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Solitary waves for nonlinear Klein–Gordon equations coupled with Born–Infeld theory

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Abstract

We consider the nonlinear Klein–Gordon equations coupled with the Born–Infeld theory under the electrostatic solitary wave ansatz. The existence of the least-action solitary waves is proved in both bounded smooth domain case and $\mathbb{R}^3$ case. In particular, for bounded smooth domain case, we study the asymptotic behaviors and profiles of the positive least-action solitary waves with respect to the frequency parameter ω. We show that when κ and ω are suitably large, the least-action solitary waves admit only one local maximum point. When $\omega \to \infty$, the point-condensation phenomenon occurs if we consider the normalized least-action solitary waves.

1. Introduction

1.1. Background

The Born–Infeld geometric theory of electromagnetism is a nonlinear generalization of the classical Maxwell theory. It was introduced to overcome the infinite energy problem associated with a point-charge source in the original Maxwell theory. Based on the action principle of special relativity, Born proposed the *first* Born–Infeld theory (cf. [2] and [3]). It is defined by the action density

$$\mathcal{L}_{BI,1} = b^2\left(1 - \sqrt{1 + \frac{F_{\mu\nu}F^{\mu\nu}}{2b^2}}\right)\sqrt{-\det(g_{\mu\nu})} \tag{1.1}$$

where $g_{\mu\nu}$ is the metric tensor of a $(3+1)$-dimensional Minkowskian space–time of the signature $(+---)$, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength curvature induced from a gauge potential (connection 1-form) A_μ. Later, by considering the invariance principle, Born and Infeld reconsidered (1.1) and introduced the *second* Born–Infeld theory in [8] and [9], which is defined by the action density

$$\mathcal{L}_{BI,2} = b^2\left(\sqrt{-\det(g_{\mu\nu})} - \sqrt{-\det\left(g_{\mu\nu} + \frac{F_{\mu\nu}}{b}\right)}\right). \tag{1.2}$$

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It is clear that when the Born–Infeld free parameter $b \to \infty$, both (1.1) and (1.2) reduce to the classical Maxwell theory.

In recent years, the Born–Infeld nonlinear electromagnetism has regained its importance due to its relevance in the theory of superstrings and membranes (cf. [29]). It has received much attention from both theoretic physicists and mathematicians (cf. [32,27,1,18,34,24]). Mathematically, motivated from Gibbons' work (cf. [18]), in [34], Yang proposed an extended Born–Infeld equation, which can be used to unify the minimal surface equations and maximal hypersurface equations. Meanwhile, the author studied the existence of magnetostatic minimum-energy solutions in the Born–Infeld–Higgs model. Later, in Lin and Yang's work (cf. [24]), the authors studied the gauged harmonic maps by extending the scalar Higgs fields in [34] to maps from a 2-surface into the standard 2-sphere. The coexistence of vortices and antivortices were obtained by studying the first-order system of self-dual and anti-self-dual equations. The existence of cosmic strings induced by these vortices were also established.

1.2. Electrostatic solitary wave ansatz

As we know, the gauge potential A_μ can be coupled to a complex order parameter ψ through the minimal coupling rule. That is the formal substitution

$$\frac{\partial}{\partial t} \mapsto \frac{\partial}{\partial t} - iA_0, \qquad \nabla \mapsto \nabla - iA$$

where $A = (A_1, A_2, A_3)$ is a magnetic vector potential, and A_0 is an electric potential. Therefore, in a flat Minkowskian space–time with metric $(g_{\mu\nu}) = \mathrm{diag}[1, -1, -1, -1]$, we can define the Klein–Gordon–Maxwell Lagrangian density $\mathcal{L}_{KGM}$ as

$$\mathcal{L}_{KGM} = \frac{1}{2}\left[\left|\frac{\partial \psi}{\partial t} - i\psi A_0\right|^2 - |\nabla\psi - iA\psi|^2 - m^2|\psi|^2\right] + \frac{1}{p}|\psi|^p \tag{1.3}$$

where m is the mass of a particle, and $p \in (2, 6)$ is a constant describing the nonlinearity in (1.3). The nonlinear Born–Infeld–Klein–Gordon equations (NBIKG for short) are the Euler–Lagrange equations of the total action

$$\mathcal{S} = \int \mathcal{L}_{BI,1} + \mathcal{L}_{KGM}. \tag{1.4}$$

Under the electrostatic solitary wave ansatz

$$\psi(x,t) = u(x)e^{i\omega t}, \qquad A_0 = -\phi(x), \qquad A = 0$$

where u and ϕ are real-valued functions defined on a subset U of $\mathbb{R}^3$, and ω is a positive frequency parameter, the total action in (1.4) takes the form

$$F_U(u,\phi) := \int_U \frac{1}{2}|\nabla u|^2 + \frac{1}{2}\left(m^2 - \omega^2\right)u^2 - \frac{1}{p}|u|^p\,dx - E_{u,U}(\phi) \tag{1.5}$$

with

$$E_{u,U}(\phi) := \int_U b^2\left(1 - \sqrt{1 - \frac{1}{b^2}|\nabla\phi|^2}\right) + \omega u^2\phi + \frac{1}{2}\phi^2 u^2. \tag{1.6}$$

The critical point (ϕ, u) of F_U satisfies the Euler–Lagrange equations associated with (1.5). By standard calculations, we get

$$\begin{cases} \nabla \cdot \dfrac{\nabla\phi}{\sqrt{1 - \frac{1}{b^2}|\nabla\phi|^2}} = u^2(\omega + \phi), \\ \Delta u = \left(m^2 - (\omega + \phi)^2\right)u - |u|^{p-2}u. \end{cases} \tag{1.7}$$

In this article, U is assumed to be a bounded smooth domain in $\mathbb{R}^3$ or the whole $\mathbb{R}^3$ space. When $U = \mathbb{R}^3$, we omit the subscription U from (1.5) and (1.6). That is, we use F and E_u to denote $F_{\mathbb{R}^3}$ and $E_{u,\mathbb{R}^3}$, respectively. As a convention in this paper, in an integral expression, if its integration domain is $\mathbb{R}^3$, we will omit $\mathbb{R}^3$ from the integral expression.

1.3. Main results

The existence of solitary waves has been well studied in different systems (cf. [4–7,10–13,16,22,25]). Particularly in [13] and [25], the authors considered the system of nonlinear Klein–Gordon equation coupled with the Born–Infeld type equations. Here, the Born–Infeld type equations refer to the second order expansion of the original Born–Infeld equations when the Born–Infeld parameter $b \to \infty$. In this article, we study the original Born–Infeld equations.

Firstly, for any fixed $u \in L^2(U)$, we study the variational problem

$$\min_{\phi \in M} E_{u,U}(\phi). \tag{1.8}$$

The configuration space

$$M := D(U) \cap \{\phi \mid \|\nabla\phi\|_{L^\infty(U)} \leqslant b\} \tag{1.9}$$

naturally arises from the physical constraint (quantity under the square root in (1.6) should be nonnegative) and the finite energy condition ($E_{u,U}(\phi) < \infty$) of (1.6). In (1.9), $D(U)$ denotes the completion of $C_0^\infty(U;\mathbb{R})$ with respect to the norm

$$\|\phi\|_{D(U)} := \|\nabla\phi\|_{L^2(U)} + \|\nabla\phi\|_{L^4(U)}.$$

It is embedded into $L^\infty(U)$ continuously (cf. Proposition 8 in [17]). Therefore, M is a topological space by equipping with the uniform norm topology when U is a bounded smooth domain or the locally uniform norm topology when $U = \mathbb{R}^3$. The existence of global minimizer for the variational problem (1.8) will be studied in Section 2.1 by an application of the direct method in the Calculus of Variations. The minimizer is unique because of the convexity of the functional $E_{u,U}$. Therefore, we can define a nonlinear operator

$$\Phi : L^2(U) \mapsto M, \tag{1.10}$$

which sends one $L^2(U)$ function u to the unique minimizer of the variational problem (1.8). In Section 2.3, the operator Φ is proved to be a continuous map between $L^2(U)$ and M if we equip $L^2(U)$ with its strong topology.

Plug the minimizer $\Phi(u)$ into (1.5), the functional

$$J[u] := F_U\big(u, \Phi(u)\big), \tag{1.11}$$

which is defined on $H^1(U)$ or a subspace of $H^1(U)$, is strongly indefinite. In Proposition 3.1, the functional J will be proved to be C^1 differentiable in the sense of Fréchet. We emphasize here that, with the loss of the C^1 regularity on Φ, our method relies only on the continuity of the operator Φ. In Section 3.2, we will assume $U = \mathbb{R}^3$. By the $\mathbb{Z}_2$ Mountain Pass Theorem, we will prove the existence of infinite many critical points of J with radial symmetry when

$$\kappa = \frac{m}{\omega} \tag{1.12}$$

is suitably large (see Theorem 3.3). Furthermore, we will show that among all nonzero critical points of J with radial symmetry, there exists one that attains the least-J-action (see Theorem 3.8). In Section 3.3, U is a bounded smooth domain. We will study the positive critical points of the functional J with boundary values set to be 0. Therefore, we work on the functional

$$J_+[u] := \int_U \frac{1}{2}|\nabla u|^2 + \frac{1}{2}\big(m^2 - \omega^2\big)u^2 - \frac{1}{p}u_+^p\,dx - E_{u,U}\big(\Phi(u)\big). \tag{1.13}$$

J_+ is defined on $H_0^1(U)$. Based on the well-known Mountain Pass Lemma due to Ambrosetti and Rabinowitz, we will show in Theorem 3.18 and Theorem 3.19 that if κ is suitably large, then among all nonzero critical points of J_+, there exists one that attains the least-J-action. We call it the positive least-J-action critical point of J. Compared to the magnetostatic minimum-energy solutions for the Born–Infeld–Higgs model or the gauged harmonic map model in [34] and [24], our results in Theorems 3.3, 3.8, 3.18 and 3.19 verify the existence of the least-action solutions in the context of electrostatic fields. Moreover, in Theorem 3.18, we generalize the existence of the Mountain Pass solution in [26] and [33] to a coupled system combining a semilinear elliptic equation with an electric potential governed by the Born–Infeld theory.

Denote $u_{\kappa,\omega}$ by one positive least-J-action critical point of the functional J_+. With respect to the parameters κ and ω, in Section 4, we focus on the asymptotic behaviors and profiles of the functions in relation to $u_{\kappa,\omega}$ and $\Phi_\omega(u_{\kappa,\omega})$. We adopt a singular perturbation method. In fact, in Section 4.3, we will prove the point-condensation phenomenon for the normalized function $v_{\kappa,\omega} = \omega^{\frac{2}{2-p}} u_{\kappa,\omega}$. More precisely, it will be proved that when κ is a suitably large constant, in the limit of $\omega \to \infty$, $v_{\kappa,\omega}$ admits only one local maximum point $P_{\kappa,\omega}$. In particular,

$$v_{\kappa,\omega}(P_{\kappa,\omega}) \geqslant \left(\kappa^2 - 1\right)^{\frac{1}{p-2}} \tag{1.14}$$

is uniformly bounded from below by a constant depending only on κ. Meanwhile, the normalized function $v_{\kappa,\omega} \to 0$ in $C^{1,\alpha}_{\mathrm{loc}}(U \setminus P_{\kappa,\omega})$.

2. Minimizer of $E_{u,U}$

2.1. Existence of the minimizer $\Phi(u)$

Denote $D^{1,2}(U)$ by the completion of $C_0^\infty(U;\mathbb{R})$ with respect to the norm

$$\|\phi\|_{D^{1,2}(U)} := \|\nabla\phi\|_{L^2(U)}.$$

$D(U)$ is continuously embedded into both $D^{1,2}(U)$ and $L^\infty(U)$. Moreover, $D^{1,2}(U)$ is continuously embedded into $L^6(U)$ by Sobolev inequality.

In the following, we begin to study the variational problem (1.8) in this section. First, let us state a lemma.

Lemma 2.1. *If $\{\phi_n\} \subseteq M$ and $\phi_n \rightharpoonup \phi$ in $D^{1,2}(U)$, then $\phi_n \to \phi$ uniformly in $\bar U$ if U is a bounded smooth domain; $\phi_n \to \phi$ locally uniformly in $\mathbb{R}^3$, if $U = \mathbb{R}^3$.*

Proof. Assume that $\{\phi_{n_k}\}$ is any subsequence of $\{\phi_n\}$. Since $\{\phi_{n_k}\} \subseteq M$ and $\phi_{n_k} \rightharpoonup \phi$ weakly in $D^{1,2}(U)$, $\{\nabla\phi_{n_k}\}$ is uniformly bounded in $L^2(U)$ and $L^4(U)$. If U is a bounded smooth domain, by Morrey's inequality, we then have

$$\|\phi_{n_k}\|_{C^{0,\frac14}(\bar U)} \leqslant C\|\phi_{n_k}\|_{D(U)},$$

which implies that $\{\phi_{n_k}\}$ is equicontinuous on $\bar U$. Apply Arzelá–Ascoli's theorem, we can extract a subsequence, which is denoted by $\{\phi_{n_{k_l}}\}$, such that $\phi_{n_{k_l}} \rightrightarrows \phi$ in $\bar U$. Because the subsequence $\{\phi_{n_k}\}$ is arbitrary, we have $\phi_n \rightrightarrows \phi$ in $\bar U$. $\mathbb{R}^3$ case is similar. We omit the proof. □

In the following, we use the direct method in the Calculus of Variations (cf. Theorem 1.2 in [31]) to show that

Theorem 2.2. *For every $u \in L^2(U)$, there exists a unique $\Phi(u) \in M$ such that*

$$E_{u,U}\big(\Phi(u)\big) = \inf_{\phi\in M} E_{u,U}(\phi).$$

In fact, we need to verify that:

(1) M is a weakly closed subset of $D^{1,2}(U)$;
(2) $E_{u,U}$ is coercive and weakly sequentially lower semicontinuous on M with respect to $D^{1,2}(U)$.

