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Ann. I. H. Poincaré – AN 29 (2012) 171–198

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Convergence of approximate deconvolution models to the mean Navier–Stokes equations

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Received 7 April 2011; accepted 3 October 2011

Available online 12 October 2011

Abstract

We consider a 3D Approximate Deconvolution Model ADM which belongs to the class of Large Eddy Simulation (LES) models. We aim at proving that the solution of the ADM converges towards a dissipative solution of the mean Navier–Stokes equations. The study holds for periodic boundary conditions. The convolution filter we first consider is the Helmholtz filter. We next consider generalized convolution filters for which the convergence property still holds.

MSC: 76D05; 35Q30; 76F65; 76D03

Keywords: Navier–Stokes equations; Large eddy simulation; Deconvolution models

1. Introduction

Kolmogorov's theory predicts that simulating incompressible turbulent flows by using the incompressible Navier–Stokes equations,

$$\begin{aligned}
&\partial_t \mathbf{u} + \nabla \cdot (\mathbf{u} \otimes \mathbf{u}) - \nu \Delta \mathbf{u} + \nabla p = \mathbf{f}, \\
&\nabla \cdot \mathbf{u} = 0, \\
&\mathbf{u}(0, \mathbf{x}) = \mathbf{u}_0(\mathbf{x}),
\end{aligned} \tag{1.1}$$

requires $\mathcal{N} = O(Re^{9/4})$ degrees of freedom, where $Re = UL\nu^{-1}$ denotes the Reynolds number, U and L being typical velocity and length scales. This number $\mathcal{N}$ is too large, in comparison with memory capacities of actual computers, to perform a Direct Numerical Simulation (DNS). Indeed, for realistic flows, such as geophysical flows, the Reynolds number is of order 10^8, yielding $\mathcal{N}$ of order $10^{18}\ldots$. This is why one aims at computing at least the "mean values" of the flow fields, the velocity field $\mathbf{u} = (u^1, u^2, u^3)$ and the scalar pressure field p.

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0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2011.10.001

In Large Eddy Simulation model, means of the fields are computed by

$$\bar{\mathbf{u}}(t,\mathbf{x}) = \int G_\alpha(\mathbf{x},\mathbf{y})\mathbf{u}(\mathbf{y})\,d\mathbf{y}, \qquad \overline{p} = \int G_\alpha(\mathbf{x},\mathbf{y})p(\mathbf{y})\,d\mathbf{y}.$$

For homogeneous turbulent flows, one may take

$$G_\alpha(\mathbf{x},\mathbf{y}) = G_\alpha\big(|\mathbf{x}-\mathbf{y}|\big).$$

This filter is a convolution filter, the case that we consider throughout the paper. The scale α is in general chosen to be of the order of the mesh size in a practical computation [17]. Moreover, the kernel G_α is smooth to get a real smoothing effect and satisfies $G_\alpha \to \delta$ when $\alpha \to 0$, where δ is the Dirac function.

In THE PERIODIC CASE or in THE WHOLE SPACE with suitable decay conditions at infinity, the homogeneous assumption leads the filter operation to COMMUTE with differential operators. Therefore, when we formally filter the incompressible Navier–Stokes equations, we obtain what we call the "mean Navier–Stokes equations",

$$\begin{aligned}
&\partial_t \bar{\mathbf{u}} + \nabla\cdot(\overline{\mathbf{u}\otimes\mathbf{u}}) - \nu\Delta\bar{\mathbf{u}} + \nabla\overline{p} = \bar{\mathbf{f}},\\
&\nabla\cdot\bar{\mathbf{u}} = 0,\\
&\bar{\mathbf{u}}(0,\mathbf{x}) = \overline{\mathbf{u}_0}(\mathbf{x}).
\end{aligned}\tag{1.2}$$

This raises the question of the *interior closure problem*, that is the modelling of the tensor $R(\mathbf{u}) = \overline{\mathbf{u}\otimes\mathbf{u}}$.

Large Eddy Simulations (LES) models consider an approximation $(\mathbf{w},q)$ of the means $(\bar{\mathbf{u}},\overline{p})$ and a system satisfied by these fields, thanks to a suitable definition of R. The tensor R is often defined in terms of $(\mathbf{w},q)$ by

$$R = \mathbf{w}\otimes\mathbf{w} - \nu_T(\mathbf{k}/\mathbf{k}_c)\mathcal{D}(\mathbf{w}), \quad \mathcal{D}(\mathbf{w}) = (1/2)\big(\nabla\mathbf{w} + \nabla\mathbf{w}^T\big).$$

In the formula above, ν_T is an eddy viscosity based on a cut-off wave number $\mathbf{k}_c \approx O(1/\alpha)$ defining the resolved scales (see a general setting in [27]). This yields a model for the approximate fields $(\mathbf{w},q)$ supposed to fit with the field $(\bar{\mathbf{u}},\overline{p})$ for large scales, with the constraint that the total energy dissipation of both fields remains the same. Moreover, it is expected that $(\mathbf{w},q)$ converges towards a solution of the Navier–Stokes equations when α goes to zero (see in [14]).

Another way that avoids eddy viscosities, consists in approaching R by a quadratic term of the form $B(\mathbf{w},\mathbf{w})$. J. Leray [18] introduced in 1934 the approximation $B(\mathbf{w},\mathbf{w}) = \overline{\mathbf{w}}\otimes\mathbf{w}$ to get smooth approximation to the incompressible Navier–Stokes equations. This approximation yields the recent Leray-alpha fashion models, considered to be LES models, and a broad class of related models (see e.g. [2,5,10,11,19]), in which the convolution is defined thanks to the Helmholtz kernel in the periodic case, a kernel considered below.

The model we study in this paper, is the Approximate Deconvolution Model (ADM), first introduced by Adams and Stolz [1,28,29], as far as we know. This model is defined by

$$B(\mathbf{w},\mathbf{w}) = \overline{D_N(\mathbf{w})\otimes D_N(\mathbf{w})}, \quad D_N = \sum_{n=0}^{N}(I-G_\alpha)^n,$$

where we still denote $G_\alpha(\mathbf{w}) = \overline{\mathbf{w}} = G_\alpha \star \mathbf{w}$ and N is a given integer that we call "the order of the deconvolution". This yields the initial value problem:

$$\begin{aligned}
&\partial_t \mathbf{w} + \nabla\cdot\big(\overline{D_N(\mathbf{w})\otimes D_N(\mathbf{w})}\big) - \nu\Delta\mathbf{w} + \nabla q = \bar{\mathbf{f}},\\
&\nabla\cdot\mathbf{w} = 0,\\
&\mathbf{w}(0,\mathbf{x}) = \overline{\mathbf{u}_0}(\mathbf{x}).
\end{aligned}\tag{1.3}$$

Throughout the paper, $\alpha > 0$ is fixed. In [13,15], the case $N = 0$ was carefully studied for the Helmholtz kernel. We are now interested in the question of "N large".

There are cases and kernels G_α for which $D_N \to G_\alpha^{-1} = A_\alpha$ in some sense, when $N \to \infty$. This is why we call the operator D_N a *deconvolution operator*, where we denote the kernel likewise the corresponding convolution operator.

The principle of deconvolution initially comes from image processing [3]. The idea is to reconstruct a noised field thanks to a deconvolution operator for large N. We do the same thing in model (1.3): we try to re-construct as much of the field's high frequency as we can using as few degrees of freedom as possible in a numerical simulation.

Indeed, when N grows and α is fixed, we expect a numerical simulation by (1.3) to approach a Direct Numerical Simulation of the mean field that satisfies (1.2), while keeping numerical stability. We studied the question of feasibility of such an ADM in [16]. The issue is that under suitable assumptions, the ADM needs less degrees of freedom than a DNS in a practical computation for a fixed N and yields reasonable accuracy in terms of resolved scales.

The question of the asymptotic behavior of the model when N goes to infinity and the scale α is fixed, was open up to now, and is the main aim of the present paper.

As we shall see later, there are cases such that the model (1.3) has a unique solution $(\mathbf{w}_N, q_N)$ for a fixed N, in a sense to be defined, solution that satisfies estimates uniform in N. Let us define $(\mathbf{w}, q)$ to be an eventual limit of a subsequence of $\{(\mathbf{w}_N, q_N)\}_{N\in\mathbb{N}}$. This raises the question of the equation satisfied by $(\mathbf{w}, q)$ and especially the behavior of the quadratic sequence $D_N(\mathbf{w}_N) \otimes D_N(\mathbf{w}_N)$ when $N \to \infty$, a question that we study carefully in the remainder.

Notice that when $D_N \to G_\alpha^{-1} = A_\alpha$, we expect that $D_N(\mathbf{w}_N) \otimes D_N(\mathbf{w}_N)$ converges towards $A_\alpha(\mathbf{w}) \otimes A_\alpha(\mathbf{w})$ when N goes to infinity and that the limit satisfies

$$\begin{aligned} &\partial_t \mathbf{w} + \nabla \cdot \big(\overline{A_\alpha(\mathbf{w}) \otimes A_\alpha(\mathbf{w})}\big) - \nu \Delta \mathbf{w} + \nabla q = \bar{\mathbf{f}}, \\ &\nabla \cdot \mathbf{w} = 0, \end{aligned} \tag{1.4}$$

with initial data $\mathbf{w}(0, \mathbf{x}) = \overline{\mathbf{u}_0}(\mathbf{x})$. Therefore, let us set $(\mathbf{u}, p) = (A_\alpha(\mathbf{w}), A_\alpha(q))$ or equivalently $\mathbf{w} = G_\alpha(\mathbf{u}) = \bar{\mathbf{u}}$, $q = \bar{p}$. If we are in a case where the convolution operator commutes with the differentiation, we obtain that $(\mathbf{u}, p)$ is a solution of the mean incompressible Navier–Stokes equations (1.2).

We prove in this paper a series of convergence results like this, the first one being Theorem 4.1 which is our main result. These results are consistency and stability results, that are a partial mathematical validation of experimental/numerical results initially displayed by Adams and Stolz for some special cases.

We consider in this paper the case of periodic boundary conditions. The equations are set in a 3D torus $\mathbb{T}_3$ of size L, $\mathbb{T}_3 = \mathbb{R}^3/\mathcal{T}_3$, with $\mathcal{T}_3 := 2\pi\mathbb{Z}^3/L$. We carefully detail the question raised above when the convolution filter is specified by the Helmholtz equation,

$$-\alpha^2 \Delta \bar{\mathbf{u}} + \bar{\mathbf{u}} + \nabla \pi = \mathbf{u}, \qquad \nabla \bar{\mathbf{u}} = 0 \quad \text{in } \mathbb{T}_3.$$

The corresponding convolution function G_α, denoted by G for simplicity, is given in terms of a Fourier series

$$G(\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \frac{1}{1+\alpha^2|\mathbf{k}|^2} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

We start with this filter mainly for historical reasons. In all the above quoted mathematical references about Leray-alpha and/or Bardina and/or ADM, this filter is the one that is always studied for practical reasons.

We first show an existence and uniqueness result of what we call a "regular weak solution" to model (1.3) for a fixed N, that satisfies estimates uniform in N (see Definition 3.1 and Theorem 3.1). Then as we already said, we prove that the corresponding sequence of solutions converges to a solution of the mean Navier–Stokes equations when N goes to infinity (Theorem 4.1).

Notice that Dunca and Epshteyn [9] have proved an existence result for the ADM already. Their proof does not include uniform estimates in N and did not allow to take the limit when N goes to infinity. This is why we have to seek another existence's proof that precisely allows to take the limit.

We next consider another filter that we call "the generalized Helmholtz filter", which is specified thanks to the following PDE,

$$-\alpha^{2p} \Delta^p \bar{\mathbf{u}} + \bar{\mathbf{u}} + \nabla \pi = \mathbf{u}, \qquad \nabla \bar{\mathbf{u}} = 0 \quad \text{in } \mathbb{T}_3,$$

$p \in \mathbb{R}_+^\star$, the convolution function G of which being

$$G(\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \frac{1}{1+\alpha^{2p}|\mathbf{k}|^{2p}} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

In this case, we prove the existence and uniqueness of a "generalized regular weak solution" to the ADM of order N when $p > 3/4$ (see Definition 5.1 and Theorem 5.1). Moreover, we prove that the corresponding sequence converges to a solution of the mean Navier–Stokes equations, when N goes to infinity (see Theorem 5.2). The case $3/4 < p < 1$ is rather academic. In practice, one prefers to take $p > 1$ to increase the regularity, as shows the analysis carried out in Section 5.3.

The above results naturally lead us to consider more general filters that are not specified by a PDE (see Section 6), but only through their general convolution kernel G,

$$G(\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \widehat{G}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

When the coefficients $\widehat{G}_{\mathbf{k}}$ satisfy the growth condition

$$\forall\mathbf{k}\in\mathcal{T}_3, \quad \frac{C_1}{1+\alpha^{2q}|\mathbf{k}|^{2q}} \leqslant \widehat{G}_{\mathbf{k}} \leqslant \frac{C_2}{1+\alpha^{2p}|\mathbf{k}|^{2p}},$$

where $C_1 > 0$ and $C_2 > 0$, $p > 3/4$, our method still applies and the same results hold as for Helmholtz filters. Notice that in all cases we considered above, we always have the convergence property $D_N \to G^{-1}$, when $N \to \infty$, in a suitable sense to be precised, where we recall that D_N denotes the deconvolution operator. This is always the property, among others, that we use to take the limit in the equations when N goes to infinity.

We also consider in Section 6, the case of the Fejér convolution kernel, which is natural since this is one of the main convolution filters used for theoretical results about Fourier series. It is mainly defined thanks to trigonometric polynomials of the form

$$G(\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{T}_3^\star,\ |\mathbf{k}|\leqslant J} \widehat{G}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}},$$

where the “cut off” J is naturally taken of order $1/\alpha$. As we easily see (Section 6.2) the corresponding deconvolution operator D_N is such that the amplitude of frequencies larger than J are of order $N+1$, and we do not have “$D_N \to G^{-1}$”. Therefore, our method fails and we are not able to take the limit in the terms $D_N(\mathbf{w}_N)\otimes D_N(\mathbf{w}_N)$. However, a “compactness by compensation” (a principle due initially to F. Murat and L. Tartar, see in [24,25,31,32]) might occur, but this is an open problem.

We only consider periodic boundary conditions in this work, like in most of mathematical papers about LES models. This raises the issue of the setting and of the mathematical analysis of the ADM, and more generally of LES models, with other types of boundary conditions. First, we conjecture that our results may be easily extended to the case of the whole space $\mathbb{R}^3$, where the boundary conditions are specified through integrability conditions. It remains the case of domains with boundaries.

Many of the reported LES of wall bounded flows have no filtering at all in the wall normal direction, because of the strong inhomogeneity of the turbulence near the wall [7]. Furthermore, LES near walls is one of the great challenges of turbulence modelling. One may find in [26] a broad review of the major difficulties of such modelling.

The deconvolution procedure was already used for LES of an incompressible channel flow with walls in [29], with filtering in all coordinate directions. In these simulations, filtering is carried out from the numerical scheme, using a discrete filter G that depends on the numerical grid. Up to our knowledge, there is no mathematical analysis of the convergence of this numerical procedure towards weak solutions of the Navier–Stokes equation, which is an open problem.

In [7,4,23], another way to perform general LES in bounded domains is considered. The flow domain Ω_f is embedded in a larger computational domain Ω_c, and the velocity $\mathbf{u}$ is extended by $\mathbf{0}$ in the buffer region $\mathcal{B} = \Omega_c \setminus \Omega_f$. The commutation default term in the mean equations due to the filtering is modelled, as well as the boundary conditions for the filtered field $\bar{\mathbf{u}}$, by minimizing its energy in the buffer,

$$E = \int_{\mathcal{B}} \left(|\bar{\mathbf{u}}|^2 + \alpha \left| \frac{\partial \bar{\mathbf{u}}}{\partial t} \right|^2 \right) dx,$$

α being of the order of the time step. Up to our knowledge, there is no mathematical setting and analysis of such procedure, as well as its possible deconvolution, opening other problems.

2. General background

2.1. Orientation

This section is first devoted to definitions of: the function spaces that we use, the filter through the Helmholtz equation, and the "deconvolution operator". There is nothing new here that is not already introduced in former papers. This is why we restrict ourselves to what we need for our display and we skip proofs and technical details. Those details can be proved by standard analysis and the reader can check them in several references already quoted in the introduction and also quoted below in the text.

