

# Isoperimetric inequalities for the first Neumann eigenvalue in Gauss space

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## Abstract

We provide isoperimetric Szegő–Weinberger type inequalities for the first nontrivial Neumann eigenvalue  $\mu_1(\Omega)$  in Gauss space, where  $\Omega$  is a possibly unbounded domain of  $\mathbb{R}^N$ . Our main result consists in showing that among all sets  $\Omega$  of  $\mathbb{R}^N$  symmetric about the origin, having prescribed Gaussian measure,  $\mu_1(\Omega)$  is maximum if and only if  $\Omega$  is the Euclidean ball centered at the origin.

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## 1. Introduction

Let us consider the classical eigenvalue problem for the free membrane

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.1)$$

where  $\Omega$  is a smooth connected subset of  $\mathbb{R}^N$  and  $\nu$  is the outward normal to  $\partial\Omega$ .

In [26] Kornhauser and Stakgold conjectured that among all planar simply connected domains, with fixed measure,  $\mu_1(\Omega)$ , the first nontrivial eigenvalue of (1.1), achieves its maximum value if and only if  $\Omega$  is a disk.

This conjecture was proved by Szegő in [33], by means of tools from complex analysis, in particular he used the invariance of Dirichlet integrals under conformal transplantation.

Soon after (see [30]) Weinberger generalized this result to any bounded smooth domain  $\Omega$  of  $\mathbb{R}^N$ . Weinberger obtained from the eigenfunctions of the unit ball of  $\mathbb{R}^N$ ,  $B_1$ , test functions admissible in the variational characterization

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of  $\mu_1(\Omega)$ . His idea was to extend radially such eigenfunctions in  $\mathbb{R}^N$ , just setting their value constant outside  $B_1$ . Via the so-called “center of mass” arguments, he obtained  $N$  different functions having mean value zero on  $\Omega$ . At this point he is allowed to use all these functions as trial functions for  $\mu_1(\Omega)$  and the result is finally achieved by symmetrization arguments.

This last method turned out to be rather flexible. We recall indeed that, adapting Weinberger arguments, similar inequalities for spaces of constant sectional curvature are derived. For instance in [3], see also [11], it is shown that if  $\Omega$  is a domain of  $\mathbb{S}^N$ , contained in a hemisphere, then

$$\mu_1(\Omega) \leq \mu_1(\Omega^\sharp),$$

where  $\Omega^\sharp$  is the cap (i.e. the geodesic ball in  $\mathbb{S}^N$ ) having the same measure as  $\Omega$ .

On the other hand in [27] it is proved that the first nonzero Neumann eigenvalue is maximal for the equilateral triangle among all triangles of given perimeter, and hence among all triangles of given area.

For further references see, e.g., the monographs [4,11,25] and the survey paper [2].

The present paper deals with the Neumann eigenvalue problem in Gauss space. More precisely we study the problem

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i} \left( \varphi_N(x) \frac{\partial u}{\partial x_i} \right) = \mu \varphi_N(x) u & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \quad (1.2)$$

where  $\Omega$  is a Lipschitz domain of  $\mathbb{R}^N$ ,  $N \geq 1$  and  $\varphi_N(x) = (2\pi)^{-\frac{N}{2}} e^{-\frac{|x|^2}{2}}$  is the density of normalized  $N$ -dimensional Gaussian measure  $d\gamma_N = \varphi_N(x) dx$ .

Since the first half of the last century problems of the type (1.2) have attracted attention among both pure mathematicians and physicists. There is indeed a tight connection between the eigenvalues of (1.2) and the energy levels of the  $N$ -dimensional quantum harmonic oscillator. Related references are the classical Courant–Hilbert monographs [16] (see also [20]). On the other hand the interest in probability is motivated, for instance, from the fact that the differential operator  $L = -\Delta + x \cdot \nabla$  appearing at the left-hand side of (1.2) is the generator of the Ornstein–Uhlenbeck semigroup, see, e.g., [8] and the references therein. Finally problems of the type (1.2) are related to some functional inequalities as the well-known Gross’s Theorem on the Sobolev Logarithmic embedding (see [21,1,29,15,19,7]).

If  $\Omega$  is the whole space  $\mathbb{R}^N$  the eigenfunctions of (1.2) are the Hermite polynomials. If instead  $\Omega \subsetneq \mathbb{R}^N$  then sharp estimates for the eigenvalues and eigenfunctions of (1.2) with zero boundary conditions are contained, e.g., in [17,6,5].

Our aim is to prove isoperimetric Szegő–Weinberger type inequalities for the first eigenvalue of (1.2) that, with an abuse of notation, we still denote by  $\mu_1(\Omega)$ .

In this setting it appears natural to maximize  $\mu_1(\Omega)$  keeping fixed the Gaussian measure of  $\Omega$ . We recall that the Gaussian measure in  $\mathbb{R}^N$  can be obtained as a limit, as  $k$  goes to infinity, of normalized surface measures on  $\mathbb{S}^{\frac{k+N+1}{\sqrt{k}}}$ , the sphere in  $\mathbb{R}^{k+N+2}$  of radius  $\sqrt{k}$ , (a process known in literature as “Poincaré limit”). Using this limit process many properties for the Gauss space (i.e.  $\mathbb{R}^N$  equipped with the measure  $d\gamma_N$ ) can be deduced from analogous properties which hold true for the sphere. One of the most remarkable example is the Gaussian isoperimetric inequality, which asserts that among all subsets  $G$  of  $\mathbb{R}^N$  with fixed Gaussian measure, the half-spaces achieve the smallest Gaussian perimeter (see, e.g., [9,32,10]). We recall indeed, see, e.g., [18], that the half-spaces are the “Poincaré limit” of the caps, which are in turn the optimal sets in the isoperimetric problem on the sphere. Another example is the Faber–Krahn type inequality for Gauss space: the first Dirichlet Gaussian eigenvalue is minimum on the half-space (see, e.g., [18] and [6]).

There are therefore two clues that might lead one to think that the half-space would be a good candidate to maximize  $\mu_1$ . One reason is that the caps maximize the first Neumann eigenvalue on the sphere, the other is that in all the classical situations, described before, it is always the isoperimetric set to maximize  $\mu_1$ .

This phenomenon here does not occur.

In one dimension we provide a detailed description of the behavior of  $\mu_1$ . Let  $\Omega = (a, b)$  with  $-\infty \leq a < b \leq +\infty$  and  $\gamma_1(a, b) = L \in (0, 1)$ . We prove that  $\mu_1(a, b)$  is minimum when the interval reduces to a half-line, it is maximum

when it is centered at the origin and finally  $\mu_1(a, b)$  is strictly monotone as  $(a, b)$  slides between these extreme positions. Therefore the set which gives the highest eigenvalue is the one which maximizes the weighted perimeter and vice versa.

Our main result, which goes in the same direction of the previous one, concerns the  $N$ -dimensional case. We show that among all connected and possibly unbounded domain  $\Omega$  of  $\mathbb{R}^N$ , symmetric about the origin and with fixed Gaussian measure,  $\mu_1(\Omega)$  achieves its maximum value if and only if  $\Omega$  is the Euclidean ball.

Since, obviously, the half-spaces are not symmetric about the origin, the above result cannot exclude the possibility that the half-spaces maximize  $\mu_1(\Omega)$  in dimension greater than one. We are able to exclude this possibility providing a suitable counterexample, see Remark 4.3.

We finally note that, as well known, a distinctive feature of Gauss measure is that its density has both radial symmetry and product structure. Hence in the problem under consideration the former feature prevails on the latter.