Lemma 2.3. *M is a weakly closed subset of $D^{1,2}(U)$.*

Proof. Note that, $D^{1,2}(U)$ is reflexive and M is convex. We only need to prove the strong closedness of M in $D^{1,2}(U)$. Choose $\{\phi_n\} \subseteq M$, $\phi_n \to \phi$ in $D^{1,2}(U)$. Up to a subsequence, we can assume $\nabla\phi_n \to \nabla\phi$ almost everywhere in U. Then $\|\nabla\phi\|_{L^\infty(U)} \leqslant b$. We also have

$$\int_U |\nabla\phi_n - \nabla\phi|^4 \leqslant 4b^2 \int_U |\nabla\phi_n - \nabla\phi|^2 \to 0.$$

Hence, $\phi \in D(U)$. Under the definition of M in (1.9), $\phi \in M$. □

Pay attention to the fact that

$$F(x,\phi,p) = b^2\left(1-\sqrt{1-\frac{1}{b^2}|p|^2}\right) + \omega u^2(x)\phi + \frac{1}{2}u^2(x)\phi^2 \geqslant -\frac{1}{2}\omega^2 u^2(x)$$

is convex in p. Then, F is a Caratheodory function. In the following, we apply Theorem 1.6 in [31] and show that

Lemma 2.4. *$E_{u,U}(\cdot)$ is coercive and weakly sequentially lower semicontinuous with respect to $D^{1,2}(U)$.*

Proof. The coercivity of $E_{u,U}$ with respect to $D^{1,2}(U)$ is a result of the fact that

$$E_{u,U}(\phi) \geqslant \int_U \frac{1}{2}|\nabla\phi|^2\,dx - \frac{1}{2}\omega^2 \int_U u^2\,dx.$$

Let us now prove the weakly sequentially lower semicontinuity of $E_{u,U}$. Assume that $\{\phi_n\} \subseteq M$ and $\phi_n \rightharpoonup \phi$ weakly in $D^{1,2}(U)$. From Lemma 2.1, we know that $\phi_n \rightrightarrows \phi$ locally in U. That is, $\forall U' \Subset U$, we have $\phi_n \rightrightarrows \phi$ in U'. Hence, $\phi_n \to \phi$ in $L^1(U')$. Since $\nabla\phi_n \rightharpoonup \nabla\phi$ weakly in $L^2(U)$, we get $\nabla\phi_n \rightharpoonup \nabla\phi$ weakly in $L^1(U')$. Apply Theorem 1.6 in [31],

$$E_{u,U}(\phi) \leqslant \liminf_{n\to\infty} E_{u,U}(\phi_n). \qquad \square$$

Remark 2.5. Use the same method as the above, we can prove that if $\{\phi_n\} \subseteq M$ and $\phi_n \rightharpoonup \phi$ weakly in $D^{1,2}(U)$, then

$$\int_U b^2\left(1-\sqrt{1-\frac{1}{b^2}|\nabla\phi|^2}\right) \leqslant \liminf_{n\to\infty} \int_U b^2\left(1-\sqrt{1-\frac{1}{b^2}|\nabla\phi_n|^2}\right).$$

Now we complete this section by the proof of Theorem 2.2.

Proof of Theorem 2.2. Lemma 2.3 and Lemma 2.4 together with Theorem 1.2 in [31] imply that there exists a minimizer of $E_{u,U}$ on M. The convexity of the functional $E_{u,U}$ ensures the uniqueness of the minimizer. □

2.2. Some properties of the operator Φ

From Theorem 2.2, we can construct the operator Φ, which is defined as in (1.10). Since for a fixed $u \in L^2(U)$, $\Phi(u)$ is the unique minimizer of the variational problem (1.8), then

Proposition 2.6. *$\forall u \in L^2(U)$ fixed, $\Phi = \Phi(u)$, we have*

$$\int_U \frac{|\nabla\Phi|^2}{\sqrt{1-\frac{1}{b^2}|\nabla\Phi|^2}} \leqslant \int_U -u^2(\omega+\Phi)\Phi. \tag{2.1}$$

Proof. $\forall\lambda \in (0,1)$, $\lambda\Phi \in M$. Consider $E_{u,U}(\lambda\Phi)$ as a function of λ, then

$$\frac{d}{d\lambda}E_{u,U}(\lambda\Phi) = \int_U \lambda\frac{|\nabla\Phi|^2}{\sqrt{1-\frac{1}{b^2}|\nabla\Phi|^2\lambda^2}} + \omega u^2\Phi + \lambda u^2\Phi^2. \tag{2.2}$$

Since $\varPhi$ attains the minimum of $E_{u,U}$ in M, we have

$$\liminf_{\lambda\to 1^-}\frac{d}{d\lambda}E_{u,U}(\lambda\varPhi)\leqslant 0. \tag{2.3}$$

Apply Fatou's Lemma, (2.2) and (2.3) may imply (2.1). □

Since $D(U)$ is embedded into $L^\infty(U)$ continuously, then $(\omega+\varPhi)\varPhi\in L^\infty(U)$. By the assumption that $u\in L^2(U)$, we can conclude from Proposition 2.6 that the minimizer $\varPhi=\varPhi(u)$ of the variational problem (1.8) satisfies

$$\int_U\frac{|\nabla\varPhi|^2}{\sqrt{1-\frac{1}{b^2}|\nabla\varPhi|^2}}<+\infty.$$

Therefore, we can define $K\subseteq M$ as

$$K:=\left\{\psi\in M\ \Big|\ \int_U\frac{|\nabla\psi|^2}{\sqrt{1-\frac{1}{b^2}|\nabla\psi|^2}}<+\infty\right\}. \tag{2.4}$$

The minimizer $\varPhi=\varPhi(u)$ of the variational problem (1.8) is also the minimizer of the problem to minimize the energy functional $E_{u,U}$ on K. Therefore, we may get the following proposition.

Proposition 2.7 *(Variational Inequality). $\forall u\in L^2(U)$ fixed, $\varPhi=\varPhi(u)$ is the minimizer of $E_{u,U}$ on M. Then, $\forall\psi\in K$,*

$$\int_U\frac{\nabla\varPhi}{\sqrt{1-\frac{1}{b^2}|\nabla\varPhi|^2}}\cdot\nabla(\varPhi-\psi)+(\omega+\varPhi)(\varPhi-\psi)u^2\leqslant 0. \tag{2.5}$$

Proof. $\forall\psi\in K$ and $0<\lambda<1$, $\lambda\varPhi+(1-\lambda)\psi\in K$. We know that $\varPhi(u)$ attains the minimum of $E_{u,U}$ on K, thus,

$$\liminf_{\lambda\to 1^-}\frac{d}{d\lambda}E_{u,U}\big(\lambda\varPhi+(1-\lambda)\psi\big)\leqslant 0.$$

Apply Lebesgue's Dominated Convergence theorem. We complete the proof. □

As an application of the variational inequality (2.5), we prove the a priori upper and lower bounds of $\varPhi(u)$ for a fixed $u\in L^2(U)$.

Proposition 2.8. *$\forall u\in L^2(U)$, it results in $\varPhi(u)\leqslant 0$. Moreover, $\varPhi(u)(x)\geqslant-\omega$ if $u(x)\neq 0$.*

Proof. Apply the variational inequality (2.5) and set $\psi=-\varPhi^-$, we have

$$\int_U\frac{\nabla\varPhi}{\sqrt{1-\frac{1}{b^2}|\nabla\varPhi|^2}}\cdot\nabla\big(\varPhi+\varPhi^-\big)+\omega\big(\varPhi+\varPhi^-\big)u^2+\big(\varPhi+\varPhi^-\big)\varPhi u^2\leqslant 0.$$

That is,

$$\int_{\varPhi\geqslant 0}\frac{\nabla\varPhi^+}{\sqrt{1-\frac{1}{b^2}|\nabla\varPhi^+|^2}}\cdot\nabla\varPhi^++\omega\varPhi^+u^2+\varPhi^+\varPhi^+u^2\leqslant 0.$$

So we get $\nabla\varPhi^+=0$. Hence, $\varPhi\leqslant 0$. If we set $\psi=-\omega+(\omega+\varPhi)^+$, then we have

$$\int_{\varPhi\leqslant-\omega}\frac{|\nabla\varPhi|^2}{\sqrt{1-\frac{1}{b^2}|\nabla\varPhi|^2}}+u^2(\varPhi+\omega)^2\leqslant 0. \tag{2.6}$$

Hence, $\varPhi(u)(x)\geqslant-\omega$ whenever $u(x)\neq 0$. □

In the rest of this section, we assume that $U = \mathbb{R}^3$. Let f be a function defined on $\mathbb{R}^3$. With respect to fixed $x_0 \in \mathbb{R}^3$ and $g \in O(3)$, we can define translation and rotation on f as

$$T_{x_0} f(x) := f(x + x_0), \qquad T_g f(x) := f(gx), \quad \forall x \in \mathbb{R}^3. \tag{2.7}$$

From the definition of K in (2.4), it is easy to prove that K is translation–rotation invariant. Pay attention to the uniqueness result in Theorem 2.2. We prove the fact that Φ is even and commutes with the group of rototranslations in Proposition 2.9 below.

Proposition 2.9. *$\forall u \in L^2(\mathbb{R}^3)$, $\forall g \in O(3)$, $\forall x_0 \in \mathbb{R}^3$, we have*

$$\Phi(T_{x_0} u) = T_{x_0}\Phi(u), \tag{2.8}$$

$$\Phi(T_g u) = T_g \Phi(u), \tag{2.9}$$

$$\Phi(u) = \Phi(-u). \tag{2.10}$$

Proof. We know that

$$E_u\big(\Phi(u)\big) = E_{T_{x_0}u}\big(T_{x_0}\Phi(u)\big).$$

Since K is translation–rotation invariant, we have $T_{x_0}\Phi(u) \in K$. Note that, $\Phi(T_{x_0}u)$ attains the minimum of $E_{T_{x_0}u}$ on M. Then,

$$E_u\big(\Phi(u)\big) \geqslant E_{T_{x_0}u}\big(\Phi(T_{x_0}u)\big).$$

Since x_0 is arbitrary, we conclude that

$$E_u\big(\Phi(u)\big) = E_{T_{x_0}u}\big(\Phi(T_{x_0}u)\big) = E_{T_{x_0}u}\big(T_{x_0}\Phi(u)\big).$$

The uniqueness result in Theorem 2.2 immediately implies (2.8). The proofs for (2.9) and (2.10) are similar. We omit the discussions here. □

As an application of (2.8) in Proposition 2.9 and the variational inequality (2.5) in Proposition 2.7, let us consider a $W^{2,2}$ estimate on the minimizer $\Phi(u)$.

Lemma 2.10. *If $\nabla\Phi(u) \in L^{p_1}(\mathbb{R}^3)$, $\nabla u \in L^2(\mathbb{R}^3) \cap L^{p_2}(\mathbb{R}^3)$ and $u \in L^2(\mathbb{R}^3) \cap L^{p_3}(\mathbb{R}^3)$ where $1 \leqslant p_i \leqslant \infty$ $(i = 1, 2, 3)$ satisfying $\frac{1}{p_1} + \frac{1}{p_2} + \frac{1}{p_3} = 1$, then*

$$\int \sum_{i,j} \big|\partial_{ij}\Phi(u)\big|^2 \leqslant 6\omega \big\|\nabla\Phi(u)\big\|_{L^{p_1}(\mathbb{R}^3)} \|\nabla u\|_{L^{p_2}(\mathbb{R}^3)} \|u\|_{L^{p_3}(\mathbb{R}^3)}. \tag{2.11}$$

Proof. Assume that $\Phi = \Phi(u)$, $\Phi_{1,\epsilon} := \Phi(x + \epsilon e_1)$ and $u_{1,\epsilon} := u(x + \epsilon e_1)$ where $e_1 = (1, 0, 0)$. Since K is translation–rotation invariant, $\Phi_{1,\epsilon} \in K$. By Proposition 2.7, we have

$$\int \frac{\nabla\Phi}{\sqrt{1 - \frac{1}{b^2}|\nabla\Phi|^2}} \cdot \nabla(\Phi - \Phi_{1,\epsilon}) + \omega(\Phi - \Phi_{1,\epsilon})u^2 + (\Phi - \Phi_{1,\epsilon})\Phi u^2 \leqslant 0. \tag{2.12}$$

From (2.8) in Proposition 2.9, we know that $\Phi_{1,\epsilon} = \Phi(u(\cdot + \epsilon e_1))$. Apply Proposition 2.7 again, we have

$$\int \frac{\nabla\Phi_{1,\epsilon}}{\sqrt{1 - \frac{1}{b^2}|\nabla\Phi_{1,\epsilon}|^2}} \cdot \nabla(\Phi_{1,\epsilon} - \Phi) + \omega(\Phi_{1,\epsilon} - \Phi)u_{1,\epsilon}^2 + (\Phi_{1,\epsilon} - \Phi)\Phi_{1,\epsilon}u_{1,\epsilon}^2 \leqslant 0. \tag{2.13}$$

By adding the inequalities (2.12) and (2.13), we have

$$\int \left|\nabla \frac{\Phi_{1,\epsilon} - \Phi}{\epsilon}\right|^2 + \big(\omega + \Phi(u)\big)\left(\frac{\Phi_{1,\epsilon} - \Phi}{\epsilon}\right)\left(\frac{u^2(x + \epsilon e_1) - u^2(x)}{\epsilon}\right) \leqslant 0.$$

Therefore,

$$\int \left|\nabla \frac{\Phi_{1,\epsilon}-\Phi}{\epsilon}\right|^2 + 2u\big(\omega+\Phi(u)\big)\left(\frac{\Phi_{1,\epsilon}-\Phi}{\epsilon}\right)\left(\frac{u(x+\epsilon e_1)-u(x)}{\epsilon}\right)$$
$$+\int \big(\omega+\Phi(u)\big)\left(\frac{\Phi_{1,\epsilon}-\Phi}{\epsilon}\right)\left(\frac{u(x+\epsilon e_1)-u(x)}{\epsilon}\right)\big(u(x+\epsilon e_1)-u(x)\big) \leqslant 0. \tag{2.14}$$

Since we know that if $1 \leqslant p \leqslant \infty$,

$$\left\|\frac{\Phi_{1,\epsilon}-\Phi}{\epsilon}\right\|_{L^p(\mathbb{R}^3)} \leqslant \|\nabla\Phi\|_{L^p(\mathbb{R}^3)}.$$

In particular, if $p=\infty$,

$$\left\|\frac{\Phi_{1,\epsilon}-\Phi}{\epsilon}\right\|_{L^\infty(\mathbb{R}^3)} \leqslant b. \tag{2.15}$$