2.2. Function spaces

In what follows, we will use the customary Lebesgue L^p and Sobolev $W^{k,p}$ and $W^{s,2} = H^s$ spaces. Since we work with periodic boundary conditions we can better characterize the divergence-free spaces we need. In fact, the spaces we consider are well-defined by using Fourier series on the 3D torus $\mathbb{T}_3$ defined just below. Let $L \in \mathbb{R}_+^\star = \{x \in \mathbb{R}\colon\ x > 0\}$ be given. We denote by $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ the orthonormal basis of $\mathbb{R}^3$, and by $\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3$ the standard point in $\mathbb{R}^3$.

We put $\mathcal{T}_3 := 2\pi\mathbb{Z}^3/L$.

Let $\mathbb{T}_3$ be the torus defined as the compact quotient manifold, $\mathbb{T}_3 = \mathbb{R}^3/\mathcal{T}_3$.

We use $\|\cdot\|$ to denote the $L^2(\mathbb{T}_3)$ norm and associated operator norms.

We always impose the zero mean condition $\int_\Omega \phi\, d\mathbf{x} = 0$ on every field we consider, $\phi = \mathbf{w}$, p, $\mathbf{f}$, or $\mathbf{w}_0$.

We define, for a general exponent $s \geqslant 0$,

$$\mathbf{H}_s = \left\{\mathbf{w}\colon\ \mathbb{T}_3 \to \mathbb{R}^3,\ \mathbf{w} \in H^s(\mathbb{T}_3)^3,\ \nabla\cdot\mathbf{w} = 0,\ \int_{\mathbb{T}_3} \mathbf{w}\, d\mathbf{x} = \mathbf{0}\right\},$$

where $H^s(\mathbb{T}_3)^k = [H^s(\mathbb{T}_3)]^k$, for all $k \in \mathbb{N}$. (If $0 \leqslant s < 1$ the condition $\nabla\cdot\mathbf{w} = 0$ must be understood in a weak sense.)

A vector field $\mathbf{w} \in \mathbf{H}_s$ being given, we can expand it as a Fourier series

$$\mathbf{w}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \widehat{\mathbf{w}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad \text{where } \mathbf{k} \in \mathcal{T}_3^\star \text{ is the wave number,}$$

and the Fourier coefficients are given by

$$\widehat{\mathbf{w}}_{\mathbf{k}} = \frac{1}{L^3}\int_{\mathbb{T}_3} \mathbf{w}(\mathbf{x}) e^{-i\mathbf{k}\cdot\mathbf{x}}\, d\mathbf{x}.$$

The magnitude of $\mathbf{k}$ is defined by

$$k := |\mathbf{k}| = \left\{|k_1|^2 + |k_2|^2 + |k_3|\right\}^{\frac{1}{2}}.$$

We define the $\mathbf{H}_s$ norms by

$$\|\mathbf{w}\|_s^2 = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} |\mathbf{k}|^{2s} |\widehat{\mathbf{w}}_{\mathbf{k}}|^2,$$

where of course $\|\mathbf{w}\|_0^2 = \|\mathbf{w}\|^2$. The inner products associated with these norms are

$$(\mathbf{w}, \mathbf{v})_{\mathbf{H}_s} = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} |\mathbf{k}|^{2s} \widehat{\mathbf{w}}_{\mathbf{k}} \cdot \overline{\widehat{\mathbf{v}}_{\mathbf{k}}}, \tag{2.1}$$

where here, without risk of confusion with the filter defined later, $\overline{\widehat{\mathbf{v}}_{\mathbf{k}}}$ denotes the complex conjugate of $\widehat{\mathbf{v}}_{\mathbf{k}}$. This means that if $\widehat{\mathbf{v}}_{\mathbf{k}} = (v_{\mathbf{k}}^1, v_{\mathbf{k}}^2, v_{\mathbf{k}}^3)$, then $\overline{\widehat{\mathbf{v}}_{\mathbf{k}}} = (\overline{v_{\mathbf{k}}^1}, \overline{v_{\mathbf{k}}^2}, \overline{v_{\mathbf{k}}^3})$.

Since we are looking for real valued vector fields, we have the natural relation, for any field denoted by $\mathbf{w} \in \mathbf{H}_s$:

$$\widehat{\mathbf{w}}_{\mathbf{k}} = \overline{\widehat{\mathbf{w}}_{-\mathbf{k}}}, \quad \forall\mathbf{k} \in \mathcal{T}_3^\star.$$

Therefore, our space $\mathbf{H}_s$ is a closed subset of the space $\mathbb{H}_s$ made of complex valued functions and defined by

$$\mathbb{H}_s = \left\{ \mathbf{w} = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \widehat{\mathbf{w}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\colon \sum_{\mathbf{k}\in\mathcal{T}_3^\star} |\mathbf{k}|^{2s} |\widehat{\mathbf{w}}_{\mathbf{k}}|^2 < \infty,\ \mathbf{k}\cdot\widehat{\mathbf{w}}_{\mathbf{k}} = 0 \right\},$$

equipped with the Hilbertian structure given by (2.1). It can be shown (see e.g. [8]) that when s is an integer, $\|\mathbf{w}\|_s^2 = \|\nabla^s \mathbf{w}\|^2$. One also can prove that for general $s \in \mathbb{R}$, $(\mathbb{H}_s)' = \mathbb{H}_{-s}$ (see in [21]).

2.3. *About the filter*

We now recall the main properties of the Helmholtz filter. In the following, $\alpha > 0$ denotes a given number and $\mathbf{w} \in \mathbf{H}_s$. We consider the Stokes-like problem for $s \geqslant -1$:

$$\begin{aligned} &-\alpha^2 \Delta \overline{\mathbf{w}} + \overline{\mathbf{w}} + \nabla \pi = \mathbf{w} \quad \text{in } \mathbb{T}_3, \\ &\nabla \cdot \overline{\mathbf{w}} = 0 \quad \text{in } \mathbb{T}_3, \end{aligned} \tag{2.2}$$

and in addition, $\int_{\mathbb{T}_3} \pi \, d\mathbf{x} = 0$ to have a uniquely defined Lagrange multiplier.

It is clear that this problem has a unique solution $(\overline{\mathbf{w}}, \pi) \in \mathbf{H}_{s+2} \times H^{s+1}(\mathbb{T}_3)$, for any $\mathbf{w} \in \mathbf{H}_s$. We put $G(\mathbf{w}) = \overline{\mathbf{w}}$, $A = G^{-1}$. Notice that even if we work with real valued fields, $G = A^{-1}$ maps more generally $\mathbb{H}_s$ onto $\mathbb{H}_{s+2}$. Observe also that – in terms of Fourier series – when one inserts

$$\mathbf{w} = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \widehat{\mathbf{w}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$$

in (2.2), one easily gets that, by searching $(\overline{\mathbf{w}}, \pi)$ in terms of Fourier series,

$$\overline{\mathbf{w}}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \frac{1}{1+\alpha^2|\mathbf{k}|^2} \widehat{\mathbf{w}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} = G(\mathbf{w}), \quad \text{and} \quad \pi = 0. \tag{2.3}$$

With a slight abuse of notation, for a scalar function χ we still denote by $\overline{\chi}$ the solution of the pure Helmholtz problem

$$A\overline{\chi} = -\alpha^2 \Delta \overline{\chi} + \overline{\chi} = \chi \quad \text{in } \mathbb{T}_3, \qquad G(\chi) = \overline{\chi} \tag{2.4}$$

and of course there are not vanishing-mean conditions to be imposed for such cases. This notation – which is nevertheless historical – is motivated from the fact that in the periodic setting and for divergence-free vector fields the Stokes filter (2.2) is exactly the same as (2.4). Observe in particular that in the LES model (1.3) and in the filtered equations (1.2)–(4.3), the symbol "$\overline{}$" denotes the pure Helmholtz filter, applied component-by-component to the tensor fields $D_N(\mathbf{w}) \otimes D_N(\mathbf{w})$, $\mathbf{u} \otimes \mathbf{u}$, and $A\mathbf{w} \otimes A\mathbf{w}$ respectively.

2.4. *The deconvolution operator*

We start this section with a useful definition that we shall use several times in the remainder, to understand the relevant properties of the LES model.

Definition 2.1. Let K be an operator acting on $\mathbf{H}_s$. Assume that $e^{-i\mathbf{k}\cdot\mathbf{x}}$ are eigen-vectors of K with corresponding eigenvalues $\widehat{K}_{\mathbf{k}}$. Then we shall say that $\widehat{K}_{\mathbf{k}}$ is the symbol of K.

For instance, the symbol of the operator A is $\widehat{A}_{\mathbf{k}} = 1 + \alpha^2|\mathbf{k}|^2$ and the one of G is $\widehat{G}_{\mathbf{k}} = \widehat{A}_{\mathbf{k}}^{-1}$. We now turn to the definition and various properties of the deconvolution operator.

The deconvolution operator D_N is constructed thanks to the Van-Cittert algorithm, and is formally defined by

$$D_N := \sum_{n=0}^{N} (I - G)^n. \tag{2.5}$$

The reader will find a complete description and analysis of the Van-Cittert algorithm and its variants in [20]. Here we only report the properties we need for the description of the model.

Starting from (2.5), we can express the deconvolution operator in terms of Fourier series by the formula

$$D_N(\mathbf{w}) = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \widehat{D}_N(\mathbf{k})\widehat{\mathbf{w}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}},$$

where

$$\widehat{D}_N(\mathbf{k}) = \sum_{n=0}^{N}\left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2}\right)^n = \left(1+\alpha^2|\mathbf{k}|^2\right)\rho_{N,\mathbf{k}}, \quad \rho_{N,\mathbf{k}} = 1-\left(\frac{\alpha^2|\mathbf{k}|^2}{1+\alpha^2|\mathbf{k}|^2}\right)^{N+1}. \tag{2.6}$$

The symbol $\widehat{D}_N(\mathbf{k})$ of the operator D_N satisfies the following crucial convergence property.

Lemma 2.1. *For each fixed* $\mathbf{k}\in\mathcal{T}_3$,

$$\widehat{D}_N(\mathbf{k}) \to 1+\alpha^2|\mathbf{k}|^2 = \widehat{A}_{\mathbf{k}}, \quad \textit{as } N\to+\infty, \tag{2.7}$$

even if not uniformly in $\mathbf{k}$.

This means that $\{D_N\}_{N\in\mathbb{N}}$ converges to A in some sense when $N\to\infty$. We need to specify this convergence in order to take the limit better than in "a formal way", to go from (1.3) (the ADM model) to (1.4) (the limit, which is equivalent to the "mean Navier–Stokes equations" (1.2)). One general aim of the paper is to fix the notion of "$D_N \to A$" and to obtain enough estimates for the solution $\mathbf{w}$ of (1.3) to take the limit.

The basic properties satisfied by $\widehat{D}_N$ that we need are summarized in the following lemma.

Lemma 2.2. *For each* $N\in\mathbb{N}^\star$, *the operator* $D_N:\mathbf{H}_s\to\mathbf{H}_s$

- *is self-adjoint,*
- *commutes with differentiation,*

and satisfies:

$$1 \leqslant \widehat{D}_N(\mathbf{k}) \leqslant N+1 \quad \forall\mathbf{k}\in\mathcal{T}_3, \tag{2.8}$$

$$\widehat{D}_N(\mathbf{k}) \approx (N+1)\frac{1+\alpha^2|\mathbf{k}|^2}{\alpha^2|\mathbf{k}|^2} \quad \textit{for large } |\mathbf{k}|, \tag{2.9}$$

$$\lim_{|\mathbf{k}|\to+\infty} \widehat{D}_N(\mathbf{k}) = N+1 \quad \textit{for fixed } \alpha>0, \tag{2.10}$$

$$\widehat{D}_N(\mathbf{k}) \leqslant 1+\alpha^2|\mathbf{k}|^2 = \widehat{A}_{\mathbf{k}} \quad \forall\mathbf{k}\in\mathcal{T}_3,\ \alpha>0. \tag{2.11}$$

All these claims are straightforward thanks to definition (2.6). Nevertheless, they call for some comments. Observe first that (2.10) is a direct consequence of (2.9), which says that the $\mathbf{H}_s$'s are stable under D_N's action. More precisely, for all $s\geqslant 0$, the map

$$\mathbf{w} \mapsto D_N(\mathbf{w})$$

is an isomorphism which satisfies

$$\|D_N\|_{\mathbf{H}_s} = O(N+1).$$

Moreover, the term $\overline{D_N(\mathbf{w})\otimes D_N(\mathbf{w})}$ in model (1.3) has more regularity than the convective term $\overline{A\mathbf{w}\otimes A\mathbf{w}}$ in the classical filtered Navier–Stokes equations. This is why we get what we call a unique "regular weak solution" for model (1.3) (see Definition 3.1 in the next section), which satisfies an energy equality.

As suggested by its name, a "regular weak solution" is more regular than a usual weak solution "à la Leray", because each D_N is a zero-order differential operator, while A is a second-order operator.

It is however hard to take the limit when N goes to infinity, since high frequency modes of the solution are not under direct control, and may generate what we call a "sliding peak".

3. Existence results

The aim of this section is:

- to give a definition of what is called a "regular weak solution" to model (1.3),
- to prove an existence and uniqueness result of a regular weak solution to model (1.3) for a fixed N.

3.1. Definition of regular weak solution

Recall that $\alpha > 0$ is fixed, and we assume that the data are such that

$$\mathbf{u}_0 \in \mathbf{H}_0, \qquad \mathbf{f} \in L^2\big([0,T] \times \mathbb{T}_3\big), \tag{3.1}$$

which naturally yields

$$\overline{\mathbf{u}_0} \in \mathbf{H}_2, \qquad \bar{\mathbf{f}} \in L^2\big([0,T]; \mathbf{H}_2\big). \tag{3.2}$$

Definition 3.1 *("Regular weak" solution).* We say that the couple $(\mathbf{w}, q)$ is a "regular weak" solution to system (1.3) if and only if the three following items are satisfied:

1) *Regularity*

$$\mathbf{w} \in L^2\big([0,T]; \mathbf{H}_2\big) \cap C\big([0,T]; \mathbf{H}_1\big), \tag{3.3}$$

$$\partial_t \mathbf{w} \in L^2\big([0,T]; \mathbf{H}_0\big), \tag{3.4}$$

$$q \in L^2\big([0,T]; H^1(\mathbb{T}_3)\big). \tag{3.5}$$

2) *Initial data*

$$\lim_{t\to 0} \big\| \mathbf{w}(t,\cdot) - \overline{\mathbf{u}_0} \big\|_{\mathbf{H}_1} = 0. \tag{3.6}$$

3) *Weak formulation*

$$\forall \mathbf{v} \in L^2\big([0,T]; H^1(\mathbb{T}_3)^3\big), \tag{3.7}$$

$$\int_0^T\!\!\int_{\mathbb{T}_3} \partial_t \mathbf{w} \cdot \mathbf{v} - \int_0^T\!\!\int_{\mathbb{T}_3} \overline{D_N(\mathbf{w}) \otimes D_N(\mathbf{w})} : \nabla \mathbf{v} + \nu \int_0^T\!\!\int_{\mathbb{T}_3} \nabla \mathbf{w} : \nabla \mathbf{v} + \int_0^T\!\!\int_{\mathbb{T}_3} \nabla q \cdot \mathbf{v} = \int_0^T\!\!\int_{\mathbb{T}_3} \bar{\mathbf{f}} \cdot \mathbf{v}. \tag{3.8}$$

All terms in (3.8) are obviously well-defined thanks to (3.3)–(3.4)–(3.5)–(3.7), except the convective term that must be checked carefully.

Recall first that D_N maps $\mathbf{H}_s$ onto itself, and the Sobolev embedding implies that $\mathbf{w} \in C([0,T]; \mathbf{H}_1) \subset L^\infty([0,T]; L^6(\mathbb{T}_3)^3)$, which yields $D_N(\mathbf{w}) \in C([0,T]; \mathbf{H}_1) \subset L^\infty([0,T]; L^6(\mathbb{T}_3)^3)$. In particular,

$$D_N(\mathbf{w}) \otimes D_N(\mathbf{w}) \in L^\infty\big([0,T]; L^3(\mathbb{T}_3)^3\big)^2.$$

Consequently, we have

$$\overline{D_N(\mathbf{w}) \otimes D_N(\mathbf{w})} \in L^\infty\big([0,T]; H^2(\mathbb{T}_3)^3\big)^2 \subset L^\infty\big([0,T] \times \mathbb{T}_3\big)^9,$$

which yields the integrability of $\overline{D_N(\mathbf{w}) \otimes D_N(\mathbf{w})} : \nabla \mathbf{v}$ for any $\mathbf{v} \in L^2([0,T]; H^1(\mathbb{T}_3)^3)$.

