|\,dx|_{t=T}$ is already available, thus $\inf_{x\in\Omega}\theta(T, x)$, which is only used in estimating the former, is not necessary to be estimated in terms of $\inf_{x\in\Omega}\theta_0(x)$.

If $T + T_1 > T^*$, then it contradicts the maximality of T^*. Otherwise, we can continue to extend the solution by taking the values of the solution at $t = T + T_1$ as initial data again. Since the a priori estimates are independent of any $t < T^*$, the solution can be extended to $t = T + 2T_1$. Here we remark that by applying Proposition 1.1, the solution can be extended from $t = T + T_1$ to $t = T + 2T_1$, since the local existence interval depends only on the initial data which, in our case, are bounded in any time interval $[0, \overline{T}]$ with a bound depending on $\overline{T}$ only. Utilizing Proposition 1.1 repeatedly, there must exist a positive integer m, such that $T + mT_1 > T^*$. This also leads to the contradiction to the maximality of T^*. Therefore, the assumption (2.1) does not hold. This completes the proof of Theorem 1.1.

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