## 2. Notation and preliminary results

Here and in the sequel  $\Omega$  will denote a connected, smooth, open subset of  $\mathbb{R}^N$  such that  $\gamma_N(\Omega) := \int_{\Omega} d\gamma_N < 1$ . The natural functional space associated to problem (1.2) is  $H^1(\Omega, \gamma_N)$  which is the weighted Sobolev space defined as follows

$$H^1(\Omega, \gamma_N) = \{u \in W_{loc}^{1,1}(\Omega) : (u, |Du|) \in L^2(\Omega, \gamma_N) \times L^2(\Omega, \gamma_N)\},$$

endowed with the norm

$$\|u\|_{H^1(\Omega, \gamma_N)} = \|u\|_{L^2(\Omega, \gamma_N)} + \|Du\|_{L^2(\Omega, \gamma_N)} = \left( \int_{\Omega} u^2 d\gamma_N \right)^{\frac{1}{2}} + \left( \int_{\Omega} |Du|^2 d\gamma_N \right)^{\frac{1}{2}}. \tag{2.1}$$

In [19], among other things, it is proved that the subspace of  $H^1(\Omega, \gamma_N)$  made of those functions having mean value zero in  $\Omega$  is compactly embedded in  $L^2(\Omega, \gamma_N)$ . This circumstance allows us to use standard spectral theory for self-adjoint compact operator. In particular the variational characterization of  $\mu_1(\Omega)$  will be used throughout

$$\mu_1(\Omega) = \min_{\substack{v \neq 0 \\ \int_{\Omega} v d\gamma_N = 0}} \frac{\int_{\Omega} |Dv|^2 d\gamma_N}{\int_{\Omega} v^2 d\gamma_N}. \tag{2.2}$$

We recall, see, e.g., [16], that when  $\Omega = \mathbb{R}^N$  the eigenfunctions to problem (1.2) are combinations of homogeneous Hermite polynomials. The Hermite polynomials in one variable are defined by

$$H_n(t) = (-1)^n e^{t^2/2} \frac{d^n}{dt^n} e^{-t^2/2}, \quad n \in \mathbb{N} \cup \{0\}, \tag{2.3}$$

and they constitute a complete set of eigenfunctions to problem (1.2) with  $\Omega = \mathbb{R}$ , more precisely it holds

$$-(\varphi_1(t)H_n'(t))' = n\varphi_1(t)H_n(t).$$

Since  $\Omega$  is a smooth set, its Gaussian perimeter is simply given by

$$P_{\gamma_N}(\Omega) = \int_{\partial\Omega} \gamma_N(x) \mathcal{H}_{N-1}(dx),$$

where  $\mathcal{H}_{N-1}(x)$  is the  $(N - 1)$ -dimensional Hausdorff measure.

As already mentioned in the introduction, for the Gaussian measure an isoperimetric inequality holds true. Consider the half-space

$$\Omega^\star = \{x \in \mathbb{R}^N : x_1 > \Phi^{-1}(\gamma_N(\Omega))\}, \tag{2.4}$$

where  $\Phi(t)$  is the complementary error function

$$\Phi(t) = \frac{1}{\sqrt{2\pi}} \int_t^\infty e^{-\frac{s^2}{2}} ds. \tag{2.5}$$

In other words  $\Omega^\star$  is the half-space orthogonal to the  $x_1$ -axis having the same Gaussian measure as  $\Omega$ .

The isoperimetric inequality for Gaussian measure (see [32,9,17,10]) states that

$$P_{\gamma_N}(\Omega) \geq P_{\gamma_N}(\Omega^\star), \quad (2.6)$$

where equality holds in (2.6) if and only if  $\Omega = \Omega^\star$ , modulo a rotation.

Now we recall a few definitions and properties about Gaussian rearrangement, whose notion was introduced by Ehrhard in [17]. For exhaustive treatment on rearrangements we refer, e.g., to [4,14,24,31].

Let  $u : x \in \Omega \rightarrow \mathbb{R}$  be a measurable function. We denote by  $\mu(t)$  the distribution function of  $|u(x)|$  i.e.

$$\mu(t) = \gamma_N(\{x \in \Omega : |u(x)| > t\}), \quad t \geq 0,$$

while the decreasing rearrangement and the increasing rearrangement of  $u$ , with respect to the Gaussian measure, are defined respectively by

$$u^*(s) = \inf\{t \geq 0 : \mu(t) \leq s\}, \quad s \in ]0, \gamma_N(\Omega)]$$

and

$$u_*(s) = u^*(\gamma_N(\Omega) - s), \quad s \in [0, \gamma_N(\Omega)[.$$

Finally  $u^\star$ , the Gaussian rearrangement of  $u$ , is given by

$$u^\star(x) = u^*(\Phi(x_1)), \quad x \in \Omega^\star.$$

By its very definition  $u^\star$  depends on one variable only and it is an increasing function, therefore its level sets are parallel half-spaces. Since, by definition,  $u$  and  $u^\star$  are equimeasurable, Cavalieri's principle ensures

$$\|u\|_{L^p(\Omega, \gamma_N)} = \|u^\star\|_{L^p(\Omega^\star, \gamma_N)}, \quad \forall p \geq 1.$$

We will also make use of the Hardy–Littlewood inequality, which states that

$$\int_0^{\gamma_N(\Omega)} u^*(s)v_*(s) ds \leq \int_\Omega |u(x)v(x)| d\gamma_N \leq \int_0^{\gamma_N(\Omega)} u^*(s)v^*(s) ds. \quad (2.7)$$

We finally recall the Polya–Szegő principle which asserts that the weighted  $L^2$ -norms of a nonnegative function vanishing on  $\partial\Omega$  decreases under Gaussian symmetrization. More precisely let  $H_0^1(\Omega, \gamma_N)$  be the closure of  $C_0^\infty(\Omega)$  in  $H^1(\Omega, \gamma_N)$ . It holds that

$$\int_\Omega |Du(x)|^2 d\gamma_N \geq \int_{\Omega^\star} |Du^\star(x)|^2 d\gamma_N,$$

for any nonnegative  $u$  in  $H_0^1(\Omega, \gamma_N)$ .

### 3. The one-dimensional case

Let  $a, b \in \overline{\mathbb{R}}$  with  $-\infty \leq a < b \leq +\infty$  and  $\gamma_1(a, b) < 1$ . In this case problem (1.2) becomes

$$\begin{cases} -u'' + xu' = \mu u & \text{in } (a, b), \\ u'(a) = u'(b) = 0. \end{cases} \quad (3.1)$$

We will denote by  $\mu_1(a, b)$  the first nontrivial eigenvalue of (3.1), clearly its value is given by

$$\mu_1(a, b) = \min_{u \neq 0: \int_a^b u d\gamma_1 = 0} \frac{\int_a^b (u')^2 d\gamma_1}{\int_a^b u^2 d\gamma_1}. \quad (3.2)$$

Here we are interested in studying the behavior of  $\mu_1(a, b)$  when the interval  $(a, b)$  slides along the  $x$ -axis, keeping fixed its Gaussian measure. In other words, we impose the constraint

$$\gamma_1(a, b) = L \in (0, 1). \tag{3.3}$$

Obviously, under these conditions,  $b$  can be expressed in terms of  $a$  as follows

$$b(a) = \sqrt{2} \operatorname{erf}^{-1} \left[ 2L + \operatorname{erf} \left( \frac{\sqrt{2}}{2} a \right) \right], \tag{3.4}$$

where

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$$

is the error function.

Since condition (3.3) is in force, the function

$$f : a \in \mathbb{R} \rightarrow \mu_1(a, b(a)) \tag{3.5}$$

is defined on the interval  $[-\infty, \sqrt{2} \operatorname{erf}^{-1}(1 - 2L)]$  and it is even with respect to  $x = -\sqrt{2} \operatorname{erf}^{-1}(L)$ .

The following result holds.

**Theorem 3.1.** *Let  $L \in (0, 1)$  and let  $-\infty \leq a < b \leq +\infty$ , with  $\gamma_1(a, b(a)) = L$ . Then*

$$\min_a \mu_1(a, b(a)) = \mu_1(-\infty, \sqrt{2} \operatorname{erf}^{-1}(2L - 1)) = \mu_1(\sqrt{2} \operatorname{erf}^{-1}(1 - 2L), +\infty), \tag{3.6}$$

and

$$\max_a \mu_1(a, b(a)) = \mu_1(-\sqrt{2} \operatorname{erf}^{-1}(L), \sqrt{2} \operatorname{erf}^{-1}(L)). \tag{3.7}$$

Furthermore the function  $f$  defined in (3.5) is increasing in the interval  $[-\infty, -\sqrt{2} \operatorname{erf}^{-1}(L)]$ .

