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Water waves over a rough bottom in the shallow water regime

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Abstract

This is a study of the Euler equations for free surface water waves in the case of varying bathymetry, considering the problem in the shallow water scaling regime. In the case of rapidly varying periodic bottom boundaries this is a problem of homogenization theory. In this setting we derive a new model system of equations, consisting of the classical shallow water equations coupled with nonlocal evolution equations for a periodic corrector term. We also exhibit a new resonance phenomenon between surface waves and a periodic bottom. This resonance, which gives rise to secular growth of surface wave patterns, can be viewed as a nonlinear generalization of the classical Bragg resonance. We justify the derivation of our model with a rigorous mathematical analysis of the scaling limit and the resulting error terms. The principal issue is that the shallow water limit and the homogenization process must be performed simultaneously. Our model equations and the error analysis are valid for both the two- and the three-dimensional physical problems.

MSC: 76B15; 35Q35

Keywords: Water waves; Shallow water; Rough bathymetry

1. Introduction

Studies of the Euler equations for free surface water waves are important to understanding the dynamics of ocean waves. The case of an idealized flat bottom and the resulting model equations has been widely studied for many years. The more realistic situation of varying bathymetry is less well known, despite its fundamental importance to studies of ocean wave dynamics in coastal regions, and there is not a complete consensus as to the appropriate model equations. In the case of topography there are many asymptotic scaling regimes of interest, including long-wave of modulational hypotheses for the evolution of the free surface, and short scale and/or long scale variations in the variable bottom fluid boundary.

In this paper we address the evolution of waves in the shallow water regime, for which we investigate the effect of the roughness of the bottom topography. The simplest situation is where the bottom varies periodically and rapidly

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with respect to the typical surface wavelength, a regime which can be described in the context of homogenization theory. Ideally, wave motion in this regime of rapid periodic bottom variations is described in terms of a long wave *effective* component, which is then adjusted by a smaller multi-scale *corrector* at the next order of approximation. In terms of the initial value problem for this regime, initial configurations consisting of large scale data with a multi-scale corrector term are expected to give rise to solutions with the same character, up to a smaller error term. In this paper we derive a system of model equations for such multi-scale approximate solutions. While other authors have looked at similar situations, as far as we know this system is new, consisting of a version of the shallow water equations for a mean field or effective components of the surface elevation and the fluid velocity, which then drive a nonlocal system of two additional equations for the evolution of a more rapidly oscillating corrector term. Because of the number of other models that have been proposed to describe this setting, we justify the derivation of our system with a rigorous analysis, giving error estimates for our approximate solutions. In cases in which there is a resonance between the effective velocity and the periodic bottom, the solution of the corrector equation can exhibit secular growth at a linear rate. This phenomenon can be viewed as a nonlinear generalization of the classical Bragg resonance between the bottom topography and the free surface. This is a local phenomenon, which may occur when the local Froude number is subcritical. In the absence of resonances, our analysis is valid over time intervals of existence of the effective component.

The literature on models of free surface water waves over a variable depth is extensive, including the paper of Miles [21] on its Hamiltonian formulation, and that of Wu [29] on models which are valid in long wave scaling regimes. The paper of Rosales and Papanicolaou [27] studies the long wave regime in which the bottom is rapidly varying, in the sense that the typical wavelength of surface waves is taken to be much longer than typical lengthscale of the variations of the bottom depth. When the latter are periodic, or more generally when they are given by a stationary ergodic process, the techniques of homogenization theory are used to obtain effective long wave model equations. The two most important examples are of periodic bottom topography, and of topography given by a stationary random process. Recently there has been a renewal of interest in this problem, both from the point of view of modeling of water waves in asymptotic scaling regimes, and of mathematical analysis. A central question is the validity of the homogenization approximation, and the character of the resulting model equations. Following [27], the paper of Nachbin and Sølna [22] studies the deformation of surface waves by the effects of propagation over a rough bottom, taken in the shallow water scaling regime. In this work the bottom is given by a random process, and the authors treat both the two- and three-dimensional cases. The paper of Craig et al. [10] considers large periodic bottom variations, again for dimensions $n = 1 + d$ ($d = 1, 2$), deriving model equations to quite high order of accuracy for the profiles which describe weak limits of surface waves in the homogenization limit of the nonlinear long wave regime. Similarly, the paper of Garnier, Kraenkel and Nachbin [12] studies the long wave scaling limits of water waves over a periodic bottom (for $d = 1$), deriving an effective KdV equation, for which they describe the dependence of the coefficients of nonlinearity and dispersion on the topography of the bottom. This study continues in Garnier, Grajales and Nachbin [13] in the case of random bathymetry. There are other studies of surface wave propagation over periodic bathymetry, that focus on regimes which are not homogenization theoretic. Namely, there is the case in which the typical wavelength of surface waves is comparable or smaller than the typical bottom variations. Among these, Choi and Milewski [6] consider periodic solutions of systems of KdV equations which are coupled through resonant interactions with a periodic bottom. The paper by Nakoulima et al. [23] considers shallow water theory with and without dispersive corrections, for a periodic and piecewise constant bottom of very long wavelength.

The paper of Grataloup and Mei [14] considers the propagation of modulational solutions over a random seabed in dimension $d = 1$, which is extended to the case $d = 2$ in Pihl, Mei and Hancock [26]. In this work, the typical wavelengths represented in the surface and the topography are comparable, and the effort is to derive envelope equations for the free surface and to understand its statistical properties, given the ensemble of realizations of the random bathymetry.

There is also a long history of study of resonant interaction between water surface waves with periodic bottom. The paper of Mei [20] gives the theory of linear Bragg resonances between surface waves and bottom variations of the same spatial scale. This is extended to nonlinear resonances in Liu and Yue [19]. The difference between these references and our work is that, in the latter, short scales perturbations of the free surface are generated by interaction of the bottom with long waves on the free surface, a feature typical of homogenization theory.

None of the references above, however, give a mathematical theorem which justifies on a rigorous basis the model equations that are derived. After the derivation of the shallow water model in the present paper, the second main point of our work is to provide a rigorous justification of this derivation. There is a history of results on the mathematical verification of the model equations for free surface water waves, starting in fact with the papers of Ovsjannikov [24, 25] and Kano and Nishida [17] which give existence theorems for the full water wave equations and as well a proof of convergence of solutions in the shallow water scaling limit. In both cases the bottom is assumed to be flat, and the authors work with initial data given in spaces of analytic functions. Results on long wave scaling limits of the water waves problem in dispersive regimes include Craig [8] and Schneider and Wayne [28] and their treatment of the two-dimensional problem, a long-time existence theory, and the Boussinesq and KdV limits in Sobolev spaces. More recently, the paper of Lannes [18] gives an existence theory for solutions of the water wave problem for fluid domains with smooth variable bathymetry, and the further paper of Alvarez-Samaniego and Lannes [2] gives rigorous results on a number of long wave scaling limits of the same problem (see also Iguchi [15] and earlier papers of Bona et al. [4] and Chazel [5]), all papers working with Sobolev space initial data. In the context of this body of work, what distinguishes the present paper is the oscillatory nature of the bottom boundary of the fluid domain, which has the implications that the solutions themselves are oscillatory, and principally, that the homogenization theory Ansatz giving the form of solutions must be justified. Our analysis has several features in common with the results of [11] on the justification of the nonlinear Schrödinger equation and the Davey–Stewartson system as envelope equations for modulation theory, the most important of which being that the principal theorem is a consistency result rather than a full fledged limit theorem for solutions. Nonetheless, as far as we know this is the first rigorous result which justifies with a rigorous analytic argument the application of homogenization theory to the water wave problem with rapidly varying periodic bathymetry. In the present framework, precise error estimates are needed because the shallow water limit and the homogenization limit do not commute. More precisely, shallow water expansions are derived for slowly varying bottoms, neglecting some terms that are relevant for rough bottoms. Conversely, homogenization limits are usually performed with low regularity estimates on solutions, that place them outside of the regime of high order shallow water asymptotics (see for instance [7] for a recent homogenization result at leading order for the Dirichlet–Neumann operator). The point of our work and the source of many of its technical difficulties is that we perform the homogenization and shallow water limit simultaneously, thereby retaining the full complement of relevant terms from the original water waves equations. The (local) effects of this infinity of terms neglected in previous studies add up to create the nonlocal effects present in our approximation.

1.1. General setting

The time-dependent fluid domain consists of the fluid domain $\Omega(b,\zeta)=\{(x,z)\in\mathbb{R}^{d+1},\ -H_0+b(x)<z<\zeta(x,t)\}$ in which the fluid velocity is represented by the gradient of a velocity potential Φ. The dependent variable $\zeta(x,t)$ denotes the surface elevation and $b(x)$ denotes the variation of the bottom of the fluid domain from its mean value. We use the Hamiltonian formulation due to Zakharov [30] and Craig and Sulem [9] in the form of a coupled system for the surface elevation ζ and the trace of the velocity potential at the surface $\psi=\Phi_{|z=\zeta}$, namely

$$\begin{cases} \partial_t\zeta - G[\zeta,b]\psi = 0, \\ \partial_t\psi + g\zeta + \dfrac{1}{2}|\nabla\psi|^2 - \dfrac{(G[\zeta,b]\psi+\nabla\zeta\cdot\nabla\psi)^2}{2(1+|\nabla\zeta|^2)} = 0. \end{cases} \tag{1.1}$$

The quantity $G[\zeta,b]\cdot$ is the Dirichlet–Neumann operator, defined by

$$G[\zeta,b]\psi = \sqrt{1+|\nabla\zeta|^2}\,\partial_n\Phi_{|z=\zeta}, \tag{1.2}$$

where Φ is the solution of the elliptic boundary value problem

$$\begin{cases} \Delta\Phi + \partial_z^2\Phi = 0 \quad \text{in } \Omega(b,\zeta), \\ \Phi_{|z=\zeta} = \psi, \qquad \partial_n\Phi_{|z=-H_0+b} = 0. \end{cases} \tag{1.3}$$

Writing the equations of evolution in terms of nondimensional variables, different asymptotic regimes of this problem are identified by scaling regimes of the associated dimensionless parameters. Denote by A the typical amplitude of surface waves, with λ their typical wavelength. Similarly let B denote the typical amplitude of the variations of the bottom from its mean value H_0, with ℓ their typical wavelength. From these quantities we define the dimensionless variables as follows:

$$x = \lambda X', \qquad z = H_0 z', \qquad t = \frac{\lambda}{\sqrt{gH_0}} t',$$
$$\zeta = A\zeta', \qquad \Phi = \frac{A}{H_0}\lambda\sqrt{gH_0}\Phi', \qquad b = Bb'\left(\frac{x}{\ell}\right). \tag{1.4}$$

Stemming from this change of variables there are four dimensionless parameters:

$$\mu = \frac{H_0^2}{\lambda^2}, \qquad \varepsilon = \frac{A}{H_0}, \qquad \beta = \frac{B}{H_0}, \qquad \gamma = \frac{\ell}{\lambda}. \tag{1.5}$$

Our analysis is concerned with the shallow water regime $\mu \ll 1$. The relative amplitude of solutions is governed by ε. In addition to this, the relative amplitude of the bathymetry is given by β, the parameter γ determines the relative length of bottom perturbations with respect to the typical wavelength of surface waves, and the bottom variations $b'(\cdot)$ are assumed to be 2π-periodic in all variables. We consider relatively large amplitude surface waves, meaning that no smallness assumption is made on ε. As usual for this regime, we therefore set $\varepsilon = 1$ for the sake of simplicity. With regard to the bottom variations, we set

$$\beta = \sqrt{\mu} = \gamma \ll 1. \tag{1.6}$$

The fact that $\beta = \gamma$ corresponds to small bathymetry slope in this regime, while the *roughness strength* is $\rho := \sqrt{\mu}/\gamma = 1$. For clarity of notation we drop this 'prime' notation for the remainder of the paper.