Remark 3.1. We use the name "regular weak" solution:

- "weak" since in point 3), $(\mathbf{w}, q)$ is defined to be a solution in the sense of distributions,
- "regular" because of the spaces involved in point 1), that in particular yields uniqueness.

Moreover, as we shall see later, this solution satisfies an energy-like equality instead of only an energy inequality in the usual Navier–Stokes equations. We stress that this is one possible choice of definition among many others.

3.2. Existence result

The main result of Section 3 is the following.

Theorem 3.1. *Assume that* (3.1) *holds,* $\alpha > 0$ *and* $N \in \mathbb{N}$ *are given and fixed. Then problem* (1.3) *has a unique regular weak solution.*

Proof. We use the usual Galerkin method, using space vector fields having zero divergence (see [22]). This yields the construction of the velocity part of the solution. The pressure is then recovered by De Rham Theorem. The proof is divided into five steps:

STEP 1: we construct approximate solutions $\mathbf{w}_m$, solving ordinary differential equations on finite dimensional spaces (see the definition in (3.9) below);
STEP 2: we look for bounds on $\{\mathbf{w}_m\}_{m\in\mathbb{N}}$ and $\{\partial_t \mathbf{w}_m\}_{m\in\mathbb{N}}$, uniform with respect to $m \in \mathbb{N}$, in suitable spaces. To do so, we use an energy equality satisfied by $A^{1/2}D_N^{1/2}(\mathbf{w}_m)$. The most important thing is that these bounds are almost all *uniform in* N, where $N \in \mathbb{N}$ is the index related to the order of deconvolution of the model;
STEP 3: we apply usual results to get compactness properties about the sequence $\{\mathbf{w}_m\}_{n\in\mathbb{N}}$. Then we take the limit when $m \to \infty$ and N is fixed, to obtain a solution to the model;
STEP 4: we check the question of the initial data (point 2)) in Definition 3.1);
STEP 5: we show uniqueness of the solution thanks to Gronwall's lemma.

Since Steps 1 and 3 are very classical, we will only sketch them, as well as Step 4 which is very close from what has already been done in [6,21,33]. On the other hand, Step 2 is one of the main original contributions in the paper and will also be useful in the next section. Indeed, we obtain many estimates, uniform in N, that allow us to take the limit when N goes to infinity and then to prove Theorem 4.1. Also Step 4 needs some application of classical tools in a way that is less standard than usual. We also point out that Theorem 3.1 greatly improves the corresponding existence result in [9] and it is not a simple restatement of those results.

Step 1: Construction of the velocity's approximations.

Let $m \in \mathbb{N}^\star$ be given and let $\mathbf{V}_m$ be the space of real valued trigonometric polynomial vector fields of degree less than or equal to m, with zero divergence and zero mean value on the torus $\mathbb{T}_3$,

$$\mathbf{V}_m := \left\{\mathbf{w} \in \mathbf{H}_1\colon \int_{\mathbb{T}_3} \mathbf{w}(\mathbf{x})e^{-i\mathbf{k}\cdot\mathbf{x}} = \mathbf{0},\ \forall\mathbf{k},\ \text{with } |\mathbf{k}| > m\right\}. \tag{3.9}$$

We put $d_m = \dim \mathbf{V}_m$. We have $\mathbf{V}_m \subset \mathbf{V}_{m+1}$ and

$$\mathbf{H}_1 = \overline{\bigcup_{m\in\mathbb{N}^\star} \mathbf{V}_m}.$$

We notice that $\mathbf{V}_m$ is a subset of the finite dimensional space

$$\mathbf{W}_m := \left\{\mathbf{w}: \mathbb{T}_3 \to \mathbb{C}^3,\ \mathbf{w} = \sum_{\mathbf{k}\in\mathcal{T}_3,\ |\mathbf{k}|\leqslant m} \widehat{\mathbf{w}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}\right\},$$

and we have:

$$\mathbf{V}_m := \mathbf{W}_m \cap \mathbf{H}_0. \tag{3.10}$$

Let $(\mathbf{e}_1, \ldots, \mathbf{e}_{d_m})$ be an orthogonal basis of $\mathbf{V}_m$. Let us remark that this basis is not made of the $e^{i\mathbf{k}\cdot\mathbf{x}}$'s. However, we do not need to explicit this basis. Moreover, the family $\{\mathbf{e}_j\}_{j\in\mathbb{N}}$ is an orthogonal basis of $\mathbf{H}_0$ as well as of $\mathbf{H}_1$. As we shall see in the following, the $\mathbf{e}_j$'s can be chosen to be eigen-vectors of A, with $\|\mathbf{e}_j\| = 1$.

Let $\mathbb{P}_m$ be the orthogonal projection from $\mathbf{H}_s$ ($s = 0, 1$) onto $\mathbf{V}_m$. For instance, for $\mathbf{w}_0 = \overline{\mathbf{u}_0} = \sum_{j=1}^{\infty} w_j^0 \mathbf{e}_j$,

$$\mathbb{P}_m(\overline{\mathbf{u}_0}) = \sum_{j=1}^{d_m} w_j^0 \mathbf{e}_j.$$

In order to use classical tools for ordinary differential equations, we approximate the external force by means of a standard Friedrichs mollifier, see e.g. [30,33]. Let ρ be an even function such that $\rho \in C_0^\infty(\mathbb{R})$, $0 \leqslant \rho(s) \leqslant 1$, $\rho(s) = 0$ for $|s| \geqslant 1$, and $\int_{\mathbb{R}} \rho(s)\, ds = 1$. Then, set $\mathbf{F}(t) = \bar{\mathbf{f}}(t)$ if $t \in [0, T]$ and zero elsewhere and for all positive ϵ define $\bar{\mathbf{f}}_\epsilon$, the smooth (with respect to time) approximation of $\bar{\mathbf{f}}$, by

$$\bar{\mathbf{f}}_\epsilon(t) := \frac{1}{\epsilon} \int_{\mathbb{R}} \rho\left(\frac{t-s}{\epsilon}\right) \mathbf{F}(s)\, ds.$$

Well-known results imply that if (3.1) is satisfied, then $\bar{\mathbf{f}}_\epsilon \to \bar{\mathbf{f}}$ in $L^2([0,T]; \mathbf{H}_1)$.

Thanks to Cauchy–Lipschitz Theorem, we know that there exist:

- $T_m > 0$,
- a unique $\mathbf{w}_m(t, \mathbf{x}) = \sum_{j=1}^{d_m} w_{m,j}(t) \mathbf{e}_j(\mathbf{x})$, where $\forall j = 1, \dots, m$, $w_{m,j} \in C^1([0, T_m])$,

such that

- $w_{m,j}(0) = w_j^0$,
- $\forall \mathbf{v} \in \mathbf{V}_m$, $\forall t \in [0, T_m]$,

$$\int_{\mathbb{T}_3} \partial_t \mathbf{w}_m(t, \mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x} - \int_{\mathbb{T}_3} \big(\overline{D_N(\mathbf{w}_m) \otimes D_N(\mathbf{w}_m)}\big)(t, \mathbf{x}) : \nabla \mathbf{v}(\mathbf{x})\, d\mathbf{x} + \nu \int_{\mathbb{T}_3} \nabla \mathbf{w}_m(t, \mathbf{x}) : \nabla \mathbf{v}(\mathbf{x})\, d\mathbf{x}$$
$$= \int_{\mathbb{T}_3} \bar{\mathbf{f}}_{1/m}(t, \mathbf{x}) \cdot \mathbf{v}(\mathbf{x})\, d\mathbf{x}, \tag{3.11}$$

where

$$\partial_t \mathbf{w}_m = \sum_{j=1}^{d_m} \frac{d w_{m,j}(t)}{dt} \mathbf{e}_j.$$

As we shall see it in Step 2, we can take $T_m = T$. This ends the local-in-time construction of the approximate solutions $\mathbf{w}_m(t, \mathbf{x})$.

Remark 3.2. We should write

$$\mathbf{w}_{m,N,\alpha},$$

instead of $\mathbf{w}_m$. This simplification aims to avoid a too heavy notation, since in this section both N and α are fixed.

Step 2: Estimates.

We need estimates on the $\mathbf{w}_m$'s and the $\partial_t \mathbf{w}_m$'s for compactness properties, to take the limit when $m \to \infty$ and N is still kept fixed.

We must identify suitable test vector fields in (3.11) such that, the scalar product with the nonlinear term vanishes (if such a choice does exist). The natural candidate is $AD_N(\mathbf{w}_m)$. Indeed, since A is self-adjoint and commutes with differential operators, we have:

$$\int_{\mathbb{T}_3} \big(\overline{D_N(\mathbf{w}_m) \otimes D_N(\mathbf{w}_m)}\big) : \nabla\big(AD_N(\mathbf{w}_m)\big)\, d\mathbf{x} = \int_{\mathbb{T}_3} G\big(D_N(\mathbf{w}_m) \otimes D_N(\mathbf{w}_m)\big) : \nabla\big(AD_N(\mathbf{w}_m)\big)\, d\mathbf{x}$$
$$= \int_{\mathbb{T}_3} (AG)\big(D_N(\mathbf{w}_m) \otimes D_N(\mathbf{w}_m)\big) : \nabla\big(D_N(\mathbf{w}_m)\big)\, d\mathbf{x} = 0,$$

because $A \circ G = \mathrm{Id}$ on $\mathbf{H}_s$, $\nabla \cdot (D_N(\mathbf{w}_m)) = 0$, and thanks to the periodicity. This yields the equality

$$\big(\partial_t \mathbf{w}_m, AD_N(\mathbf{w}_m)\big) - \nu\big(\Delta \mathbf{w}_m, AD_N(\mathbf{w}_m)\big) = \big(\bar{\mathbf{f}}_{1/m}, AD_N(\mathbf{w}_m)\big). \tag{3.12}$$

This formal computation asks for two clarifications:

(i) We must check that $AD_N(\mathbf{w}_m)$ is a "legal" test vector field, to justify the formal procedure above. That means that for any fixed time t, $AD_N(\mathbf{w}_m) \in \mathbf{V}_m$.
(ii) Equality (3.12) does not give a direct information about $\mathbf{w}_m$ itself and/or $\partial_t \mathbf{w}_m$. Therefore we must find how to deduce suitable estimates from (3.12).

Point (i) follows from general properties of the operator G. Indeed, we already know that $G(\mathbf{H}_0) = \mathbf{H}_2 \subset \mathbf{H}_0$. Moreover, formula (2.3) yields $G(\mathbf{W}_m) \subset \mathbf{W}_m$. Therefore by (3.10) we get $G(\mathbf{V}_m) \subset \mathbf{V}_m$. Finally, it is clear that $\mathrm{Ker}(G) = \mathbf{0}$ and since $\mathbf{V}_m$ has a finite dimension, G is an linear isomorphism on it. Then we have $A(\mathbf{V}_m) \subset \mathbf{V}_m$ as well as $D_N(\mathbf{V}_m) \subset \mathbf{V}_m$. Therefore, $AD_N(\mathbf{w}_m)(t,\cdot) \in \mathbf{V}_m$ a "legal" multiplier in formulation (3.11), for each fixed t. Moreover, since A and D_N are self-adjoint operators that commute, one can choose the basis $(\mathbf{e}_1, \ldots, \mathbf{e}_{d_m}, \ldots)$ such that each $\mathbf{e}_j$ is still an eigen-vector of the operator A and D_N together. Therefore, the projection $\mathbb{P}_m$ commutes with A as well as with all by-products of A, such as D_N for instance. We shall use this remark later in the estimates.

Let us turn to point (ii). The following identities hold:

$$\big(\partial_t \mathbf{w}_m, AD_N(\mathbf{w}_m)\big) = \frac{1}{2}\frac{d}{dt}\big\|A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}_m)\big\|^2, \tag{3.13}$$

$$\big(-\Delta \mathbf{w}_m, AD_N(\mathbf{w}_m)\big) = \big\|\nabla A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}_m)\big\|^2, \tag{3.14}$$

$$\big(\bar{\mathbf{f}}_{1/m}, AD_N(\mathbf{w}_m)\big) = \big(A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\bar{\mathbf{f}}_{1/m}), A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}_m)\big). \tag{3.15}$$

These equalities are straightforward because A and D_N both commute, as well as they do with all differential operators. Therefore (3.12), that we write as

$$\frac{1}{2}\frac{d}{dt}\big\|A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}_m)\big\|^2 + \big\|\nabla A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}_m)\big\|^2 = \big(A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\bar{\mathbf{f}}_{1/m}), A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}_m)\big), \tag{3.16}$$

shows that we get an estimate about $A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}_m)$. As we shall see in the remainder, norms of this quantity do control $\mathbf{w}_m$, as well as the natural key variable $D_N(\mathbf{w}_m)$. Finally, this yields an estimate for $\partial_t \mathbf{w}_m$.

We are now in position to get estimates for the sequence $(\mathbf{w}_m)_{m\in\mathbb{N}}$ and related sequences. Since we need to display many estimates, for the reader's convenience we organize the results in the following table (3.17), that is organized as follows.

Label	Variable	Bound	Order
(a)	$A^{1/2} D_N^{1/2}(\mathbf{w}_m)$	$L^\infty([0,T];\mathbf{H}_0) \cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
(b)	$D_N^{1/2}(\mathbf{w}_m)$	$L^\infty([0,T];\mathbf{H}_0) \cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
(c)	$D_N^{1/2}(\mathbf{w}_m)$	$L^\infty([0,T];\mathbf{H}_1) \cap L^2([0,T];\mathbf{H}_2)$	$O(\alpha^{-1})$
(d)	$\mathbf{w}_m$	$L^\infty([0,T];\mathbf{H}_0) \cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
(e)	$\mathbf{w}_m$	$L^\infty([0,T];\mathbf{H}_1) \cap L^2([0,T];\mathbf{H}_2)$	$O(\alpha^{-1})$
(f)	$D_N(\mathbf{w}_m)$	$L^\infty([0,T];\mathbf{H}_0) \cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
(g)	$D_N(\mathbf{w}_m)$	$L^\infty([0,T];\mathbf{H}_1) \cap L^2([0,T];\mathbf{H}_2)$	$O(\alpha^{-1} \cdot (N+1)^{1/2})$
(h)	$\partial_t \mathbf{w}_m$	$L^2([0,T];\mathbf{H}_0)$	$O(\alpha^{-1})$

(3.17)

In the first column we have labeled the estimates. The second column precises the variable. The third one explains the bound in term of function spaces, where to shorten

"$E_m \in F$" = "the sequence $\{E_m\}_{m\in\mathbb{N}}$ is bounded in the space F".

Finally the fourth column precises the order in terms of α, m, and N for each bound. The quantity

$$\|\mathbf{u}_0\|_{L^2} + \frac{1}{\nu}\|\mathbf{f}\|_{L^2([0,T];L^2)}$$

is involved in all estimates and therefore we do not quote it. All bounds are

– uniform in m,
– uniform in N, except (3.17)(g),
– uniform in T yielding $T_m = T$ for each T.

We mention that we can take $T = \infty$ if $\mathbf{f}$ is defined on $[0, \infty[$.

We now prove all estimates in table (3.17) one after each other.

Checking (3.17)(a): We assumed for the simplicity that $\mathbf{f} \in L^2([0,T] \times \mathbb{T}_3)^3$, but the proof remains the same when $\mathbf{f} \in L^2([0,T]; \mathbf{H}_{-1})$.[1] Let us integrate (3.16) on the time interval $[0,t]$ for any time $t \leqslant T_m$ and let us use (3.13)–(3.15) together with Cauchy–Schwartz inequality. We obtain

$$\frac{1}{2}\big\|A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}_m)(t,\cdot)\big\|^2 + \nu \int_0^t \big\|\nabla A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}_m)\big\|^2 \, d\tau$$
$$\leqslant \frac{1}{2}\big\|A^{\frac{1}{2}} D_N^{\frac{1}{2}} \mathbb{P}_m \overline{\mathbf{u}_0}\big\|^2 + \int_0^t \big\|A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\bar{\mathbf{f}}_{1/m})\big\| \cdot \big\|A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}_m)\big\| \, d\tau. \tag{3.18}$$

• Notice that $A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\bar{\mathbf{f}}_\epsilon) = A^{-\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{f}_\epsilon)$. Since the operator $A^{-\frac{1}{2}} D_N^{\frac{1}{2}}$ has for symbol $\rho_{N,\mathbf{k}}^{1/2} \leqslant 1$, then $\|A^{\frac{1}{2}} D_N^{\frac{1}{2}} \bar{\mathbf{f}}_\epsilon\| \leqslant C\|\mathbf{f}\|$.