Then, we know that

$$\int \big(\omega+\Phi(u)\big)\left(\frac{\Phi_{1,\epsilon}-\Phi}{\epsilon}\right)\left(\frac{u(x+\epsilon e_1)-u(x)}{\epsilon}\right)^2 \leqslant b\big\|\big(\omega+\Phi(u)\big)\big\|_{L^\infty(\mathbb{R}^3)}\|\nabla u\|^2_{L^2(\mathbb{R}^3)}.$$

Therefore, from (2.14) and Proposition 2.8, we have

$$\int \left|\nabla \frac{\Phi_{1,\epsilon}-\Phi}{\epsilon}\right|^2 \leqslant 2\int \left|\big(\omega+\Phi(u)\big)\left(\frac{\Phi_{1,\epsilon}-\Phi}{\epsilon}\right)\left(\frac{u(x+\epsilon e_1)-u(x)}{\epsilon}\right)u\right| + O(\epsilon)$$
$$\leqslant 2\omega\int \left|\left(\frac{\Phi_{1,\epsilon}-\Phi}{\epsilon}\right)\left(\frac{u(x+\epsilon e_1)-u(x)}{\epsilon}\right)u\right| + O(\epsilon). \tag{2.16}$$

By applying Hölder's inequality on (2.16) and letting $\epsilon \to 0$, we get

$$\int \big|\nabla\partial_1\Phi(u)\big|^2 \leqslant 2\omega\|\nabla\Phi\|_{L^{p_1}(\mathbb{R}^3)}\|\nabla u\|_{L^{p_2}(\mathbb{R}^3)}\|u\|_{L^{p_3}(\mathbb{R}^3)}$$

where $1 \leqslant p_i \leqslant \infty$ $(i=1,2,3)$ satisfying $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=1$. □

Apply Lemma 2.10 by setting $p_1=\infty$, $p_2=p_3=2$. The following proposition holds:

Proposition 2.11. *If $u \in H^1(\mathbb{R}^3)$, then $\nabla\Phi(u) \in H^1(\mathbb{R}^3;\mathbb{R}^3)$, which satisfies*

$$\int \sum_{i,j}\big|\partial_{ij}\Phi(u)\big|^2 \leqslant 6\omega b\|\nabla u\|_{L^2(\mathbb{R}^3)}\|u\|_{L^2(\mathbb{R}^3)}. \tag{2.17}$$

2.3. Continuity of the operator Φ

In this section, we study the continuity of the operator Φ in two cases. In Proposition 2.12, we prove that $\Phi: L^2(U) \mapsto M$ is a continuous map if we equip $L^2(U)$ with the strong $L^2(U)$ topology. If we restrict the operator Φ on

$$L := L^2(U) \cap L^{\frac{12}{5}}(U) \cap L^3(U) \tag{2.18}$$

and equip L with the strong $L^{\frac{12}{5}}(U) \cap L^3(U)$ topology, then in Proposition 2.13, we show that $\Phi: L \mapsto M$ is a continuous map. Here, $u_n \to u$ under the strong $L^{\frac{12}{5}}(U) \cap L^3(U)$ topology means that $u_n \to u$ in both $L^{\frac{12}{5}}(U)$ and $L^3(U)$ strongly. As before, M is a topological space equipped with the uniform norm topology if U is a bounded smooth domain or with the locally uniform norm topology if $U=\mathbb{R}^3$.

Proposition 2.12. *$\Phi: L^2(U) \mapsto M$ is a continuous map.*

Proof. Assume that $u_n \to u$ strongly in $L^2(U)$. In the following, we show that $\forall\{u_{n_k}\} \subseteq \{u_n\}$, there exists a subsequence $\{u_{n_{k_l}}\}$, such that when U is a bounded smooth domain, $\Phi(u_{n_{k_l}}) \rightrightarrows \Phi(u)$ in $\bar{U}$; when $U=\mathbb{R}^3$, $\Phi(u_{n_{k_l}}) \rightrightarrows \Phi(u)$ locally in $\mathbb{R}^3$.

Note that, $\{u_{n_k}\}$ is uniformly bounded in $L^2(U)$. By Propositions 2.6 and 2.8, $\{\Phi(u_{n_k})\}$ is uniformly bounded in $D(U)$. Hence, it is also uniformly bounded in $L^\infty(U)$. Then, there exist a subsequence $\{u_{n_{k_l}}\}$ and $f \in M$, such that $\Phi(u_{n_{k_l}}) \rightharpoonup f$ in $D^{1,2}(U)$. By Lemma 2.1, $\Phi(u_{n_{k_l}}) \rightrightarrows f$ in $\bar{U}$ when U is a bounded smooth domain or $\Phi(u_{n_{k_l}}) \rightrightarrows f$ locally in $\mathbb{R}^3$ when $U=\mathbb{R}^3$. For the rest of the proof, we only need to show that $f=\Phi(u)$.

Because $\Phi(u_{n_{k_l}})$ is the minimizer of $E_{u_{n_{k_l}},U}$ on M and $u_{n_{k_l}} \to u$ in $L^2(U)$, we have

$$E_{u_{n_{k_l}},U}\big(\Phi(u_{n_{k_l}})\big) \leqslant E_{u_{n_{k_l}},U}\big(\Phi(u)\big) \to E_{u,U}\big(\Phi(u)\big).$$

Note that, $\{\Phi(u_{n_{k_l}})\}$ is uniformly bounded in $L^\infty(U)$ and converges to f pointwisely. When $u_{n_{k_l}} \to u$ in $L^2(U)$, we have

$$\int_U \Phi(u_{n_{k_l}})u_{n_{k_l}}^2 \to \int_U fu^2, \qquad \int_U \Phi(u_{n_{k_l}})^2 u_{n_{k_l}}^2 \to \int_U f^2u^2. \tag{2.19}$$

From Remark 2.5 and (2.19), we get

$$E_{u,U}(f) \leqslant \liminf_{l\to\infty} E_{u_{n_{k_l}},U}\big(\Phi(u_{n_{k_l}})\big) \leqslant \lim_{l\to\infty} E_{u_{n_{k_l}},U}\big(\Phi(u)\big) = E_{u,U}\big(\Phi(u)\big).$$

By the uniqueness result in Theorem 2.2, $f=\Phi(u)$. □

Proposition 2.13. *$\Phi : L \mapsto M$ is a continuous map. L is defined in* (2.18).

Proof. Assume that $u_n \to u$ in L. When U is a bounded smooth domain, the result has already been included in Proposition 2.12. Now, we assume $U=\mathbb{R}^3$. Notice Proposition 2.8 and apply Hölder's inequality on the right-hand side of (2.1). We have

$$\int \big|\nabla\Phi(u_n)\big|^2 + \frac{1}{2b^2}\big|\nabla\Phi(u_n)\big|^4 \leqslant \omega\|u_n\|^2_{L^{\frac{12}{5}}(\mathbb{R}^3)}\big\|\Phi(u_n)\big\|_{L^6(\mathbb{R}^3)}. \tag{2.20}$$

Since $u_n \to u$ in L, we have $\{u_n\}$ uniformly bounded in $L^{\frac{12}{5}}(\mathbb{R}^3)$. From (2.20), $\{\Phi(u_n)\}$ is uniformly bounded in $D(\mathbb{R}^3)$. Hence, it is also uniformly bounded in $L^\infty(\mathbb{R}^3)$ and $L^6(\mathbb{R}^3)$. Then, there exist a subsequence, still denoted by $\{\Phi(u_n)\}$, and $f \in M$, such that $\Phi(u_n) \rightharpoonup f$ in $D^{1,2}(\mathbb{R}^3)$. Apply Lemma 2.1, $\Phi(u_n) \rightrightarrows f$ locally in $\mathbb{R}^3$. For the rest of the proof, we only need to show that $f=\Phi(u)$.

Note that,

$$\left|\int u_n^2\Phi(u) - u^2\Phi(u)\right| \leqslant \big\|u_n^2-u^2\big\|_{L^{\frac{6}{5}}}\big\|\Phi(u)\big\|_{L^6} \leqslant C\big\|\nabla\Phi(u)\big\|_{L^2}\|u_n-u\|_{L^{\frac{12}{5}}} \to 0, \tag{2.21}$$

$$\left|\int u_n^2\Phi(u)^2 - u^2\Phi(u)^2\right| \leqslant \big\|u_n^2-u^2\big\|_{L^{\frac{3}{2}}}\big\|\Phi(u)\big\|^2_{L^6} \leqslant C\big\|\nabla\Phi(u)\big\|^2_{L^2}\|u_n-u\|_{L^3} \to 0. \tag{2.22}$$

Hence, we have

$$\lim_{n\to\infty} E_{u_n}\big(\Phi(u)\big) = E_u\big(\Phi(u)\big).$$

Since $E_{u_n}(\Phi(u_n)) \leqslant E_{u_n}(\Phi(u))$, then

$$\limsup_n E_{u_n}\big(\Phi(u_n)\big) \leqslant E_u\big(\Phi(u)\big). \tag{2.23}$$

In another way, note that, $\{\Phi(u_n)\}$ is uniformly bounded in $L^6(\mathbb{R}^3)$, $L^\infty(\mathbb{R}^3)$ and converges to f pointwisely. When $u_n \to u$ in L, we can apply the Lebesgue's Dominated Convergence theorem and similar arguments as in (2.21) and (2.22) to show that

$$\int u_n^2\Phi(u_n) \to \int u^2 f, \qquad \int u_n^2\Phi(u_n)^2 \to \int u^2 f^2. \tag{2.24}$$

Because of Remark 2.5 and (2.24), we have

$$\liminf_{n} E_{u_n}\big(\Phi(u_n)\big) \geqslant E_u(f). \tag{2.25}$$

Obviously, (2.23) and (2.25) imply that $E_u(f) = E_u(\Phi(u))$. Therefore, we have $f = \Phi(u)$ by the uniqueness result in Theorem 2.2. □

Remark 2.14. If $u_n \to u$ in $L^2(U)$ or L, then

$$\lim_{n\to\infty} E_{u_n,U}\big(\Phi(u_n)\big) = E_{u,U}\big(\Phi(u)\big). \tag{2.26}$$

3. Existence of solitary wave solutions

In this section, we prove the existence of critical points of J and J_+. Firstly, let us study the C^1 differentiability of J and J_+ in the sense of Fréchet.

3.1. *Differentiability of J and J_+*

In this section, the functional J is defined on $\mathcal{H}$, which is a subspace of $H^1(U)$. Naturally, $\mathcal{H}$ is endowed with the metric from $H^1(U)$. The main purpose of this section is to prove

Proposition 3.1. $J \in C^1(\mathcal{H}; \mathbb{R})$ *in the sense of Fréchet.*

Proof. It is sufficient to prove

$$\lim_{\|v\|_{H^1}\to 0}\big(J[u+v] - J[u] - DJ[u]v\big)/\|v\|_{H^1} = 0$$

where

$$DJ[u]v := \int_U \nabla u\cdot\nabla v + \big(m^2 - \big(\omega+\Phi(u)\big)^2\big)uv - |u|^{p-2}uv. \tag{3.1}$$

We split $J[u+v] - J[u] - DJ[u]v$ into three parts. That is,

$$J[u+v] - J[u] - DJ[u]v = \mathrm{A} + \mathrm{B} + \mathrm{C}$$

where

$$\mathrm{A} = \int_U \frac{1}{2}\big|\nabla(u+v)\big|^2 - \frac{1}{2}|\nabla u|^2 - \nabla u\cdot\nabla v,$$

$$\mathrm{B} = -\frac{1}{p}\int_U |u+v|^p - |u|^p - p|u|^{p-2}uv,$$

$$\mathrm{C} = Eu, U\big(\Phi(u)\big) - E_{u+v,U}\big(\Phi(u+v)\big) + \int_U \frac{1}{2}\big(m^2-\omega^2\big)v^2 + \big(2\omega+\Phi(u)\big)\Phi(u)uv.$$

By Sobolev embedding theorem, it is easy to prove A, $\mathrm{B} = o(\|v\|_{H^1})$. We thus focus on the proof of $\mathrm{C} = o(\|v\|_{H^1})$. In the following, we will prove $\mathrm{C} \geqslant o(\|v\|_{H^1})$ and $\mathrm{C} \leqslant o(\|v\|_{H^1})$ successively. Then we complete the proof of Proposition 3.1.

We first prove $\mathrm{C} \geqslant o(\|v\|_{H^1(\mathbb{R}^3)})$. Since $\Phi(u+v)$ minimizes $E_{u+v,U}$ on M, we have

$$\mathrm{C} \geqslant \mathrm{D} := E_{u,U}\big(\Phi(u)\big) - E_{u+v,U}\big(\Phi(u)\big) + \int_U \frac{1}{2}\big(m^2-\omega^2\big)v^2 + \big(2\omega+\Phi(u)\big)\Phi(u)uv.$$

Therefore, $\mathrm{C} \geqslant o(\|v\|_{H^1})$ by the fact that

$$\mathrm{D} = \int_U \frac{1}{2}\left(m^2 - \omega^2\right)v^2 - \omega\Phi(u)v^2 - \frac{1}{2}\Phi(u)^2 v^2 = o\left(\|v\|_{H^1}\right).$$

Let us now prove $\mathrm{C} \leqslant o(\|v\|_{H^1})$. Since $\Phi(u)$ minimizes $E_{u,U}$ on M,

$$\mathrm{C} \leqslant E_{u,U}\left(\Phi(u+v)\right) - E_{u+v,U}\left(\Phi(u+v)\right) + \int_U \left(2\omega + \Phi(u)\right)\Phi(u)uv + \frac{1}{2}\left(m^2 - \omega^2\right)v^2.$$

Set

$$\Theta := uv\left(\Phi(u+v) - \Phi(u)\right)\left(2\omega + \Phi(u) + \Phi(u+v)\right),$$

we get

$$\mathrm{C} \leqslant o\left(\|v\|_{H^1}\right) - \int_U \Theta. \tag{3.2}$$

In the following, we prove

$$\lim_{\|v\|_{H^1}\to 0} \frac{1}{\|v\|_{H^1}} \left|\int_U \Theta\right| = 0 \tag{3.3}$$

in two cases, then from (3.2), we get $\mathrm{C} \leqslant o(\|v\|_{H^1})$.