**Proof.** We denote by  $\lambda_1(a, b(a))$  the first eigenvalue of the problem

$$\begin{cases} -v'' + xv' = \lambda v & \text{in } (a, b(a)), \\ v(a) = v(b(a)) = 0. \end{cases} \tag{3.8}$$

It is easy to verify that

$$\lambda_1(a, b(a)) = \mu_1(a, b(a)) - 1. \tag{3.9}$$

Since they differ by a constant, in place of the Neumann eigenvalue we can equivalently study the behavior of the Dirichlet eigenvalue.

As a first consequence of this observation we note that the Faber–Krahn inequality for Gaussian measure (see [18] and [6]) directly gives (3.6).

The isoperimetric properties of the half-space (see, e.g., [9] and [32]) reads as follows

$$\min_a P_{\gamma_1}(a, b(a)) = P_{\gamma_1}(-\infty, \sqrt{2} \operatorname{erf}^{-1}(2L - 1)) = P_{\gamma_1}(\sqrt{2} \operatorname{erf}^{-1}(1 - 2L), +\infty).$$

A straightforward application of Lagrange multipliers rule tells us that the function  $P_{\gamma_1}(a, b)$  admits just one stationary point on the constraint  $\gamma_1(a, b) - L = 0$ . Moreover, as it is immediate to verify, such a point occurs at  $a = -b = -\sqrt{2} \operatorname{erf}^{-1}(L)$ . Now since the function  $P_{\gamma_1}(a, b(a))$  is smooth on the interval  $(-\infty, \sqrt{2} \operatorname{erf}^{-1}(1 - 2L))$ , from (3.6) we get

$$\max_a P_{\gamma_1}(a, b(a)) = P_{\gamma_1}(-\sqrt{2} \operatorname{erf}^{-1}(L), \sqrt{2} \operatorname{erf}^{-1}(L)). \tag{3.10}$$

These considerations allow us to say that

$$\frac{d}{da} P_{\gamma_1}(a, b(a)) > 0, \quad \forall a \in (-\infty, -\sqrt{2} \operatorname{erf}^{-1}(L)) \tag{3.11}$$

and by symmetry reasons

$$\frac{d}{da} P_{\gamma_1}(a, b(a)) < 0, \quad \forall a \in (-\sqrt{2} \operatorname{erf}^{-1}(L), \sqrt{2} \operatorname{erf}^{-1}(1 - 2L)). \tag{3.12}$$

Now we can finally turn our attention on the monotonicity properties of the eigenvalue  $\mu_1(a, b(a))$ . Let  $a_1, a_2 \in (-\infty, -\sqrt{2} \operatorname{erf}^{-1}(L))$  with  $a_1 < a_2$ . Our aim is to prove that

$$\mu_1(a_1, b(a_1)) < \mu_1(a_2, b(a_2)) \tag{3.13}$$

or equivalently

$$\lambda_1(a_1, b(a_1)) < \lambda_1(a_2, b(a_2)).$$

Let us denote by  $\phi_i(x)$ , with  $i = 1, 2$ , the first Dirichlet eigenfunctions corresponding to  $\lambda_1(I_i)$ , where  $I_i = (a_i, b(a_i))$ , with  $i = 1, 2$ , normalized in such a way that they are positive and

$$\int_{a_i}^{b(a_i)} \phi_i^2(x) d\gamma_1 = 1.$$

For any fixed  $t \in (0, 1)$  we denote by  $I_2^t$  the set  $\{x \in I_2 : \phi_2(x) > t\}$ . From the level sets of  $\phi_2(x)$  we want to build a function defined in  $I_1$  admissible as test function for  $\lambda_1(I_1)$ . This auxiliary function, denoted with  $\tilde{\phi}(x)$ , is the function uniquely defined by the following relationships:

- (i)  $\tilde{\phi} : x \in I_1 \rightarrow [0, \max \phi_2]$ ,
- (ii)  $\gamma_1\{x : \tilde{\phi}(x) > t\} = \gamma_1\{x : \phi_2(x) > t\}, \forall t \in [0, \max \phi_2]$ ,
- (iii)  $\{x : \tilde{\phi}(x) > t\}$  are intervals  $(\tilde{a}_t, \tilde{b}_t)$ , denoted with  $\tilde{I}_t$ , centered at  $\frac{a_1+b(a_1)}{2}, \forall t \in [0, \max \phi_2]$ .

By construction  $\tilde{\phi}$  is even with respect to  $\frac{a_1+b(a_1)}{2}$  and it is increasing in  $(a_1, \frac{a_1+b(a_1)}{2})$ . Furthermore it is equimeasurable with  $\phi_2$ , therefore  $\mu_{\tilde{\phi}}(t) = \mu_{\phi_2}(t)$  and

$$\int_{a_1}^{b(a_1)} \tilde{\phi}^2 d\gamma_1 = \int_{a_2}^{b(a_2)} \phi_2^2 d\gamma_1 = 1.$$

Coarea formula and Cauchy–Schwarz inequality ensure that

$$\begin{aligned} \lambda_1(I_2) &= \frac{1}{\sqrt{2\pi}} \int_{a_2}^{b(a_2)} \left(\frac{d\phi_2}{dx}\right)^2 e^{-\frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_0^{\max \phi_2} \left(\int_{\{\phi_2=t\}} \left|\frac{d\phi_2}{dx}\right| e^{-\frac{x^2}{2}} d\mathcal{H}^0\right) dt \\ &\geq \frac{1}{\sqrt{2\pi}} \int_0^{\max \phi_2} \frac{(\int_{\{\phi_2=t\}} e^{-\frac{x^2}{2}} d\mathcal{H}^0)^2}{\int_{\{\phi_2=t\}} \left|\frac{d\phi_2}{dx}\right|^{-1} e^{-\frac{x^2}{2}} d\mathcal{H}^0} dt = \int_0^{\max \phi_2} \frac{(P_{\gamma_1}\{\phi_2 > t\})^2}{-\mu'_{\phi_2}(t)} dt. \end{aligned} \tag{3.14}$$

At this point we note that by (3.11) and by the construction of  $\tilde{\phi}$  we have

$$P_{\gamma_1}\{\phi_2 > t\} > P_{\gamma_1}\{\tilde{\phi} > t\} = P_{\gamma_1}(\tilde{I}_t), \quad \forall t \in (0, \max \phi_2) \tag{3.15}$$

and

$$\mu'_{\phi_2}(t) = \mu'_{\tilde{\phi}}(t), \quad \text{a.e. } t \in (0, L). \tag{3.16}$$

So by (3.14), (3.15) and (3.16) we have

$$\lambda_1(I_2) > \int_0^{\max \tilde{\phi}} \frac{(P_{\gamma_1}\{\tilde{\phi} > t\})^2}{-\mu'_{\tilde{\phi}}(t)} dt. \tag{3.17}$$

Since the function  $\tilde{\phi}$  is, by construction, even with respect to  $\frac{a_1+b(a_1)}{2}$  we have

$$\left| \frac{d\tilde{\phi}}{dx}(\tilde{a}^t) \right| = \left| \frac{d\tilde{\phi}}{dx}(\tilde{b}^t) \right|, \quad \text{a.e. } t \in (0, \max \phi_2),$$

and therefore the Cauchy–Schwarz inequality used in (3.14) for  $\tilde{\phi}$  reduces to an equality. This consideration together with (3.17), yields

$$\lambda_1(I_2) \geq \frac{1}{\sqrt{2\pi}} \int_{a_1}^{b(a_1)} \left( \frac{d\tilde{\phi}}{dx} \right)^2 e^{-\frac{x^2}{2}} dx \geq \lambda_1(I_1). \tag{3.18}$$