• Since $\mathbb{P}_m$ commutes with A and D_N, we have

$$\big\|A^{\frac{1}{2}} D_N^{\frac{1}{2}} \mathbb{P}_m \overline{\mathbf{u}_0}\big\| = \big\|\mathbb{P}_m A^{\frac{1}{2}} D_N^{\frac{1}{2}} \overline{\mathbf{u}_0}\big\| \leqslant \big\|A^{\frac{1}{2}} D_N^{\frac{1}{2}} \overline{\mathbf{u}_0}\big\| \leqslant \|\mathbf{u}_0\|.$$

• By Poincaré's inequality and Young's inequality, and standard properties of mollifiers, we get

$$\frac{1}{2}\big\|A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}_m)(t,\cdot)\big\|^2 + \frac{\nu}{2} \int_0^t \big\|\nabla A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}_m)\big\|^2 \, d\tau \leqslant C\big(\|\mathbf{u}_0\|, \|\mathbf{f}\|_{L^2([0,T];\mathbf{H}_{-1})}\big), \tag{3.19}$$

that gives (3.17)(a).

In addition, we check here that we can take $T_m = T$. Indeed, insert the definition of $\mathbf{w}_m$ in (3.19) and use that the $\mathbf{e}_j$'s are eigen-vectors for both A and D_N and therefore also for $A^{1/2} D_N^{1/2}$. Then we get in particular,

$$\sum_{j=1}^{d_m} \rho_{N,j} w_{m,j}(t)^2 \leqslant C\big(\|\mathbf{u}_0\|, \|\mathbf{f}\|_{L^2([0,T];\mathbf{H}_{-1})}\big).$$

Therefore since no $\rho_{N,j}$ vanishes, then no $w_{m,j}(t)$ blows up. Therefore, we can take $T_m = T$ for any $T < \infty$, and the approximate solutions are well-defined on $[0, \infty[$.

Checking (3.17)(b)–(c): Let $\mathbf{v} \in \mathbf{H}_2$. Then, with obvious notations one has

$$\big\|A^{\frac{1}{2}} \mathbf{v}\big\|^2 = \sum_{\mathbf{k} \in \mathcal{T}_3^\star} \big(1 + \alpha^2 |\mathbf{k}|^2\big) |\widehat{\mathbf{v}}_{\mathbf{k}}|^2 = \|\mathbf{v}\|^2 + \alpha^2 \|\nabla \mathbf{v}\|^2.$$

It suffices to apply this identity to $\mathbf{v} = D_N^{\frac{1}{2}}(\mathbf{w}_m)$ and to $\mathbf{v} = \partial_i D_N^{\frac{1}{2}}(\mathbf{w}_m)$ $(i = 1,2,3)$ in (3.18) to get the claimed result.

Checking (3.17)(d)–(e): This is a direct consequence of (3.17)(b)–(c) combined with (2.8), that can also be understood as

$$\|\mathbf{w}\|_s \leqslant \big\|D_N(\mathbf{w})\big\|_s \leqslant (N+1)\|\mathbf{w}\|_s,$$

for general $\mathbf{w}$ and for any $s \geqslant 0$. This explains why it is crucial to have a "lower bound" for the operator D_N.

[1] Substitute in (3.15) the integral over $\mathbb{T}_3$ with the duality pairing $\langle . \rangle$ between $\mathbf{H}_1$ and $\mathbf{H}_{-1}$ and estimate in a standard way the quantity $\langle \bar{\mathbf{f}}_{1/m}, AD_N(\mathbf{w}_m)\rangle = \langle A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\bar{\mathbf{f}}_{1/m}), A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{w}_m)\rangle$.

Checking (3.17)(f): The operator $A^{1/2}D_N^{1/2}$ has for symbol $(1+\alpha^2|\mathbf{k}|^2)\rho_{N,\mathbf{k}}^{1/2}$ while the one of D_N is $(1+\alpha^2|\mathbf{k}|^2)\rho_{N,\mathbf{k}}$. Since $0 \leqslant \rho_{N,\mathbf{k}} \leqslant 1$, then $\|D_N(\mathbf{w})\|_s \leqslant \|A^{1/2}D_N^{1/2}(\mathbf{w})\|_s$ for general $\mathbf{w}$ and for any $s \geqslant 0$. Therefore, the estimate (3.17)(f) is still a consequence of (3.17)(a).

Checking (3.17)(g): This follows directly from (3.17)(e) together with (2.8). This also explains why the result depends on N because we use here the *upper bound* on the norm of the operator D_N, that depends on N.

Checking (3.17)(h): Let us take $\partial_t \mathbf{w}_m \in \mathbf{V}_m$ as test vector field in (3.11). We get

$$\|\partial_t \mathbf{w}_m\|^2 + \int_{\mathbb{T}_3} \mathbf{A}_{N,m} \cdot \partial_t \mathbf{w}_m + \frac{\nu}{2}\frac{d}{dt}\|\nabla \mathbf{w}_m\|^2 = \int_{\mathbb{T}_3} \bar{\mathbf{f}}_{1/m} \cdot \partial_t \mathbf{w}_m,$$

where

$$\mathbf{A}_{N,m} := \overline{\nabla \cdot \big(D_N(\mathbf{w}_m) \otimes D_N(\mathbf{w}_m)\big)}. \tag{3.20}$$

So far $\mathbf{w}_m(0,\cdot) = \mathbb{P}_m(\overline{\mathbf{u}_0}) \in \mathbf{H}_2$ and obviously $\|\mathbb{P}_m(\overline{\mathbf{u}_0})\|_2 \leqslant C\alpha^{-1}\|\mathbf{u}_0\|$, we only have to check that $\mathbf{A}_{N,m}$ is bounded in $L^2([0,T]\times\mathbb{T}_3)^3$ and that the bound does not depend neither on m nor on N.

- Thanks to (3.17)(f), it is easily checked that $D_N(\mathbf{w}_m) \in L^4([0,T]; L^3(\mathbb{T}_3)^3)$ to conclude, where the bound depends neither on m nor on N. Therefore, $D_N(\mathbf{w}_m) \otimes D_N(\mathbf{w}_m) \in L^2([0,T]; L^{3/2}(\mathbb{T}_3)^9)$.
- Because the operator $(\nabla\cdot)\circ G$ makes to "gain one derivative", we deduce that $\mathbf{A}_{N,m} \in L^2([0,T]; W^{1,3/2}(\mathbb{T}_3)^3)$, which yields $\mathbf{A}_{N,m} \in L^2([0,T]\times\mathbb{T}_3)^3$ since $W^{1,3/2}(\mathbb{T}_3) \subset L^3(\mathbb{T}_3) \subset L^2(\mathbb{T}_3)$ and $L^2([0,T]; L^2(\mathbb{T}_3)^3)$ is isomorphic to $L^2([0,T]\times\mathbb{T}_3)^3$ (see [21]). Moreover, the bound is of order $O(\alpha^{-1})$ as well, because the norm of the operator $(\nabla\cdot)\circ G$ is of order $O(\alpha^{-1})$.

Notice that this bound is not optimal, but fits with our requirements.

Step 3: Taking the limit in the equations when $m \to \infty$, and N is fixed.

Thanks to the bounds (3.17), we can extract from the sequence $\{\mathbf{w}_m\}_{n\in\mathbb{N}}$ a sub-sequence which converges to a $\mathbf{w} \in L^\infty([0,T]; \mathbf{H}_1) \cap L^2([0,T]; \mathbf{H}_2)$. Using Aubin–Lions lemma thanks to (3.17)(d) and (3.17)(h), this convergence is such that:

$$\mathbf{w}_m \to \mathbf{w} \quad \text{weakly in } L^2\big([0,T]; \mathbf{H}_2\big), \tag{3.21}$$

$$\mathbf{w}_m \to \mathbf{w} \quad \text{strongly in } L^p\big([0,T]; \mathbf{H}_1\big), \quad \forall p < \infty, \tag{3.22}$$

$$\partial_t \mathbf{w}_m \to \partial_t \mathbf{w} \quad \text{weakly in } L^2\big([0,T]; \mathbf{H}_0\big). \tag{3.23}$$

This already implies that $\mathbf{w}$ satisfies (3.3)–(3.4). Note that the continuity of $\mathbf{w}$ with values in $\mathbf{H}_1$ is a consequence of $\mathbf{w} \in L^2([0,T]; \mathbf{H}_2)$ together with $\partial_t \mathbf{w} \in L^2([0,T]; \mathbf{H}_0)$.

Let us determine which equation is satisfied by $\mathbf{w}$. By (3.22) and the continuity of D_N in $\mathbf{H}_s$, $D_N(\mathbf{w}_m)$ converges strongly to $D_N(\mathbf{w})$ in $L^4([0,T]\times\mathbb{T}_3)$. Hence, $D_N(\mathbf{w}_m)\otimes D_N(\mathbf{w}_m)$ converges strongly to $D_N(\mathbf{w})\otimes D_N(\mathbf{w})$ in $L^2([0,T]\times\mathbb{T}_3)$. This convergence result together with the strong convergence of $(\bar{\mathbf{f}}_{1/m})_{m\in\mathbb{N}}$ to $\bar{\mathbf{f}}$, implies that for all $\mathbf{v} \in L^2([0,T]; \mathbf{H}_1)$

$$\int_0^T\!\!\int_{\mathbb{T}_3} \partial_t \mathbf{w}\cdot\mathbf{v}\,d\mathbf{x}\,d\tau - \int_0^T\!\!\int_{\mathbb{T}_3} \overline{D_N(\mathbf{w})\otimes D_N(\mathbf{w})} : \nabla\mathbf{v}\,d\mathbf{x}\,d\tau + \nu\int_0^T\!\!\int_{\mathbb{T}_3} \nabla\mathbf{w} : \nabla\mathbf{v}\,d\mathbf{x}\,d\tau = \int_0^T\!\!\int_{\mathbb{T}_3} \bar{\mathbf{f}}\cdot\mathbf{v}\,d\mathbf{x}\,d\tau. \tag{3.24}$$

Arguing similarly to [20], we easily get that $\mathbf{w}$ satisfies (3.6).

We must now introduce the pressure. We take test vector fields in $L^2([0,T]; \mathbf{H}_1)$ to be in accordance with classical presentations. However, the regularity of $\mathbf{w}$ leads to $\overline{\nabla\cdot(D_N(\mathbf{w})\otimes D_N(\mathbf{w}))} \in L^2([0,T]\times\mathbb{T}_3)^3$ as well as $\Delta\mathbf{w} \in L^2([0,T]\times\mathbb{T}_3)^3$. Consequently, one can take vector test fields $\mathbf{v} \in L^2([0,T]; \mathbf{H}_0)$ in formulation (3.24) that we can rephrase as: $\forall \mathbf{v} \in L^2([0,T]; \mathbf{H}_0)$,

$$\int_0^T\!\!\int_{\mathbb{T}_3} (\partial_t \mathbf{w} + \mathbf{A}_N - \nu\Delta\mathbf{w} - \bar{\mathbf{f}})\cdot\mathbf{v}\,d\mathbf{x}\,d\tau = 0, \tag{3.25}$$

where for convenience, we have set

$$\mathbf{A}_N := \overline{\nabla\cdot\big(D_N(\mathbf{w})\otimes D_N(\mathbf{w})\big)}.$$

Therefore, for almost every $t \in [0, T]$,

$$\mathbb{F}(t, \cdot) = (\partial_t \mathbf{w} + \mathbf{A}_N - \nu \Delta \mathbf{u} - \bar{\mathbf{f}})(t, \cdot) \in L^2(\mathbb{T}_3)^3$$

is orthogonal to divergence-free vector fields in $L^2(\mathbb{T}_3)^3$ and De Rham's Theorem applies. From (3.25), we deduce that for each Lebesgue point t of $\mathbb{F}$, there is a scalar function $q(t, \cdot) \in H^1(\mathbb{T}_3)$, such that $\mathbb{F} = -\nabla q$. This yields the following equation, satisfied in the sense of the distributions:

$$\partial_t \mathbf{w} + \mathbf{A}_N - \nu \Delta \mathbf{w} + \nabla q = \bar{\mathbf{f}}. \tag{3.26}$$

It remains to check the regularity of q. Without loss of generality, one can assume that $\nabla \cdot \mathbf{f} = 0$. Therefore, taking the divergence of Eq. (3.26) yields

$$\Delta q = \nabla \cdot \mathbf{A}_N,$$

which easily yields $q \in L^2([0, T]; H^1(\mathbb{T}_3))$. We already knew about (3.3)–(3.4) (regularity of $\mathbf{w}$) by the previous section. We now know about (3.5) (existence and regularity of the pressure), (3.7)–(3.8) (weak formulation).

Step 4: About the initial data.

We already know that $\mathbf{w}(0, \cdot) \in \mathbf{H}_1$ because $\mathbf{w} \in C([0, T]; \mathbf{H}_1)$. Moreover, we have

$$\lim_{t \to 0^+} \|\mathbf{w}(t, \cdot) - \mathbf{w}(0, \cdot)\|_{\mathbf{H}_1} = 0.$$

It remains to identify $\mathbf{w}(0, \cdot)$. The construction displayed in Step 1 yields for $m \in \mathbb{N}$,

$$\mathbf{w}_m(t, \mathbf{x}) = \mathbb{P}_m(\overline{\mathbf{u}_0})(\mathbf{x}) + \int_0^t \partial_t \mathbf{w}_m(s, \mathbf{x})\, ds, \tag{3.27}$$

an identity that holds in $C^1([0, T] \times \Omega)$. Because of the weak convergence of $(\partial_t \mathbf{w}_m)_{m \in \mathbb{N}}$ to $\partial_t \mathbf{w}$ in $L^2([0, T]; \mathbf{H}_0)$ and thanks to usual properties of $\mathbb{P}_m$, one easily can pass to the limit in (3.27) in a weak sense in the space $L^2([0, T]; \mathbf{H}_0)$, to obtain

$$\mathbf{w}(t, \mathbf{x}) = \overline{\mathbf{u}_0}(\mathbf{x}) + \int_0^t \partial_t \mathbf{w}(s, \mathbf{x})\, ds.$$

Therefore, $\mathbf{w}(0, \mathbf{x}) = \overline{\mathbf{u}_0}(\mathbf{x})$ and (3.6) is satisfied.

Step 5: Uniqueness.

Let $\mathbf{w}_1$ and $\mathbf{w}_2$ be two solutions and consider $\mathbf{W} := \mathbf{w}_1 - \mathbf{w}_2$. We want to take $AD_N(\mathbf{W})$ as test function in the equation satisfied by $\mathbf{W}$, because it is the natural multiplier for this specific question, and next apply Gronwall's lemma.