Case 1 *(Bounded Smooth Domain).* Under definition of Θ, we have by applying Hölder's inequality that

$$\frac{1}{\|v\|_{H^1}} \left|\int_U \Theta\right| \leqslant \left\|2\omega + \Phi(u) + \Phi(u+v)\right\|_{L^\infty(U)} \left\|\Phi(u+v) - \Phi(u)\right\|_{L^\infty(U)} \|u\|_{L^2(U)}.$$

From Propositions 2.6, 2.8 and the fact that $D(U)$ is embedded into $L^\infty(U)$ continuously, we know that, as $v \to 0$ in $H^1(U)$, $\|2\omega + \Phi(u) + \Phi(u+v)\|_{L^\infty(U)}$ is uniformly bounded by a constant depending on b, ω and $\|u\|_{L^2(U)}$. By Proposition 2.12, we get (3.3) in the bounded smooth domain case.

Case 2 *(The whole Euclidean space $\mathbb{R}^3$).* Similarly as in Case 1, when $v \to 0$ in $H^1(\mathbb{R}^3)$, $\|\Phi(u+v)\|_{L^\infty(\mathbb{R}^3)}$ is uniformly bounded by a constant $C = C(b, \omega, \|u\|_{L^2(\mathbb{R}^3)})$. Since $u \in L^2(\mathbb{R}^3)$, $\forall\epsilon > 0$, there exists $R > 0$ large enough, such that $\|u\|_{L^2(B_R^c)} < \epsilon$. Therefore, by Hölder's inequality,

$$\frac{1}{\|v\|_{H^1(\mathbb{R}^3)}} \left|\int_{B_R^c} \Theta\right| \leqslant C\|u\|_{L^2(B_R^c)} < C\epsilon. \tag{3.4}$$

Fix R above. Since we know from Proposition 2.12 that if $v \to 0$ in $L^2(\mathbb{R}^3)$, then $\Phi(u+v) \rightrightarrows \Phi(u)$ in $\bar{B}_R$. We thus get

$$\frac{1}{\|v\|_{H^1(\mathbb{R}^3)}} \left|\int_{B_R} \Theta\right| \leqslant C\left\|\Phi(u+v) - \Phi(u)\right\|_{L^\infty(\bar{B}_R)} \|u\|_{L^2(\mathbb{R}^3)} \to 0, \quad \text{as } \|v\|_{H^1(\mathbb{R}^3)} \to 0. \tag{3.5}$$

By (3.4) and (3.5), we get (3.3) when $U = \mathbb{R}^3$. □

Remark 3.2. If we consider the functional J_+ defined as in (1.13), then by using the same method as in the proof of Proposition 3.1, we get $J_+ \in C^1(\mathcal{H}; \mathbb{R})$ in the sense of Fréchet.

3.2. Solitary wave solutions – $U=\mathbb{R}^3$

The functional J in (1.11) is defined on $H^1(\mathbb{R}^3)$. According to Proposition 3.1, J is C^1 differentiable in the sense of Fréchet. In this section, we study the existence of critical points of J by the $\mathbb{Z}_2$ Mountain Pass Theorem. More precisely, we prove

Theorem 3.3. *If $(\frac{p}{2}-1)m^2 > \frac{p}{2}\omega^2$, then in $H^1(\mathbb{R}^3)$, there exist infinitely many critical points of J with radial symmetry. Moreover, the critical points of J satisfy the equation*

$$-\Delta u + \left(m^2 - \left(\omega + \Phi(u)\right)^2\right)u - |u|^{p-2}u = 0. \tag{3.6}$$

Before we prove Theorem 3.3, let us state two lemmas, which are simple applications of (2.9) and (2.10).

Lemma 3.4. $\forall u \in H^1(\mathbb{R}^3)$, $J[u] = J[-u]$.

Lemma 3.5. $\forall u \in H^1(\mathbb{R}^3)$, $\forall g \in O(3)$, *we have* $J[T_g u] = J[u]$.

Lemma 3.4 and Lemma 3.5 imply that the functional J is even and T_g invariant. Therefore, by the principle of symmetric criticality (cf. [28]), we may restrict J on the radially symmetric subspace $H_r^1(\mathbb{R}^3) \subset H^1(\mathbb{R}^3)$. That is,

Lemma 3.6. *If $u \in H_r^1(\mathbb{R}^3)$ is a critical point of $J|_{H_r^1(\mathbb{R}^3)}$, then u is a critical point of J.*

Because J is invariant under translations, there is a lack of compactness on $H^1(\mathbb{R}^3)$. For this reason, we restrict J to the subspace $H_r^1(\mathbb{R}^3)$, which is a natural constraint for J in the sense of Lemma 3.6. Then, there is no lack of compactness. Indeed, we have

Lemma 3.7. *If $(\frac{p}{2}-1)m^2 > \frac{p}{2}\omega^2$, the functional $J|_{H_r^1(\mathbb{R}^3)}$ satisfies the Palais–Smale condition.*

After some slight modifications, the proof of Lemma 3.7 is almost the same as the proof of Lemma 6 in [13]. We omit the discussion here. Now we begin to prove Theorem 3.3.

Proof of Theorem 3.3. J is even. According to Lemma 3.7, we know that $J|_{H_r^1}$ satisfies the Palais–Smale condition. Therefore, by Theorem 9.12 in [30], we only need to show that $J|_{H_r^1}$ satisfies the following two geometric hypothesis:

(G_1) $\exists \rho > 0$ and $\alpha > 0$ such that $J[u] \geqslant \alpha$, $\forall u$ with $\|u\|_{H_r^1} = \rho$;
(G_2) for every finite dimensional subspace V of H_r^1, $\exists R = R(V) > 0$ such that $J[u] \leqslant 0$, $\forall u \in V$ with $\|u\|_{H_r^1} \geqslant R$.

Since we know that $\forall u \in H^1(\mathbb{R}^3)$,

$$C(m,\omega)\|u\|_{H^1}^2 - \frac{1}{p}\|u\|_{L^p}^p \leqslant J[u] \leqslant C(m)\|u\|_{H^1}^2 - \frac{1}{p}\|u\|_{L^p}^p.$$

Sobolev inequality implies that $\|u\|_{L^p} \leqslant C\|u\|_{H^1}$. This helps us to prove (G_1). In finite dimensional subspace V of $H_r^1(\mathbb{R}^3)$, the L^p norm and H^1 norm are equivalent. Thus in V, $\|u\|_{H^1} \leqslant C(V)\|u\|_{L^p}$, $\forall u \in V$. This helps us to complete the proof of G_2. □

Theorem 3.3 implies the existence of infinitely many critical points of the functional J with the radial symmetry. In Theorem 3.8 below, we prove the existence of least-J-action critical point among all nonzero critical points of J with the radial symmetry. Note that J is strongly indefinite. It is not bounded from above or below, we need to restrict our functional J on the manifold

$$\Sigma := \left\{u \in H_r^1\left(\mathbb{R}^3\right) \,\middle|\, u \not\equiv 0,\ u \text{ is a critical point of } J\right\}.$$

Theorem 3.8. *If* $(\frac{p}{2}-1)m^2 > \frac{p}{2}\omega^2$, *then*

$$F := \inf_{u\in\Sigma} J[u] = \min_{u\in\Sigma} J[u] > 0. \tag{3.7}$$

Proof. First, let us prove $F > 0$. If $u \in \Sigma$, then

$$\int |\nabla u|^2 + \left(m^2 - \left(\omega + \Phi(u)\right)^2\right)u^2 - |u|^p = 0. \tag{3.8}$$

Apply Sobolev embedding theorem,

$$\int |u|^p = \int |\nabla u|^2 + \left(m^2 - \left(\omega + \Phi(u)\right)^2\right)u^2 \geqslant C\left(\int |u|^p\right)^{\frac{2}{p}} \tag{3.9}$$

where $C > 0$ is a constant depending on m and ω. Since $u \not\equiv 0$, we have

$$\int |u|^p \geqslant C^{\frac{p}{p-2}} > 0. \tag{3.10}$$

That is, Σ keeps strictly away from 0. By (3.8) and the fact that $E_u(\Phi(u)) \leqslant 0$, we have

$$J[u] \geqslant \int \left(\frac{1}{2} - \frac{1}{p}\right)|\nabla u|^2 + \left[\left(\frac{1}{2} - \frac{1}{p}\right)m^2 - \frac{1}{2}\omega^2\right]u^2 \geqslant C\|u\|_{L^p}^2 \geqslant C(p,\omega,m) > 0. \tag{3.11}$$

Hence, $F > 0$.

If $\{u_n\} \subseteq \Sigma$ such that $J[u_n] \to F$, then $\{J[u_n]\}$ is bounded and from (3.11), $\{u_n\}$ is uniformly bounded in $H_r^1(\mathbb{R}^3)$. Therefore, we can extract a subsequence, denoted also by $\{u_n\}$, such that $u_n \rightharpoonup u_0$ in $H_r^1(\mathbb{R}^3)$. Since $H_r^1(\mathbb{R}^3) \hookrightarrow L^s(\mathbb{R}^3)$ compactly with $s \in (2,6)$, we can further assume that $u_n \to u_0$ in $L^s(\mathbb{R}^3)$ with $s = \frac{12}{5}$, 3, p. By Proposition 2.13, we know that $\Phi(u_n) \rightrightarrows \Phi(u_0)$ locally in $\mathbb{R}^3$. Since

$$\int \nabla u_n \cdot \nabla\phi + \left(m^2 - \left(\omega + \Phi(u_n)\right)^2\right)u_n\phi = \int |u_n|^{p-2}u_n\phi, \quad \forall\phi \in C_0^\infty\left(\mathbb{R}^3\right),$$

let $n \to \infty$,

$$\int \nabla u_0 \cdot \nabla\phi + \left(m^2 - \left(\omega + \Phi(u_0)\right)^2\right)u_0\phi = \int |u_0|^{p-2}u_0\phi.$$

That is, u_0 is a weak radial solution of (3.6). Regarding (3.10) and L^p convergence of $\{u_n\}$, we imply that $u_0 \not\equiv 0$. Hence, $u_0 \in \Sigma$. In addition, notice Remark 2.14. We have

$$\begin{aligned} F \leqslant J[u_0] &= \int \frac{1}{2}|\nabla u_0|^2 + \frac{1}{2}\left(m^2 - \omega^2\right)u_0^2 - \frac{1}{p}|u_0|^p - E_{u_0}\left(\Phi(u_0)\right) \\ &\leqslant \liminf_n \int \frac{1}{2}|\nabla u_n|^2 + \frac{1}{2}\left(m^2 - \omega^2\right)u_n^2 - \lim_n \int \frac{1}{p}|u_n|^p - \lim_n E_{u_n}\left(\Phi(u_n)\right) = F. \end{aligned}$$

i.e. F can be attained by the function $u_0 \in \Sigma$. □

3.3. Solitary wave solutions – bounded smooth domain

In this section, U is a bounded smooth domain. We are interested in the existence of the positive least-J-action critical point of J. Therefore, we study the functional J_+, which is defined on $H_0^1(U)$. Due to Remark 3.2, J_+ is C^1 differentiable in the sense of Fréchet.

Follow the notations as in [26]. Given $e \geqslant 0$, $e \not\equiv 0$, $e \in H_0^1(U)$ with $J_+[e] = 0$, we define

$$\mathcal{C} := \inf_{h\in\Gamma} \max_{0\leqslant t\leqslant 1} J_+\left[h(t)\right] \tag{3.12}$$

where Γ is the set of all continuous paths joining the origin and this given e. In addition, we define

$$M[v] := \sup_{t\geqslant 0} g_v(t), \quad \forall v \in H_0^1(U),\ v \geqslant 0,\ v \not\equiv 0 \tag{3.13}$$

where

$$g_v(t) := J_+[tv], \quad \forall t \geqslant 0. \tag{3.14}$$

In the following, we study $g_v(t)$. The key point is to understand the nonlinear operator Φ defined as in (1.10).

Lemma 3.9. *$\forall s, t \in \mathbb{R}$, $\forall v \in L^2(U)$, we have*

$$\int_U \left|\frac{\nabla\Phi(sv) - \nabla\Phi(tv)}{s-t}\right|^2 + \omega v^2(s+t)\frac{\Phi(sv)-\Phi(tv)}{s-t} + v^2\frac{\Phi(sv)-\Phi(tv)}{s-t}\left(s^2\frac{\Phi(sv)-\Phi(tv)}{s-t} + (s+t)\Phi(tv)\right) \leqslant 0. \tag{3.15}$$

Proof. Apply the variational inequality in Proposition 2.7 repeatedly by setting $u = sv$, $\psi = \Phi(tv)$ and $u = tv$, $\psi = \Phi(sv)$. We can imply (3.15). □

Lemma 3.10. (3.15) *implies that for a given $v \in H_0^1(U)$, $\forall s_n, l_n \to t$, there exist a subsequence $\{s_{n_k}\}, \{l_{n_k}\}$ and $\zeta \in D^{1,2}(U)$, such that*

$$\frac{\nabla\Phi(s_{n_k}v) - \nabla\Phi(l_{n_k}v)}{s_{n_k} - l_{n_k}} \rightharpoonup \nabla\zeta \quad \text{in } L^2(U), \tag{3.16}$$

$$\frac{\Phi(s_{n_k}v) - \Phi(l_{n_k}v)}{s_{n_k} - l_{n_k}} \to \zeta \quad \text{in } L^s(U),\ s = \frac{12}{5}, 3. \tag{3.17}$$

Proof. By Hölder inequality, Sobolev inequality and Proposition 2.8, we imply from (3.15) that $(\Phi(s_nv) - \Phi(l_nv))/(s_n - l_n)$ is uniformly bounded in $D^{1,2}(U)$ by a constant depending on ω, t, U and the given function v. Thus implies (3.16). (3.17) is a result of the fact that $D^{1,2}(U)$ is compactly embedded into $L^s(U)$ with $s = \frac{12}{5}, 3$. □

From Lemma 3.9 and Lemma 3.10, as well as Proposition 2.12, we know that

Lemma 3.11. *For a given $v \in H_0^1(U)$, $\forall s_n, l_n \to t$, there exist a subsequence $\{s_{n_k}\}, \{l_{n_k}\}$ and $\zeta \in D^{1,2}(U)$, such that* (3.16), (3.17) *hold true and in addition, we have this ζ satisfies*

$$\int_U v^2t^2\zeta^2 + 2t\zeta v^2\big(\omega + \Phi(tv)\big) \leqslant 0. \tag{3.18}$$

On the basis of the preparations above, in the following, we consider the first and in some sense the second derivatives of the function g_v.