1.2. Presentation of results

The first result of this paper is the construction of an approximate solution (ζ_a, ψ_a) of the water waves problem in the form of the Ansatz

$$\zeta_a = \zeta_0(X,t) + \gamma\zeta_1(X,t,X/\gamma,t/\gamma), \tag{1.7}$$
$$\psi_a = \psi_0(X,t) + \gamma^2\psi_1(X,t,X/\gamma,t/\gamma). \tag{1.8}$$

Remark 1.1. The factor of γ^2 in front of the corrector ψ_1 is natural; indeed, this yields a $O(\gamma)$ corrector for the velocity, which is the physical relevant quantity.

Setting $V_0 = \nabla\psi_0$ and $h_0 = 1 + \zeta_0$, we show that (ζ_0, V_0) satisfies the classical shallow water system with flat bottom,

$$\begin{cases} \partial_t\zeta_0 + \nabla\cdot(h_0V_0) = 0, \\ \partial_t V_0 + \nabla\zeta_0 + (V_0\cdot\nabla)V_0 = 0, \end{cases} \tag{1.9}$$

while the corrector terms (ζ_1, ψ_1) satisfy a linear nonlocal coupled system of equations in the fast variables ($\tau = t/\gamma$, $Y = X/\gamma$)

$$\begin{cases} \partial_\tau\zeta_1 + V_0\cdot\nabla_Y\zeta_1 - |D_Y|\tanh\big(h_0|D_Y|\big)\psi_1 = V_0\cdot\nabla_Y \operatorname{sech}\big(h_0\,|D_Y|\big)b, \\ \partial_\tau\psi_1 + V_0\cdot\nabla_Y\psi_1 + \zeta_1 = 0. \end{cases} \tag{1.10}$$

In system (1.10), the functions ζ_1, ψ_1 are periodic in the variables Y, while the variables (t, X) are to be treated as parameters. The above system represents the linearized water wave equations in a fluid region of depth h_0, with a background flow given by the velocity field V_0. The source term of the RHS is due to the effect of scattering of the background flow from the variable bottom.

The second result of this paper is a mathematical justification of the derivation of the above system of model equations (1.9)–(1.10). Our proof is in the form of a consistency analysis of the Euler equations of free surface water waves, for which we show that the functions (ζ_a, ψ_a) whose constituents satisfy (1.9)–(1.10) are approximate solutions of the Euler equations. They are not in general an exact solution, but they satisfy Eqs. (1.1) up to an error term E_a, and we show that this error is small. Namely, we prove that

$$|E_a|_{H^*} < C\gamma^{3/8},$$

where the appropriate norm $|\cdot|_{H^*}$ is defined as $|E_a|_{H^*} = |E_{a1}|_{L^2} + \gamma^{-3/8}|E_{a2}|_{H^{1/2}}$, and $E_a = (E_{a1}, E_{a2})$. In particular, the error is small for the usual Hamiltonian norm of the water waves equations. The most striking point of our analysis is that this result is valid for the natural time scale $t = \mathcal{O}(1)$ associated to (1.9) only if the free surface does not resonate with the rapidly varying bottom. Such a resonance is obtained if there exists (t, X) such that

$$\big(k \cdot V_0(X,t)\big)^2 = |k| \tanh\big(h_0(X,t)|k|\big)$$

for some $k \in \mathbb{Z}$ corresponding to a nonzero mode of the Fourier decomposition of the bottom parametrization b. This condition can be viewed as a nonlinear generalization of the classical Bragg resonance which is obtained when the wavelengths of the free surface and of the bottom are of the same order, while here, the latter is much smaller. In absence of such resonances, it is possible to find *locally stationary* solutions for the corrector terms, that is, solutions to (1.10) that do not depend on the fast time variable τ. When such resonances occur, the dependence of the correctors on τ cannot be removed, and this induces secular growth effects that destroy the accuracy of the approximation (it is only valid on a much smaller time scale, $t = o(1)$, than the relevant one). It is likely that in this case, the dynamics of the leading term (ζ_0, V_0) is affected, but this point is left for a future study.

The Ansatz (1.7)–(1.8) and the error estimates for the quantity E_a represent a problem in homogenization theory. The principal terms (ζ_0, ψ_0) are solutions of an effective equation, and the multiscale terms (ζ_1, ψ_1) are the first corrector terms. The dynamics of the Euler equations require solving an elliptic equation at each instant of time, on an unknown domain $\Omega(b_\gamma, \zeta)$ whose boundaries are defined by oscillatory functions. The approach we take in this paper to the analysis of this elliptic problem and its asymptotic behavior is to transform this domain to a reference domain Ω_0, resulting in an elliptic problem with rapidly varying periodic coefficients. The principal (effective) term and the correctors are derived from this problem, with the principal term solving an effective equation, and the corrector solving an appropriate cell problem. These are then used to express the Dirichlet–Neumann operator on the free surface of the fluid domain, which in turn is used to express the evolution equations (1.1). The dynamics of the short spatial scales are separated from the evolution of the long scales using the concept of convergence on two scales [1]. The principal part of our mathematical analysis is to control the error estimates of the homogenization approximation (1.7)–(1.8) in describing the solutions of this elliptic boundary value problem and the associated expression for the Dirichlet–Neumann problem.

2. Euler equations

Zakharov showed that the water wave problem can be written in the Hamiltonian form [30]

$$\partial_t \begin{pmatrix} \zeta \\ \psi \end{pmatrix} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \begin{pmatrix} \delta_\zeta H \\ \delta_\psi H \end{pmatrix}, \tag{2.1}$$

where the canonical variables are the surface elevation ζ and the trace of the velocity potential on the free surface $\psi = \Phi_{|z=\zeta}$, and the Hamiltonian H is given by

$$H(\zeta, \psi) = \frac{1}{2} \int_{\mathbb{R}^d} \psi G[\zeta, b]\psi + g\zeta^2 \, dX. \tag{2.2}$$

The system for (ζ, ψ) is written as (1.1), which in dimensionless form becomes

$$\begin{cases} \partial_t \zeta - \dfrac{1}{\mu} G_\mu[\zeta, \beta b_\gamma]\psi = 0, \\ \partial_t \psi + \zeta + \dfrac{1}{2}|\nabla\psi|^2 - \mu \dfrac{(\frac{1}{\mu} G_\mu[\zeta, \beta b_\gamma]\psi + \nabla\zeta \cdot \nabla\psi)^2}{2(1 + \mu|\nabla\zeta|^2)} = 0, \end{cases} \tag{2.3}$$

where $b_\gamma(\cdot) = b(\cdot/\gamma)$ and where $G_\mu[\zeta, \beta b_\gamma]$ is the nondimensionalized Dirichlet–Neumann operator defined by

$$G_\mu[\zeta, \beta b_\gamma]\psi = \sqrt{1 + |\nabla\zeta|^2}\,\partial_n \Phi_{|z=\zeta} \tag{2.4}$$

and where Φ is the potential function, satisfying

$$\begin{cases} \mu\Delta\Phi + \partial_z^2\Phi = 0 & \text{in } \Omega, \\ \Phi_{|z=\zeta} = \psi, & \partial_n\Phi_{|z=-1+\beta b_\gamma} = 0, \end{cases} \tag{2.5}$$

in the fluid domain $\Omega(b_\gamma, \zeta)$,

$$\Omega(b_\gamma, \zeta) = \{(X, z) \in \mathbb{R}^{d+1}, -1 + \beta b(X/\gamma) < z < \zeta(X)\}.$$

The operator ∂_n is the outwards *conormal* derivative associated with the operator $\mu\Delta + \partial_z^2$. One can rewrite (2.3) in Hamiltonian form (2.1), replacing the Hamiltonian H given by (2.2) by its nondimensional form

$$H(\zeta, \psi) = \frac{1}{2}\int_{\mathbb{R}^d}\left(\psi\frac{1}{\mu}G_\mu[\zeta, \beta b_\gamma]\psi + \zeta^2\right)dX. \tag{2.6}$$

2.1. Notation

We denote by $d = 1$ or 2 the horizontal dimension of the fluid domain, and by $X \in \mathbb{R}^d$ the horizontal variables, while z is the vertical variable. We denote by $\mathbf{e}_z$ the unit upward vertical vector.

The domain and the potential function will depend upon both regular and rapidly oscillating variables, which we denote $X \in \mathbb{R}^d$ and $Y \in \mathbb{T}^d = \mathbb{R}^d/(2\pi\mathbb{Z})^d$, respectively. That is, we will give data for the water wave problem which is of a multiscale nature, with the fixed multiscale bottom variations as well, and we will seek solutions which have a well-defined asymptotic expansion in terms of multiscale quantities. To express this, we use the classical notation of a multiscale function that is, a function $f(X, Y)$ defined on $\mathbb{R}^d \times \mathbb{T}^d$, for which the realization is the trace $f|_{Y=\frac{X}{\gamma}} = f(X, X/\gamma)$ [3]. In the problem we consider, there are other variables as well, such as the vertical variable $z \in [-1, 0]$, for which $f = f(X, Y, z)$ is a multiscale function whose realization is $f(X, X/\gamma, z)$.

The differential operators ∇ and Δ act on functions of the horizontal variable X. The operator Λ is defined by $\Lambda := (1 - \Delta)^{1/2}$. We use the standard notation for Fourier multipliers, namely $D = \frac{1}{i}\nabla$ and $\widehat{f(D)u}(k) = f(k)\hat{u}(k)$. When applied to multiscale functions, we distinguish this fact using the notation ∇_Y, Δ_Y, D_Y, when differential operators act specifically on the fast variables Y, and ∇_X, Δ_X, D_X when they act on the long scale X variables. Finally, the notation ∇^μ stands for $\nabla^\mu = (\sqrt{\mu}\nabla^T, \partial_z)^T$.