We must first check that $AD_N(\mathbf{W}) \in L^2([0, T] \times \mathbb{T}_3)^3$ to be convinced that this is a "legal" multiplier. Notice that AD_N has for symbol

$$\left(1 + \alpha^2 |\mathbf{k}|^2\right)^2 \rho_{N,\mathbf{k}} \approx (N + 1)\left(1 + \alpha^2 |\mathbf{k}|^2\right)^2 / \alpha^2 |\mathbf{k}|^2 \approx (N + 1)\alpha^2 |\mathbf{k}|^2$$

for large $|\mathbf{k}|$. Therefore, for each fixed $N \in \mathbb{N}$, AD_N is "like a Laplacian" and "makes lose" two derivatives in space. Fortunately, $\mathbf{W} \in L^2([0, T]; \mathbf{H}_2)$ and therefore $AD_N(\mathbf{W}) \in L^2([0, T] \times \mathbb{T}_3)^3$. Therefore we can take $AD_N(\mathbf{W})$ as multiplier and integrate by parts. After applying rules many times used in this paper, we get

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \left\| A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{W}) \right\|^2 + \nu \left\| \nabla A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{W}) \right\|^2 &\leqslant \left| \left(\left(D_N(\mathbf{W}) \cdot \nabla \right) D_N(\mathbf{w}_2), D_N(\mathbf{W}) \right) \right| \\ &\leqslant \left\| D_N(\mathbf{W}) \right\|_{L^4(\mathbb{T}_3)}^2 \left\| \nabla D_N(\mathbf{w}_2) \right\| \\ &\leqslant \left\| D_N(\mathbf{W}) \right\|^{1/2} \left\| \nabla D_N(\mathbf{W}) \right\|^{3/2} \left\| \nabla D_N(\mathbf{w}_2) \right\|, \end{aligned} \tag{3.28}$$

where the last line is obtained thanks to the well-known "Ladyženskaya inequality" for interpolation of L^4 with L^2 and H^1, see [12, Chapter 1]. Starting from the last line of (3.28), we combine the following known facts:

$$\begin{array}{ll} \|D_N(\mathbf{W})\| \leqslant \|A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{W})\|, & \|D_N(\nabla\mathbf{W})\| \leqslant \|A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\nabla\mathbf{W})\|, \\ D_N \text{ and } \nabla \text{ commute}, & \|D_N\| = (N+1), \\ \text{the bound of } \mathbf{w}_2 \text{ in } L^\infty([0,T];\mathbf{H}_1), & \text{Young's inequality.} \end{array}$$

We obtain

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{W})\|^2 + \nu\|\nabla A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{W})\|^2 \\ &\quad \leqslant \frac{27(N+1)^4 \sup_{t\geqslant 0}\|\nabla \mathbf{w}_2\|}{32\nu^3}\|A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{W})\|^2 + \frac{\nu}{2}\|\nabla A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{W})\|^2. \end{aligned}$$

In particular, we get

$$\frac{1}{2}\frac{d}{dt}\|A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{W})\|^2 \leqslant \frac{27(N+1)^4 \sup_{t\geqslant 0}\|\nabla \mathbf{w}_2\|}{32\nu^3}\|A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{W})\|^2.$$

We deduce from Gronwall's Lemma that $A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{W}) = \mathbf{0}$ because $A^{\frac{1}{2}} D_N^{\frac{1}{2}}(\mathbf{W})(0,\cdot) = \mathbf{0}$. To conclude that $\mathbf{W} = \mathbf{0}$, we must show that the kernel of the operator $A^{\frac{1}{2}} D_N^{\frac{1}{2}}$ is reduced to $\mathbf{0}$. This operator has for symbol $(1+\alpha^2|\mathbf{k}|^2)\rho_{N,\mathbf{k}} \approx \alpha|\mathbf{k}|$ for large values of $\mathbf{k}$. This symbol never vanishes and the equivalence at infinity shows that $A^{\frac{1}{2}} D_N^{\frac{1}{2}}$ is of the same order of $\alpha|\nabla|$. Therefore, it is an isomorphism that maps $\mathbf{H}_s$ onto $\mathbf{H}_{s-1}$ and its kernel is reduced to zero, which concludes the question of uniqueness. □

Remark 3.3. As we have seen, we can use $AD_N(\mathbf{w})$ as a test in Eq. (3.26). Therefore, the following energy equality is satisfied by $A^{1/2}D_N^{1/2}(\mathbf{w})$,

$$\frac{1}{2}\frac{d}{dt}\|A^{1/2}D_N^{1/2}(\mathbf{w})\|^2 + \nu\|\nabla A^{1/2}D_N^{1/2}(\mathbf{w})\|^2 = \left(A^{-1/2}D_N^{1/2}(\mathbf{f}), A^{1/2}D_N^{1/2}(\mathbf{w})\right). \tag{3.29}$$

4. Taking the limit when $N \to \infty$ and energy inequality

The aim of this section is the proof of our main result, Theorem 4.1 below, that states the sequence of regular weak solutions converges to a solution of the mean Navier–Stokes equations, as N goes to infinity.

We divide this section into two subsections. One is devoted to the proof of the theorem. An additional subsection is devoted to the study of the energy inequality satisfied by the limit.

4.1. Taking the limit when $N \to \infty$

Let $(\mathbf{w}_N, q_N)$ be the "regular weak" solution to problem (1.3):

$$\begin{aligned} &\partial_t \mathbf{w}_N + \nabla\cdot\left(\overline{D_N(\mathbf{w}_N)\otimes D_N(\mathbf{w}_N)}\right) - \nu\Delta\mathbf{w}_N + \nabla q_N = \bar{\mathbf{f}} \quad \text{in } [0,T]\times\mathbb{T}_3, \\ &\nabla\cdot\mathbf{w}_N = 0 \quad \text{in } [0,T]\times\mathbb{T}_3, \\ &\mathbf{w}_N(0,\mathbf{x}) = \overline{\mathbf{u}_0}(\mathbf{x}) \quad \text{in } \mathbb{T}_3. \end{aligned} \tag{4.1}$$

Recall that the scale $\alpha > 0$ is fixed. We aim to prove Theorem 4.1

Theorem 4.1. *From the sequence $\{(\mathbf{w}_N, q_N)\}_{N\in\mathbb{N}}$ one can extract a sub-sequence (still denoted $\{(\mathbf{w}_N, q_N)\}_{N\in\mathbb{N}}$) such that*

$$\begin{aligned} &\mathbf{w}_N \to \mathbf{w} \quad \begin{cases} \textit{weakly in } L^2([0,T];H^2(\mathbb{T}_3)^3) \cap L^\infty([0,T];H^1(\mathbb{T}_3)^3), \\ \textit{strongly in } L^p([0,T];H^1(\mathbb{T}_3)^3), \quad \forall 1\leqslant p<+\infty, \end{cases} \\ &q_N \to q \quad \textit{weakly in } L^2([0,T];W^{1,2}(\mathbb{T}_3)) \cap L^{5/3}([0,T];W^{2,5/3}(\mathbb{T}_3)), \end{aligned} \tag{4.2}$$

and such that the system

$$\partial_t \mathbf{w} + \nabla \cdot (\overline{A\mathbf{w} \otimes A\mathbf{w}}) - \nu \Delta \mathbf{w} + \nabla q = \bar{\mathbf{f}},$$
$$\nabla \cdot \mathbf{w} = 0,$$
$$\mathbf{w}(0, \mathbf{x}) = \overline{\mathbf{u}_0}(\mathbf{x}) \tag{4.3}$$

holds in the sense of the distributions.

It is straightforward to check that $(A\mathbf{w}, Aq)$ is therefore a distributional solution to the Navier–Stokes equations. We divide the proof into two steps:

1. We seek additional estimates uniform in N, to get compactness properties for the sequences $\{D_N(\mathbf{w}_N)\}_{N\in\mathbb{N}}$ and $\{\mathbf{w}_N\}_{N\in\mathbb{N}}$.
2. We take the limit in Eq. (4.1) when $N \to \infty$.

The challenge is to take the limit in the nonlinear term $D_N(\mathbf{w}_N) \otimes D_N(\mathbf{w}_N)$. This is why we want to get a bound on the sequence $\{\partial_t D_N(\mathbf{w}_N)\}_{N\in\mathbb{N}}$ in a suitable space, since we already know estimates for $\{D_N(\mathbf{w}_N)\}_{N\in\mathbb{N}}$. The goal is to prove a compactness property satisfied by $\{D_N(\mathbf{w}_N)\}_{N\in\mathbb{N}}$, to take the limit in the nonlinear term.

Step 1: Additional estimates.

We quote in the following table the estimates useful to take the limit. Table (4.4) is organized as the previous one (3.17).

Label	Variable	Bound	Order
(a)	$\mathbf{w}_N$	$L^\infty([0,T];\mathbf{H}_0) \cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
(b)	$\mathbf{w}_N$	$L^\infty([0,T];\mathbf{H}_1) \cap L^2([0,T];\mathbf{H}_2)$	$O(\alpha^{-1})$
(c)	$D_N(\mathbf{w}_N)$	$L^\infty([0,T];\mathbf{H}_0) \cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
(d)	$\partial_t \mathbf{w}_N$	$L^2([0,T] \times \mathbb{T}_3)^3$	$O(\alpha^{-1})$
(e)	q_N	$L^2([0,T];H^1(\mathbb{T}_3)) \cap L^{5/3}([0,T];W^{2,5/3}(\mathbb{T}_3))$	$O(\alpha^{-1})$
(f)	$\partial_t D_N(\mathbf{w}_N)$	$L^{4/3}([0,T];\mathbf{H}_{-1})$	$O(1)$

(4.4)

Estimates (4.4)(a), (b), (c), and (d) have already been obtained in the previous section. Therefore, we just have to check (4.4)(e) and (4.4)(f).

Checking (4.4)(e): Let us take the divergence of (3.26):

$$-\Delta q_N = \nabla \cdot \mathbf{A}_N - \nabla \cdot \bar{\mathbf{f}},$$

where we recall that

$$\mathbf{A}_N = \overline{\nabla \cdot \big(D_N(\mathbf{w}_N) \otimes D_N(\mathbf{w}_N)\big)}.$$

Next, since $\mathbf{f} \in L^2([0,T] \times \mathbb{T}_3)^3$, then we get $\nabla \cdot \bar{\mathbf{f}} \in L^2([0,T]; H^1(\mathbb{T}_3))$. We now investigate the regularity of $\mathbf{A}_N$. We already know from the estimates proved in the previous section that $\mathbf{A}_N \in L^2([0,T] \times \mathbb{T}_3)^3$. This yields the first bound in $L^2([0,T]; H^1(\mathbb{T}_3))$ for q_N.

We now seek for the other estimate for q_N. Classical interpolation inequalities combined with (4.4)(c) yield $D_N(\mathbf{w}_N) \in L^{10/3}([0,T] \times \mathbb{T}_3)^3$. Therefore, $\mathbf{A}_N \in L^{5/3}([0,T]; W^{1,5/3}(\mathbb{T}_3))$. Consequently, we obtain

$$q_N \in L^2\big([0,T]; H^1(\mathbb{T}_3)\big) \cap L^{5/3}\big([0,T]; W^{2,5/3}(\mathbb{T}_3)\big).$$

Checking (4.4)(f): Let $\mathbf{v} \in L^4([0,T]; \mathbf{H}_1)$ be given. We use $D_N(\mathbf{v}) \in L^4([0,T]; \mathbf{H}_1)$ as test function in the equation satisfied by $(\mathbf{w}_N, q_N)$, Eq. (4.1), that is now (by the results previously proved) a completely justified computation. We get, thanks to:

– $\partial_t \mathbf{w}_N \in L^2([0,T] \times \mathbb{T}_3)^3$ (as well as all other terms in the equation),
– D_N commutes with differential operators,

– G and D_N are self-adjoint,
– the pressure term vanishes because $\nabla \cdot D_N(\mathbf{v}) = 0$,

$$\begin{aligned}\big(\partial_t \mathbf{w}_N, D_N(\mathbf{v})\big) &= \big(\partial_t D_N(\mathbf{w}_N), \mathbf{v}\big) \\ &= \nu\big(\Delta \mathbf{w}_N, D_N(\mathbf{v})\big) - \big(D_N(\mathbf{w}_N) \otimes D_N(\mathbf{w}_N), \overline{D_N(\nabla \mathbf{v})}\big) - \big(D_N(\bar{\mathbf{f}}), \mathbf{v}\big). \end{aligned} \tag{4.5}$$

We first observe that

$$\big|\big(\Delta \mathbf{w}_N, D_N(\mathbf{v})\big)\big| = \big|\big(\nabla D_N(\mathbf{w}_N), \nabla \mathbf{v}\big)\big| \leqslant C_1(t)\|\mathbf{v}\|_1, \tag{4.6}$$

and we use the $L^2([0,T]; H^1(\mathbb{T}_3)^3)$ bound for $D_N(\mathbf{w}_N)$, to infer that the function $C_1(t) \in L^2([0,T])$, with a bound uniform in $N \in \mathbb{N}$. Using $\|D_N(\bar{\mathbf{f}})\| \leqslant \|\mathbf{f}\|$ already proved in the previous section and Poincaré's inequality, we handle the term the external forcing is involved in as follows:

$$\big|\big(D_N(\bar{\mathbf{f}}), \mathbf{v}\big)\big| \leqslant C\|\mathbf{f}\|\|\mathbf{v}\|_1, \tag{4.7}$$

C being Poincaré's constant. Finally, from (4.4)(c) and usual interpolation inequalities, we obtain that $D_N(\mathbf{w}_N)$ belongs to $L^{8/3}([0,T]; L^4(\mathbb{T}_3)^3)$, which yields

$$D_N(\mathbf{w}_N) \otimes D_N(\mathbf{w}_N) \in L^{4/3}\big([0,T]; L^2(\mathbb{T}_3)^9\big).$$

Therefore, when we combine the latter estimate with $\|\overline{D_N(\nabla \mathbf{v})}\| \leqslant \|\nabla \mathbf{v}\|$, we get

$$\big|\big(D_N(\mathbf{w}_N) \otimes D_N(\mathbf{w}_N), \overline{D}_N(\nabla \mathbf{v})\big)\big| \leqslant C_2(t)\|\mathbf{v}\|_1, \tag{4.8}$$

where $C_2(t) \in L^{4/3}([0,T])$ and it is uniform in $N \in \mathbb{N}$. The final result is a consequence of (4.5) combined with (4.6)–(4.8), and $C_1(t) + \|\mathbf{f}(t,\cdot)\| + C_2(t) = C(t) \in L^{4/3}([0,T])$, uniformly in $N \in \mathbb{N}$. Therefore, (4.5) yields

$$\big|(\partial_t D_N \mathbf{w}_N, \mathbf{v})\big| = \big|\big(\partial_t \mathbf{w}_N, D_N(\mathbf{v})\big)\big| \leqslant C(t)\|\mathbf{v}\|_1,$$

hence estimate (3.3)(e) follows.

Step 2: Taking the limit.

Estimates in table (4.4) yield the existence of

$$\begin{aligned}&\mathbf{w} \in L^\infty\big([0,T]; \mathbf{H}_1\big) \cap L^2\big([0,T]; \mathbf{H}_2\big), \\ &\mathbf{z} \in L^\infty\big([0,T]; \mathbf{H}_0\big) \cap L^2\big([0,T]; \mathbf{H}_1\big), \\ &q \in L^2\big([0,T]; H^1(\mathbb{T}_3)\big) \cap L^{5/3}\big([0,T]; W^{2,5/3}(\mathbb{T}_3)\big),\end{aligned}$$

such that, up to sub-sequences,

$$\begin{cases} \mathbf{w}_N \to \mathbf{w} & \begin{cases} \text{weakly in } L^2\big([0,T]; \mathbf{H}_2\big), \\ \text{weakly}* \text{ in } L^\infty\big([0,T]; \mathbf{H}_1\big), \\ \text{strongly in } L^p\big([0,T]; \mathbf{H}_1\big), \quad \forall p < \infty, \end{cases} \\ \partial_t \mathbf{w}_N \to \partial_t \mathbf{w} & \text{weakly in } L^2\big([0,T] \times \mathbb{T}_3\big), \\ D_N(\mathbf{w}_N) \to \mathbf{z} & \begin{cases} \text{weakly in } L^2\big([0,T]; \mathbf{H}_1\big), \\ \text{weakly}* \text{ in } L^\infty\big([0,T]; \mathbf{H}_0\big), \\ \text{strongly in } L^p\big([0,T] \times \mathbb{T}_3\big)^3, \quad \forall p < 10/3, \end{cases} \\ \partial_t D_N(\mathbf{w}_N) \to \partial_t \mathbf{z} & \text{weakly in } L^{4/3}\big([0,T]; \mathbf{H}_{-1}\big), \\ q_N \to q & \text{weakly in } L^2\big([0,T]; H^1(\mathbb{T}_3)\big) \cap L^{5/3}\big([0,T]; W^{2,5/3}(\mathbb{T}_3)\big). \end{cases} \tag{4.9}$$

We especially have

$$D_N \mathbf{w}_N \otimes D_N \mathbf{w}_N \to \mathbf{z} \otimes \mathbf{z} \quad \text{strongly in } L^p\big([0,T] \times \mathbb{T}_3\big)^9, \quad \forall p < 5/3. \tag{4.10}$$

It is straightforward to take the limit in the equations. It remains to prove that

$$\mathbf{z} = A\mathbf{w} = \lim_{N \to \infty} D_N(\mathbf{w}_N) \tag{4.11}$$

thanks to (4.10).