Proposition 3.12. *For a fixed $v \in H_0^1(U)$, g_v is C^1 differentiable and*

$$g_v'(t) = \int_U t|\nabla v|^2 + tv^2\big(m^2 - \big(\omega + \Phi(tv)\big)^2\big) - t^{p-1}v_+^p, \quad \forall t > 0. \tag{3.19}$$

Proof. We need only to prove

$$h_v(t) := E_{tv,U}\big(\Phi(tv)\big)$$

is C^1 differentiable. Assume $s \to t^-$. Since $\Phi(tv)$ is a minimizer of $E_{tv,U}$ on M, $h_v(t) \leqslant E_{tv,U}(\Phi(sv))$. So

$$h_v(t) - h_v(s) \leqslant \big(t^2 - s^2\big)\int_U \omega v^2\Phi(sv) + \frac{1}{2}v^2\Phi(sv)^2.$$

Let $s \to t^-$ and notice that $\Phi(sv) \rightrightarrows \Phi(tv)$ in $\bar{U}$ (see Proposition 2.12). We have

$$\limsup_{s \to t^-} \frac{h_v(t) - h_v(s)}{t - s} \leqslant 2t \int_U \omega v^2 \Phi(tv) + \frac{1}{2} v^2 \Phi(tv)^2.$$

Note that, $h_v(s) \leqslant E_{sv,U}(\Phi(tv))$, similarly, we have

$$\liminf_{s \to t^-} \frac{h_v(t) - h_v(s)}{t - s} \geqslant 2t \int_U \omega v^2 \Phi(tv) + \frac{1}{2} v^2 \Phi(tv)^2.$$

Hence,

$$\lim_{s \to t^-} \frac{h_v(t) - h_v(s)}{t - s} = 2t \int_U \omega v^2 \Phi(tv) + \frac{1}{2} v^2 \Phi(tv)^2.$$

In the case of $s \to t^+$, we can apply the same argument as the above. Therefore,

$$h_v'(t) = 2t \int_U \omega v^2 \Phi(tv) + \frac{1}{2} v^2 \Phi(tv)^2. \qquad \square$$

Remark 3.13. Assume $v \in H_0^1(U)$, $v \geqslant 0$, $v \not\equiv 0$. By Proposition 2.8 and (3.19), we get

$$\int_U t|\nabla v|^2 + tv^2(m^2 - \omega^2) - t^{p-1} v^p \leqslant g_v'(t) \leqslant \int_U t|\nabla v|^2 + tm^2 v^2 - t^{p-1} v^p, \quad \forall t > 0.$$

Set

$$t_1 = \frac{1}{\|v\|_{L^p(U)}^{\frac{p}{p-2}}} \left[\int_U |\nabla v|^2 + (m^2 - \omega^2) v^2 \right]^{\frac{1}{p-2}}, \qquad t_2 = \frac{1}{\|v\|_{L^p(U)}^{\frac{p}{p-2}}} \left[\int_U |\nabla v|^2 + m^2 v^2 \right]^{\frac{1}{p-2}}.$$

We have if $t < t_1$, $g_v'(t) > 0$; if $t > t_2$, $g_v'(t) < 0$.

Let us study in some sense the second derivative of $g_v(t)$ for $t \in [t_1, t_2]$.

Proposition 3.14. *If $v \in H_0^1(U)$, $v \geqslant 0$, $v \not\equiv 0$, then $\forall t \in [t_1, t_2]$, $\forall s_n, l_n \to t$, we can extract a subsequence $s_{n_k}, l_{n_k} \to t$, such that*

$$\lim_{k \to \infty} \frac{g_v'(s_{n_k}) - g_v'(l_{n_k})}{s_{n_k} - l_{n_k}} \leqslant (2 - p) \int_U |\nabla v|^2 \, dx + \left[(2 - p)m^2 + (p + 4)\omega^2\right] \int_U v^2. \tag{3.20}$$

In particular, if κ defined in (1.12) satisfies $\kappa^2 > \frac{p+4}{p-2}$, then

$$\lim_{k \to \infty} \frac{g_v'(s_{n_k}) - g_v'(l_{n_k})}{s_{n_k} - l_{n_k}} < 0. \tag{3.21}$$

Proof. We can find ζ and subsequence $\{s_{n_k}\}, \{l_{n_k}\}$ as in Lemma 3.11 such that (3.16)–(3.18) hold true. By (3.19), we have

$$\lim_{k \to \infty} \frac{g_v'(s_{n_k}) - g_v'(l_{n_k})}{s_{n_k} - l_{n_k}} = \int_U |\nabla v|^2 + \left(m^2 - \left(\omega + \Phi(tv)\right)^2\right) v^2 - (p - 1) t^{p-2} v^p - 2t\zeta v^2 \left(\omega + \Phi(tv)\right).$$

Since $t \geqslant t_1$, by (3.18) and Cauchy–Schwartz inequality, we complete the proof. $\square$

Proposition 3.14 implies that

Proposition 3.15. *If v is assumed as in Proposition 3.14 and $\kappa^2 > \frac{p+4}{p-2}$, then g_v' is strictly decreasing in $[t_1, t_2]$.*

Proof. Let us start from $t = t_1$. We claim that $\exists\delta > 0$, such that g_v' is strictly decreasing in $[t_1, t_1 + \delta]$. Otherwise, there exists $\{s_n\}$, $\{l_n\}$, such that $t_1 < s_n < l_n$, $s_n, l_n \to t_1^+$, but

$$g_v'(s_n) \leqslant g_v'(l_n). \tag{3.22}$$

By Proposition 3.14, there exists a subsequence $s_{n_k}, l_{n_k} \to t_1^+$ such that

$$\lim_{k\to\infty} \frac{g_v'(s_{n_k}) - g_v'(l_{n_k})}{s_{n_k} - l_{n_k}} < 0.$$

Since $s_{n_k} < l_{n_k}$, we have $g_v'(s_{n_k}) > g_v'(l_{n_k})$ when k is large. This is a contradiction to (3.22). Set

$$s := \sup\bigl\{\delta \leqslant t_2 - t_1 \bigm| g_v' \text{ strictly decreasing in } [t_1, t_1 + \delta]\bigr\}. \tag{3.23}$$

We claim that $s = t_2 - t_1$. Otherwise, we can prove that for some $\delta > 0$, g_v' is strictly decreasing on $[t_1 + s, t_1 + s + \delta]$. This is a contradiction to the definition of s in (3.23). □

Notice that $g_v' > 0$ on $(0, t_1)$, $g_v' < 0$ on (t_2, ∞). Therefore, Proposition 3.15 implies that there exists only one point $t_0 \in [t_1, t_2]$, such that $g_v'(t_0) = 0$. More precisely,

Proposition 3.16. *If $\kappa^2 > \frac{p+4}{p-2}$, g_v has only one local maximum point on $(0, +\infty)$. Actually, this local maximum point is the absolute maximum point of g_v. In addition, we know that,*

$$g_v'(t_0) = 0 \quad \Leftrightarrow \quad g_v(t_0) = \sup_{t\geqslant 0} J_+[tv]. \tag{3.24}$$

Remark 3.17. If u solves

$$\begin{cases} -\Delta u + \bigl(m^2 - (\omega + \Phi(u))^2\bigr)u = u_+^{p-1} & \text{in } U, \\ u > 0 & \text{in } U, \\ u = 0 & \text{on } \partial U, \end{cases} \tag{3.25}$$

weakly, then we know that $g_u'(1) = 0$. From (3.24), we get $J_+[u] = M[u]$.

The preparations above lead to the following theorem.

Theorem 3.18. *If $\kappa^2 > \frac{p+4}{p-2}$, then*

$$\mathcal{C} = \inf\bigl\{M[v] \bigm| v \in H_0^1(U), v \geqslant 0, v \not\equiv 0\bigr\}. \tag{3.26}$$

In addition, $\mathcal{C}$ is a critical value of J_+, which is independent of the choice of e.

Notice Proposition 3.15 and Remark 3.17. The proof of Theorem 3.18 is almost the same as in [26]. We omit it here. Assume that u is the critical point of J_+ corresponding to the critical value $\mathcal{C}$ in (3.12). $v \not\equiv 0$ is a solution of (3.25). From Remark 3.17, $M[v] = J_+[v]$. By (3.26), $\mathcal{C} = J_+[u] \leqslant M[v] = J_+[v]$. Therefore,

Theorem 3.19. *The critical points corresponding to the critical value in (3.12) are the positive least-J-action critical points of J.*

4. The asymptotic behaviors and profiles of the positive least-J-action solutions

In order to emphasize on the dependence with κ and ω, in the following, $E_{u,U}$, Φ and J_+ in the previous sections are denoted by $E_{\omega,u,U}$, Φ_ω and $J_{+,\kappa,\omega}$, respectively. κ is assumed to be a suitably large constant. Therefore, from Theorems 3.18 and 3.19, there exists a positive least-$J_{+,\kappa,\omega}$-action critical point $u_{\kappa,\omega}$ corresponding to the critical value in Theorem 3.18. Now, we focus on the asymptotic behaviors and profiles of $\{u_{\kappa,\omega}\}$ when $\omega \to \infty$.

Firstly, let us introduce some notations that will be used in the following sections. After scaling the functional $J_{+,\kappa,\omega}$,

$$J_{\kappa,\omega}[v] := \omega^{\frac{2p}{2-p}} J_{+,\kappa,\omega}\big[\omega^{\frac{2}{p-2}} v\big] \tag{4.1}$$

is a functional defined on $H_0^1(U)$ with its critical points solving the elliptic problem

$$\begin{cases} -\dfrac{1}{\omega^2}\Delta u + \left(\kappa^2 - \left(1 + \dfrac{\Phi_\omega(\omega^{\frac{2}{p-2}} u)}{\omega}\right)^2\right) u = u_+^{p-1} & \text{in } U, \\ u > 0 & \text{in } U, \\ u = 0 & \text{on } \partial U. \end{cases} \tag{4.2}$$

If $e \in H_0^1(U)$, $e \geqslant 0$, $e \not\equiv 0$, $J_{+,\kappa,\omega}[e] = 0$, then $J_{\kappa,\omega}[\omega^{\frac{2}{2-p}} e] = 0$. Assume that $\Gamma_{\kappa,\omega}$ is a set of all continuous paths connecting 0 and $\omega^{\frac{2}{2-p}} e$ while Γ still denotes the set of all continuous paths connecting 0 and e. Theorems 3.18 and 3.19 imply the following proposition.

Proposition 4.1. *If $\kappa^2 > \frac{p+4}{p-2}$, then*

$$\mathcal{C}_{\kappa,\omega} := \inf_{h \in \Gamma_{\kappa,\omega}} \max_{0 \leqslant t \leqslant 1} J_{\kappa,\omega}\big[h(t)\big] = \omega^{\frac{2p}{2-p}} \inf_{h \in \Gamma} \max_{0 \leqslant t \leqslant 1} J_{+,\kappa,\omega}\big[h(t)\big], \tag{4.3}$$

$$\mathcal{C}_{\kappa,\omega} = \inf\Big\{\sup_{t \geqslant 0} J_{\kappa,\omega}[tv] \;\Big|\; v \in H_0^1(U),\ v \geqslant 0,\ v \not\equiv 0\Big\}. \tag{4.4}$$

Moreover, $\mathcal{C}_{\kappa,\omega}$ is the least critical value of $J_{\kappa,\omega}$ among all solutions of (4.2).

From Proposition 4.1, we know that the normalized function $\omega^{\frac{2}{2-p}} u_{\kappa,\omega}$ is a least-$J_{\kappa,\omega}$-action solution of the problem (4.2). More generally, in the following, we assume $v_{\kappa,\omega}$ is a critical point of $J_{\kappa,\omega}$ corresponding to the critical value $\mathcal{C}_{\kappa,\omega}$ defined as in (4.3).

4.1. Asymptotic behaviors of $V_{\kappa,\omega}$ and $\phi_{\kappa,\omega}$

First, we assume $P_{\kappa,\omega}$ is a local maximum point of $v_{\kappa,\omega}$. After translating and dilating the domain U, we get

$$U_{\kappa,\omega} := \big\{y = \omega(x - P_{\kappa,\omega}):\ x \in U\big\}. \tag{4.5}$$

$\forall y \in U_{\kappa,\omega}$, we define

$$V_{\kappa,\omega}(y) := v_{\kappa,\omega}\left(\frac{1}{\omega} y + P_{\kappa,\omega}\right), \qquad \phi_{\kappa,\omega}(y) := \frac{1}{\omega}\Phi_\omega\big(\omega^{\frac{2}{p-2}} v_{\kappa,\omega}\big)\left(\frac{1}{\omega} y + P_{\kappa,\omega}\right).$$

In order to study the asymptotic behaviors of $V_{\kappa,\omega}$ and $\phi_{\kappa,\omega}$ when $\omega \to \infty$, we first consider the uniform boundedness of $\{v_{\kappa,\omega}\}$. We assume in the following that B is one constant depending on p, κ and the domain U. Notice the a priori upper and lower bounds of Φ_ω in Proposition 2.8. The following three lemmas are easily proved after slight modifications of the proofs of Theorem 2 and Corollary 2.1 in [23].

Lemma 4.2. *$\exists \omega_0 = \omega_0(p, \kappa, U) > 0$, such that $\forall \omega > \omega_0$,*

$$\mathcal{C}_{\kappa,\omega} = J_{\kappa,\omega}[v_{\kappa,\omega}] \leqslant B\omega^{-3}. \tag{4.6}$$

Note that,

$$J_{\kappa,\omega}[v_{\kappa,\omega}] \geqslant \int\limits_U \left(\frac{1}{2}-\frac{1}{p}\right)\frac{1}{\omega^2}|\nabla v_{\kappa,\omega}|^2 + \left[\left(\frac{1}{2}-\frac{1}{p}\right)\kappa^2-\frac{1}{2}\right]v_{\kappa,\omega}^2.$$

Hence, Lemma 4.2 implies that

Lemma 4.3. *$\exists \omega_0=\omega_0(p,\kappa,U)>0$, such that $\forall \omega>\omega_0$,*

$$\int\limits_U |\nabla v_{\kappa,\omega}|^2 \leqslant B\omega^{-1}, \qquad \int\limits_U v_{\kappa,\omega}^2 \leqslant B\omega^{-3}. \tag{4.7}$$

Therefore, $\|V_{\kappa,\omega}\|_{H_0^1(U_{\kappa,\omega})}$ is uniformly bounded if $\omega>\omega_0$.