That is the claim (3.13). Note finally that if  $\lambda_1(a_1, b(a_1)) = \lambda_1(a_2, b(a_2))$  then all the above inequalities reduce to equalities. In particular equality in (3.15) implies that  $a_1 = a_2$  and  $b(a_1) = b(a_2)$ .  $\square$

**Remark 3.1.** Theorem 3.1, together with the shape derivative formula for one-dimensional Neumann eigenvalues, allows to get some qualitative information on  $u_1$ . Let us consider two smooth functions  $a(t)$  and  $b(t)$ , such that  $\gamma_1(a(t), b(t)) = L$  and  $(a(0), b(0)) = (a, b)$ . Let us denote by  $\mu_1(t) = \mu_1(a(t), b(t))$  the first eigenvalue of problem

$$\begin{cases} -\frac{d^2}{dx^2}u(x, t) + x \frac{d}{dx}u(x, t) = \mu(t)u(x, t) & \text{in } (a(t), b(t)), \\ \frac{d}{dx}u(x, t) \Big|_{x=a(t), b(t)} = 0, \end{cases}$$

and by  $u_1(x, t)$  a corresponding eigenfunction such that  $\int_{a(t)}^{b(t)} u_1^2(x, t) d\gamma_1 = 1$ . Then, see, e.g., [22,23], it is easy to verify that

$$\mu'_1(0) = \mu_1(a, b)e^{-\frac{a^2}{2}}(u^2(a) - u^2(b)). \tag{3.19}$$

Therefore if  $a < -\sqrt{2} \operatorname{erf}^{-1}(L)$  then, by Theorem 3.1, we have that  $|u(a)| > |u(b)|$ , conversely if  $a \in (-\sqrt{2} \operatorname{erf}^{-1}(L), \sqrt{2} \operatorname{erf}^{-1}(1 - 2L))$  then  $|u(a)| < |u(b)|$ .

#### 4. The $N$ -dimensional case

Let us examine, by means of the separation of variables method, problem (1.2) when  $\Omega$  is the ball of  $\mathbb{R}^N$  centered at the origin of radius  $R$ , throughout denoted by  $B_R$ , that is

$$\begin{cases} -\Delta u + x \cdot Du = \mu u & \text{in } B_R, \\ \frac{\partial u}{\partial r} = 0 & \text{on } \partial B_R. \end{cases} \tag{4.1}$$

The equation in (4.1) can be rewritten, using polar coordinates, as

$$\frac{1}{r^{N-1}} \frac{\partial}{\partial r} \left( r^{N-1} \frac{\partial u}{\partial r} \right) + \frac{1}{r^2} \Delta_{\mathbb{S}^{N-1}}(u|_{\mathbb{S}_r^{N-1}}) - r \frac{\partial u}{\partial r} + \mu u = 0, \tag{4.2}$$

where  $\mathbb{S}_r^{N-1}$  is the sphere of radius  $r$  in  $\mathbb{R}^N$ ,  $u|_{\mathbb{S}_r^{N-1}}$  is the restriction of  $u$  on  $\mathbb{S}_r^{N-1}$  and finally  $\Delta_{\mathbb{S}^{N-1}}(u|_{\mathbb{S}_r^{N-1}})$  is the standard Laplace–Beltrami operator relative to the manifold  $\mathbb{S}_r^{N-1}$ .

Setting  $u(x) = Y(\theta)f(r)$  in Eq. (4.2), where  $\theta$  belongs to  $\mathbb{S}_1^{N-1}$ , we have

$$Y \frac{1}{r^{N-1}} (r^{N-1} f')' + \Delta_{\mathbb{S}^{N-1}} Y \frac{f}{r^2} - Y r f' + \mu Y f = 0,$$

and hence

$$\frac{1}{r^{N-3} f} (r^{N-1} f')' - r^3 \frac{f'}{f} + \mu r^2 = -\frac{\Delta_{\mathbb{S}^{N-1}} Y}{Y} = \bar{k}. \quad (4.3)$$

As well known, see, e.g., [28] and [12], the last equality is fulfilled if and only if

$$\bar{k} = k(k + N - 2) \quad \text{with } k \in \mathbb{N} \cup \{0\}.$$

Multiplying the left-hand side of Eq. (4.3) by  $\frac{f}{r^2}$ , we get

$$f'' + f' \left( \frac{N-1}{r} - r \right) + \mu f - k(k + N - 2) \frac{f}{r^2} = 0 \quad \text{in } (0, R).$$

The eigenfunctions are either purely radial

$$u_i(r) = f_0(\mu_i; r), \quad \text{if } k = 0, \quad (4.4)$$

or in the form

$$u_i(r, \theta) = f_k(\mu_i; r) Y(\theta), \quad \text{if } k \in \mathbb{N}. \quad (4.5)$$

The functions  $f_k$ , with  $k \in \mathbb{N} \cup \{0\}$ , clearly satisfy

$$\begin{cases} f_k'' + f_k' \left( \frac{N-1}{r} - r \right) + \mu_i f_k - k(k + N - 2) \frac{f_k}{r^2} = 0 & \text{in } (0, R), \\ f_k(0) = 0, \quad f_k'(R) = 0. \end{cases} \quad (4.6)$$

In the sequel we will denote by  $\tau_n(R)$ , with  $n \in \mathbb{N} \cup \{0\}$ , the sequence of eigenvalues of (4.1) whose corresponding eigenfunctions are purely radial, i.e. in the form (4.4) or equivalently solutions to problem (4.6) with  $k = 0$ . Clearly in this case the first eigenfunction is constant and the corresponding eigenvalue  $\tau_0(R)$  is trivially zero. We will denote by  $\nu_n(R)$ , with  $n \in \mathbb{N}$ , the remaining eigenvalues of (4.1).

**Lemma 4.1.** *It holds that*

$$\nu_1(R) < \tau_1(R), \quad \forall R > 0. \quad (4.7)$$

**Proof.** We recall that  $\tau_1 = \tau_1(R)$  is the first nontrivial eigenvalue of

$$\begin{cases} g'' + g' \left( \frac{N-1}{r} - r \right) + \tau g = 0 & \text{in } (0, R), \\ g'(0) = g'(R) = 0, \end{cases} \quad (4.8)$$

and  $\nu_1 = \nu_1(R)$  is the first eigenvalue of

$$\begin{cases} w'' + w' \left( \frac{N-1}{r} - r \right) + \nu w - (N-1) \frac{w}{r^2} = 0 & \text{in } (0, R), \\ w(0) = w'(R) = 0. \end{cases} \quad (4.9)$$

First of all we observe that the first eigenfunction  $w_1$  of (4.9) does not change its sign in  $(0, R)$ , thus we can assume that  $w_1 > 0$  in  $(0, R)$ .

Moreover  $w_1' \geq 0$  in  $(0, R)$ . Indeed, assume, by contradiction, that we can find two values  $r_1, r_2$ , with  $r_1 < r_2$ , such that  $w_1''(r_1) \leq 0$ ,  $w_1'(r_1) = 0$  and  $w_1''(r_2) \geq 0$ ,  $w_1'(r_2) = 0$ . By evaluating the equation in (4.9)

$$\frac{w_1''}{w_1} + \frac{w_1'}{w_1} \left( \frac{N-1}{r} - r \right) + \nu_1 - \frac{N-1}{r^2} = 0$$



at  $r_1$  and  $r_2$ , we get

$$v_1 - \frac{N-1}{r_2^2} \leq 0 \quad \text{and} \quad v_1 - \frac{N-1}{r_1^2} \geq 0,$$

that means  $r_1 \geq r_2$  and this is a contradiction.

On the other hand, the first nontrivial eigenfunction of problem (4.8),  $g_1 = g_1(r)$ , has mean value zero i.e.

$$\int_{B_R} g_1 d\gamma_N = \frac{N\omega_N}{(2\pi)^{N/2}} \int_0^R g_1(r) e^{-\frac{r^2}{2}} r^{N-1} dr = 0,$$

where, here and in the sequel,  $\omega_N$  will denote the volume of the unit ball in  $\mathbb{R}^N$ .