Let $\mathbf{v} \in L^2([0,T];\mathbf{H}_2)$. We have $(D_N(\mathbf{w}_N),\mathbf{v}) = (\mathbf{w}_N, D_N(\mathbf{v}))$. We claim that

$$D_N(\mathbf{v}) \to A\mathbf{v} \quad \text{strongly in } L^2\big([0,T]\times\mathbb{T}_3\big)^3, \tag{4.12}$$

which suffices to conclude the proof. Indeed, assume that such a convergence result holds. Then by (4.9), still keeping the notation $(\cdot,\cdot)$ for the scalar product in $L^2([0,T]\times\mathbb{T}_3)^3$, as long as no risk of confusion occurs,

$$\begin{tikzcd}
(D_N(\mathbf{w}_N),\mathbf{v}) \arrow[r, equal] \arrow[d] & (\mathbf{w}_N, D_N(\mathbf{v})) \arrow[d] \\
(\mathbf{z},\mathbf{v}) \arrow[d, equal] & (\mathbf{w}, A\mathbf{v}) \arrow[d, equal] \\
(\mathbf{z},\mathbf{v}) \arrow[r, equal] & (A\mathbf{w},\mathbf{v})
\end{tikzcd}$$

yielding $\mathbf{z} = A\mathbf{w}$, since for all $\mathbf{v} \in L^2([0,T];\mathbf{H}_2)$, $(\mathbf{z},\mathbf{v}) = (A\mathbf{w},\mathbf{v})$.

It remains to prove (4.12). We can write

$$\mathbf{v} = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \widehat{\mathbf{v}}_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}},$$

and consequently (see in [21])

$$\|\mathbf{v}\|^2_{L^2([0,T];\mathbf{H}_2)} = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} |\mathbf{k}|^4 \int_0^T \big|\widehat{\mathbf{v}}_{\mathbf{k}}(t)\big|^2\,dt < \infty.$$

Let $\varepsilon > 0$ be given. Then, there exists $0 < K = K(\mathbf{v}) \in \mathbb{N}$ such that

$$\sum_{|\mathbf{k}|>K} 2\big(1+\alpha^4|\mathbf{k}|^4\big) \int_0^T \big|\widehat{\mathbf{v}}_{\mathbf{k}}(t)\big|^2\,dt < \frac{\varepsilon}{2}.$$

Since $0 \leqslant (1-\rho_{N,\mathbf{k}}) \leqslant 1$, we have

$$\begin{aligned}
\int_0^T \big\|(A-D_N)\mathbf{v}\big\|^2 &= \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \big(1+\alpha^2|\mathbf{k}|^2\big)^2 (1-\rho_{N,\mathbf{k}})^2 \int_0^T \big|\widehat{\mathbf{v}}_{\mathbf{k}}(t)\big|^2\,dt \\
&= \sum_{0<|\mathbf{k}|\leqslant K} \big(1+\alpha^2|\mathbf{k}|^2\big)^2 (1-\rho_{N,\mathbf{k}})^2 \int_0^T \big|\widehat{\mathbf{v}}_{\mathbf{k}}(t)\big|^2\,dt \\
&\quad + \sum_{|\mathbf{k}|>K} \big(1+\alpha^2|\mathbf{k}|^2\big)^2 (1-\rho_{N,\mathbf{k}})^2 \int_0^T \big|\widehat{\mathbf{v}}_{\mathbf{k}}(t)\big|^2\,dt \\
&< \sum_{0<|\mathbf{k}|\leqslant K} \big(1+\alpha^2|\mathbf{k}|^2\big)^2 (1-\rho_{N,\mathbf{k}})^2 \int_0^T \big|\widehat{\mathbf{v}}_{\mathbf{k}}(t)\big|^2\,dt + \frac{\varepsilon}{2}.
\end{aligned}$$

Observe that – for each given $\mathbf{k} \in \mathcal{T}_3^\star$ – we have $\rho_{N,\mathbf{k}} \to 1$ when $N \to \infty$. Therefore, there exists $N_0 \in \mathbb{N}$ (obviously depending on $\mathbf{v}$ and on K) such that for all $N > N_0$,

$$\sum_{|\mathbf{k}|\leqslant K} \big(1+\alpha^2|\mathbf{k}|^2\big)^2 (1-\rho_{N,\mathbf{k}})^2 \int_0^T \big|\widehat{\mathbf{v}}_{\mathbf{k}}(t)\big|^2\,dt < \frac{\varepsilon}{2},$$

hence,

$$\forall\varepsilon>0 \quad \exists N_0 = N_0(\mathbf{v}) \in \mathbb{N}: \quad \big\|(A-D_N)\mathbf{v}\big\|^2_{L^2([0,T];\mathbf{H}_0)} < \varepsilon, \quad \forall N > N_0,$$

ending the proof.

Remark 4.1. Let $(\mathbf{u}_N, p_N) = (D_N(\mathbf{w}_N), D_N(q_N))$, and define $(\mathbf{u}, p) := (A\mathbf{w}, Aq)$. Our proof also shows that the field $(\mathbf{u}_N, p_N)$ satisfies the system

$$\begin{aligned}
&\partial_t \mathbf{u}_N + (D_N \circ G)\big(\nabla \cdot (\mathbf{u}_N \otimes \mathbf{u}_N)\big) - \nu \Delta \mathbf{u}_N + \nabla p_N = (D_N \circ G)(\mathbf{f}),\\
&\nabla \cdot \mathbf{u}_N = 0,\\
&\mathbf{u}_N(0, \mathbf{x}) = (D_N \circ G)(\mathbf{u}_0)(\mathbf{x}).
\end{aligned} \tag{4.13}$$

This equation is consistent with the convergence result, since $D_N \circ G \to \mathrm{Id}$, and the proof contains the fact that $(\mathbf{u}, p) = \lim_{N\to+\infty}(A(\mathbf{w}_N), A(q_N))$ is at least a distributional solution of the Navier–Stokes equations (1.1). We also recall that the energy equality holds (see Remark 3.3), for the solution $(\mathbf{w}_N, q_N)$ of the ADM (4.1).

Remark 4.2. Things that make our convergence result true are essentially:

- The decay in $O(|\mathbf{k}|^{-2})$ of $\widehat{G}_{\mathbf{k}}$ and the growth in $O(|\mathbf{k}|^2)$ of $\widehat{A}_{\mathbf{k}}$.
- The convergence property (2.7), inequalities (2.8) and (2.11).

4.2. Energy inequality

We now prove that the solution $\mathbf{u} = A(\mathbf{w})$ satisfies an "energy inequality". We still assume that (3.1) holds, i.e. $\mathbf{u}_0 \in \mathbf{H}_0$ and $\mathbf{f} \in L^2([0, T]; \mathbf{H}_0)$. Moreover, $\{(\mathbf{w}_N, q_N)\}_{N\in\mathbb{N}}$ is a (possibly relabelled) sequence of regular weak solutions that converges to a weak solution $(\mathbf{w}, q)$ of the filtered Navier–Stokes equations.

Proposition 4.1. *Let* $\mathbf{u} = A(\mathbf{w})$. *Then the field* $\mathbf{u}$ *satisfies the energy inequality*:

$$\frac{1}{2}\big\|\mathbf{u}(t)\big\|^2 + \nu \int_0^t \big\|\nabla \mathbf{u}(s)\big\|^2 ds \leqslant \frac{1}{2}\big\|\mathbf{u}(0)\big\|^2 + \int_0^t \big(\mathbf{f}(s), \mathbf{u}(s)\big)\, ds \quad \forall t \in [0, T]. \tag{4.14}$$

Remark 4.3. The energy inequality above can also be rephrased as follows:

$$\frac{1}{2}\frac{d}{dt}\|A\mathbf{w}\|^2 + \nu\|\nabla A\mathbf{w}\|^2 \leqslant (\mathbf{f}, A\mathbf{w}), \tag{4.15}$$

which holds in the sense of the distributions: for all $\phi \in C_0^\infty(0, T)$ such that $\phi \geqslant 0$,

$$-\frac{1}{2}\int_0^T \big\|A\mathbf{w}(s)\big\|^2 \phi'(s)\, ds + \nu \int_0^T \big\|\nabla A\mathbf{w}(s)\big\|^2 \phi(s)\, ds \leqslant \int_0^T \big(\mathbf{f}(s), A\mathbf{w}(s)\big)\phi(s)\, ds.$$

This implies that $\mathbf{w}$ is the average of the velocity part $\mathbf{u}$ of a dissipative solution of the Navier–Stokes equation (1.1) in the sense of Leray–Hopf, that one also can read as

$$\frac{1}{2}\frac{d}{dt}\|\mathbf{u}\|^2 + \nu\|\nabla A\mathbf{u}\|^2 \leqslant (\mathbf{f}, \mathbf{u}).$$

Remark 4.4. If we assume less regularity on the external force, for instance $\mathbf{f} \in L^2([0, T]; \mathbf{H}_{-1})$, the proof remains the same and we obtain the corresponding inequality

$$\frac{1}{2}\frac{d}{dt}\|A\mathbf{w}\|^2 + \nu\|\nabla A\mathbf{w}\|^2 \leqslant \langle \mathbf{f}, A\mathbf{w}\rangle,$$

where $\langle\cdot,\cdot\rangle$ denotes the duality pairing.

Proof of Proposition 4.1. The starting point is the energy equality originally labeled (3.29),

$$\frac{1}{2}\frac{d}{dt}\big\|A^{1/2}D_N^{1/2}(\mathbf{w}_N)\big\|^2 + \nu\big\|\nabla A^{1/2}D_N^{1/2}(\mathbf{w}_N)\big\|^2 = \big(A^{-1/2}D_N^{1/2}(\mathbf{f}), A^{1/2}D_N^{1/2}(\mathbf{w}_N)\big). \tag{4.16}$$

We integrate (4.16) on the time interval $[0, t]$ for any given t,

$$\frac{1}{2}\left\|A^{1/2}D_N^{1/2}(\mathbf{w}_N)(t)\right\|^2 + \nu\int_0^t \left\|\nabla A^{1/2}D_N^{1/2}(\mathbf{w}_N)(s)\right\|^2 ds$$
$$= \frac{1}{2}\left\|A^{1/2}D_N^{1/2}(\mathbf{w}_N)(0)\right\|^2 + \int_0^t \left(A^{-1/2}D_N^{1/2}(\mathbf{f}), A^{1/2}D_N^{1/2}(\mathbf{w}_N)\right) ds. \tag{4.17}$$

We must take the limit in (4.17) when $N \to \infty$, and we first focus to its r.h.s. We claim that, up to a sub-sequence,

$$\begin{aligned} &A^{1/2}D_N^{1/2}(\mathbf{w}_N) \to A(\mathbf{w}) \quad \text{weakly in } L^2\big([0,T];\mathbf{H}_1\big),\\ &A^{1/2}D_N^{1/2}(\mathbf{w}_N) \to A(\mathbf{w}) \quad \text{weakly}{*} \text{ in } L^\infty\big([0,T];\mathbf{H}_0\big). \end{aligned} \tag{4.18}$$

Thanks to the bound (3.17)(a), we can extract from $\{A^{1/2}D_N^{1/2}(\mathbf{w}_N)\}_{N\in\mathbb{N}}$ a sub-sequence (without changing the notation) such that

$$A^{1/2}D_N^{1/2}(\mathbf{w}_N) \to \mathbf{z} \quad \begin{cases} \text{weakly in } L^2\big([0,T];\mathbf{H}_1\big),\\ \text{weakly}{*} \text{ in } L^\infty\big([0,T];\mathbf{H}_0\big). \end{cases} \tag{4.19}$$

We must prove $\mathbf{z} = A(\mathbf{w})$. We already have proved that when $N \to \infty$ (see (4.9)–(4.11) above)

$$D_N(\mathbf{w}_N) \to A\mathbf{w} \quad \begin{cases} \text{strongly in } L^p\big([0,T]\times\mathbb{T}_3\big)^3, \quad \forall p < 10/3,\\ \text{weakly in } L^2\big([0,T];\mathbf{H}_0\big), \end{cases}$$

the same kind of proof applies to the sequence $\{D_N^{1/2}(\mathbf{w}_N)\}_{N\in\mathbb{N}}$ (we skip the details here), and we have at least

$$D_N^{1/2}(\mathbf{w}_N) \to A^{1/2}(\mathbf{w}) \quad \text{weakly in } L^2\big([0,T];\mathbf{H}_1\big) \text{ (and much better)}.$$

By the continuity of A and the uniqueness of the weak limit, we finally get $\mathbf{z} = A(\mathbf{w})$, hence (4.18).

Next, due to the assumptions on $\mathbf{f}$ it is easy proved using arguments already detailed before, that when $N \to \infty$,

$$A^{-1/2}D_N^{1/2}\mathbf{f} \to \mathbf{f} \quad \text{strongly in } L^2\big([0,T];\mathbf{H}_0\big).$$

In addition, since for all $N \in \mathbb{N}$, $\mathbf{w}_N(0) = \mathbf{w}(0) = \bar{\mathbf{u}}(0) \in \mathbf{H}_2$, we can take the limit in the r.h.s. of (4.17) when $N \to \infty$, and we get

$$\lim_{N\to+\infty}\left[\frac{1}{2}\left\|A^{1/2}D_N^{1/2}(\mathbf{w}_N)(0)\right\|^2 + \int_0^t \left(A^{-1/2}D_N^{1/2}(\mathbf{f}), A^{1/2}D_N^{1/2}(\mathbf{w}_N)\right) ds\right]$$
$$= \frac{1}{2}\left\|A\mathbf{w}(0)\right\|^2 + \int_0^t (\mathbf{f}, A\mathbf{w})\, ds.$$

The previous limit implies that the left-hand side of (4.17) is bounded uniformly in $N \in \mathbb{N}$, and the following inequality holds,

$$\limsup_{N\to+\infty}\left[\frac{1}{2}\left\|A^{1/2}D_N^{1/2}(\mathbf{w}_N)(t)\right\|^2 + \nu\int_0^t \left\|\nabla A^{1/2}D_N^{1/2}(\mathbf{w}_N)(s)\right\|^2 ds\right]$$
$$\leqslant \frac{1}{2}\left\|A\mathbf{w}(0)\right\|^2 + \int_0^t \big(\mathbf{f}(s), A\mathbf{w}(s)\big)\, ds.$$

Next, we use the elementary inequality for the real valued sequences $\{a_N\}_{n\in\mathbb{N}}$ and $\{b_N\}_{n\in\mathbb{N}}$

$$\limsup_{N\to+\infty} a_N + \liminf_{N\to+\infty} b_N \leqslant \limsup_{N\to+\infty}(a_N + b_N),$$

with

$$a_N := \frac{1}{2}\left\|A^{1/2}D_N^{1/2}(\mathbf{w}_N)(t)\right\|^2 \quad \text{and} \quad b_N = \nu\int_0^t \left\|\nabla A^{1/2}D_N^{1/2}(\mathbf{w}_N)(s)\right\|^2 ds.$$

(The inequality holds since we know in advance that the right-hand side is finite.) We infer that

$$\limsup_{N\to+\infty}\frac{1}{2}\left\|A^{1/2}D_N^{1/2}(\mathbf{w}_N)(t)\right\|^2 + \liminf_{N\to+\infty}\nu\int_0^t \left\|\nabla A^{1/2}D_N^{1/2}(\mathbf{w}_N)(s)\right\|^2 ds$$
$$\leqslant \frac{1}{2}\left\|A\mathbf{w}(0)\right\|^2 + \int_0^t \big(\mathbf{f}(s), A\mathbf{w}(s)\big)\, ds.$$

By lower semi-continuity of the norm this implies that

$$\int_0^t \left\|\nabla A\mathbf{w}(s)\right\|^2 ds \leqslant \liminf_{N\to+\infty}\int_0^t \left\|\nabla A^{1/2}D_N^{1/2}\mathbf{w}_N(s)\right\|^2 ds.$$

On the other hand, since $D^{1/2}\mathbf{w}_N \to A^{1/2}\mathbf{w}$ weakly$*$ in $L^\infty([0,T];\mathbf{H}_0)$ we get, again by identification of the weak limit,

$$\left\|A\mathbf{w}(t)\right\|^2 \leqslant \limsup_{N\to+\infty}\left\|A^{1/2}D_N^{1/2}\mathbf{w}_N(t)\right\|^2.$$

By collecting all the estimates, we have finally proved that, for all $t\in[0,T]$,

$$\frac{1}{2}\left\|A\mathbf{w}(t)\right\|^2 + \nu\int_0^t \left\|\nabla A\mathbf{w}(s)\right\|^2 ds \leqslant \frac{1}{2}\left\|A\mathbf{w}(0)\right\|^2 + \int_0^t \big(\mathbf{f}(s), A\mathbf{w}(s)\big)\, ds. \tag{4.20}$$

This can be read as the standard energy inequality for $\mathbf{u} = A\mathbf{w}$,

$$\frac{1}{2}\left\|\mathbf{u}(t)\right\|^2 + \nu\int_0^t \left\|\nabla\mathbf{u}(s)\right\|^2 ds \leqslant \frac{1}{2}\left\|\mathbf{u}(0)\right\|^2 + \int_0^t \big(\mathbf{f}(s), \mathbf{u}(s)\big)\, ds \quad \forall t\in[0,T],$$

which ends the proof of the energy inequality. □

5. Generalized Helmholtz filter

We aim in this section to study the case of "generalized Helmholtz filters". We call "generalized Helmholtz filter" a filter defined thanks to the equations

$$\begin{aligned} &-\alpha^{2p}\Delta^p\overline{\mathbf{w}} + \overline{\mathbf{w}} + \nabla\pi = \mathbf{w} \quad \text{in } \mathbb{T}_3, \\ &\nabla\cdot\overline{\mathbf{w}} = 0 \quad \text{in } \mathbb{T}_3, \end{aligned} \tag{5.1}$$

π having a zero mean on $\mathbb{T}_3$. The symbol of the operator Δ^p is $|\mathbf{k}|^{2p}$ and $\alpha > 0$ is still fixed. We introduce the deconvolution like model that corresponds to this filter (see (5.6) below) and we define the suitable notion of generalized weak solution.