By applying a standard iteration method used in the proof of Corollary 2.1 in [23], we have

Lemma 4.4. *$\exists \omega_0=\omega_0(p,\kappa,U)>0$, such that*

$$\sup_{\omega>\omega_0} \|v_{\kappa,\omega}\|_{L^\infty(U)} \leqslant B. \tag{4.8}$$

On the basis of the preparations above, we begin to study the uniform convergence of $\phi_{\kappa,\omega}$ when $\omega\to\infty$. Since we are interested in the asymptotic behaviors of $v_{\kappa,\omega}$ when $\omega\to\infty$, we assume $\omega>\omega_0$ in the following.

Proposition 4.5.

$$\lim_{\omega\to\infty} \|\phi_{\kappa,\omega}\|_{L^\infty(U_{\kappa,\omega})}=0. \tag{4.9}$$

Proof. From

$$E_{\omega,\omega^{\frac{2}{p-2}}v_{\kappa,\omega},U}\left(\Phi_\omega\left(\omega^{\frac{2}{p-2}}v_{\kappa,\omega}\right)\right)\leqslant 0,$$

we get

$$\int\limits_U b^2\left(1-\sqrt{1-\frac{1}{b^2}\left|\nabla\Phi_\omega\left(\omega^{\frac{2}{p-2}}v_{\kappa,\omega}\right)\right|^2}\right)\leqslant \frac{1}{2}\omega^{\frac{2p}{p-2}}\int\limits_U v_{\kappa,\omega}^2. \tag{4.10}$$

From Lemma 4.3, we have, $\forall q>0$, q is an even number,

$$\int\limits_U \left|\nabla\Phi_\omega\left(\omega^{\frac{2}{p-2}}v_{\kappa,\omega}\right)\right|^q \leqslant C(p,q,\kappa,b,U)\omega^{-3+\frac{2p}{p-2}}.$$

This is equivalent to

$$\int\limits_{U_{\kappa,\omega}} |\nabla\phi_{\kappa,\omega}|^q \leqslant C(p,q,\kappa,b,U)\omega^{\frac{2p}{p-2}-2q}. \tag{4.11}$$

By Hölder inequality, we then get from (4.11) that

$$\int\limits_{U_{\kappa,\omega}} |\nabla\phi_{\kappa,\omega}|^2 \leqslant \left(\int\limits_{U_{\kappa,\omega}} |\nabla\phi_{\kappa,\omega}|^q\right)^{\frac{2}{q}}|U_{\kappa,\omega}|^{1-\frac{2}{q}} \leqslant C(p,q,\kappa,b,U)|U|^{1-\frac{2}{q}}\omega^{(\frac{4p}{p-2}-6-q)\frac{1}{q}},$$

$$\int\limits_{U_{\kappa,\omega}} |\nabla\phi_{\kappa,\omega}|^4 \leqslant C(p,q,\kappa,b,U)|U|^{1-\frac{4}{q}}\omega^{(\frac{8p}{p-2}-12-5q)\frac{1}{q}}.$$

Fix a large q such that $\frac{4p}{p-2} - 6 - q < 0$ and $\frac{8p}{p-2} - 12 - 5q < 0$. Then, $\|\phi_{\kappa,\omega}\|_{D(\mathbb{R}^3)} \to 0$ as $\omega \to \infty$. Since $D(\mathbb{R}^3) \hookrightarrow L^\infty(\mathbb{R}^3)$ continuously, (4.9) holds. □

Before proceeding, let us introduce the elliptic problem

$$\begin{cases} \Delta w - cw + w^{p-1} = 0 & \text{in } \mathbb{R}^3, \\ w > 0 & \text{in } \mathbb{R}^3, \end{cases} \tag{4.12}$$

where $c > 0$ is a constant and w satisfies

$$\begin{cases} w(0) = \max_{x\in\mathbb{R}^3} w(x), \\ w(z) \to 0 \quad \text{as } z \to \infty. \end{cases} \tag{4.13}$$

It is well known that there exists a unique solution, denoted by w_c, for the problem (4.12)–(4.13). Moreover, w_c must be spherically symmetric about the origin and strictly decreasing in $r = |z|$. Since the solution of (4.12)–(4.13) is unique, we get

$$w_c(x) = c^{\frac{1}{p-2}} w(\sqrt{c}x), \quad \forall x \in \mathbb{R}^3,\ c > 0. \tag{4.14}$$

Here, w denotes the solution of (4.12)–(4.13) when $c = 1$.

Now, we begin to study the $C^2_{\mathrm{loc}}(\mathbb{R}^3)$ convergence of $V_{\kappa,\omega}$ when $\omega \to \infty$. Notice that if K is a compact subset of $\mathbb{R}^3$, when ω is large enough, $K \subseteq U_{\kappa,\omega}$ (see Lemma 4.8 in Section 4.2). Therefore, from Lemma 4.3, we can assume, up to a subsequence, that for some $v \in H^1_0(\mathbb{R}^3)$,

$$V_{\kappa,\omega} \rightharpoonup v \quad \text{in } H^1_{0,\mathrm{loc}}(\mathbb{R}^3), \qquad V_{\kappa,\omega} \to v \quad \text{in } L^p_{\mathrm{loc}}(\mathbb{R}^3). \tag{4.15}$$

Moreover, note that, $V_{\kappa,\omega}$ satisfies

$$-\Delta V_{\kappa,\omega} + \big(\kappa^2 - (1+\phi_{\kappa,\omega})^2\big) V_{\kappa,\omega} = V^{p-1}_{\kappa,\omega}. \tag{4.16}$$

From Lemma 4.4 and Proposition 2.8, we know that

$$\Delta V_{\kappa,\omega} = \big(\kappa^2 - (1+\phi_{\kappa,\omega})^2\big) V_{\kappa,\omega} - V^{p-1}_{\kappa,\omega} \in L^\infty(U_{\kappa,\omega})$$

is uniformly bounded. Thus, by using Caldéron–Zygmund inequality, Sobolev embedding theorem and Arzelá–Ascoli's theorem, we can assume, up to a subsequence, that

$$V_{\kappa,\omega} \to v \quad \text{in } C^1_{\mathrm{loc}}(\mathbb{R}^3), \text{ as } \omega \to \infty. \tag{4.17}$$

In fact, we can have a stronger convergence than (4.17). As we know, $V_{\kappa,\omega}$ satisfies (4.16). Hence, $\partial_i V_{\kappa,\omega}$ satisfies the equation

$$-\Delta \partial_i V_{\kappa,\omega} + \big(\kappa^2 - (1+\phi_{\kappa,\omega})^2\big)\partial_i V_{\kappa,\omega} - 2V_{\kappa,\omega}(1+\phi_{\kappa,\omega})\partial_i \phi_{\kappa,\omega} = (p-1)V^{p-2}_{\kappa,\omega}\partial_i V_{\kappa,\omega}$$

weakly. Apply Caldéron–Zygmund inequality, Sobolev embedding theorem and Arzelá–Ascoli's theorem again. We can assume, up to a subsequence that

$$V_{\kappa,\omega} \to v \quad \text{in } C^2_{\mathrm{loc}}(\mathbb{R}^3). \tag{4.18}$$

From the analysis above, we can assume, up to a subsequence, that $V_{\kappa,\omega}$ satisfies (4.15) and (4.18) simultaneously. Note that $V_{\kappa,\omega}$ is a weak solution of (4.16) and $\phi_{\kappa,\omega}$ satisfies Proposition 4.5. By letting $\omega \to \infty$, we conclude that v satisfies the elliptic equation in (4.12) with $c = \kappa^2 - 1$. Moreover, note that, $V_{\kappa,\omega}(0) = v_{\kappa,\omega}(P_{\kappa,\omega}) \geqslant (\kappa^2-1)^{\frac{1}{p-2}}$. By strong maximum principle, we conclude that $v > 0$ in $\mathbb{R}^3$. Furthermore, it is known that (see [14, Theorem 5] and [35]) if $v \in H^1(\mathbb{R}^3)$ solves (4.12), then $v(z) \to 0$ as $|z| \to \infty$. Therefore, one can apply Theorem 2 of [19] and conclude that (i) v is spherically symmetric with respect to some point $z_0 \in \mathbb{R}^3$; (ii) $v_r < 0$ for $r > 0$. Note that,

$$\nabla v(0) = \lim_{\omega\to\infty} \nabla V_{\kappa,\omega}(0) = 0.$$

Therefore, v is spherically symmetric with respect to the origin and

$$v(0) = \max_{x\in\mathbb{R}^3} v(x).$$

Apply the uniqueness theorem in [21], we get $v = w_{\kappa^2-1}$. Therefore,

Proposition 4.6. *$V_{\kappa,\omega} \to w_{\kappa^2-1}$ in $C^2_{\text{loc}}(\mathbb{R}^3)$, when $\omega \to \infty$.*

4.2. Geometric lemmas about local maximum points of $v_{\kappa,\omega}$

We prove two geometric lemmas in this section. They will be applied in the next section to determine the number of local maximum points of $v_{\kappa,\omega}$.

Lemma 4.7. *Assume that $x^1_{\kappa,\omega}$, $x^2_{\kappa,\omega}$ are two local maximum points of $v_{\kappa,\omega}$. Then*

$$\omega\left|x^1_{\kappa,\omega} - x^2_{\kappa,\omega}\right| \to +\infty, \quad \text{as } \omega \to +\infty. \tag{4.19}$$

Proof. If not, $\exists C > 0$, $\omega_n \to \infty$, such that

$$\omega_n\left|x^1_{\kappa,\omega_n} - x^2_{\kappa,\omega_n}\right| \leqslant C, \quad \forall n \in \mathbb{N}. \tag{4.20}$$

From (4.20), there must exist a constant $R > C$ such that $V'_{\kappa,\omega_n} := v_{\kappa,\omega_n}(\frac{1}{\omega_n}y + x^1_{\kappa,\omega_n})$ has at least two local maximum points in $B_R(0)$ for n large. That is, $y = 0$ and $y = \omega_n(x^2_{\kappa,\omega_n} - x^1_{\kappa,\omega_n})$ are two local maximum points of V'_{κ,ω_n}, at least. In addition, since $w''_{\kappa^2-1}(0) < 0$ and w_{κ^2-1} decay to 0 when $x \to \infty$, we can choose two numbers a, b $(0 < a < b)$ such that $w_{\kappa^2-1}(b) < (\kappa^2 - 1)^{\frac{1}{p-2}}$ and

$$w''_{\kappa^2-1}(r) < 0, \quad \forall r \in [0, a].$$

Set

$$C_* = \min\left\{\left|w'_{\kappa^2-1}(r)\right| \,\middle|\, r \in [a, b]\right\}, \quad \epsilon < C_*.$$

Assume that $A_{p,q}$ is the annulus centered at 0 with radius $r \in [p, q]$. In the following, we contradict the assumption (4.20) so that (4.19) holds true.

(i) If x_n is one local maximum point of V'_{κ,ω_n}, then $\Delta V'_{\kappa,\omega_n}(x_n) \leqslant 0$. Hence, from (4.16),

$$V'_{\kappa,\omega_n}(x_n) \geqslant \left(\kappa^2 - 1\right)^{\frac{1}{p-2}}. \tag{4.21}$$

Notice Proposition 4.6 and the fact that $w_{\kappa^2-1}(r) < (\kappa^2 - 1)^{\frac{1}{p-2}}$ when $R \geqslant r \geqslant b$. Then, when n is large enough, $V'_{\kappa,\omega_n} < (\kappa^2 - 1)^{\frac{1}{p-2}}$ on $A_{b,R}$. Therefore, from (4.21), we know that when n is large enough, there is no local maximum point of V'_{κ,ω_n} on $A_{b,R}$.

(ii) From Proposition 4.6, we know that $\nabla V'_{\kappa,\omega_n} \rightrightarrows \nabla w_{\kappa^2-1}$ on $A_{a,b}$. Hence, we have $|\nabla V'_{\kappa,\omega_n}| \geqslant |\nabla w_{\kappa^2-1}| - |\nabla V'_{\kappa,\omega_n} - \nabla w_{\kappa^2-1}| \geqslant C_* - \epsilon > 0$ when n is large. That is, when n is large enough, V'_{κ,ω_n} has no local maximum point on $A_{a,b}$.

(iii) Note that, $w''_{\kappa^2-1} < 0$ on the ball $\bar{B}_a(0)$, $\nabla V'_{\kappa,\omega_n}(0) = 0$ and $\Delta V'_{\kappa,\omega_n} - \Delta w_{\kappa^2-1} = V'^{p-1}_{\kappa,\omega_n} - w^{p-1}_{\kappa^2-1} - [(\kappa^2 - (1 + \phi_{\kappa,\omega_n})^2)V'_{\kappa,\omega_n} - (\kappa^2 - 1)w_{\kappa^2-1}] \rightrightarrows 0$ on $\bar{B}_a(0)$. Then we can imply from Lemma 3.3 in [36] that on $\bar{B}_a(0)$, $\nabla V'_{\kappa,\omega_n}(x) \neq 0$ for $x \neq 0$.

From the analysis above, we know that when n is large enough, V'_{κ,ω_n} can have only one local maximum point $y = 0$ in $B_R(0)$. This argument contradicts the assumption (4.20) since we know that when (4.20) holds, V'_{κ,ω_n} has at least two local maximum points in $B_R(0)$. □

Lemma 4.8. *Assume that $P_{\kappa,\omega}$ is a local maximum point of $v_{\kappa,\omega}$. Then*

$$d(P_{\kappa,\omega}, \partial U)\omega \to +\infty, \quad \text{as } \omega \to +\infty. \tag{4.22}$$

Proof. Suppose on the contrary that there exist a constant $C^* > 0$ and $\omega_n \to +\infty$ such that

$$d(P_{\kappa,\omega_n}, \partial U)\omega_n \leqslant C^*, \quad \forall n \in \mathbb{N}. \tag{4.23}$$

Passing to a subsequence, we may assume that $P_{\kappa,\omega_n} \to P_0 \in \partial U$. Through translation and rotation of the coordinate system, we may assume that P_0 is the origin and the inner normal to ∂U at P_0 is pointing in the direction of the positive x_3 axis. Then there exists a smooth function $\omega_{P_0}(x')$ ($x' = (x_1, x_2)$ sufficiently small) such that in a small neighborhood $\mathcal{N}$ of P_0,

(i) $\omega_{P_0}(0) = 0$ and $\nabla\omega_{P_0}(0) = 0$;
(ii) $\partial U \cap \mathcal{N} = \{(x', x_3) \mid x_3 = \omega_{P_0}(x')\}$ and $U \cap \mathcal{N} = \{(x', x_3) \mid x_3 > \omega_{P_0}(x')\}$.