We encounter functions defined on the fluid domain $\Omega(b_\gamma, \zeta)$ or the reference domain $\Omega_0 = \mathbb{R}^d \times (0, 1)$, as well as functions defined on the free surface, parametrized by $X \in \mathbb{R}^d$. The notation used for function space norms is that $\|\cdot\|_{L^2}$, $\|\cdot\|_{H^r}$ is used for the classical Sobolev space norms over Ω_0, while for norms defined over the boundary $X \in \mathbb{R}^d$ we use the notation $|\cdot|_{L^2}$, $|\cdot|_{H^r}$. Norms of multiscale functions are given similarly, for example $|\cdot|_{L^2(C_Y^1)}$.

For all $r_1, r_2 \geqslant 0$, we also define the space $H^{r_1,r_2} = H^{r_1,r_2}(\mathbb{R}^d \times \mathbb{T}^d)$ by

$$H^{r_1,r_2}(\mathbb{R}^d \times \mathbb{T}^d) = \{f \in L^2(\mathbb{R}^d \times \mathbb{T}^d),\ |f|_{H^{r_1,r_2}} < \infty\}, \tag{2.7}$$

with $|f|^2_{H^{r_1,r_2}} = |(1 - \Delta_X)^{r_1/2}(1 - \Delta_Y)^{r_2/2}f|^2_{L^2(\mathbb{R}^d\times\mathbb{T}^d)}$.

2.2. Change of variables and domain

The first component of the Hamiltonian (2.6) corresponds to the nondimensionalized kinetic energy. It follows from the definition of $G_\mu[\zeta, \beta b_\gamma]$ and Green's identity that

$$\int_{\mathbb{R}^d}\psi\frac{1}{\mu}G_\mu[\zeta, \beta b_\gamma]\psi\,dX = \frac{1}{\mu}\int_\Omega|\nabla^\mu\Phi|^2\,dz\,dX, \tag{2.8}$$

where Φ is the velocity potential (2.5). Since this expression depends on β and γ through the domain of integration $\Omega(b_\gamma, \zeta)$, it is convenient to transform it into an integral over a fixed domain independent of the parameters and of the perturbations ζ and b. Under the assumption that the fluid height $h = 1 + \zeta - \beta b_\gamma$ is always non-negative, namely

$$\exists\alpha > 0, \quad 1 + \zeta - \beta b_\gamma \geqslant \alpha \quad \text{on } \mathbb{R}^d, \tag{2.9}$$

an explicit diffeomorphism S mapping the flat strip Ω_0 onto the fluid domain Ω is given by

$$S: \quad \begin{array}{rcl} \Omega_0 & \to & \Omega, \\ (X,z) & \mapsto & \big(X, z+\sigma(X,z)\big), \end{array} \tag{2.10}$$

where $\sigma(X,z) = (z+1)\zeta(X) - z\beta b_\gamma(X)$. We have in particular $h = 1 + \partial_z\sigma$.

Defining ϕ on Ω_0 by $\phi = \Phi \circ S$, one can check (see Proposition 2.7 of [18] and §2.2 of [2]) that the new potential function ϕ solves

$$\begin{cases} \nabla^\mu \cdot P[\sigma]\nabla^\mu \phi = 0, \\ \phi_{|z=0} = \psi, \qquad \partial_n \phi_{|z=-1} = 0, \end{cases} \tag{2.11}$$

where $\partial_n \phi_{|z=-1}$ is the outward conormal derivative in the new variables

$$\partial_n \phi_{|z=-1} = -\mathbf{e}_z \cdot P[\sigma]\nabla^\mu \phi_{|z=-1}$$

and where the matrix $P[\sigma]$ is given by

$$P[\sigma] = \begin{pmatrix} hI & -\sqrt{\mu}\nabla\sigma \\ -\sqrt{\mu}\nabla\sigma^T & \frac{1+\mu|\nabla\sigma|^2}{h} \end{pmatrix}, \quad \text{with} \quad h = 1 + \zeta - \beta b_\gamma. \tag{2.12}$$

3. Multiple scale asymptotic expansions

3.1. Ansatz and decomposition of the solutions

This section is devoted to the study of the elliptic problem (2.11) where (ζ, ψ) are given; the time is fixed and appears as a parameter. We pose the multiple-scale Ansatz on (ζ, ψ):

$$\zeta = \zeta_0(X) + \gamma\zeta_1(X, X/\gamma), \qquad \psi = \psi_0(X) + \gamma^2\psi_1(X, X/\gamma). \tag{3.1}$$

Recalling that $\beta = \gamma = \sqrt{\mu}$, this leads to the decomposition of the height function of the fluid domain

$$h = h_0 + \beta h_1, \quad \text{where} \quad h_0 = 1 + \zeta_0 \quad \text{and} \quad h_1 = \zeta_1 - b_\gamma.$$

Similarly, the new vertical deformations are posed in terms of this Ansatz

$$\sigma = \sigma_0 + \beta\sigma_1, \quad \text{where} \quad \sigma_0 = (z+1)\zeta_0 \quad \text{and} \quad \sigma_1 = (z+1)\zeta_1 - zb_\gamma.$$

The coefficients $P[\sigma]$ are then written as

$$P[\sigma] = P_0 + \beta P_1,$$

with

$$P_0 = P[\sigma_0] \quad \text{and} \quad \beta P_1 = P[\sigma] - P_0. \tag{3.2}$$

Explicitly

$$P_1 = \begin{pmatrix} (\zeta_1 - b)I & -\sqrt{\mu}\nabla\sigma_1 \\ -\sqrt{\mu}\nabla\sigma_1^T & \frac{1}{\beta}\big(\frac{1+\mu|\nabla\sigma|^2}{h} - \frac{1+\mu|\nabla\sigma_0|^2}{h_0}\big) \end{pmatrix}.$$

We accordingly decompose the potential function ϕ as

$$\phi = \phi_0(X,z) + \beta\gamma\chi(X,z;\gamma) = \phi_0(X,z) + \mu\chi(X,z;\gamma) \tag{3.3}$$

where all the contributions coming from the roughness are contained in χ. This section is devoted to deriving asymptotic expansions, with accompanying error estimates on the two components ϕ_0 and χ, in the limit $\mu \to 0$. In order to do so, we must augment (2.9) with the assumption that

$$\exists\alpha_0 > 0, \quad 1 + \zeta_0 \geqslant \alpha_0 \quad \text{on } \mathbb{R}^d. \tag{3.4}$$

This ensures that the water depth does not vanish for the averaged fluid domain that arises when all the fluctuations due to the roughness are neglected. Assumption (3.4) ensures the coercivity of P_0.

Proposition 3.1. *Let $\zeta, b \in W^{1,\infty}(\mathbb{R}^d)$ and assume that* (2.9) *and* (3.4) *are satisfied. Then for all ψ such that $\nabla\psi \in H^{1/2}(\mathbb{R}^d)^d$, there exists a unique solution ϕ to* (2.11) *such that $\nabla^\mu\phi \in H^1(\Omega_0)^{d+1}$. Moreover, ϕ_0 and χ solve*

$$\begin{cases} \nabla^\mu \cdot P_0\nabla^\mu\phi_0 = 0, \\ \phi_{0|_{z=0}} = \psi_0, \qquad -\mathbf{e}_z \cdot P_0\nabla^\mu\phi_{0|_{z=-1}} = 0, \end{cases} \tag{3.5}$$

and

$$\begin{cases} \nabla^\mu \cdot P[\sigma]\nabla^\mu\chi = -\dfrac{1}{\gamma}\nabla^\mu \cdot P_1\nabla^\mu\phi_0, \\ \chi_{|_{z=0}} = \psi_1, \qquad -\mathbf{e}_z \cdot P[\sigma]\nabla^\mu\chi_{|_{z=-1}} = \dfrac{1}{\gamma}\mathbf{e}_z \cdot P_1\nabla^\mu\phi_{0|_{z=-1}}. \end{cases} \tag{3.6}$$

Proof. The existence of a unique solution ϕ such that $\nabla^\mu\phi \in H^1(\Omega_0)^{d+1}$ to (2.11) is a classical result, and we thus omit the proof. Similarly, there exists a unique solution ϕ_0 such that $\nabla^\mu\phi_0 \in H^1(\Omega_0)^{d+1}$ to (3.5) since the boundary condition on the lower boundary is the conormal derivative associated to the elliptic operator $\nabla^\mu \cdot P_0\nabla^\mu$. It remains to prove that χ solves (3.6). A calculation gives that

$$\begin{aligned} \partial_n\phi_{|_{z=-1}} &:= -\mathbf{e}_z \cdot P[\sigma]\nabla^\mu\phi_{|_{z=-1}} \\ &= -\beta\gamma\mathbf{e}_z \cdot P[\sigma]\nabla^\mu\chi_{|_{z=-1}} - \mathbf{e}_z \cdot P_0\nabla^\mu\phi_{0|_{z=-1}} - \beta\mathbf{e}_z \cdot P_1\nabla^\mu\phi_{0|_{z=-1}}. \end{aligned}$$

Since by assumption one also has $\partial_n\phi_{|_{z=-1}} = 0$ and $-\mathbf{e}_z \cdot P_0\nabla^\mu\phi_{0|_{z=-1}} = 0$, one has

$$-\mathbf{e}_z \cdot P[\sigma]\nabla^\mu\chi_{|_{z=-1}} = \frac{1}{\gamma}\mathbf{e}_z \cdot P_1\nabla^\mu\phi_{0|_{z=-1}}.$$

It is straightforward to check that $\chi_{|_{z=0}} = \psi_1$ and that

$$\begin{aligned} \nabla^\mu \cdot P[\sigma]\nabla^\mu\chi &= \frac{1}{\gamma\beta}\big(\nabla^\mu \cdot P[\sigma]\nabla^\mu\phi - \nabla^\mu \cdot P[\sigma]\nabla^\mu\phi_0\big) \\ &= -\frac{1}{\gamma}\nabla^\mu \cdot P_1\nabla^\mu\phi_0, \end{aligned}$$

and the result follows. □

3.2. Asymptotic analysis with estimates of $\nabla^\mu\phi_0$

In this section we prove an estimate on $\nabla^\mu\phi_0$, and we give the first terms of its asymptotic expansion in the limit as $\mu \to 0$. For purposes of understanding the $H^{-1/2}$-norm of the trace of $\nabla^\mu\phi_0$ on the free surface $\{z = 0\}$, we use L^2 estimates on Ω_0 of both $\nabla^\mu\phi_0$ and its generalized Riesz transform, given by $\Lambda^{-1}\partial_z\nabla^\mu\phi_0$. This is generalized to higher order norms.