This implies that  $g_1(r)$  must change its sign in  $(0, R)$ . Let us suppose  $g_1(r) > 0$  in  $(0, r_0)$  and  $g_1(r_0) = 0$ . We observe that  $g_1'(r) < 0$  in  $(0, R)$ . Moreover evaluating the equation of problem (4.8) at  $r_0$ , we have

$$g_1''(r_0) + g_1'(r_0) \left( \frac{N-1}{r_0} - r_0 \right) = 0. \tag{4.10}$$

Now we consider the following intervals  $J_1 := (0, \sqrt{N-1}]$ ,  $J_2 := (\sqrt{N-1}, \sqrt{N-1} + \frac{\pi}{\sqrt{8}}]$  and  $J_3 := (\sqrt{N-1} + \frac{\pi}{\sqrt{8}}, +\infty)$ . Clearly  $\bigcup_{i=1}^3 J_i = (0, +\infty)$  for any  $N \in \mathbb{N}$ . The proof of (4.7) requires different arguments depending on the interval  $J_i$  in which the radius  $R$  of the ball  $B_R$  lies.

Case 1:  $R \in J_1 = (0, \sqrt{N-1}]$ .

Since  $r_0 < R \leq \sqrt{N-1}$ , from (4.10) we get

$$g_1''(r_0) \geq 0. \tag{4.11}$$

Moreover if we set  $\psi = g_1'$ , then problem (4.8) becomes

$$\begin{cases} \psi'' + \psi' \left( \frac{N-1}{r} - r \right) + \psi \left( -\frac{N-1}{r^2} - 1 \right) + \tau_1 \psi = 0 & \text{in } (0, R), \\ \psi(0) = \psi(R) = 0, \end{cases}$$

and in particular

$$\begin{cases} \psi'' + \psi' \left( \frac{N-1}{r} - r \right) - \frac{N-1}{r^2} \psi + \tau_1 \psi \leq 0 & \text{in } (0, r_0), \\ \psi(0) = 0, \quad \psi'(r_0) \geq 0. \end{cases} \tag{4.12}$$

Now we multiply equation in (4.9) by  $r^{N-1}\psi \varphi_N$  and equation in (4.12) by  $r^{N-1}w_1 \varphi_N$ , respectively. Hence, by subtracting, we obtain

$$r^{N-1}\varphi_N(\psi w_1'' - w_1\psi'') + r^{N-1}\varphi_N \left( \frac{N-1}{r} - r \right) (\psi w_1' - w_1\psi') + (v_1 - \tau_1)w_1 r^{N-1}\psi \varphi_N \geq 0 \quad \text{in } (0, r_0).$$

Integrating by parts the above inequality on  $(0, r_0)$ , we get

$$\begin{aligned} (v_1 - \tau_1) \int_0^{r_0} w_1 r^{N-1} \psi \varphi_N &> \int_0^{r_0} \varphi_N w_1 (r^{N-1}\psi')' - \varphi_N \psi (r^{N-1}w_1')' + r^N \varphi_N (\psi w_1' - w_1\psi') dr \\ &= r_0^{N-1} \varphi_N(r_0) (\psi'(r_0)w(r_0) - \psi(r_0)w'(r_0)) > 0. \end{aligned}$$

In other words

$$v_1(R) < \tau_1(R), \quad \forall R \in J_1.$$

Case 2:  $R \in J_2 = (\sqrt{N-1}, \sqrt{N-1} + \frac{\pi}{\sqrt{8}}]$ .

The above proof does not work in  $J_2$ . This because when  $r > \sqrt{N-1}$  one cannot exclude a priori that  $r_0 > \sqrt{N-1}$  too. Hence (4.10) does no longer guarantee (4.11). Clearly we may assume here that

$$\sqrt{N-1} < r_0 < R, \tag{4.13}$$

indeed, if not (i.e. if  $r_0 \leq \sqrt{N-1}$ ), we can get the claim by repeating the arguments of Case 1.

By (4.8) we get

$$\begin{cases} g_1'' + \tau_1 g_1 < 0 & \text{in } (r_0, R), \\ g_1(r_0) = g_1'(R) = 0. \end{cases} \tag{4.14}$$

Multiplying the equation in (4.14) by  $g_1(r) < 0$  and integrating between  $r_0$  and  $R$ , we get

$$\int_{r_0}^R (g_1')^2 dr < \tau_1 \int_{r_0}^R (g_1)^2 dr,$$

that implies

$$\tau_1(R) > \min_{v \neq 0: v(r_0)=v'(R)=0} \frac{\int_{r_0}^R (v')^2 dr}{\int_{r_0}^R v^2 dr} = \frac{\pi^2}{4(R-r_0)^2}.$$

Finally, taking into account that we are under the assumption (4.13), we get the following:

$$\tau_1(R) > \frac{\pi^2}{4(R-r_0)^2} > \frac{\pi^2}{4(R-\sqrt{N-1})^2} := h(R), \quad \forall R \in J_2. \tag{4.15}$$

Now we want to provide an estimate from above for  $v_1(R)$ , namely  $v_1(R) < k(R)$ . To this aim we firstly note that for the values of  $R \in J_2$  such that  $v_1(R) \leq \tau_1(R)$  we have

$$v_1 = \min_{\substack{v \in H^1(B_R), v \neq 0 \\ \int_{B_R} v d\gamma_N = 0}} \frac{\int_{B_R} |Dv|^2 d\gamma_N}{\int_{B_R} |v|^2 d\gamma_N}. \tag{4.16}$$

While for the remaining values of  $R$  we have to impose also the orthogonality with  $g_1$ , that is

$$v_1 = \min_{\substack{v \in H^1(B_R), v \neq 0 \\ \int_{B_R} v d\gamma_N = 0, \int_{B_R} v g_1 d\gamma_N = 0}} \frac{\int_{B_R} |Dv|^2 d\gamma_N}{\int_{B_R} |v|^2 d\gamma_N}. \tag{4.17}$$

In both cases  $v = x_i$  for  $i = 1, \dots, N$  are admissible trial functions for  $v_1$  and hence

$$v_1 \leq \frac{\gamma_N(B_R)}{\int_{B_R} x_1^2 d\gamma_N}, \dots, v_1 \leq \frac{\gamma_N(B_R)}{\int_{B_R} x_N^2 d\gamma_N}.$$

So

$$\frac{N}{v_1} \geq \frac{\int_{B_R} (x_1^2 + \dots + x_N^2) d\gamma_N}{\gamma_N(B_R)};$$

and

$$v_1 = v_1(R) \leq \frac{N \int_0^R e^{-\frac{s^2}{2}} s^{N-1} ds}{\int_0^R e^{-\frac{s^2}{2}} s^{N+1} ds} := k(R). \tag{4.18}$$

At this point we observe that  $k(R)$  is a decreasing function, indeed

$$k'(R) = \frac{N e^{-\frac{R^2}{2}} R^{N-1}}{(\int_0^R e^{-\frac{s^2}{2}} s^{N+1} ds)^2} \left[ \int_0^R e^{-\frac{s^2}{2}} s^{N+1} ds - R^2 \int_0^R e^{-\frac{s^2}{2}} s^{N-1} ds \right] < 0,$$

where the quantity in the square brackets is negative because

$$\int_0^R e^{-\frac{s^2}{2}} s^{N+1} ds = \int_0^R s^2 e^{-\frac{s^2}{2}} s^{N-1} ds < R^2 \int_0^R e^{-\frac{s^2}{2}} s^{N-1} ds.$$

Furthermore the function  $h(R)$ , defined in (4.15), is obviously a decreasing function.