By the ω_{P_0} above, we can construct a diffeomorphism $\mathcal{F}^{-1}$ mapping from an open set $\mathcal{V} \subset \{y_3 > 0\}$ to an open set $\mathcal{O} \subset U \cap \mathcal{N}$ by

$$\mathcal{F}^{-1}(y) = \left(y_1 - y_3\frac{\partial\omega_{P_0}}{\partial x_1}\big(y'\big), y_2 - y_3\frac{\partial\omega_{P_0}}{\partial x_2}\big(y'\big), y_3 + \omega_{P_0}\big(y'\big)\right) \in \mathcal{O}, \quad \forall y \in \mathcal{V}. \tag{4.24}$$

Without loss of generality, we assume that $\bar{B}^+_{3\varrho} \subset \mathcal{V}$. When n is large enough, we have $P_{\kappa,\omega_n} \in \mathcal{O}$ and $Q_{\kappa,\omega_n} = (q'_{\kappa,\omega_n}, \frac{\alpha_{\kappa,\omega_n}}{\omega_n}) = \mathcal{F}(P_{\kappa,\omega_n}) \in B^+_{\varrho} := \{y \in B_\varrho \mid y_3 > 0\}$ for some $\alpha_{\kappa,\omega_n} \geqslant 0$. Since $\mathcal{F}$ is a diffeomorphism, from (4.23), $\{\alpha_{\kappa,\omega_n}\}$ is uniformly bounded. Hence, up to a subsequence, we can assume $\alpha_{\kappa,\omega_n} \to \alpha \geqslant 0$. Set

$$\mathbb{R}^3_{\alpha,+} = \{y \mid y_3 > -\alpha\}$$

and

$$\tilde{v}_{\kappa,\omega_n}(y) := v_{\kappa,\omega_n}\big(\mathcal{F}^{-1}(y)\big), \quad y \in \bar{B}^+_{2\varrho},$$
$$\tilde{w}_{\kappa,\omega_n}(z) := \tilde{v}_{\kappa,\omega_n}\left(Q_{\kappa,\omega_n} + \frac{1}{\omega_n}z\right), \quad z \in \bar{B}_{\varrho\omega_n} \cap \{z_3 \geqslant -\alpha_{\kappa,\omega_n}\}.$$

It is clear from (4.2) that in $B_{\varrho\omega_n} \cap \{z_3 > -\alpha_{\kappa,\omega_n}\}$, $\tilde{w}_{\kappa,\omega_n}$ satisfies

$$\sum_{i,j=1}^{3} \frac{\partial}{\partial z_i}\left(a^{\omega_n}_{ij}\frac{\partial\tilde{w}_{\kappa,\omega_n}}{\partial z_j}\right) + \sum_{j=1}^{3}\left(\frac{1}{\omega_n}b^{\omega_n}_j - \sum_i \frac{\partial a^{\omega_n}_{ij}}{\partial z_i}\right)\frac{\partial\tilde{w}_{\kappa,\omega_n}}{\partial z_j}$$
$$- \left\{\kappa^2 - \left(1 + \frac{1}{\omega_n}\Phi_\omega\big(\omega_n^{\frac{2}{p-2}}v_{\kappa,\omega_n}\big)\right)^2\left[\mathcal{F}^{-1}\left(Q_{\kappa,\omega_n} + \frac{1}{\omega_n}z\right)\right]\right\}\tilde{w}_{\kappa,\omega_n} + \tilde{w}^{p-1}_{\kappa,\omega_n} = 0 \tag{4.25}$$

where

$$a^{\omega_n}_{ij}(z) = \sum_{l=1}^{3}\frac{\partial\mathcal{F}_i}{\partial x_l}\left(\mathcal{F}^{-1}\left(Q_{\kappa,\omega_n} + \frac{1}{\omega_n}z\right)\right)\frac{\partial\mathcal{F}_j}{\partial x_l}\left(\mathcal{F}^{-1}\left(Q_{\kappa,\omega_n} + \frac{1}{\omega_n}z\right)\right), \quad 1 \leqslant i, j \leqslant 3,$$
$$b^{\omega_n}_j = (\Delta\mathcal{F}_j)\left(\mathcal{F}^{-1}\left(Q_{\kappa,\omega_n} + \frac{1}{\omega_n}z\right)\right), \quad j = 1, 2, 3.$$

By a similar argument as in the proof of Proposition 4.6, we can assume that

$$\tilde{w}_{\kappa,\omega_n} \to w_0 \quad \text{in } C^2_{\mathrm{loc}}\big(\mathbb{R}^3_{\alpha,+}\big).$$

We claim that this w_0 solves the following elliptic problem

$$\begin{cases} \Delta u - \big(\kappa^2 - (1+\tau)^2\big)u + u^{p-1} = 0 & \text{in } \mathbb{R}^3_{\alpha,+}, \\ u > 0 & \text{in } \mathbb{R}^3_{\alpha,+}, \\ u = 0 & \text{on } \partial\mathbb{R}^3_{\alpha,+}. \end{cases} \tag{4.26}$$

Here, $\tau \in [-1, 0]$ is a constant. But from Theorem 1.1 in [15], the system (4.26) has no solution. We get a contradiction. So the sequence $\{\omega_n\}$ satisfying (4.23) does not exist.

In the rest, let us prove the claim that w_0 solves the system (4.26). Assume

$$\tilde{\phi}_{\kappa,\omega_n}(z) := \frac{\Phi_{\omega_n}(\omega_n^{\frac{2}{p-2}}v_{\kappa,\omega_n})}{\omega_n}\left(\mathcal{F}^{-1}\left(Q_{\kappa,\omega_n} + \frac{1}{\omega_n}z\right)\right).$$

By using the fact that $\Phi_{\omega_n}(\omega_n^{\frac{2}{p-2}} v_{\kappa,\omega_n}) \in M$, it is simple to see that

$$\|\nabla \tilde{\phi}_{\kappa,\omega_n}\|_{L^\infty(\bar{B}_{\varrho\omega_n} \cap \{z_3 > -\alpha_{\kappa,\omega_n}\})} \leqslant \frac{C}{\omega_n^2},$$

C is a constant depending on the Born–Infeld parameter b and the diffeomorphism $\mathcal{F}$. Thus,

$$\int\limits_{\bar{B}_{\varrho\omega_n} \cap \{z_n > -\alpha_{\kappa,\omega_n}\}} |\nabla \tilde{\phi}_{\kappa,\omega_n}|^2 \leqslant \frac{C\varrho^3}{\omega_n} \to 0, \quad \text{as } \omega_n \to \infty. \tag{4.27}$$

We can choose a sequence of bounded convex open sets $\{\Omega_n\}$, such that $\Omega_n \Subset \Omega_{n+1} \Subset \mathbb{R}^3_{\alpha,+}$ and $\{\Omega_n\}$ makes a covering of $\mathbb{R}^3_{\alpha,+}$. By applying Poincaré inequality on Ω_n, we know that for s large

$$\int\limits_{\Omega_n} \big|\tilde{\phi}_{\kappa,\omega_s} - (\tilde{\phi}_{\kappa,\omega_s})_{\Omega_n}\big|^2 \leqslant C(n) \int\limits_{\Omega_n} |\nabla \tilde{\phi}_{\kappa,\omega_s}|^2 \tag{4.28}$$

where $C(n)$ depends on Ω_n only. Notice Proposition 2.8. $\forall s$, $|(\tilde{\phi}_{\kappa,\omega_s})_{\Omega_n}| \leqslant 1$. Up to a subsequence, we can assume that $\tilde{\phi}_{\kappa,\omega_s} \to \tau$ in $L^2(\Omega_n)$ where $\tau \in [-1, 0]$ is a constant. Moreover, by diagonal process, we can assume, up to a subsequence, that

$$\tilde{\phi}_{\kappa,\omega_s} \to \tau \quad \text{in } L^2_{\mathrm{loc}}\big(\mathbb{R}^3_{\alpha,+}\big). \tag{4.29}$$

See from (4.25), (4.29) and the fact that $\tilde{w}_{\kappa,\omega_s} \to w_0$ in $C^2_{\mathrm{loc}}(\mathbb{R}^3_{\alpha,+})$, we know that w_0 is a solution of

$$\sum_{i,j} \frac{\partial}{\partial z_i}\left(a_{ij}(0)\frac{\partial w_0}{\partial z_j}\right) - \big(\kappa^2 - (1+\tau)^2\big) w_0 + w_0^{p-1} = 0.$$

Here, $a_{ij}(0)$ is the limit of $a_{ij}^{\omega_n}(z)$ as $n \to \infty$. In view of

$$D\mathcal{F}(0) = \big[D\mathcal{F}^{-1}(0)\big]^{-1} = I,$$

we know that w_0 satisfies the equation in (4.26). In order to prove w_0 satisfying the boundary value condition in (4.26), we define

$$\bar{w}_{\kappa,\omega_n}(z) = \tilde{w}_{\kappa,\omega_n}(z - p_n), \quad \forall z \in \bar{B}_{\varrho\omega_n}(p_n) \cap \{z_3 \geqslant -\alpha\}$$

where $p_n = (0, 0, \alpha_{\kappa,\omega_n} - \alpha)$. Denote $p_0 = (0, 0, -\alpha)$. For a fixed $R > 0$,

$$B_R^+(p_0) := \big\{x \in B_R(p_0) \mid z_3 > -\alpha\big\}.$$

Since $\{\alpha_{\kappa,\omega_n}\}$ is bounded, we have $B_R^+(p_0) \Subset B_{\varrho\omega_n}(p_n) \cap \{z_3 \geqslant -\alpha\}$ when n is large enough. Apply a standard L^p estimate on (4.25). We conclude that $\forall q > 1$, $\bar{w}_{\kappa,\omega_n}$ is uniformly bounded in $W^{2,q}(B_R^+(p_0))$. Note that, $\alpha_{\kappa,\omega_n} \to \alpha$ and $W^{2,q}(B_R^+(p_0))$ is compactly embedded into $C^{1,\beta}(\bar{B}_R^+(p_0))$ for some $\beta \in (0,1)$. Then $\bar{w}_{\kappa,\omega_n} \rightrightarrows w_0$ in $\bar{B}_R^+(p_0)$. Because $\bar{w}_{\kappa,\omega_n}(z) = 0$ if $z_3 = -\alpha$, w_0 satisfies the boundary value condition in (4.26). Note that $\tilde{w}_{\kappa,\omega_n}(0) = v_{\kappa,\omega_n}(P_{\kappa,\omega_n}) \geqslant (\kappa^2 - 1)^{\frac{1}{p-2}}$. By strong maximum principle, we get $w_0 > 0$ in $\mathbb{R}^3_{\alpha,+}$. Therefore, w_0 is a solution of (4.26). □

4.3. Profiles of the least-J-action solutions

In this section, we show that (1) when κ and ω are suitably large, $v_{\kappa,\omega}$ has exactly one local maximum point $P_{\kappa,\omega}$ and (2) outside this local maximum point, $v_{\kappa,\omega} \to 0$ in $C^{1,\alpha}_{\mathrm{loc}}$.

Proposition 4.9. *$v_{\kappa,\omega}$ has exactly one local maximum point if ω and κ are suitably large.*

Proof. Assume that there exists a sequence $\omega_n \to \infty$ such that $\forall n \in \mathbb{N}$, v_{κ,ω_n} has at least l local maximum points. By Lemmas 4.7 and 4.8, we know that for a fixed $R > 0$, there exists $k(R) > 0$, when $s > k(R)$,

$$B_{\frac{1}{\omega_s}R}\big(x^i_{\kappa,\omega_s}\big) \subset U,\ B_{\frac{1}{\omega_s}R}\big(x^i_{\kappa,\omega_s}\big) \cap B_{\frac{1}{\omega_s}R}\big(x^j_{\kappa,\omega_s}\big) = \emptyset \tag{4.30}$$

where x^i_{κ,ω_s} and x^j_{κ,ω_s} are two different local maximum points of v_{κ,ω_s}.

Note that,

$$J_{\kappa,\omega}[v_{\kappa,\omega}] \geqslant \int_U \left(\frac{1}{2} - \frac{1}{p}\right)\frac{1}{\omega^2}|\nabla v_{\kappa,\omega}|^2 + \left[\frac{1}{2}\big(\kappa^2 - 1\big) - \frac{1}{p}\kappa^2\right]v^2_{\kappa,\omega}.$$

If we define

$$V^i_{\kappa,\omega_s}(x) := v_{\kappa,\omega_s}\left(\frac{1}{\omega_s}x + x^i_{\kappa,\omega_s}\right)$$

where $\{x^i_{\kappa,\omega_s} \mid i = 1, 2, \dots, l\}$ are l local maximum points of v_{κ,ω_s}, then for s large,

$$\begin{aligned} J_{\kappa,\omega_s}[v_{\kappa,\omega_s}] &\geqslant \sum_{i=1}^{l} \int_{B_{\frac{1}{\omega_s}R}(x^i_{\kappa,\omega_s})} \left(\frac{1}{2} - \frac{1}{p}\right)\frac{1}{\omega_s^2}|\nabla v_{\kappa,\omega_s}|^2 + \left[\frac{1}{2}\big(\kappa^2 - 1\big) - \frac{1}{p}\kappa^2\right]v^2_{\kappa,\omega_s} \\ &= \omega_s^{-3}\sum_{i=1}^{l}\int_{B_R}\left(\frac{1}{2} - \frac{1}{p}\right)\big|\nabla V^i_{\kappa,\omega_s}\big|^2 + \left[\frac{1}{2}\big(\kappa^2 - 1\big) - \frac{1}{p}\kappa^2\right]\big(V^i_{\kappa,\omega_s}\big)^2. \end{aligned} \tag{4.31}$$

In another way, by Proposition 4.1 and Proposition 2.8, we get

$$\mathcal{C}_{\kappa,\omega_s} \leqslant \kappa^{\frac{6-p}{p-2}} \inf_{v\in P}\left\{\sup_{t\geqslant 0} t^2 \int_{U_\kappa} \frac{1}{2\omega_s^2}|\nabla v|^2 + \frac{1}{2}v^2\,dy - \frac{1}{p}t^p \int_{U_\kappa} v_+^p\,dy\right\} \tag{4.32}$$

where

$$P = \big\{v \;\big|\; v \geqslant 0,\ v \in H^1_0(U_\kappa),\ v \not\equiv 0\big\}, \qquad U_\kappa = \{y \mid y = \kappa x,\ x \in U\}.$$

Apply Lemma 3.1 in [26] on the right-hand side of (4.32). We know that if w is the unique solution of the elliptic problem (4.12)–(4.13) when $c = 1$, then

$$\mathcal{C}_{\kappa,\omega_s} = J_{\kappa,\omega_s}[v_{\kappa,\omega_s}] \leqslant \kappa^{\frac{6-p}{p-2}}\omega_s^{-3}\left[\left(\frac{1}{2} - \frac{1}{p}\right)\|w\|^2_{H^1(\mathbb{R}^3)} + o(1)_s\right] \tag{4.33}$$

where $o(1)_s \to 0$ when κ is fixed and $s \to \infty$.