Proposition 3.2. *Let $r \in \mathbb{N}$ and $\zeta_0 \in W^{1+r,\infty} \cap W^{2,\infty}(\mathbb{R}^d)$ and assume that* (3.4) *is satisfied for some $\alpha_0 > 0$. Then*:

(i) *For all $\mu \in (0, 1)$ and all ψ_0 such that $\nabla\psi_0 \in H^r(\mathbb{R}^d)^d$, the solution ϕ_0 to* (3.5) *satisfies*

$$\big\|\Lambda^r\nabla^\mu\phi_0\big\|_{L^2} \leqslant \sqrt{\mu}C\left(\frac{1}{\alpha_0}, |\zeta_0|_{W^{1+r,\infty}}\right)|\nabla\psi_0|_{H^r};$$

$$\big\|\Lambda^{r-1}\partial_z\nabla^\mu\phi_0\big\|_{L^2} \leqslant \mu C\left(\frac{1}{\alpha_0}, |\zeta_0|_{W^{1+r,\infty}}, |\zeta_0|_{W^{2,\infty}}\right)|\nabla\psi_0|_{H^r}.$$

(ii) *If $\nabla\psi_0 \in H^{r+1}(\mathbb{R}^d)^d$, one also has*

$$\big\|\Lambda^r\big(\nabla^\mu\phi_0 - \nabla^\mu\psi_0\big)\big\|_{L^2} \leqslant \mu C\left(\frac{1}{\alpha_0}, |\zeta_0|_{W^{1+r,\infty}}\right)|\nabla\psi_0|_{H^{r+1}};$$

$$\big\|\Lambda^{r-1}\partial_z\big(\nabla^\mu\phi_0 - \nabla^\mu\psi_0\big)\big\|_{L^2} \leqslant \mu C\left(\frac{1}{\alpha_0}, |\zeta_0|_{W^{1+r,\infty}}, |\zeta_0|_{W^{2,\infty}}\right)|\nabla\psi_0|_{H^{r+1}}.$$

(iii) *Suppose that* $\zeta_0 \in W^{2+r,\infty}(\mathbb{R}^d)$ *and* $\Delta\psi_0 \in H^{2+r}(\mathbb{R}^d)$, *and set*

$$\phi_0^{(1)} = -h_0^2\left(\frac{z^2}{2} + z\right)\Delta\psi_0 \quad (h_0 := 1 + \zeta_0),$$

as the next term of the asymptotic expansion. Then there are estimates of the remainder, in the form

$$\left\| \Lambda^r\left(\nabla^\mu\phi_0 - \nabla^\mu\left(\psi_0 - \mu\phi_0^{(1)}\right)\right)\right\|_{L^2} \leqslant \mu^2 C\left(\frac{1}{\alpha_0}, |\zeta_0|_{W^{2+r,\infty}}\right)|\Delta\psi_0|_{H^{2+r}};$$

$$\left\| \Lambda^{r-1}\partial_z\nabla^\mu\left(\phi_0 - \psi_0 - \mu\phi_0^{(1)}\right)\right\|_{L^2} \leqslant \mu^2 C\left(\frac{1}{\alpha_0}, |\zeta_0|_{W^{2+r,\infty}}\right)|\Delta\psi_0|_{H^{2+r}}.$$

Proof. The first inequality of the proposition is obtained by standard elliptic estimates for $\|\nabla^\mu\phi_0\|_{L^2}$ (see Corollary 2.2 of [2]). The Riesz transform of $\nabla^\mu\phi_0$ has components $\Lambda^{-1}\partial_z\nabla^\mu\phi_0 = (\sqrt{\mu}\Lambda^{-1}\nabla\partial_z\phi_0, \Lambda^{-1}\partial_z^2\phi_0)$; estimates of the first component come from the first inequality of (i), since $\Lambda^{-1}\nabla$ is L^2 bounded. Estimates of the second component are obtained through Eq. (3.5) itself. Namely one has an expression for $\partial_z^2\phi_0$ in the form

$$\partial_z^2\phi_0 = \mu\frac{h_0}{1+\mu|\nabla\sigma_0|^2}\left[-\frac{\partial_z|\nabla\sigma_0|^2}{h_0}\partial_z\phi_0 + \partial_z(\nabla\sigma_0\cdot\nabla\phi_0) + \nabla\cdot(\nabla\sigma_0\partial_z\phi_0) - \nabla\cdot(h_0\nabla\phi_0)\right]. \tag{3.7}$$

In order to get an estimate on $\|\Lambda^{r-1}\partial_z^2\phi_0\|_2$, we need the following lemma.

Lemma 3.3. *Let* $r \in \mathbb{N}$ *and* F *and* $G \geqslant 0$ *be such that* $\Lambda^{r-1}F \in L^2(\Omega_0)$ *and* $G \in L^\infty((-1,0); W^{|r-1|,\infty}(\mathbb{R}^d))$. *Then*

$$\left\|\Lambda^{r-1}\left(\frac{F}{1+G}\right)\right\|_{L^2} \leqslant C\left(\|G\|_{L_z^\infty W_X^{|r-1|,\infty}}\right)\left\|\Lambda^{r-1}F\right\|_{L^2}.$$

Proof. Just write $\frac{F}{1+G} = F - F\frac{G}{1+G}$. Recalling that one has $|fg|_{H^{r-1}} \lesssim |f|_{H^{r-1}}|g|_{W^{|r-1|,\infty}}$, we get

$$\begin{aligned}\left\|\Lambda^{r-1}\left(\frac{F}{1+G}\right)\right\|_{L^2} &\lesssim \left\|\Lambda^{r-1}F\right\|_{L^2}\left(1 + \left\|\frac{G}{1+G}\right\|_{L_z^\infty W_X^{|r-1|,\infty}}\right)\\ &\lesssim C\left(\|G\|_{L_z^\infty W_X^{|r-1|,\infty}}\right)\left\|\Lambda^{r-1}F\right\|_{L^2}. \qquad \square\end{aligned}$$

Applying this lemma to (3.7) with $G = \mu|\nabla\sigma_0|^2$, one easily gets

$$\left\|\Lambda^{r-1}\partial_z^2\phi_0\right\|_{L^2} \leqslant \mu C\left(\frac{1}{\alpha_0}, |\zeta_0|_{W^{r+1,\infty}}\right)\frac{1}{\sqrt{\mu}}\left\|\Lambda^r\nabla^\mu\phi_0\right\|_{L^2},$$

and the estimate follows from the control on $\|\Lambda^r\nabla^\mu\phi_0\|_{L^2}$ established above.

For the second point of the proposition, we write $\phi_0 = \psi_0 + \mu\chi_0^{(1)}$. The resulting system for $\chi_0^{(1)}$ is

$$\begin{cases} \nabla^\mu\cdot P_0\nabla^\mu\chi_0^{(1)} = -\frac{1}{\mu}\nabla^\mu\cdot P_0\nabla^\mu\psi_0, \\ \chi_0^{(1)}{}_{|z=0} = 0, \qquad -\mathbf{e}_z\cdot P_0\nabla^\mu\chi_0^{(1)}{}_{|z=-1} = \frac{1}{\mu}\mathbf{e}_z\cdot P_0\nabla^\mu\psi_{0|z=-1}. \end{cases} \tag{3.8}$$

A calculation shows that

$$-\frac{1}{\mu}\nabla^\mu\cdot P_0\nabla^\mu\psi_0 = -\nabla\cdot(h_0\nabla\psi_0) + \partial_z(\nabla\sigma_0\cdot\nabla\psi_0) = -h_0\Delta\psi_0$$

and that $\mathbf{e}_z\cdot P_0\nabla^\mu\psi_{0|z=-1} = 0$, we obtain

$$\begin{cases} \nabla^\mu\cdot P_0\nabla^\mu\chi_0^{(1)} = -h_0\Delta\psi_0, \\ \chi_0^{(1)}{}_{|z=0} = 0, \qquad -\mathbf{e}_z\cdot P_0\nabla^\mu\chi_0^{(1)}{}_{|z=-1} = 0. \end{cases} \tag{3.9}$$

Multiplying the equation by $\chi_0^{(1)}$ and integrating by parts, we get

$$\int_{\Omega_0} \nabla^\mu \chi_0^{(1)} \cdot P_0 \nabla^\mu \chi_0^{(1)}\, dz\, dX = \int_{\Omega_0} \chi_0^{(1)} h_0 \Delta\psi_0\, dz\, dX.$$

Using the coercivity of the matrix P_0 (Proposition 2.3(iii) of [2]), the Cauchy–Schwarz inequality and Poincaré inequality (in order to control $\|\chi_0^{(1)}\|_{L^2}$ by $\|\nabla^\mu \chi_0^{(1)}\|_{L^2}$), one gets

$$\|\nabla^\mu \chi_0^{(1)}\|_{L^2} \leqslant C\left(\frac{1}{\alpha_0}, |\zeta_0|_{W^{1,\infty}}\right)|\Delta\psi_0|_{L^2}$$

and the result follows. Higher order estimates are handled similarly and only require the control of additional commutator estimates. We omit these classical details. The estimate of the generalized Riesz transform $\Lambda^{-1}\partial_z\nabla^\mu(\phi_0 - \psi_0)$ is similar to the analog estimate in (i) of this proposition.