Let us consider the case  $N = 2$  first. Let  $\bar{R}$  be the unique positive zero of the function  $f(t) = t^2 + 1 - e^{\frac{t^2}{2}}$  ( $\bar{R} \simeq 1.585$ ). If  $1 < R < \bar{R}$  then by (4.15), (4.18) and by the monotonicity properties of the functions  $k(R)$  and  $h(R)$  we get

$$\nu_1(R) < k(R) < \sup_{(1, \bar{R})} k(R) = k(1), \quad \forall R \in (1, \bar{R}) \tag{4.19}$$

and

$$h(\bar{R}) = \inf_{(1, \bar{R})} h(R) < h(R) < \tau_1(R), \quad \forall R \in (1, \bar{R}). \tag{4.20}$$

Now since

$$k(1) = \frac{2 \int_0^1 e^{-\frac{t^2}{2}} t dt}{\int_0^1 e^{-\frac{t^2}{2}} t^3 dt} = \frac{2 - 2e^{-\frac{1}{2}}}{2 - 3e^{-\frac{1}{2}}} \simeq 4.362 \tag{4.21}$$

and

$$h(\bar{R}) = \frac{\pi^2}{4(\bar{R} - 1)^2} \simeq 7.210, \tag{4.22}$$

taking into account of (4.19) and (4.20), we get

$$\nu_1(R) < \tau_1(R), \quad \forall R \in (1, \bar{R}).$$

Let us consider the remaining interval  $[\bar{R}, 1 + \frac{\pi}{\sqrt{8}}]$ . Since

$$k(\bar{R}) = \frac{2 - 2e^{-\frac{\bar{R}}{2}}}{2 - 3e^{-\frac{\bar{R}}{2}}} \simeq 1.705 < h\left(1 + \frac{\pi}{\sqrt{8}}\right) = 2,$$

arguing as before we get

$$\nu_1(R) < \tau_1(R), \quad \forall R \in \left[\bar{R}, 1 + \frac{\pi}{\sqrt{8}}\right].$$

Now let  $N \geq 3$ . If  $\sqrt{N-1} < R < \sqrt{N+2}$ , by (4.15) and (4.18) we get

$$\nu_1(R) < k(R) < \sup_{(\sqrt{N-1}, \sqrt{N+2})} k(R) = k(\sqrt{N-1}). \tag{4.23}$$

We claim that

$$k(\sqrt{N-1}) \leq \frac{2N+1}{N-1}. \tag{4.24}$$

Indeed by an integration by parts the claim becomes

$$k(\sqrt{N-1}) = \frac{(N-1)^{\frac{N}{2}} e^{-\frac{N-1}{2}} + \int_0^{\sqrt{N-1}} e^{-\frac{s^2}{2}} s^{N+1} ds}{\int_0^{\sqrt{N-1}} e^{-\frac{s^2}{2}} s^{N+1} ds} \leq \frac{2N+1}{N-1}.$$

In order to prove the above inequality it suffices to show that

$$e^{\frac{N-1}{2}} \int_0^{\sqrt{N-1}} e^{-\frac{s^2}{2}} s^{N+1} ds \geq \frac{(N-1)^{\frac{N}{2}+1}}{N+2}. \tag{4.25}$$

Inequality (4.25), and hence the claim (4.24), easily follows by observing that

$$e^{\frac{N-1}{2}} \int_0^{\sqrt{N-1}} e^{-\frac{s^2}{2}} s^{N+1} ds > e^{\frac{N-1}{2}} \int_0^{\sqrt{N-1}} s^{N+1} ds = e^{\frac{N-1}{2}} \frac{(N-1)^{\frac{N+2}{2}}}{N+2} > \frac{(N-1)^{\frac{N}{2}+1}}{N+2}.$$

Now we want to prove that

$$\frac{2N+1}{N-1} < h(\sqrt{N+2}) = \frac{\pi^2}{4(\sqrt{N+2} - \sqrt{N-1})^2}, \quad \forall N \geq 3. \tag{4.26}$$

It is elementary to verify that (4.26) is true for  $N = 3$ . On the other hand observe that (4.26) is false for  $N = 2$ , that is the reason we were forced to split Case 2 in the proof of Lemma 4.1 in these subcases.

Finally we get (4.26) since the sequences  $\frac{2N+1}{N-1}$  and  $\frac{\pi^2}{4(\sqrt{N+2} - \sqrt{N-1})^2}$  are decreasing and increasing respectively. Therefore, from the monotonicity of the functions  $k(R)$  and  $h(R)$ , (4.23), (4.24) and (4.26) yield

$$v_1(R) < k(R) < k(\sqrt{N-1}) \leq \frac{2N+1}{N-1} < h(\sqrt{N+2}) < h(R) < \tau_1(R), \quad \forall R \in (\sqrt{N-1}, \sqrt{N+2}).$$

Finally let  $R \in [\sqrt{N+2}, \sqrt{N-1} + \frac{\pi}{\sqrt{8}}]$ . We claim that

$$k(\sqrt{N+2}) \leq 2. \tag{4.27}$$

Indeed arguing as before we have

$$\begin{aligned} k(\sqrt{N+2}) - 2 &= \frac{(N+2)^{\frac{N}{2}} e^{-\frac{N+2}{2}} - \int_0^{\sqrt{N+2}} e^{-\frac{s^2}{2}} s^{N+1} ds}{\int_0^{\sqrt{N+2}} e^{-\frac{s^2}{2}} s^{N+1} ds} \\ &< \frac{(N+2)^{\frac{N}{2}} e^{-\frac{N+2}{2}} - \int_0^{\sqrt{N+2}} s^{N+1} ds}{\int_0^{\sqrt{N+2}} e^{-\frac{s^2}{2}} s^{N+1} ds} = \frac{(N+2)^{\frac{N}{2}} (e^{-\frac{N+2}{2}} - 1)}{\int_0^{\sqrt{N+2}} e^{-\frac{s^2}{2}} s^{N+1} ds} < 0. \end{aligned}$$

Finally we have

$$\begin{aligned} v_1(R) < k(R) < k(\sqrt{N+2}) < 2 &= h\left(\sqrt{N-1} + \frac{\pi}{\sqrt{8}}\right) < h(R) < \tau_1(R), \\ \forall R \in \left[\sqrt{N+2}, \sqrt{N-1} + \frac{\pi}{\sqrt{8}}\right]. \end{aligned}$$

Case 3:  $R \in J_3 = (\sqrt{N-1} + \frac{\pi}{\sqrt{8}}, +\infty)$ .

Before addressing this last case let us remark that the above method cannot be used for large values of  $R$ . Indeed when  $N = 2$ , for instance, we have

$$\lim_{R \rightarrow +\infty} k(R) = \lim_{R \rightarrow +\infty} \frac{2 - 2e^{-\frac{R^2}{2}}}{2 - (R^2 + 2)e^{-\frac{R^2}{2}}} = 1 \quad \text{and} \quad \lim_{R \rightarrow +\infty} h(R) = 0.$$

Therefore the inequality  $k(R) < h(R)$ , we have used in Case 2, does not hold for any  $R \in J_3$ .

In order to analyze the problem for large value of the radius  $R$  it appears natural to consider the solution to problem (4.8) with  $R = +\infty$ . Its first radial eigenfunction, as well known, is

$$g_\infty(r) = \sum_{i=1}^N H_2(x_i) = r^2 - N,$$

where  $H_2$  is the Hermite polynomial defined in (2.3). More explicitly we have

$$\begin{cases} g''_\infty + g'_\infty \left( \frac{N-1}{r} - r \right) + \tau_1(\infty)g_\infty = 0 & \text{in } (0, +\infty), \\ g'_\infty(0) = \lim_{r \rightarrow +\infty} (g'_\infty(r)e^{-\frac{r^2}{2}}) = 0, \end{cases} \tag{4.28}$$

where  $\tau_1(\infty) = 2$ .