We show that when $p > 3/4$ this model has a unique generalized regular weak solution that converges towards a solution of the mean Navier–Stokes equation. The analysis carried out in this section highlights a strong regularizing effect when $p > 1$.

Remark 5.1. The exponent "3/4" looks like a "critical exponent". We conjecture that we can get an existence and uniqueness result for lower exponents, but concerning the convergence towards the mean Navier–Stokes equations, we think that it is the best exponent, but this question remains an open one, although it remains an academic issue.

The plan of this section follows the previous scheme:

- Definition of the deconvolution operator of order N and main properties.
- Definition of a generalized regular weak solution.
- *A priori* estimates and existence result.
- Study of the convergence as N goes to infinity.

When arguments are similar to those for the case $p = 1$ already studied, we shall skip them to focus on essential features of this generalized situation.

5.1. The generalized deconvolution operator

We first notice that for any given $\mathbf{w} \in \mathbf{H}_s$ $(s > 0)$, $\mathbf{w} = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \widehat{\mathbf{w}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}$, then (5.1) has a unique solution $(\overline{\mathbf{w}}, 0)$, where $\overline{\mathbf{w}} \in \mathbf{H}_{s+2p}$ and

$$\overline{\mathbf{w}} = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \frac{1}{1+\alpha^{2p}|\mathbf{k}|^{2p}} \widehat{\mathbf{w}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{5.2}$$

We write $A_p(\overline{\mathbf{w}}) = \mathbf{w}$, that defines an isomorphism between $\mathbf{H}_{s+2p}$ and $\mathbf{H}_s$, and similarly $G_p = A_p^{-1}$. We still denote by A_p and G_p (written with overbars too) the same operator also acting on scalar and/or matrix fields. Let us formally define our deconvolution operator $D_{N,p}$ by

$$D_{N,p} := \sum_{n=0}^{N} (I - G_p)^n. \tag{5.3}$$

The symbol of $D_{N,p}$ is

$$\widehat{D_{N,p}}(\mathbf{k}) = \sum_{n=0}^{N} \left(\frac{\alpha^{2p}|\mathbf{k}|^{2p}}{1+\alpha^{2p}|\mathbf{k}|^{2p}} \right)^n = \left(1+\alpha^{2p}|\mathbf{k}|^{2s}\right)\rho_{N,p,\mathbf{k}}, \quad \rho_{N,p,\mathbf{k}} = 1 - \left(\frac{\alpha^{2p}|\mathbf{k}|^{2p}}{1+\alpha^{2p}|\mathbf{k}|^{2p}} \right)^{N+1}.$$

All operators A_p, G_p and $D_{N,p}$ are self-adjoint, commute with each other as well as with differential operators. They have a common basis of eigen-vectors. Moreover, the crucial analogous property to (2.7) holds,

$$\text{for each } \mathbf{k} \in \mathcal{T}_3 \text{ fixed} \quad \widehat{D}_{N,p}(\mathbf{k}) \to 1 + \alpha^{2p}|\mathbf{k}|^{2p} = \widehat{(A_p)}_{\mathbf{k}}, \quad \text{as } N \to +\infty. \tag{5.4}$$

Even if this convergence is not uniform in N, it is the property that makes possible the ADM (5.6) below to converge to the mean Navier–Stokes equations. Moreover, elementary calculus yields the same properties as when $p = 1$,

$$\begin{cases} 1 \leqslant \widehat{D_{N,p}}(\mathbf{k}) \leqslant N+1, & \forall \mathbf{k} \in \mathcal{T}_3, \\ \widehat{D_{N,p}}(\mathbf{k}) \approx (N+1)\dfrac{1+\alpha^{2p}|\mathbf{k}|^{2p}}{\alpha^{2p}|\mathbf{k}|^{2p}}, & \text{for large } |\mathbf{k}|, \\ \lim\limits_{|\mathbf{k}|\to+\infty} \widehat{D_{N,p}}(\mathbf{k}) = N+1, & \\ \widehat{D_{N,p}}(\mathbf{k}) \leqslant \left(1+\alpha^{2p}|\mathbf{k}|^{2p}\right), & \forall \mathbf{k} \in \mathcal{T}_3. \end{cases} \tag{5.5}$$

5.2. "Generalized regular weak solution"

The problem we consider is the problem

$$\begin{aligned} &\partial_t \mathbf{w} + \nabla \cdot \left(\overline{D_{N,p}(\mathbf{w}) \otimes D_{N,p}(\mathbf{w})}\right) - \nu\Delta\mathbf{w} + \nabla q = \bar{\mathbf{f}}, \\ &\nabla \cdot \mathbf{w} = 0, \\ &\mathbf{w}(0,\mathbf{x}) = \overline{\mathbf{u}_0}(\mathbf{x}), \end{aligned} \tag{5.6}$$

where here $\bar{\mathbf{F}} = A_p^{-1}(\mathbf{F})$ for any field $\mathbf{F}$.

Definition 5.1 *("Generalized regular weak" solution).* We say that the couple $(\mathbf{w}, q)$ is a "regular weak" solution to system (5.6) if and only if the three following items are satisfied:

1) *Regularity*

$$\mathbf{w} \in L^2\big([0,T]; \mathbf{H}_{1+p}\big) \cap C\big([0,T]; \mathbf{H}_p\big), \tag{5.7}$$

$$\partial_t \mathbf{w} \in L^2\big([0,T]; \mathbf{H}_0\big), \tag{5.8}$$

$$q \in L^2\big([0,T]; H^1(\mathbb{T}_3)\big). \tag{5.9}$$

2) *Initial data*

$$\lim_{t\to 0} \big\| \mathbf{w}(t,\cdot) - \overline{\mathbf{u}_0} \big\|_{\mathbf{H}_p} = 0. \tag{5.10}$$

3) *Weak formulation*

$$\forall \mathbf{v} \in L^2\big([0,T]; H^1(\mathbb{T}_3)^3\big), \tag{5.11}$$

$$\int_0^T\!\!\int_{\mathbb{T}_3} \partial_t \mathbf{w}\cdot\mathbf{v} - \int_0^T\!\!\int_{\mathbb{T}_3} \overline{D_{N,p}(\mathbf{w}) \otimes D_{N,p}(\mathbf{w})} : \nabla\mathbf{v} + \nu \int_0^T\!\!\int_{\mathbb{T}_3} \nabla\mathbf{w} : \nabla\mathbf{v} + \int_0^T\!\!\int_{\mathbb{T}_3} \nabla q\cdot\mathbf{v} = \int_0^T\!\!\int_{\mathbb{T}_3} \bar{\mathbf{f}}\cdot\mathbf{v}. \tag{5.12}$$

For simplicity, we still assume that (3.1) holds, that means $\mathbf{u}_0 \in \mathbf{H}_0$, $\mathbf{f} \in L^2([0,T]\times\mathbb{T}_3)$. Similar results to those in the case $p = 1$ hold, that we state below.

Theorem 5.1. *Assume that* (3.1) *holds, $\alpha > 0$ and $N \in \mathbb{N}$ are given and fixed. Assume in addition that $p > 3/4$. Then problem* (5.6) *has a unique generalized regular weak solution. Moreover, when $p \geqslant 1$,*

$$\partial_t \mathbf{w} \in L^2\big([0,T], \mathbf{H}_{p-1}\big), \qquad q \in L^2\big([0,T], H^p(\mathbb{T}_3)\big). \tag{5.13}$$

We denote by $(\mathbf{w}_{N,p}, q_{N,p})$ the regular weak solution to problem (5.6).

Theorem 5.2. *From the sequence $\{(\mathbf{w}_{N,p} q_{N,p})\}_{N\in\mathbb{N}}$ one can extract a sub-sequence (still denoted $\{(\mathbf{w}_{N,p} q_{N,p})\}_{N\in\mathbb{N}}$) such that*

$$\mathbf{w}_{N,p} \to \mathbf{w} \quad \begin{cases} \textit{weakly in } L^2\big([0,T]; H^{1+p}(\mathbb{T}_3)^3\big) \cap L^\infty\big([0,T]; H^p(\mathbb{T}_3)^3\big), \\ \textit{strongly in } L^r\big([0,T]; H^p(\mathbb{T}_3)^3\big), \quad \forall 1 \leqslant r < +\infty, \end{cases}$$

$$q_{N,p} \to q \qquad \textit{weakly in } L^2\big([0,T]; H^1(\mathbb{T}_3)\big) \cap L^{5/3}\big([0,T]; W^{2p,5/3}(\mathbb{T}_3)\big), \tag{5.14}$$

and such that the system

$$\begin{aligned} &\partial_t \mathbf{w} + \nabla\cdot(\overline{A\mathbf{w}\otimes A\mathbf{w}}) - \nu\Delta\mathbf{w} + \nabla q = \bar{\mathbf{f}}, \\ &\nabla\cdot\mathbf{w} = 0, \\ &\mathbf{w}(0,\mathbf{x}) = \overline{\mathbf{u}_0}(\mathbf{x}) \end{aligned} \tag{5.15}$$

holds in the sense of the distributions.

The construction of approximations thanks to the Galerkin method is the same as the one we detailed before in the case $p = 1$, as well as the tools to take the limit when N goes to infinity, as long as we have estimates uniform in N. Therefore, we restrict the following display to show how to get *a priori* estimates uniform in N when things are really new with respect to the case $p = 1$. Especially we explain why $3/4$ is a critical exponent here, and we highlight the distinction between the cases $3/4 < p \leqslant 1$ and $1 < p$, especially in terms of regularity and for proving uniqueness.

5.3. Estimates and sketch of proof

We formally take $A_p D_{N,p}\mathbf{w}$ as test function in (5.12), to get the following energy equality:

$$\frac{1}{2}\frac{d}{dt}\big\|A_p^{\frac{1}{2}} D_{N,p}^{\frac{1}{2}}(\mathbf{w})\big\|^2 + \big\|\nabla A_p^{\frac{1}{2}} D_{N,p}^{\frac{1}{2}}(\mathbf{w})\big\|^2 = \big(A_p^{\frac{1}{2}} D_{N,p}^{\frac{1}{2}}\mathbf{f}, A_p^{\frac{1}{2}} D_{N,p}^{\frac{1}{2}}(\mathbf{w})\big).$$

Therefore using the properties (5.5) and the same computations as for table (3.17), we get the following estimates:

Label	Variable	Bound	Order
(a)	$A_p^{\frac{1}{2}} D_{N,p}^{\frac{1}{2}}(\mathbf{w})$	$L^\infty([0,T];\mathbf{H}_0)\cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
(b)	$D_{N,p}^{1/2}(\mathbf{w})$	$L^\infty([0,T];\mathbf{H}_0)\cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
(c)	$D_{N,p}^{1/2}(\mathbf{w})$	$L^\infty([0,T];\mathbf{H}_p)\cap L^2([0,T];\mathbf{H}_{1+p})$	$O(\alpha^{-p})$
(d)	$\mathbf{w}$	$L^\infty([0,T];\mathbf{H}_0)\cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
(e)	$\mathbf{w}$	$L^\infty([0,T];\mathbf{H}_p)\cap L^2([0,T];\mathbf{H}_{1+p})$	$O(\alpha^{-p})$
(f)	$D_{N,p}(\mathbf{w})$	$L^\infty([0,T];\mathbf{H}_0)\cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
(g)	$D_{N,p}(\mathbf{w})$	$L^\infty([0,T];\mathbf{H}_p)\cap L^2([0,T];\mathbf{H}_{p+1})$	$O(\alpha^{-p}\cdot(N+1)^{1/2})$
(h)	$\partial_t\mathbf{w}$	$L^2([0,T];\mathbf{H}_0)$, for $p>\frac{3}{4}$	$O(\alpha^{-p})$

(5.16)

The only real novelty in table (5.16) (in comparison with the case $p=1$ and table (3.17)), is (5.16)(h), where the critical exponent $3/4$ is involved. Therefore, we shall be satisfied with only checking this estimate.

To get an estimate for $\partial_t\mathbf{w}$, uniform in N, it is enough to check that

$$\overline{\nabla\cdot\big(D_{N,p}(\mathbf{w})\otimes D_{N,p}(\mathbf{w})\big)} \in L^2\big([0,T]\times\mathbb{T}_3\big)^3, \tag{5.17}$$

the bound being uniform in N. Roughly speaking, $(\nabla\cdot)\circ G_p$ allows us to "gain" $2p-1$ derivatives. Therefore, the first observation is that p must be larger than $1/2$ if we want a regularization effect. Recall that by (5.16)(f), $D_{N,p}(\mathbf{w})\otimes D_{N,p}(\mathbf{w})\in L^2([0,T],L^{3/2}(\mathbb{T}_3)^9)$. Therefore

$$\overline{\nabla\cdot\big(D_{N,p}(\mathbf{w})\otimes D_{N,p}(\mathbf{w})\big)} = \big[(\nabla\cdot)\circ G_p\big]\big(D_{N,p}(\mathbf{w})\otimes D_{N,p}(\mathbf{w})\big)\in L^2\big([0,T],W^{2p-1,3/2}(\mathbb{T}_3)^3\big),$$

and we must fix p such that $W^{2p-1,3/2}(\mathbb{T}_3)\subset L^2(\mathbb{T}_3)$. Using the Sobolev embedding theorem, we know that it holds if and only if $p\geqslant 3/4$. As we lose the compactness property of the injection in the case $p=3/4$, we must retain exponents p such that $p>3/4$.

Following the same process, we easily get the following second set of estimates

Label	Variable	Bound	Order
(a)	$\mathbf{w}$	$L^\infty([0,T];\mathbf{H}_0)\cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
(b)	$\mathbf{w}$	$L^\infty([0,T];\mathbf{H}_p)\cap L^2([0,T];\mathbf{H}_{1+p})$	$O(\alpha^{-p})$
(c)	$D_{N,p}(\mathbf{w})$	$L^\infty([0,T];\mathbf{H}_0)\cap L^2([0,T];\mathbf{H}_1)$	$O(1)$
(d)	$\partial_t\mathbf{w}$	$L^2([0,T]\times\mathbb{T}_3)^3$	$O(\alpha^{-p})$
(e)	q	$L^2([0,T];H^1(\mathbb{T}_3))\cap L^{5/3}([0,T];W^{2p,5/3}(\mathbb{T}_3))$	$O(\alpha^{-p})$
(f)	$\partial_t D_{N,p}(\mathbf{w})$	$L^{4/3}([0,T];\mathbf{H}_{-1})$	$O(1)$

(5.18)

The existence of a regular weak solution follows as in Section 3.2, without significant changes. For proving uniqueness, we use the method of Section 3.2, that consists in taking $A_p D_{N,p}\mathbf{W}$ as test function in the equation, where $\mathbf{W}=\mathbf{w}_1-\mathbf{w}_2$ and $\mathbf{w}_1$, $\mathbf{w}_2$ are two generalized weak solutions. We must justify that this operation is possible. To do so, we must distinguish the cases $3/4<p\leqslant 1$ and $1<p$.