For the third point of the proposition, we solve (3.9) at lowest order in μ. We write $\chi_0^{(1)} = \phi_0^{(1)} + \mu\chi_0^{(2)}$, or equivalently

$$\phi_0 = \psi_0 + \mu\phi_0^{(1)} + \mu^2\chi_0^{(2)}$$

with $\phi_0^{(1)} = -h_0^2(\frac{z^2}{2} + z)\Delta\psi_0$. The correction $\chi_0^{(2)}$ satisfies the system

$$\begin{cases} \nabla^\mu \cdot P_0 \nabla^\mu \chi_0^{(2)} = -\nabla\cdot\left(h_0\nabla\phi_0^{(1)} - \nabla\sigma_0\partial_z\phi_0^{(1)}\right) + \partial_z\left(\nabla\sigma_0\cdot\nabla\phi_0^{(1)} - \dfrac{|\nabla\sigma_0|^2}{h_0}\partial_z\phi_0^{(1)}\right), \\ \chi_0^{(2)}{}_{|z=0} = 0, \qquad -\mathbf{e}_z\cdot P_0\nabla^\mu\chi_0^{(2)}{}_{|z=-1} = 0, \end{cases} \tag{3.10}$$

where we used the fact that $\mathbf{e}_z \cdot P_0\nabla^\mu\phi_0^{(1)}{}_{|z=-1} = 0$ to obtain the bottom boundary conditions. Proceeding as above, we get the result. □

3.3. Asymptotic analysis with estimates of χ

To find an asymptotic expansion of χ, our starting point is Eq. (3.6) for χ. Decompose the solution as the sum of a multiscale function and a correction term,

$$\chi = \phi_1^{(0)}(X, Y, z)|_{Y=\frac{X}{\gamma}} + \sqrt{\mu}\chi_1^{(1)}(X, z; \gamma). \tag{3.11}$$

When acting on a multiscale function of the variables $(X, X/\gamma)$, the operator ∇^μ becomes $\nabla_{Y,z} + \nabla_X^\mu$, where $\nabla_X^\mu = \binom{\sqrt{\mu}\nabla_X}{0}$:

$$\nabla^\mu\left(f(X,Y)|_{Y=\frac{X}{\gamma}}\right) = \left[\left(\nabla_{Y,z} + \nabla_X^\mu\right)f(X,Y)\right]\big|_{Y=\frac{X}{\gamma}}.$$

We can therefore write

$$\begin{aligned} \nabla^\mu \cdot P[\sigma]\nabla^\mu\chi = {} & \left(\nabla_{Y,z} + \nabla_X^\mu\right)\cdot P_0\left(\nabla_{Y,z} + \nabla_X^\mu\right)\phi_1^{(0)}\big|_{Y=\frac{X}{\gamma}} \\ & + \beta\nabla^\mu\cdot P_1\nabla^\mu\phi_1^{(0)} + \sqrt{\mu}\nabla^\mu\cdot P[\sigma]\nabla^\mu\chi_1^{(1)}, \end{aligned}$$

so that (3.6) becomes (recall that $\beta = \sqrt{\mu}$)

$$\begin{cases} \nabla_{Y,z}\cdot P_0\nabla_{Y,z}\phi_1^{(0)}\big|_{Y=\frac{X}{\gamma}} + \sqrt{\mu}\nabla^\mu\cdot P[\sigma]\nabla^\mu\chi_1^{(1)} \\ \quad = -\dfrac{1}{\gamma}\nabla^\mu\cdot P_1\nabla^\mu\phi_0 + \sqrt{\mu}\nabla^\mu\cdot\widetilde{A} + \sqrt{\mu}\widetilde{g}, \\ \left(\phi_1^{(0)}{}_{|_{Y=\frac{X}{\gamma}}} + \sqrt{\mu}\chi_1^{(1)}\right)_{|z=0} = \psi_1{}_{|_{Y=\frac{X}{\gamma}}}, \\ \left(-\mathbf{e}_z\cdot P_0\nabla_{Y,z}\phi_1^{(0)}\big|_{Y=\frac{X}{\gamma}}\right)_{|z=-1} - \sqrt{\mu}\mathbf{e}_z\cdot P[\sigma]\nabla^\mu\chi_1^{(1)}{}_{|z=-1} \\ \quad = \dfrac{1}{\gamma}\mathbf{e}_z\cdot P_1\nabla^\mu\phi_0{}_{|z=-1} - \sqrt{\mu}\mathbf{e}_z\cdot\widetilde{A}_{|z=-1} \end{cases} \tag{3.12}$$

with[1]

$$\widetilde{A} = -P_1 \nabla^\mu \phi_1^{(0)} - P_0 \begin{pmatrix} \nabla_X \phi_1^{(0)} \\ 0 \end{pmatrix}_{|_{Y=\frac{X}{\gamma}}}$$

and

$$\widetilde{g} = -\begin{pmatrix} \nabla_X \\ 0 \end{pmatrix} \cdot \left(P_0 \nabla_{Y,z} \phi_1^{(0)}\right)_{|_{Y=\frac{X}{\gamma}}}.$$

In order to make the leading order terms in (3.12) explicit, we further decompose P_0 as $P_0 = P_0^{(0)} + \sqrt{\mu} P_0^{(1)}$ and $P_1 = P_1^{(0)} + \sqrt{\mu} P_1^{(1)}$, with

$$P_0^{(0)} = \begin{pmatrix} h_0 I & 0 \\ 0 & \frac{1}{h_0} I \end{pmatrix}, \qquad P_0^{(1)} = \begin{pmatrix} 0 & -\nabla\sigma_0 \\ -\nabla\sigma_0^T & \frac{\sqrt{\mu}|\nabla\sigma_0|^2}{h_0} \end{pmatrix}$$

and

$$P_1^{(0)} = \begin{pmatrix} (\zeta_1 - b) I & -\nabla_Y \sigma_1 \\ -\nabla_Y \sigma_1^T & \frac{b-\zeta_1}{h_0^2} I \end{pmatrix}, \qquad P_1^{(1)} = \begin{pmatrix} 0 & -\nabla_X \sigma_1 \\ -\nabla_X \sigma_1^T & p_{22}^{(1)} \end{pmatrix}$$

where the $(2,2)$-coefficient of $P_1^{(1)}$ is

$$p_{22}^{(1)} = \mu^{-1/2}\left(\beta^{-1}\left(\frac{1+\mu|\nabla\sigma|^2}{h} - \frac{1+\mu|\nabla\sigma_0|^2}{h_0}\right) - \frac{b-\zeta_1}{h_0^2}\right).$$

Lemma 3.4. *The coefficient matrix $P_1 = P_1(\sigma)$ has multiscale functions as coefficients. Considered as $P_1 = P_1(X, X/\sqrt{\mu})$, the following estimates hold for all $r \in \mathbb{N}$:*

$$\left\| P_1^{(0)} \right\|_{L_z^\infty W^{r,\infty}} + \left\| \partial_z P_1^{(0)} \right\|_{L_z^\infty W^{r,\infty}} \leqslant \mu^{-r/2} C\left(\frac{1}{\alpha_0}, |\zeta_0|_{C^{r+1}}\right)\left(|\zeta_1|_{C^r} + |\nabla_Y \zeta_1|_{C^r} + |b|_{C^{r+1}}\right),$$

$$\left\| P_1^{(1)} \right\|_{L_z^\infty W^{r,\infty}} + \left\| \partial_z P_1^{(1)} \right\|_{L_z^\infty W^{r,\infty}} \leqslant \mu^{-r/2} C\left(\frac{1}{\alpha_0}, \frac{1}{\alpha}, |\zeta_0|_{C^{r+1}}, |\zeta_1|_{C^r}, |\nabla_X \zeta_1|_{C^r}, |\nabla_Y \zeta_1|_{C^r}, |b|_{C^{r+1}}\right).$$

Proof. The proof follows by inspecting the elements of P_1, $P_1^{(0)}$ and $P_1^{(1)}$. □

Given the above decompositions of P_0 and P_1, the first term of the LHS of (3.12) is

$$\nabla_{Y,z} \cdot P_0 \nabla_{Y,z} \phi_1^{(0)} = \nabla_{Y,z} \cdot P_0^{(0)} \nabla_{Y,z} \phi_1^{(0)} + \sqrt{\mu} \nabla_{Y,z} \cdot P_0^{(1)} \nabla_{Y,z} \phi_1^{(0)}$$

and, using that $\gamma = \sqrt{\mu}$, the first term of the RHS of (3.12) is

$$\begin{aligned} \frac{1}{\gamma} \nabla^\mu \cdot P_1 \nabla^\mu \phi_0 &= \frac{1}{\gamma} \nabla^\mu \cdot P_1^{(0)} \nabla^\mu \psi_0 + \frac{1}{\gamma} \nabla^\mu \cdot P_1^{(0)} \nabla^\mu (\phi_0 - \psi_0) + \nabla^\mu \cdot P_1^{(1)} \nabla^\mu \phi_0 \\ &= \nabla_{Y,z} \cdot P_1^{(0)} \begin{pmatrix} \nabla\psi_0 \\ 0 \end{pmatrix}\Big|_{Y=\frac{X}{\gamma}} + \nabla_X^\mu \cdot P_1^{(0)} \begin{pmatrix} \nabla\psi_0 \\ 0 \end{pmatrix}\Big|_{Y=\frac{X}{\gamma}} \\ &\quad + \frac{1}{\gamma} \nabla^\mu \cdot P_1^{(0)} \nabla^\mu (\phi_0 - \psi_0) + \nabla^\mu \cdot P_1^{(1)} \nabla^\mu \phi_0. \end{aligned}$$

[1] The operator ∇^μ always acts on multiscale functions on the two variables X and z (and not on Y). The notation $\nabla^\mu \phi_1^{(0)}$ is therefore a shortcut for $\nabla^\mu (\phi_1^{(0)}|_{Y=\frac{X}{\gamma}})$.

Extracting the principal term from these two expressions, we deduce that

$$\begin{cases} \nabla_{Y,z}\cdot P_0^{(0)}\nabla_{Y,z}\phi_1^{(0)}\big|_{Y=\frac{X}{\gamma}}+\sqrt{\mu}\nabla^\mu\cdot P[\sigma]\nabla^\mu\chi_1^{(1)} \\ \quad =-\nabla_{Y,z}\cdot P_1^{(0)}\begin{pmatrix}\nabla\psi_0\\0\end{pmatrix}\big|_{Y=\frac{X}{\gamma}}+\sqrt{\mu}\nabla^\mu\cdot A+\sqrt{\mu}g|_{Y=\frac{X}{\gamma}}, \\ \big(\phi_1^{(0)}{}_{|_{Y=\frac{X}{\gamma}}}+\sqrt{\mu}\chi_1^{(1)}\big)_{|z=0}=\psi_1|_{Y=\frac{X}{\gamma}}, \\ -\mathbf{e}_z\cdot\big(P_0^{(0)}\nabla_{Y,z}\phi_1^{(0)}\big|_{Y=\frac{X}{\gamma}}\big)_{|z=-1}-\sqrt{\mu}\mathbf{e}_z\cdot P[\sigma]\nabla^\mu\big(\chi_1^{(1)}{}_{|_{Y=\frac{X}{\gamma}}}\big)\big|_{z=-1} \\ \quad =\mathbf{e}_z\cdot\left\{P_1^{(0)}\begin{pmatrix}\nabla\psi_0\\0\end{pmatrix}-\sqrt{\mu}\mathbf{e}_z\cdot A\right\}_{|z=-1}, \end{cases} \tag{3.13}$$

with

$$A=\widetilde{A}-\frac{1}{\gamma\sqrt{\mu}}P_1^{(0)}\nabla^\mu(\phi_0-\psi_0)-\frac{1}{\sqrt{\mu}}P_1^{(1)}\nabla^\mu\phi_0+\left(2\nabla\sigma_0\cdot\nabla_Y\phi_1^{(0)}-\sqrt{\mu}\frac{|\nabla\sigma_0|^2}{h_0}\partial_z\phi_1^{(0)}\right)\mathbf{e}_z$$

and with

$$g|_{Y=\frac{X}{\gamma}}=\widetilde{g}-\nabla\partial_z\sigma_0\cdot\nabla_Y\phi_1^{(0)}-(\nabla_X\zeta_1)\cdot\nabla\psi_0-(\zeta_1-b)\Delta\psi_0. \tag{3.14}$$

In order to solve (3.13), we construct $\phi_1^{(0)}$ as a solution of a *cell problem* in the variables Y and z (the variable X being considered a parameter). The resulting solution cancels the higher order terms in (3.13), and we are left with an equation for the corrector $\chi_1^{(1)}$.