Let us denote, according to the notation used in Section 3, with  $\lambda_1(B_r)$  the first eigenvalue of the problem

$$\begin{cases} -\sum_{i=1}^N \frac{\partial}{\partial x_i} \left( \varphi_N(x) \frac{\partial u}{\partial x_i} \right) = \lambda \varphi_N(x) u & \text{in } B_r, \\ u = 0 & \text{on } \partial B_r. \end{cases}$$

We claim that

$$\tau_1(R) > \tau_1(\infty). \tag{4.29}$$

To this aim we may assume that  $r_0 \geq \sqrt{N}$ . Indeed if  $r_0 < \sqrt{N}$  we have

$$\tau_1(R) = \lambda_1(B_{r_0}) > \lambda_1(B_{\sqrt{N}}) = 2 = \tau_1(\infty).$$

Now multiplying the equation in problem (4.8) by  $r^{N-1} \varphi_N g_\infty$  and equation in problem (4.28) by  $r^{N-1} \varphi_N g_1$  respectively and hence subtracting, we get

$$\begin{aligned} r^{N-1} \varphi_N (g_\infty g''_1 - g''_\infty g_1) + r^{N-1} \varphi_N \left( \frac{N-1}{r} - r \right) (g_\infty g'_1 - g_1 g'_\infty) \\ + (\tau_1(R) - \tau_1(\infty)) g_\infty g_1 r^{N-1} \varphi_N = 0 \quad \text{in } (r_0, R). \end{aligned}$$

Integrating between  $r_0$  and  $R$ , we get

$$\begin{aligned} (\tau_1(\infty) - \tau_1(R)) \int_{r_0}^R g_\infty g_1 r^{N-1} \varphi_N dr &= \int_{r_0}^R \varphi_N g_\infty (r^{N-1} g'_1)' - \varphi_N g_1 (r^{N-1} g'_\infty)' + r^N \varphi_N (g_1 g'_\infty - g_\infty g'_1) dr \\ &= -\varphi_N(r_0) r_0^{N-1} (r_0^2 - N) g'_1(r_0) - 2R^N \varphi_N(R) g_1(R) > 0. \end{aligned}$$

The last inequality, since we are assuming that  $r_0 \geq \sqrt{N}$ , implies the claim (4.29).

Now, recalling that  $k(R)$  is a decreasing function, from (4.27), we deduce

$$k(R) < k\left(\sqrt{N-1} + \frac{\pi}{\sqrt{8}}\right) < k(\sqrt{N+2}) \leq 2, \quad \forall R > \sqrt{N-1} + \frac{\pi}{\sqrt{8}}.$$

The last inequalities and (4.18) imply

$$\nu_1(R) < k(R) < 2 < \tau_1(R) \quad \text{for } R > \sqrt{N-1} + \frac{\pi}{\sqrt{8}}. \quad \square$$

**Remark 4.1.** Note that the upper bound for  $\mu_1(R)$  given in (4.18) is asymptotically sharp, as  $R$  goes to  $+\infty$ . Indeed, as it is easy to verify, it holds

$$\lim_{R \rightarrow +\infty} \nu_1(R) = \frac{N \int_0^{+\infty} e^{-\frac{s^2}{2}} s^{N-1} ds}{\int_0^{+\infty} e^{-\frac{s^2}{2}} s^{N+1} ds} = 1 = \mu_1(\mathbb{R}^N), \quad \forall N \in \mathbb{N}.$$

Lemma 4.1 ensures that the first eigenfunction associated to the first eigenvalue of problem (1.2) with  $\Omega = B_R$ , is in the form  $u(x) = w(|x|)Y(\theta)$ , where  $\theta$  belongs to  $\mathbb{S}_1^{N-1}$  and the radial function  $w$  has one sign in  $B_R$  and it satisfies the following problem:

$$\begin{cases} w''(r) + w'(r)\left(\frac{N-1}{r} - r\right) + \mu_1(B_R)w(r) - \frac{N-1}{r^2}w(r) = 0, & \text{for } r \in (0, R), \\ w(0) = w'(R) = 0. \end{cases} \tag{4.30}$$

Multiplying the equation in (4.30) by  $w \varphi_N$  and integrating over  $B_R$ , we get

$$\begin{aligned} & \mu_1(B_R) \int_{B_R} w(|x|)^2 d\gamma_N \\ &= -N\omega_N \int_0^R (w'r^{N-1})' w(r) e^{-\frac{r^2}{2}} dr + N\omega_N \int_0^R r^N w(r) w'(r) e^{-\frac{r^2}{2}} dr + \int_{B_R} \frac{1}{|x|^2} w(|x|)^2 d\gamma_N \\ &= \int_{B_R} (w'(|x|))^2 d\gamma_N + \int_{B_R} \frac{1}{|x|^2} w(|x|)^2 d\gamma_N. \end{aligned}$$

Thus

$$\mu_1(B_R) = \frac{\int_{B_R} ((w'(|x|))^2 + \frac{N-1}{|x|^2} w(|x|)^2) d\gamma_N}{\int_{B_R} w(|x|)^2 d\gamma_N}. \tag{4.31}$$

Now we are able to prove our main result.

**Theorem 4.1.** *The ball maximizes the first Neumann eigenvalue among all Lipschitz open sets  $\Omega$  of  $\mathbb{R}^N$  of prescribed Gaussian measure and symmetric about the origin. Moreover, it is the unique maximizer in this class.*

**Proof.** Let  $B_R$  the ball centered at the origin having the same Gaussian measure as  $\Omega$ . We define

$$G(r) = \begin{cases} w(r) & \text{for } 0 < r < R, \\ w(R) & \text{for } r \geq R, \end{cases} \tag{4.32}$$

where  $w$  is the solution of (4.8) satisfying (4.31). By the results stated above the function  $G$  is nondecreasing and nonnegative. We introduce the functions

$$P_i(x) = G(|x|) \frac{x_i}{|x|} \quad \text{for } 1 \leq i \leq N.$$

The assumption on the symmetry of  $\Omega$  guarantees

$$\int_{\Omega} P_i(x) d\gamma_N = 0, \quad \forall i = 1, \dots, N. \tag{4.33}$$

Hence each function  $P_i$  is admissible in the variational formulation (2.2).

Since

$$\frac{\partial P_i}{\partial x_j} = G'(|x|) \frac{x_i x_j}{|x|^2} - G(|x|) \frac{x_i x_j}{|x|^3} + \delta_{ij} \frac{G(|x|)}{|x|},$$

where  $\delta_{ij}$  is the Kronecker symbol, summing over  $j = 1, \dots, N$ , we get

$$\mu_1(\Omega) \leq \frac{\int_{\Omega} ((G'(|x|))^2 \frac{x_i^2}{|x|^2} - G^2(|x|) \frac{x_i^2}{|x|^4} + \frac{G^2(|x|)}{|x|^2}) d\gamma_N}{\int_{\Omega} G(|x|)^2 \frac{x_i^2}{|x|^2} d\gamma_N}. \tag{4.34}$$

Set

$$N(r) = (G'(r))^2 + \frac{N-1}{r^2} G^2(r)$$

and

$$D(r) = G^2(r).$$

Summing up inequalities (4.34) over  $i = 1, \dots, N$ , the angular dependence drops out and we finally get

$$\mu_1(\Omega) \leq \frac{\int_{\Omega} ((G'(|x|))^2 + \frac{N-1}{|x|^2} G^2(|x|)) d\gamma_N}{\int_{\Omega} G(|x|)^2 d\gamma_N} = \frac{\int_{\Omega} N(|x|) d\gamma_N}{\int_{\Omega} D(|x|) d\gamma_N}. \tag{4.35}$$

It is straightforward to verify that

$$\frac{d}{dr} N(r) < 0.$$

Now we claim that

$$\int_{\Omega} N(|x|) d\gamma_N \leq \int_{B_R} N(|x|) d\gamma_N. \tag{4.36}$$

Hardy–Littlewood inequality (2.7) ensures

$$\int_{\Omega} N(|x|) d\gamma_N \leq \int_0^{\gamma_N(\Omega)} N^*(s) ds = \int_0^{\gamma_N(B_R)} N^*(s) ds, \tag{4.37}$$

where  $N^*$  is the decreasing rearrangement of  $N$ . Setting  $s = \gamma_N(B_r) = \frac{N\omega_N}{(2\pi)^{N/2}} \int_0^r e^{-\frac{s^2}{2}} s^{N-1} ds$ , we get

$$\int_0^{\gamma_N(B_R)} N^*(s) ds = \frac{N\omega_N}{(2\pi)^{N/2}} \int_0^R N^*(\gamma_N(B_r)) r^{N-1} e^{-\frac{r^2}{2}} dr.$$