Let $s \to \infty$, $R \to \infty$ successively. By (4.31), (4.33) and Proposition 4.6, we have

$$l\int_{\mathbb{R}^3}\left(\frac{1}{2} - \frac{1}{p}\right)|\nabla w_{\kappa^2-1}|^2 + \left[\frac{1}{2}\big(\kappa^2 - 1\big) - \frac{1}{p}\kappa^2\right]w^2_{\kappa^2-1} \leqslant \kappa^{\frac{6-p}{p-2}}\left(\frac{1}{2} - \frac{1}{p}\right)\|w\|^2_{H^1(\mathbb{R}^3)}. \tag{4.34}$$

By (4.14), we know that

$$w_{\kappa^2-1}(x) = \big(\kappa^2 - 1\big)^{\frac{1}{p-2}} w\big(\sqrt{\kappa^2 - 1}\,x\big), \quad \forall x \in \mathbb{R}^3. \tag{4.35}$$

Plug (4.35) into (4.34), we get

$$l\left[\frac{p-2}{2p}\|w\|^2_{H^1(\mathbb{R}^3)} - \frac{1}{p(\kappa^2 - 1)}\int_{\mathbb{R}^3} w^2\right] \leqslant \left(\frac{\kappa^2}{\kappa^2 - 1}\right)^{\frac{6-p}{2(p-2)}} \frac{p-2}{2p}\|w\|^2_{H^1(\mathbb{R}^3)}.$$

Therefore,

$$l\left[\frac{p-2}{2p}-\frac{1}{p(\kappa^2-1)}\right]\leqslant\left(\frac{\kappa^2}{\kappa^2-1}\right)^{\frac{6-p}{2(p-2)}}\frac{p-2}{2p}. \tag{4.36}$$

From (4.36), we know that there exists $\kappa_0=\kappa_0(p)>0$, when $\kappa>\kappa_0$, $l\leqslant 1$. In other words, $v_{\kappa,\omega}$ has only one local maximum point in U when κ and ω are suitably large. □

Finally, we prove the point-condensation phenomenon of $v_{\kappa,\omega}$.

Proposition 4.10. *Assume that κ and ω are suitably large. $P_{\kappa,\omega}$ is the unique local maximum point of $v_{\kappa,\omega}$. Then, for some $\alpha\in(0,1)$,*

$$v_{\kappa,\omega}(\cdot+P_{\kappa,\omega})\to 0\quad \text{in } C^{1,\alpha}_{\mathrm{loc}}\big(U-P_{\kappa,\omega}\setminus\{0\}\big),\ \text{as } \omega\to\infty.$$

Proof. Choose $\eta_0>0$ and $R_0>0$ such that

$$\kappa^2-1>(2\eta_0)^{p-2},\qquad R_0=\big(\kappa^2-1\big)^{-\frac{1}{2}}\ln\frac{C(p,\kappa)}{\eta_0} \tag{4.37}$$

where $C(p,\kappa)$ satisfies

$$w_{\kappa^2-1}(x)\leqslant C(p,\kappa)e^{-(\kappa^2-1)^{\frac{1}{2}}|x|},\quad \forall x\in\mathbb{R}^3. \tag{4.38}$$

Apply Lemma 4.8 and Proposition 4.6. We know that when ω is large enough,

$$B_{R_0}(0)\Subset U_{\kappa,\omega},\qquad \|V_{\kappa,\omega}-w_{\kappa^2-1}\|_{C^2(\bar B_{R_0}(0))}\leqslant\eta_0.$$

$V_{\kappa,\omega}$ and $U_{\kappa,\omega}$ in the following were defined at the beginning of Section 4.1. By (4.37), (4.38),

$$v_{\kappa,\omega}(x)\leqslant 2\eta_0,\quad \forall x\in\partial\bar B_{\frac{1}{\omega}R_0}(P_{\kappa,\omega}).$$

Note that $v_{\kappa,\omega}\in H^1_0(U)$ and $v_{\kappa,\omega}$ has only one local maximum point in U when ω is large enough. We then claim that $v_{\kappa,\omega}(x)<2\eta_0$ for $x\in U\setminus\bar B_{R_0\frac{1}{\omega}}(P_{\kappa,\omega})$. Otherwise, besides $P_{\kappa,\omega}$, $v_{\kappa,\omega}$ admits another local maximum point in $U\setminus\bar B_{R_0\frac{1}{\omega}}(P_{\kappa,\omega})$. Therefore, $V_{\kappa,\omega}$ satisfies

$$\begin{cases}\Delta V_{\kappa,\omega}-\big(\kappa^2-(1+\phi_{\kappa,\omega})^2-V_{\kappa,\omega}^{p-2}\big)V_{\kappa,\omega}=0 & \text{in } U_{\kappa,\omega}\setminus\bar B_{R_0}(0),\\ V_{\kappa,\omega}<2\eta_0 & \text{in } U_{\kappa,\omega}\setminus\bar B_{R_0}(0),\\ V_{\kappa,\omega}\leqslant 2\eta_0 & \text{on } \partial\bar B_{R_0}(0),\\ V_{\kappa,\omega=0} & \text{on } \partial U_{\kappa,\omega}.\end{cases} \tag{4.39}$$

In addition, by the modified Bessel function of order 1/2 (see Appendix C in [19]) and the Green's function for $-\Delta+1$ on $\mathbb{R}^3$, we can construct a function W such that

$$\begin{cases}\Delta W-\big(\kappa^2-1-(2\eta_0)^{p-2}\big)W=0 & \text{in } \mathbb{R}^3\setminus\{0\},\\ W=2\eta_0 & \text{on } \partial B_{R_0}(0).\end{cases} \tag{4.40}$$

More precisely, by setting

$$F(x)=\frac{1}{|x|}e^{-|x|},\qquad W(y)=B(p,\kappa)F\Big(\sqrt{\kappa^2-1-(2\eta_0)^{p-2}}\,y\Big)$$

where $B(p,\kappa)$ is a constant so that $W\equiv 2\eta_0$ on $\partial B_{R_0}(0)$. W satisfies (4.40).

By comparison principle, we have $V_{\kappa,\omega}\leqslant W$ on $U_{\kappa,\omega}\setminus\bar B_{R_0}(0)$. Notice the uniform boundedness of $V_{\kappa,\omega}$ in $B_{R_0}(0)$ (see Lemma 4.4). We conclude that

$$V_{\kappa,\omega}(y)\leqslant C(p,\kappa)e^{-\sqrt{\kappa^2-1-(2\eta_0)^{p-2}}|y|},\quad \forall y\in U_{\kappa,\omega}.$$

That is,

$$v_{\kappa,\omega}(x) \leqslant C(p,\kappa)e^{-\sqrt{\kappa^2-1-(2\eta_0)^{p-2}}\omega|x-P_{\kappa,\omega}|}, \quad \forall x \in U. \tag{4.41}$$

Away from $P_{\kappa,\omega}$, the right-hand side of (4.41) is exponentially decay with respect to ω. Since $v_{\kappa,\omega}$ satisfies (4.2), we then complete the proof by an application of a standard L^p estimate on $v_{\kappa,\omega}$ (see Theorem 9.11 in [20]). □

In the end, from Proposition 4.9 and Proposition 4.10, we have

Theorem 4.11. *If $u_{\kappa,\omega}$ is a least-$J_{+,\kappa,\omega}$-action solution of* (3.25)*, then it has exactly one local maximum point $P_{\kappa,\omega}$ when κ and ω are suitably large. In addition, for some $\alpha \in (0,1)$,*

$$\omega^{\frac{2}{2-p}}u_{\kappa,\omega}(\cdot + P_{\kappa,\omega}) \to 0 \quad \text{in } C^{1,\alpha}_{\text{loc}}\big(U - P_{\kappa,\omega} \setminus \{0\}\big), \text{ as } \omega \to \infty.$$

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References

[1] J.C. Brunelli, Dispersionless limit of integrable models, Brazilian J. Physics 30 (2) (June 2000) 455–468.
[2] Max Born, Modified field equations with a finite radius of the electron, Nature 132 (1933) 282.
[3] Max Born, On the quantum theory of the electromagnetic field, Proc. Roy. Soc. A 143 (1934) 410–437.
[4] V. Benci, D. Fortunato, Solitary waves of the nonlinear Klein–Gordon equation coupled with the Maxwell equations, Rev. Math. Phys. 14 (4) (2002) 409–420.
[5] V. Benci, D. Fortunato, Solitary waves in the nonlinear wave equation and in gauge theories, Fixed Point Theory Appl. 1 (2007) 61–86.
[6] V. Benci, D. Fortunato, Solitary waves in classical field theory, in: V. Benci, A. Masiello (Eds.), Nonlinear Analysis and Applications to Physical Sciences, Springer, Milano, 2004, pp. 1–50.
[7] V. Benci, D. Fortunato, On the existence of infinitely many geodesics on space–time manifolds, Adv. in Math. 105 (1994) 1–25.
[8] M. Born, L. Infeld, Foundation of the new field theory, Nature 132 (1933) 1004.
[9] M. Born, L. Infeld, Foundation of the new field theory, Proc. Roy. Soc. A 144 (1934) 425–451.
[10] V. Benci, P.H. Rabinowitz, Critical points theorems for indefinite functionals, Invent. Math. 52 (1979) 241–273.
[11] D. Cassani, Existence and non-existence of solitary waves for the critical Klein–Gordon equation coupled with Maxwell's equations, Nonlinear Anal. 58 (2004) 733–747.
[12] Teresa D'Aprile, Dimitri Mugnai, Solitary waves for nonlinear Klein–Gordon–Maxwell and Schrödinger–Maxwell equations, Proc. Roy. Soc. Edinburgh Sect. A 134 (5) (2004) 893–906.
[13] Pietro d'Avenia, Lorenzo Pisani, Nonlinear Klein–Gordon equations coupled with Born–Infeld type equations, Elect. J. Diff. Eqns. 26 (2002) 1–13.
[14] H. Egnell, Asymptotic results for finite energy solutions of semilinear elliptic equations, J. Differential Equations 98 (1992) 34–56.
[15] M. Esteban, P.L. Lions, Existence and non-existence results for semilinear elliptic problems in unbounded domanis, Proc. Roy. Soc. Edinburgh Sect. A 93 (1982) 1–14.
[16] M. Esteban, E. Séré, Stationary states of the nonlinear Dirac equation: A variational approach, Comm. Math. Phys. 171 (1995) 323–350.
[17] D. Fortunato, L. Orsina, L. Pisani, Born–Infeld type equations for electrostatic fields, J. of Math. Phys. 43 (11) (2002) 5698–5706.
[18] G.W. Gibbons, Born–Infeld particles and Dirichlet p-branes, Nucl. Phys. B 514 (1998) 603.
[19] B. Gidas, W.-M. Ni, L. Nirenberg, Symmetry of positive solutions of nonlinear elliptic equations in R^n, Adv. in Math. 7A (Suppl. Stud.) (1981) 369–402.
[20] D. Gilbarg, Neil S. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, 2001.
[21] M.K. Kwong, Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in R^n, Arch. Rational Mech. Anal. 105 (1989) 243–266.
[22] E. Long, Existence and stability of solitary waves in non-linear Klein–Gordon–Maxwell equations, Rev. Math. Phys. 18 (2006) 747–779.
[23] C.-S. Lin, W.-M. Ni, I. Takagi, Large amplitude stationary solutions to a chemotaxis system, J. Differential Equations 72 (1988) 1–27.
[24] F. Lin, Y. Yang, Gauged harmonic maps, Born–Infeld electromagnetism, and magnetic vortices, CPAM 56 (2003) 1631–1665.
[25] D. Mugnai, Coupled Klein–Gordon and Born–Infeld type equations: Looking for solitary waves, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 460 (2004) 1519–1528.
[26] W.-M. Ni, J. Wei, On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems, CPAM XLVIII (1995) 731–768.
[27] N. Ogawa, Chaplygin Gas and Brane, in: Proceedings of the 8th International Conference on Geometry Integrability & Quantization, June 2007, pp. 279–291.

[28] R.S. Palais, The principle of symmetric criticality, Comm. Math. Phys. 69 (1979) 19–30.
[29] J. Polchinski, TASI lectures on D-branes, arXiv:hep-th/9611050;
R. Argurio, Brane physics in M-theory, hep-th/9807171;
K.G. Savvidy, Born–Infeld action in string theory, hep-th/9906075.
[30] P.H. Rabinowitz, Minimax Methods in Critical Point Theory with Applications to Differential Equations, Reg. Conf. Ser. Math., vol. 65, 1986.
[31] M. Struwe, Variational Methods: Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, 3rd edition.
[32] N. Seilberg, E. Witten, String theory and noncommutative geometry, JHEP 9909 (1999) 032.
[33] W.-M. Ni, I. Takagi, On the shape of least-energy solutions to a semilinear Neumann problem, CPAM 44 (1991) 819–851.
[34] Y. Yang, Classical solutions in the Born–Infeld theory, Proceedings: Mathematical, Physical and Engineering Sciences 456 (1995) (2000) 615–640.
[35] X.-P. Zhu, Multiple entire solutions of a semilinear elliptic equation, Nonlinear Anal. 12 (1988) 1297–1316.
[36] Z. Zhang, K. Li, Spike-layered solutions of singularly perturbed quasilinear Dirichlet problems, J. Math. Anal. Appl. 283 (2003) 667–680.