3.3.1. The cell problem

We assume that b,ζ_1 are periodic with respect to the variable Y and we seek a periodic function $\phi_1^{(0)}(\cdot,Y,z)$ that solves

$$\begin{cases} \nabla_{Y,z}\cdot P_0^{(0)}\nabla_{Y,z}\phi_1^{(0)}=-\nabla_{Y,z}\cdot P_1^{(0)}\begin{pmatrix}\nabla\psi_0\\0\end{pmatrix}, \\ \phi_1^{(0)}{}_{|z=0}=\psi_1; \qquad -\mathbf{e}_z\cdot P_0^{(0)}\nabla_{Y,z}\phi_1^{(0)}{}_{|z=-1}=\mathbf{e}_z\cdot P_1^{(0)}\begin{pmatrix}\nabla\psi_0\\0\end{pmatrix}. \end{cases} \tag{3.15}$$

This choice of $\phi_1^{(0)}$ cancels the highest order terms in (3.13). Taking into account the definition of $P_0^{(0)}$ and $P_1^{(0)}$, we can further simplify (3.15) into

$$\begin{cases} \big(h_0^2\Delta_Y+\partial_z^2\big)\phi_1^{(0)}=0, \\ \phi_1^{(0)}{}_{|z=0}=\psi_1, \qquad \frac{1}{h_0}\partial_z\phi_1^{(0)}{}_{|z=-1}=\nabla_Y b\cdot\nabla\psi_0. \end{cases} \tag{3.16}$$

We recall that the spaces H^{r_1,r_2} that appear in the statement below are defined in (2.7).

Proposition 3.5. *The solution $\phi_1^{(0)}$ of the cell problem* (3.16) *is given in operator notation by the expression*

$$\phi_1^{(0)}(X,Y,z)=\frac{\cosh(h_0(z+1)|D_Y|)}{\cosh(h_0|D_Y|)}\psi_1(X,Y)+\frac{\sinh(h_0z|D_Y|)}{\cosh(h_0|D_Y|)}\frac{\nabla_Y}{|D_Y|}b(Y)\cdot\nabla\psi_0(X). \tag{3.17}$$

Assume that $h_0=h_0(X)=1+\zeta_0$ satisfies the hypotheses (3.4)*, and let $r_0>d/2$ and $r\in\mathbb{N}$. Then, for all multiindex $\alpha=(\alpha^1,\alpha^2)\in\mathbb{N}^d\times\mathbb{N}^d$ such that $|\alpha^1|+|\alpha^2|=r$, one has*

$$\begin{aligned} &\big\|\partial_X^{\alpha^1}\partial_Y^{\alpha^2}\nabla_X\phi_1^{(0)}\big\|_{L^2_XL^\infty_{Y,z}}\leqslant C\big(|h_0|_{C^{r+1}}\big)\big(|\nabla_X\psi_1|_{H^{|\alpha^1|,r_0+|\alpha^2|}}+|\nabla_X\psi_1|_{H^{0,r_0+r}}+|b|_{C^{r+2}}|\nabla\psi_0|_{H^{r+1}}\big), \\ &\big\|\partial_X^{\alpha^1}\partial_Y^{\alpha^2}\nabla_{Y,z}\phi_1^{(0)}\big\|_{L^2_XL^\infty_{Y,z}}\leqslant C\big(|h_0|_{C^{r+1}}\big)\big(|\nabla_Y\psi_1|_{H^{|\alpha^1|,r_0+|\alpha^2|}}+|\nabla_Y\psi_1|_{H^{0,r_0+r}}+|b|_{C^{r+2}}|\nabla\psi_0|_{H^{r+1}}\big). \end{aligned} \tag{3.18}$$

Moreover, derivatives of the multiscale function $\phi_1^{(0)}|_{Y=\frac{X}{\gamma}}$ *are controlled as follows*

$$\|\Lambda^r\nabla^\mu\phi_1^{(0)}\|_{L^2}\leqslant\mu^{-r/2}C(|h_0|_{C^{r+1}})\times\big(\mu^{\frac{r+1}{2}}|\nabla_X\psi_1|_{H^{r,r_0}}+|\nabla_Y\psi_1|_{H^{0,r_0+r}}+|b|_{C^{r+2}}|\nabla\psi_0|_{H^{r+1}}\big),\tag{3.19}$$

and if $r\geqslant 1$*, the same upper bound holds for* $\frac{1}{\sqrt{\mu}}\|\Lambda^{r-1}\partial_z\nabla^\mu\phi_1^{(0)}\|_{L^2}$.

It is of note that the solution of the cell problem is a multiscale expression that can be differentiated arbitrarily many times with respect to the variables (X,Y,z) without developing singular behavior in the limit as $\mu\to 0$.

Proof. Decomposing the function $\phi_1^{(0)}$ in Fourier modes $\hat{\phi}_{1k}^{(0)}$ with respect to the Y variable, we find

$$\hat{\phi}_{1k}^{(0)}=A_k e^{h_0 z|k|}+B_k e^{-h_0 z|k|}$$

with coefficients

$$A_k=\frac{1}{e^{h_0|k|}+e^{-h_0|k|}}\left(i\frac{k}{|k|}\cdot\nabla\psi_0\hat{b}_k+e^{h_0|k|}\hat{\psi}_{1k}\right),$$
$$B_k=\frac{1}{e^{h_0|k|}+e^{-h_0|k|}}\left(-i\frac{k}{|k|}\cdot\nabla\psi_0\hat{b}_k+e^{-h_0|k|}\hat{\psi}_{1k}\right).$$

After substitution, the solution $\phi_1^{(0)}$ is written using operator notation as in the statement (3.17) of the proposition.

For the proof of the estimates of the derivatives of $\phi_1^{(0)}(X,Y,z)$, the expression (3.17) is conveniently written in operator notation as

$$C_1(h_0,z,D_Y)\psi_1(X,Y)+C_2(h_0,z,D_Y)b(Y)\cdot\nabla\psi_0(X),$$

where the components are

$$C_1(h_0,z,D_Y)=\frac{\cosh(h_0(z+1)|D_Y|)}{\cosh(h_0|D_Y|)},\qquad C_2(h_0,z,D_Y)=\frac{\sinh(h_0 z|D_Y|)}{\cosh(h_0|D_Y|)}\frac{\nabla_Y}{|\nabla_Y|}.$$

For the first term, we remark that the Sobolev embedding $H_Y^{r_0}\subset L_Y^\infty$ yields

$$\|\partial_X^{\alpha^1}\partial_Y^{\alpha^2}\underline{\nabla}_k C_1(h_0,z,D_Y)\psi_1\|^2_{L_X^2L_{Y,z}^\infty}\lesssim\int_{\mathbb{R}^d}\Big[\sup_z\big(\big|\big(\partial_X^{\alpha^1}\partial_Y^{\alpha^2}\underline{\nabla}_k C_1\psi_1\big)(X,\cdot,z)\big|_{H_Y^{r_0}}\big)\Big]^2\,dX\quad(k=0,1)$$

where $\underline{\nabla}_0$ stands for ∇_X and $\underline{\nabla}_1$ for $\nabla_{Y,z}$.

Now, by Plancherel formula (with respect to Y), one easily checks that

$$\sup_z\big|\big(\partial_X^{\alpha^1}\partial_Y^{\alpha^2}\nabla_X C_1\psi_1\big)(X,\cdot,z)\big|_{H_Y^{r_0}}\leqslant C\big(|h_0|_{H^{r+1}}\big)\left(\sum_{\beta\leqslant\alpha^1}\big|\partial_X^\beta\nabla_X\psi_1(X,\cdot)\big|_{H_Y^{r_0+r-|\beta|}}+\big|\nabla_Y\psi_1(X,\cdot)\big|_{H_Y^{r_0+r}}\right),$$
$$\sup_z\big|\big(\partial_X^{\alpha^1}\partial_Y^{\alpha^2}\nabla_{Y,z}C_1\psi_1\big)(X,\cdot,z)\big|_{H_Y^{r_0}}\leqslant C\big(|h_0|_{H^{r+1}}\big)\sum_{\beta\leqslant\alpha^1}\big|\partial_X^\beta\nabla_Y\psi_1(X,\cdot)\big|_{H_Y^{r_0+r-|\beta|}}.$$

Plugging these inequalities into the integral above then yields the desired result,

$$\|\partial_X^{\alpha^1}\partial_Y^{\alpha^2}\nabla_X C_1(h_0,z,D_Y)\psi_1\|_{L_X^2L_{Y,z}^\infty}\leqslant C\big(|h_0|_{C^{r+1}}\big)\big(|\nabla_{X,Y}\psi_1|_{H^{|\alpha^1|,r_0+|\alpha^2|}}+|\nabla_{X,Y}\psi_1|_{H^{0,r_0+r}}\big),$$
$$\|\partial_X^{\alpha^1}\partial_Y^{\alpha^2}\nabla_{Y,z}C_1(h_0,z,D_Y)\psi_1\|_{L_X^2L_{Y,z}^\infty}\leqslant C\big(|h_0|_{C^{r+1}}\big)\big(|\nabla_Y\psi_1|_{H^{|\alpha^1|,r_0+|\alpha^2|}}+|\nabla_Y\psi_1|_{H^{0,r_0+r}}\big).$$

For the control of the derivatives of $C_2(h_0,z,D_Y)b(Y)\cdot\nabla\psi_0$, we easily get that

$$\|\partial_X^{\alpha^1}\partial_Y^{\alpha^2}\nabla_{X,Y,z}C_2(h_0,z,D_Y)b(Y)\cdot\nabla\psi_0\|^2_{L_X^2L_{Y,z}^\infty}\leqslant C\big(|h_0|_{C^{r+1}}\big)|b|_{C^{r+2}}|\nabla\psi_0|_{H^{r+1}},$$

where we have (somewhat non-optimally) estimated the action of singular integral operators on L_Y^∞ at the cost of one derivative. This ends the proof of (3.18).