Note that

$$N^*(\gamma_N(B_r)) = N(r),$$

since  $N^*(\gamma_N(B_r))$  and  $N(r)$  are equimeasurable and both radially decreasing functions. Therefore

$$\frac{N\omega_N}{(2\pi)^{N/2}} \int_0^R N^*(\gamma_N(B_r)) r^{N-1} e^{-\frac{r^2}{2}} dr = \frac{N\omega_N}{(2\pi)^{N/2}} \int_0^R N(r) r^{N-1} e^{-\frac{r^2}{2}} dr = \int_{B_R} N(|x|) d\gamma_N. \tag{4.38}$$

Combining (4.37) and (4.38), we obtain the claim (4.36). Analogously it is possible to prove that

$$\int_{\Omega} D(|x|) d\gamma_N \geq \int_{B_R} D(|x|) d\gamma_N. \tag{4.39}$$

Indeed since  $D$  is an increasing function, we have

$$\int_{\Omega} D(|x|) \varphi_N(|x|) dx \geq \int_0^{\gamma_N(B_R)} D_*(s) ds = \frac{N\omega_N}{(2\pi)^{N/2}} \int_0^R D_*(1 - e^{-\frac{r^2}{2}}) r^{N-1} e^{-\frac{r^2}{2}} dr = \int_{B_R} D(|x|) \varphi_N(|x|) dx,$$

where  $D_*$  is the increasing rearrangement of  $D$ . By (4.32), (4.36) and (4.39), the equality (4.35) becomes

$$\mu_1(\Omega) \leq \frac{\int_{B_R} ((w'(|x|))^2 + \frac{N-1}{|x|^2} w(|x|)^2) d\gamma_N}{\int_{B_R} w(|x|)^2 d\gamma_N} = \mu_1(B_R),$$

which is the desired inequality. Moreover, from the monotonicity properties of the functions  $N$  and  $D$ , it is easy to realize that inequalities (4.36) and (4.39) reduce to equalities only when  $\Omega$  is the ball  $B_R$ .  $\square$

**Remark 4.2.** Note that the assumption on the symmetry of  $\Omega$  is used solely to guarantee the orthogonality conditions (4.33).

**Remark 4.3.** Since the half-spaces are not symmetric about the origin, Theorem 4.1 cannot exclude the possibility that such a domain maximizes  $\mu_1(\Omega)$  in dimension greater than one. This phenomenon does not occur since any half-space has first Neumann eigenvalue equal to 1, independently of its measure. It is easy to show an example of a set which is not symmetric about the origin whose first Neumann eigenvalue is bigger than 1. Consider, for instance, in  $\mathbb{R}^2$  the square  $T = (\sqrt{3 - \sqrt{6}}, \sqrt{3 + \sqrt{6}})^2$ . As it is immediate to verify,  $\mu_1(T) = 5$  and it is a double eigenvalue. A corresponding eigenfunction is  $u_1(x, y) = u_1(x) = H_5(x) = x^5 - 10x^3 + 15x$ . This simply follows by observing that  $H'_5(x) < 0, \forall x \in (\sqrt{3 - \sqrt{6}}, \sqrt{3 + \sqrt{6}})$  and  $H'_5(\sqrt{3 - \sqrt{6}}) = H'_5(\sqrt{3 + \sqrt{6}}) = 0$ . Let us round a corner of this square by considering the family of domains

$$T_\delta = \{(x, y) \in \mathbb{R}^2: \sqrt{3 - \sqrt{6}} \leq x \leq \sqrt{3 + \sqrt{6}} \text{ and } \sqrt{3 - \sqrt{6}} \leq y \leq f_\delta(x)\},$$

with  $\delta \ll 1$  and

$$f_\delta(x) = \begin{cases} \sqrt{3 + \sqrt{6}} & \text{if } \sqrt{3 - \sqrt{6}} \leq x \leq \sqrt{3 + \sqrt{6}} - \delta, \\ \sqrt{3 + \sqrt{6}} - \delta + \sqrt{\delta^2 - (x - (\sqrt{3 + \sqrt{6}} - \delta))^2} & \text{if } \sqrt{3 + \sqrt{6}} - \delta < x \leq \sqrt{3 + \sqrt{6}}. \end{cases}$$

Now the first nontrivial Neumann eigenfunction relative to  $T_\delta$  cannot depend on one variable only. The sequence of compact sets  $T_\delta$  converges, in the Hausdorff distance, to  $T$ , and therefore, see [13], we have that  $\mu_1(T_\delta) \rightarrow \mu_1(T)$ . Therefore for  $\delta$  small enough we have

$$5 + O(1) = \mu_1(T_\delta) > 1 = \mu_1(T^\star),$$

where  $T^\star$  is the half-space having the same Gaussian measure as  $T$ .

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**Appendix A**

Here we want to show that  $\tau_k(R)$ , the nontrivial eigenvalues of (4.8), are all decreasing functions. To this aim we apply, in this simple case, the shape derivative formula for Neumann eigenvalues, see, e.g., [22] and [23].

Let  $R(t) = R + t$ , with  $t > 0$ , and let  $\mu_k(t) = \mu_k(0, R(t))$  be the  $k$ -th eigenvalue of problem

$$\begin{cases} -u_{rr}(r, t) + ru_r(r, t) - \frac{N-1}{r}u_r(r, t) = \mu_k(t)u(r, t) & \text{in } (0, R(t)), \\ u_r(r, t)|_{r=0, R(t)} = 0, \end{cases} \tag{A.1}$$

and, finally, let  $u(r, t)$  be an eigenfunction corresponding to  $\mu_k(t)$  such that

$$\|u\|_{L^2(B_{R(t)}, \gamma_N)}^2 = \frac{N\omega_N}{(2\pi)^{N/2}} \int_0^{R(t)} u^2(r, t)r^{N-1}e^{-\frac{r^2}{2}} dr = 1. \tag{A.2}$$

We have:

**Proposition A.1.** *It holds that*

$$\mu'_k(0) = -\frac{N\omega_N}{(2\pi)^{N/2}}\mu_k(0)u^2(R)R^{N-1}e^{-\frac{R^2}{2}} \tag{A.3}$$

where  $u(r) = u(r, 0)$  is the eigenfunction of problem (A.1) in  $(0, R)$ .



**Proof.** Differentiating (A.2) we have for  $t = 0$ ,

$$2 \int_0^R u(r) u_t(r, 0) r^{N-1} e^{-\frac{r^2}{2}} dr = -e^{-\frac{R^2}{2}} R^{N-1} u^2(R). \quad (\text{A.4})$$

Multiplying the equation in (A.1) by  $u(r, t) r^{N-1} e^{-\frac{r^2}{2}}$ , we get

$$\mu_k(t) u^2(r, t) r^{N-1} e^{-\frac{r^2}{2}} = -(r^{N-1} u_r(r, t))_r u(r, t) e^{-\frac{r^2}{2}} + r^N u(r, t) u_r(r, t) e^{-\frac{r^2}{2}}.$$

Integrating the above equality on  $(0, R(t))$  and recalling condition (A.2) we get

$$\mu_k(t) = \frac{N\omega_N}{(2\pi)^{N/2}} \int_0^{R(t)} u_r^2(r, t) r^{N-1} e^{-\frac{r^2}{2}} dr. \quad (\text{A.5})$$

Differentiating we obtain

$$\mu'_k(0) = \frac{2N\omega_N}{(2\pi)^{N/2}} \int_0^R u_r(r, 0) u_{rt}(r, 0) r^{N-1} e^{-\frac{r^2}{2}} dr = \frac{2N\omega_N}{(2\pi)^{N/2}} \mu_k(0) \int_0^R u(r) u_t(r, 0) r^{N-1} e^{-\frac{r^2}{2}} dr.$$

So by (A.4) we obtain the claim (A.3).  $\square$

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