Assume first $3/4<p\leqslant 1$. Since $\mathbf{W}\in L^2([0,T],\mathbf{H}_{1+p})$, then $A_p D_{N,p}\mathbf{W}\in L^2([0,T],\mathbf{H}_{1-p})$. In this case, $A_p D_{N,p}(\mathbf{W})\in L^2([0,T]\times\mathbb{T}_3)^3$ and uniqueness can be proven as in Section 3.2.

Assume now $1<p$. The estimate $A_p D_{N,p}\mathbf{W}\in L^2([0,T],\mathbf{H}_{1-p})$ does not allow to take $A_p D_{N,p}\mathbf{W}$ as test function in the equation that it satisfies, unless we prove that all terms in the equation are at least in the space $L^2([0,T],\mathbf{H}_{p-1})$,

so that the integrals in the variational formulation (5.12) become duality pairings. We then need to get more regularity about any solution $\mathbf{w}$.

Note that since $\mathbf{w} \in L^2([0,T], \mathbf{H}_{1+p})$, then $\Delta\mathbf{w} \in L^2([0,T], \mathbf{H}_{p-1})$. Let us focus on the nonlinear term. We already know from (5.18) that

$$D_{N,p}(\mathbf{w}) \otimes D_{N,p}(\mathbf{w}) \in L^2\big([0,T], L^{3/2}(\mathbb{T}_3)^9\big). \tag{5.19}$$

Therefore, we have

$$\nabla \cdot \overline{D_{N,p}(\mathbf{w}) \otimes D_{N,p}(\mathbf{w})} \in L^2\big([0,T]; W^{2p-1,3/2}(\mathbb{T}_3)^3\big) \hookrightarrow L^2\big([0,T], \mathbf{H}_{p-1}\big). \tag{5.20}$$

We conclude that $\partial_t \mathbf{w} \in L^2([0,T], \mathbf{H}_{p-1})$ thanks to (5.11)–(5.12) and the density of $L^2([0,T], \mathbf{H}_1)$ in $L^2([0,T], \mathbf{H}_{p-1})$. We get (5.13) from the De Rham Theorem, as well as uniqueness that can be proved by using classical arguments.

We can now take the limit in (5.6) when N goes to infinity as we already did for the case $p = 1$ in the previous part, without any change. Note that the analogous convergence property as (4.12) holds, which allows us to identify the limit and conclude by proving

$$\forall \mathbf{v} \in L^2\big([0,T], \mathbf{H}_{2p}\big), \quad D_{N,p}(\mathbf{v}) \to A_p\mathbf{v} \quad \text{strongly in } L^2\big([0,T] \times \mathbb{T}_3\big)^3, \tag{5.21}$$

which is one of the main ingredients of the proof.

6. Ultimate generalization and conclusions

We finish the paper by a series of remarks about generalized convolution filters that take inspiration from the previous one and for which it is possible to take the limit in the corresponding ADM, when N goes to infinity. Then we consider the well-known Fejér's filter. As we shall see, we are not able to check if the corresponding ADM converge or not toward a mean of the Navier–Stokes.

6.1. Generalized convolution filter

All filters we have considered above can be written as

$$\overline{\mathbf{w}} = G \star \mathbf{w} = G(\mathbf{w}),$$

with

$$G = \sum_{\mathbf{k} \in \mathcal{T}_3^\star} \widehat{G}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{6.1}$$

We also write $\widehat{A}_{\mathbf{k}} = \widehat{G}_{\mathbf{k}}^{-1}$. What we did before suggests to ask the $\widehat{G}_{\mathbf{k}}$'s to satisfy the following inequalities

$$\forall \mathbf{k} \in \mathcal{T}_3^\star, \quad \frac{C_1}{1 + \alpha^{2q}|\mathbf{k}|^{2q}} \leqslant \widehat{G}_{\mathbf{k}} \leqslant \frac{C_2}{1 + \alpha^{2p}|\mathbf{k}|^{2p}}, \tag{6.2}$$

where $C_1 > 0$ and $C_2 > 0$, $p > 0$, and in addition $\widehat{G}_{\mathbf{0}} = 0$. The symbol of the corresponding deconvolution operator D_N is then given by

$$\widehat{D}_N(\mathbf{k}) = \sum_{n=0}^{N} (1 - \widehat{G}_{\mathbf{k}})^n = \frac{1 - (1 - \widehat{G}_{\mathbf{k}})^{N+1}}{\widehat{G}_{\mathbf{k}}}. \tag{6.3}$$

According to the strategy above, we have to check

$$\forall \mathbf{k} \in \mathcal{T}_3^\star, \quad \lim_{N \to \infty} \widehat{D}_N(\mathbf{k}) = \widehat{A}_{\mathbf{k}}. \tag{6.4}$$

This is satisfied as long as that $\forall \mathbf{k} \in \mathcal{T}_3^\star$, $0 < \widehat{G}_{\mathbf{k}} < 1$, which is true when the constant C_2 is such that

$$C_2 < 1 + \alpha^{2p}\left(\frac{2\pi}{L}\right)^{2p}.$$

Elementary calculus yields

$$\begin{aligned}
&\forall \mathbf{k} \in \mathcal{T}_3^\star, \quad 1 \leqslant \widehat{D}_N(\mathbf{k}) \leqslant N+1,\\
&\forall \mathbf{k} \in \mathcal{T}_3^\star, \quad \widehat{D}_N(\mathbf{k}) \leqslant \widehat{A}_{\mathbf{k}},\\
&\forall \mathbf{k} \in \mathcal{T}_3^\star, \quad \widehat{D}_N(\mathbf{k}) \leqslant \frac{1+\alpha^{2q}|\mathbf{k}|^{2q}}{C_1},\\
&\lim_{|\mathbf{k}|\to\infty} \widehat{D}_N(\mathbf{k}) = N+1.
\end{aligned} \tag{6.5}$$

We may then introduce the model corresponding to (5.6). The definition of regular weak solution is similar to Definition 5.1, where the exponent p is the one that is involved in the upper bound in (6.2) above.

The same kind of proof as we did before, yields the existence and uniqueness of a "regular weak solution" to the model when $p > 3/4$, that converges to a solution of the mean Navier–Stokes equations. We stress that when $p > 1$, we gain more regularity, which is what it is expected.

Notice that the lower bound in (6.2) is necessary. Indeed, it indicates that the operator A maps $\mathbf{H}_{2q}$ onto $\mathbf{H}_0$, and so does D_N thanks to the third inequality in (6.5). This is useful to prove the analogous convergence property to (4.12) (see also (5.21)), that allows to identify the limit and conclude by proving

$$\forall \mathbf{v} \in L^2\big([0,T], \mathbf{H}_{2q}\big), \quad D_N(\mathbf{v}) \to A\mathbf{v} \quad \text{strongly in } L^2\big([0,T]\times\mathbb{T}_3\big)^3, \tag{6.6}$$

which is one of the main ingredients of the proof as we already said.

6.2. Remarks about the Fejér filter

Fejér's kernel is one of the main popular convolution kernels in the topics of periodic fields. It is used to approach periodic fields by trigonometric polynomials, yielding elementary proofs of the Stone–Weierstrass theorem and many other theorems related to Fourier series, since it appears when one writes Cesaro's means. It is given by the formula

$$F(\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{T}_3,\ |\mathbf{k}|\leqslant J} \left(1 - \frac{|k|}{J+1}\right) e^{i\mathbf{k}\cdot\mathbf{x}}, \tag{6.7}$$

for some given cut-off number $J > 0$. It seems natural to choose $J \in \mathbb{N}$ such that $J = O(1/\alpha)$, according to the α scale.

We define the regularized field thanks to the usual convolution,

$$\overline{\mathbf{w}} = F \star \mathbf{w}.$$

When we write this in Fourier series, we get

$$\mathbf{w}(\mathbf{x}) = \sum_{\mathbf{k}\in\mathcal{T}_3^\star} \widehat{\mathbf{w}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \qquad \overline{\mathbf{w}} = F(\mathbf{w}) = \sum_{\mathbf{k}\in\mathcal{T}_3^\star,\ |k|\leqslant J} \left(1 - \frac{|\mathbf{k}|}{J+1}\right) \widehat{\mathbf{w}}_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}. \tag{6.8}$$

It is easy checked that F maps $\mathbf{H}_s$ onto $\mathbf{H}_{s+2}$ and that one has

$$\|\overline{\mathbf{w}}\|_{s+2} = O\left(\frac{1}{\alpha^2}\right)\|\mathbf{w}\|_s.$$

Let D_N^F be the corresponding deconvolution operator:

$$D_N^F = \sum_{n=0}^N (\mathbf{I} - F)^n,$$

and consider the corresponding ADM

$$\begin{aligned}
&\partial_t \mathbf{w} + \nabla\cdot\big(\overline{D_N^F(\mathbf{w})\otimes D_N^F(\mathbf{w})}\big) - \nu\Delta\mathbf{w} + \nabla q = \bar{\mathbf{f}},\\
&\nabla\cdot\mathbf{w} = 0,\\
&\mathbf{w}(0,\mathbf{x}) = \overline{\mathbf{u}_0}(\mathbf{x}).
\end{aligned} \tag{6.9}$$

Then it is easy checked

$$\widehat{D_N^F}(\mathbf{k}) = \left(1 - \frac{|\mathbf{k}|}{J+1}\right)^{-1}\left(1 - \left(\frac{|\mathbf{k}|}{J+1}\right)^{N+1}\right) \quad \text{if } |\mathbf{k}| \leqslant J,$$
$$\widehat{D_N^F}(\mathbf{k}) = N+1 \quad \text{if } |\mathbf{k}| > J. \tag{6.10}$$

Of course, one might prove existence and uniqueness of some kind of regular weak solution to (6.8), say $(\mathbf{w}_N, q_N)$, that satisfies many estimates uniform in N.

Unfortunately, no property such as (2.7) and (5.4) holds: $(\widehat{D_N^F}(\mathbf{k}))_{N\in N}$ does not converge towards $\widehat{F^{-1}}(\mathbf{k})$. So the property "$D_N^F \to F^{-1}$" does not hold, even formally. Therefore we cannot use the method we developed in the paper to show that the ADM (6.8) converges toward the mean Navier–Stokes equations since we are not able to identify the limit of $D_N^F(\mathbf{w}_N) \otimes D_N^F(\mathbf{w}_N)$. This opens an exciting open problem with which we conclude the paper.

Acknowledgements

The work of Roger Lewandowski was partially supported by the ANR project 08FA300-01. Roger Lewandowski warmly thanks the Department of Applied Mathematics of the University of Pisa for the hospitality, where part of the work was done. He also thanks his colleagues Christophe Cheverry and Albert Cohen for interesting discussion about this paper. Luigi C. Berselli gratefully acknowledges the hospitality of IRMAR, Université Rennes 1, where part of the work was done. Both authors thank Matthew Paddick who accepted to read the paper and review the language, as well as the anonymous referee, whose remarks were helpful in improving this work. They also thank Argus A. Dunca for valuable comments about the proof of Theorem 5.1.

References

[1] N.A. Adams, S. Stolz, Deconvolution methods for subgrid-scale approximation in large eddy simulation, in: Modern Simulation Strategies for Turbulent Flow, R.T. Edwards, 2001.
[2] L.C. Berselli, T. Iliescu, W.J. Layton, Mathematics of Large Eddy Simulation of Turbulent Flows, Scientific Computation, Springer-Verlag, Berlin, 2006.
[3] M. Bertero, B. Boccacci, Introduction to Inverse Problems in Imaging, IOP Publishing Ltd, 1998.
[4] A. Bhattacharya, A. Das, D. Moser, A filtered-wall formulation for large-eddy simulation of wall-bounded turbulence, Phys. Fluids 20 (2008) 115104.
[5] A. Cheskidov, D.D. Holm, E. Olson, E.S. Titi, On a Leray-α model of turbulence, Proc. R. Soc. Lond. Ser. A Math. Phys. Eng. Sci. 461 (2005) 629–649.
[6] P. Constantin, C. Foias, Navier–Stokes Equations, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1988.
[7] A. Das, D. Moser, Filtering boundary conditions for les and embedded boundary conditions, in: C. Liu, L. Sakell, T. Beutner (Eds.), Proceedings of the Third International Conference on DNS-LES, Greyden Press, Columbus, 2001.
[8] C.R. Doering, J.D. Gibbon, Applied Analysis of the Navier–Stokes Equations, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 1995.
[9] A. Dunca, Y. Epshteyn, On the Stolz–Adams deconvolution model for the large-eddy simulation of turbulent flows, SIAM J. Math. Anal. 37 (6) (2006) 1890–1902 (electronic).
[10] C. Foias, D.D. Holm, E.S. Titi, The Navier–Stokes-alpha model of fluid turbulence, Physica D 152 (2001) 505–519.
[11] B.J. Geurts, Elements of Direct and Large Eddy Simulation, Edwards Publishing, Flourtown, PA, 2003.
[12] O.A. Ladyzhenskaya, The Mathematical Theory of Viscous Incompressible Flow, Mathematics and Its Applications, vol. 2, Gordon and Breach Science Publishers, New York, 1969.
[13] W. Layton, R. Lewandowski, A simple and stable scale similarity model for large eddy simulation: energy balance and existence of weak solutions, Appl. Math. Lett. 16 (2003) 1205–1209.
[14] W.J. Layton, R. Lewandowski, Analysis of an eddy viscosity model for large eddy simulation of turbulent flows, J. Math. Fluid Mech. 4 (4) (2002) 374–399.
[15] W.J. Layton, R. Lewandowski, On a well-posed turbulence model, Discrete Contin. Dyn. Syst. Ser. B 6 (1) (2006) 111–128 (electronic).
[16] W.J. Layton, R. Lewandowski, Residual stress of approximate deconvolution models of turbulence, J. Turbul. 7 (2006) 1–21.
[17] K.L. Lele, Compact finite different schemes with spectral-like resolution, J. Comput. Phys. 103 (1992) 16–42.
[18] J. Leray, Sur les mouvements d'une liquide visqueux emplissant l'espace, Acta Math. 63 (1934) 193–248.
[19] M. Lesieur, O. Métais, P. Comte, Large-Eddy Simulations of Turbulence, Cambridge University Press, New York, 2005, with a preface by James J. Riley.
[20] R. Lewandowski, On a continuous deconvolution equation for turbulence models, Lecture Notes of Neças Center for Mathematical Modeling 5 (2009) 62–102.

[21] R. Lewandowski, Approximations to the Navier–Stokes Equations, in preparation, release scheduled end 2012.
[22] J.-L. Lions, Quelques méthodes de résolution des problèmes aux limites non linéaires, Dunod, 1969.
[23] D. Moser, N. Malaya, H. Chang, P. Zandonade, P. Veluda, A. Bhattacharya, A. Haselbacher, Theoretically based optimal large-eddy simulation, Phys. Fluids 21 (2009) 105104.
[24] F. Murat, Compacité par compensation, Ann. Sc. Norm. Super. Pisa Cl. Sci. 4 (5) (1978) 489–507.
[25] F. Murat, A survey on compensated compactness, in: L. Cesari (Ed.), Contributions to Modern Calculus of Variations, in: Pitman Research Notes in Math., vol. 148, Longman, Harlow, 1987, pp. 145–183.
[26] U. Piomelli, E. Balaras, Wall-layer models for large-eddy simulations, Annu. Rev. Fluid Mech. 34 (2002) 349–374.
[27] P. Sagaut, Large Eddy Simulation for Incompressible Flows — An Introduction, Scientific Computation, Springer-Verlag, Berlin, 2001, with an introduction by Marcel Lesieur, translated from the 1998 French original by the author.
[28] S. Stolz, N.A. Adams, An approximate deconvolution procedure for large-eddy simulation, Phys. Fluids 11 (7) (1999) 1699–1701.
[29] S. Stolz, N.A. Adams, L. Kleiser, An approximate deconvolution model for large-eddy simulation with application to incompressible wall-bounded flows, Phys. Fluids 13 (4) (2001) 997–1015.
[30] L. Tartar, Nonlinear partial differential equations using compactness method, Report 1584, Mathematics Research Center, University of Wisconsin, Madison, 1975.
[31] L. Tartar, H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations, Proc. Roy. Soc. Edinburgh Sect. A 115 (3–4) (1990) 193–230.
[32] L. Tartar, Lecture Notes of the Unione Matematica Italiana, 7, Springer-Verlag/UMI, Berlin/Bologna, 2009.
[33] R. Temam, Navier–Stokes Equations: Theory and Numerical Analysis, AMS Chelsea Publishing, Providence, RI, 2001, reprint of the 1984 edition.