For the proof of (3.19), we remark that the LHS can be controlled as

$$\left\| \Lambda^r \nabla^\mu \left(\phi_1^{(0)}{}_{|_{Y=\frac{X}{\gamma}}} \right) \right\|_{L^2} \leqslant \sum_{\alpha^1,\alpha^2} \mu^{-\frac{|\alpha^2|}{2}} \left\| \partial_X^{\alpha^1} \partial_Y^{\alpha^2} \left(\nabla_{Y,z} + \nabla_X^\mu \right) \phi_1^{(0)} \right\|_{L^2_X L^\infty_{Y,z}},$$

where the summation is over $(\alpha^1, \alpha^2) \in \mathbb{N}^d \times \mathbb{N}^d$ such that $|\alpha^1| + |\alpha^2| \leqslant r$. The result follows therefore from (3.18) and a straightforward interpolation between the lowest and highest terms in terms of μ. In order to prove that the same bound holds for $\frac{1}{\sqrt{\mu}} \| \Lambda^{r-1} \partial_z \nabla^\mu \phi_1^{(0)} \|_{L^2}$ when $r \geqslant 1$, we just have to remark that

$$\left\| \Lambda^{r-1} \partial_z \nabla^\mu \left(\phi_1^{(0)}{}_{|_{Y=\frac{X}{\gamma}}} \right) \right\|_{L^2} \leqslant \sum_{\alpha^1,\alpha^2} \mu^{-\frac{|\alpha^2|}{2}} \left\| \partial_X^{\alpha^1} \partial_Y^{\alpha^2} \left(\nabla_{Y,z} + \nabla_X^\mu \right) \partial_z \phi_1^{(0)} \right\|_{L^2_X L^\infty_{Y,z}},$$

where the summation is over $(\alpha^1, \alpha^2) \in \mathbb{N}^d \times \mathbb{N}^d$ such that $|\alpha^1| + |\alpha^2| \leqslant r - 1$. From the explicit expression of $\phi_1^{(0)}$, we can check it is possible to replace $\partial_z \phi_1^{(0)}$ by $|D_Y| \phi_1^{(0)}$ in the above summation, so that the result follows as for (3.19). □

3.3.2. *Estimate on the corrector $\chi_1^{(1)}$*

With $\phi_1^{(0)}$ as in the previous section, the system (3.13) reduces to the following boundary value problem for $\chi_1^{(1)}$,

$$\begin{cases} \nabla^\mu \cdot P[\sigma] \nabla^\mu \chi_1^{(1)} = \nabla^\mu \cdot A + g, \\ \chi_1^{(1)}{}_{|_{z=0}} = 0, \qquad -\mathbf{e}_z \cdot P[\sigma] \nabla^\mu \chi_1^{(1)}{}_{|_{z=-1}} = -\mathbf{e}_z \cdot A_{|_{z=-1}}. \end{cases} \tag{3.20}$$

Proposition 3.6. *Let $r \in \mathbb{N}$ and denote $(r-1)_+ = \max\{r-1, 0\}$ and $\widetilde{r} = (r-1)_+ + 1$. The solution $\chi_1^{(1)}$ of (3.20) satisfies the estimates*

$$\left\| \Lambda^r \nabla^\mu \chi_1^{(1)} \right\|_{L^2} \leqslant \mu^{-\frac{r}{2}} M_r \left(|\nabla \psi_0|_{H^{r+1}} + \mu^{r/2} |\nabla_{X,Y} \psi_1|_{H^{r+1,r_0}} + |\nabla_{X,Y} \psi_1|_{H^{1,r_0+r}} \right),$$

$$\left\| \Lambda^{r-1} \partial_z \nabla^\mu \chi_1^{(1)} \right\|_{L^2} \leqslant \mu^{-\frac{(r-1)_+}{2}} M_r \left(|\nabla \psi_0|_{H^{r+1}} + \mu^{\frac{(r-1)_+}{2}} |\nabla_{X,Y} \psi_1|_{H^{\widetilde{r}+1,r_0}} + |\nabla_{X,Y} \psi_1|_{H^{1,r_0+\widetilde{r}}} \right),$$

with $M_r = C(\frac{1}{\alpha}, \frac{1}{\alpha_0}, |\zeta_0|_{C^{r+1} \cap C^2}, |\zeta_1|_{C^{r+1} \cap C^2}, |b|_{C^{r+1} \cap C^2})$.

Proof. The method follows the recipe of classical energy estimates, paying attention to the rapidly oscillating coefficients and their commutators with differential operators. Indeed multiplying (3.20) by $\chi_1^{(1)}$ and integrating by parts yields

$$\int_{\Omega_0} P[\sigma] \nabla^\mu \chi_1^{(1)} \cdot \nabla^\mu \chi_1^{(1)} \, dz \, dX = \int_{\Omega_0} A \cdot \nabla^\mu \chi_1^{(1)} \, dz \, dX - \int_{\Omega_0} g \chi_1^{(1)} \, dz \, dX.$$

The matrix of coefficients $P[\sigma]$ is coercive under condition (2.9) (Proposition 2.3(iii) of [2]), and is uniformly so with regard to the small parameters, as the scaling regime we are studying imposes that $\beta = \gamma$.

Using the Cauchy–Schwarz inequality and Poincaré inequality, one finds, as in the proof of Proposition 3.2, that

$$\left\| \nabla^\mu \chi_1^{(1)} \right\|_{L^2} \leqslant M_0 \left(\|A\|_{L^2} + \|g\|_{L^2} \right) \tag{3.21}$$

with M_0 as in the statement of the proposition.

For the general case $r \in \mathbb{N}$, the procedure is exactly the same replacing g by $\Lambda^r g$ and A by $A^{(r)}$, with

$$A^{(r)} = \Lambda^r A - \left[\Lambda^r, P[\sigma] \right] \nabla^\mu \chi_1^{(1)};$$

in particular, it follows from classical commutator estimates and Lemma 3.4 that

$$\left\| A^{(r)} \right\|_{L^2} \leqslant \left\| \Lambda^r A \right\|_{L^2} + C\left(\frac{1}{\alpha_0}, \frac{1}{\alpha}, |\zeta_0|_{C^{r+1}}, |\zeta_1|_{C^{r+1}}, |b|_{C^{r+1}} \right) \sum_{k=1}^{r} \mu^{-k/2} \left\| \Lambda^{r-k} \nabla^\mu \chi_1^{(1)} \right\|_{L^2}$$

so that (3.21) yields in this configuration

$$\big\|\Lambda^r\nabla^\mu\chi_1^{(1)}\big\|_{L^2}\leqslant M_0\big(\big\|\Lambda^r A\big\|_{L^2}+\big\|\Lambda^r g\big\|_{L^2}\big)+M_r\sum_{k=1}^r\mu^{-k/2}\big\|\Lambda^{r-k}\nabla^\mu\chi_1^{(1)}\big\|_{L^2}. \tag{3.22}$$

We therefore need the following lemma.

Lemma 3.7. *The following estimate holds*

$$\begin{aligned}\big\|\Lambda^r A\big\|_{L^2}+\big\|\Lambda^r g\big\|_2\leqslant{}&\mu^{-r/2}M_r\big(|\nabla\psi_0|_{H^{r+1}}+\mu^{\frac{r}{2}}\big(|\nabla_X\psi_1|_{H^{r,r_0}}+|\nabla_Y\psi_1|_{H^{r+1,r_0}}\big)\\&+|\nabla_X\psi_1|_{H^{0,r_0+r}}+|\nabla_Y\psi_1|_{H^{1,r_0+r}}\big).\end{aligned}$$

Proof. – *Control of* $\|\Lambda^r A\|_{L^2}$. Recall that

$$\begin{aligned}A={}&-P_1\nabla^\mu\phi_1^{(0)}-\frac{1}{\mu}P_1^{(0)}\nabla^\mu(\phi_0-\psi_0)-\frac{1}{\sqrt{\mu}}P_1^{(1)}\nabla^\mu\phi_0\\&-P_0\begin{pmatrix}\nabla_X\phi_1^{(0)}\\0\end{pmatrix}_{|_{Y=\frac{X}{\gamma}}}+\Big(2\nabla\sigma_0\cdot\nabla_Y\phi_1^{(0)}-\sqrt{\mu}\frac{|\nabla\sigma_0|^2}{h_0}\partial_z\phi_1^{(0)}\Big)_{|_{Y=\frac{X}{\gamma}}}\mathbf{e}_z.\end{aligned} \tag{3.23}$$

A direct application of the chain rule gives

$$\begin{aligned}\big\|\Lambda^r A\big\|_{L^2}\leqslant{}&\sum_{k=0}^r\Bigg[\|P_1\|_{L_z^\infty W^{k,\infty}}\big\|\Lambda^{r-k}\nabla^\mu\phi_1^{(0)}\big\|_{L^2}+\big\|P_1^{(0)}\big\|_{L_z^\infty W^{k,\infty}}\Big\|\frac{1}{\mu}\Lambda^{r-k}\nabla^\mu(\phi_0-\psi_0)\Big\|_{L^2}\\&+\big\|P_1^{(1)}\big\|_{L_z^\infty W^{k,\infty}}\Big\|\frac{1}{\sqrt{\mu}}\Lambda^{r-k}\nabla^\mu\phi_0\Big\|_{L^2}\Bigg]\\&+\|P_0\|_{L_z^\infty W^{r,\infty}}\big\|\Lambda^r\nabla_X\phi_1^{(0)}\big\|_{L^2}+C\Big(|\zeta_0|_{C^{r+1}},\frac{1}{\alpha_0}\Big)\big\|\Lambda^r\nabla_{Y,z}\phi_1^{(0)}\big\|_{L^2}.\end{aligned} \tag{3.24}$$

Using (2.12) and Lemma 3.4 to control the norms of P_0 and P_1, and Proposition 3.2 to control the second and third terms in the above expression, we find

$$\begin{aligned}\big\|\Lambda^r A\big\|_{L^2}\leqslant M_r\Bigg(&\mu^{-r/2}|\nabla\psi_0|_{H^{r+1}}+\sum_{k=0}^r\mu^{-k/2}\big\|\Lambda^{r-k}\nabla^\mu\phi_1^{(0)}\big\|_2\\&+\big\|\Lambda^r\big(\nabla_X\phi_1^{(0)}{}_{|_{Y=\frac{X}{\gamma}}}\big)\big\|_2+\big\|\Lambda^r\big(\nabla_{Y,z}\phi_1^{(0)}{}_{|_{Y=\frac{X}{\gamma}}}\big)\big\|_2\Bigg).\end{aligned}$$

We now control $\|\Lambda^{r-k}\nabla^\mu\phi_1^{(0)}\|_2$ through (3.19), while $\|\Lambda^r(\nabla_X\phi_1^{(0)}{}_{|_{Y=\frac{X}{\gamma}}})\|_2$ and $\|\Lambda^r(\nabla_{Y,z}\phi_1^{(0)}{}_{|_{Y=\frac{X}{\gamma}}})\|_2$ can be controlled using (3.18) and proceeding as in the proof of (3.19). This yields

$$\big\|\Lambda^r A\big\|_{L^2}\leqslant\mu^{-r/2}M_r\big(|\nabla\psi_0|_{H^{r+1}}+\mu^{\frac{r}{2}}|\nabla_X\psi_1|_{H^{r,r_0}}+|\nabla_{X,Y}\psi_1|_{H^{0,r_0+r}}\big).$$

– *Control of* $\|\Lambda^r g\|_{L^2}$. We first recall that

$$g=-\begin{pmatrix}\nabla_X\\0\end{pmatrix}\cdot P_0\big(\nabla_{Y,z}\phi_1^{(0)}\big)_{|_{Y=\frac{X}{\gamma}}}+\big(-\nabla\partial_z\sigma_0\cdot\nabla_Y\phi_1^{(0)}-\nabla_X\zeta_1\cdot\nabla\psi_0-(\zeta_1-b)\Delta\psi_0\big)_{|_{Y=\frac{X}{\gamma}}}.$$

We get therefore

$$\begin{aligned}\big\|\Lambda^r g\big\|_2&\leqslant\mu^{-r/2}M_r\big(|\nabla\psi_0|_{H^{r+1}}+\mu^{r/2}\big\|\Lambda^r\big(\nabla_{Y,z}\phi_1^{(0)}{}_{|_{Y=\frac{X}{\gamma}}}\big)\big\|_{H^1}\big)\\&\leqslant\mu^{-r/2}M_r\big(|\nabla\psi_0|_{H^{r+1}}+\mu^{r/2}|\nabla_Y\psi_1|_{H^{r+1,r_0}}+|\nabla_Y\psi_1|_{H^{1,r_0+r}}\big).\qquad\square\end{aligned}$$

Using (3.22) and the lemma, we get directly

$$\mu^{r/2}\big\|\Lambda^r\nabla^\mu\chi_1^{(1)}\big\|_{L^2}\leqslant M_r\big(|\nabla\psi_0|_{H^{r+1}}+\mu^{r/2}|\nabla_{X,Y}\psi_1|_{H^{r+1,r_0}}+|\nabla_{X,Y}\psi_1|_{H^{1,r_0+r}}\big)\\ +M_r\sum_{k=1}^{r}\mu^{(r-k)/2}\big\|\Lambda^{r-k}\nabla^\mu\chi_1^{(1)}\big\|_{L^2},$$

and the estimate on $\|\Lambda^r\nabla^\mu\chi_1^{(1)}\|$ follows from a straightforward induction. We now turn to prove the estimate on $\|\Lambda^{r-1}\partial_z\nabla^\mu\chi_1^{(1)}\|$. As for the control of $\Lambda^{r-1}\partial_z\nabla^\mu\phi_0$ in Proposition 3.2, it is enough to get an upper bound on $\|\Lambda^{r-1}\partial_z^2\chi_1^{(1)}\|$. The idea is to proceed as in Proposition 3.2, using the equation to get $\partial_z^2\chi_1^{(1)}$ in terms of quantities under control,

$$\partial_z^2\chi_1^{(1)}=\frac{h}{1+\mu|\nabla\sigma|^2}\bigg[\mu\bigg(-\frac{\partial_z|\nabla\sigma|^2}{h}\partial_z\chi_1^{(1)}+\partial_z\big(\nabla\sigma\cdot\nabla\chi_1^{(1)}\big)+\nabla\cdot\big(\nabla\sigma\,\partial_z\chi_1^{(1)}\big)-\nabla\cdot\big(h\nabla\chi_1^{(1)}\big)\bigg)\\ +RHS\ (3.20)\bigg];$$

the presence of the fast scale $X/\sqrt{\mu}$ makes things a little more complicated than in the proof of Proposition 3.2, and we need product estimates and a refinement of Lemma 3.3 for multiscale functions.

Lemma 3.8. *Let $r\in\mathbb{N}$, and denote $(r-1)_+=\max\{r-1,0\}$. Let also $G=G(X,Y,z)\in L_z^\infty W_{X,Y}^{(r-1)_+,\infty}$, and $F\in L^2(\Omega_0)$ be such that $\Lambda^{(r-1)_+}F\in L^2(\Omega_0)$. Then*

$$\big\|\Lambda^{r-1}(G_{|_{Y=\frac{X}{\gamma}}}F)\big\|_{L^2}\lesssim\|G\|_{L_z^\infty W_{X,Y}^{(r-1)_+,\infty}}\sum_{k=0}^{(r-1)_+}\mu^{-k/2}\big\|\Lambda^{(r-1)_+-k}F\big\|_{L^2}.$$

If moreover $G\geqslant 0$, then

$$\bigg\|\Lambda^{r-1}\frac{F}{1+G_{|_{Y=\frac{X}{\gamma}}}}\bigg\|_{L^2}\leqslant C\big(\|G\|_{L_z^\infty W_{X,Y}^{(r-1)_+,\infty}}\big)\sum_{k=0}^{(r-1)_+}\mu^{-k/2}\big\|\Lambda^{(r-1)_+-k}F\big\|_{L^2}.$$

Proof. The product estimates are a straightforward consequence of the chain rule; the second pair of estimates is derived as in Lemma 3.3 using these product estimates. □

Using Lemma 3.8 and the above expression for $\partial_z^2\chi_1^{(1)}$, we get, with $\widetilde{r}=(r-1)_++1$,

$$\big\|\Lambda^{r-1}\partial_z^2\chi_1^{(1)}\big\|_{L^2}\leqslant M_r\sum_{k=0}^{(r-1)_+}\mu^{-k/2}\big(\sqrt{\mu}\big\|\Lambda^{\widetilde{r}-k}\nabla^\mu\chi_1^{(1)}\big\|_2+\big\|\Lambda^{(r-1)_+-k}\ RHS\ (3.20)\big\|_2\big).\tag{3.25}$$

We can now use Lemma 3.7 to get

$$\big\|\Lambda^{(r-1)_+-k}\ RHS\ (3.20)\big\|_{L^2}\leqslant\sqrt{\mu}\big\|\Lambda^{\widetilde{r}-k}A\big\|_{L^2}+\big\|\Lambda^{(r-1)_+-k}g\big\|_{L^2}+\big\|\Lambda^{(r-1)_+-k}\partial_zA\big\|_{L^2}\\ \leqslant\mu^{-\frac{\widetilde{r}-k-1}{2}}M_r\big(|\nabla\psi_0|_{H^{r+1}}+\mu^{\frac{\widetilde{r}}{2}}|\nabla_{X,Y}\psi_1|_{H^{\widetilde{r},r_0}}+|\nabla_{X,Y}\psi_1|_{H^{0,\widetilde{r}+r_0}}\big)\\ +\big\|\Lambda^{(r-1)_+-k}\partial_zA\big\|_{L^2},\tag{3.26}$$

so that the only thing we still need to prove is a control on $\|\Lambda^{(r-1)_+-k}\partial_zA\|_{L^2}$.

Lemma 3.9. *For all $m\in\mathbb{N}$, $0\leqslant m\leqslant r-1$, one has*

$$\big\|\Lambda^m\partial_zA\big\|_{L^2}\leqslant\mu^{-\frac{m}{2}}M_r\big(|\nabla\psi_0|_{H^{r+1}}+\mu^{m/2}|\nabla_{X,Y}\psi_1|_{H^{m+1,r_0}}+|\nabla_{X,Y}\psi_1|_{H^{0,r_0+m+1}}\big).$$

Proof. From the explicit expression of A provided by (3.23), and proceeding as in the proof of Lemma 3.7, we get

$$\|\Lambda^m \partial_z A\|_{L^2} \leqslant M_r \left(\mu^{-\frac{m}{2}} |\nabla \psi_0|_{H^{r+1}} + \|\Lambda^m (\nabla_{X,Y,z} \phi_1^{(0)}{}_{|_{Y=\frac{X}{\gamma}}})\|_{H_z^1 L^2} + \sum_{k=0}^{m} \mu^{-k/2} \|\Lambda^{m-k} \partial_z \nabla^\mu \phi_1^{(0)}\|_{H_z^1 L^2} \right),$$

and the result follows from Proposition 3.5. □

The desired control on $\|\Lambda^{r-1}\partial_z^2 \chi_1^{(1)}\|_{L^2}$ is then a direct consequence of (3.25), (3.26) and the lemma. □

3.4. Asymptotic expansion of the Dirichlet–Neumann operator with estimates

We study here the asymptotic behavior of the Dirichlet–Neumann operator,

$$G[\zeta, \beta b_\gamma]\psi = \mathbf{e}_z \cdot P[\sigma] \nabla^\mu \phi_{|_{z=0}}, \tag{3.27}$$

where ϕ solves (2.11). In the previous section, we have shown that when the surface parametrization ζ and the trace of the potential at the surface ψ are of the form (3.1), one can decompose ϕ into

$$\phi = \phi_0 + \mu \chi,$$

where ϕ_0 is deduced from (2.11) by neglecting all the contributions due to the roughness. We have further decomposed ϕ_0 and the residual χ (which contains all the roughness effects) as

$$\phi_0 = \psi_0 + \mu \phi_0^{(1)} + \mu^2 \chi_0^{(2)} \quad \text{and} \quad \chi = \phi_1^{(0)} + \sqrt{\mu} \chi_1^{(1)},$$

with controls on the residuals $\chi_0^{(2)}$ and $\chi_1^{(1)}$ given in Propositions 3.2 and 3.6 respectively. We can therefore rewrite ϕ as the sum of an *effective* part and a *residual* part,

$$\phi = \phi_{eff} + \mu^{3/2} \phi_{res}$$

with

$$\phi_{eff} = \psi_0 + \mu(\phi_0^{(1)} + \phi_1^{(0)}) \quad \text{and} \quad \phi_{res} = \chi_1^{(1)} + \sqrt{\mu} \chi_0^{(2)}.$$

Similarly, we decompose the Dirichlet–Neumann operator into an effective and residual part as follows:

Proposition 3.10. *We separate the expression for the Dirichlet–Neumann operator as*

$$G[\zeta, \beta b_\gamma]\psi = (G\psi)_{eff} + \mu^{3/2} (G\psi)_{res}$$

where

$$\frac{1}{\mu}(G\psi)_{eff} = -\nabla \cdot (h_0 \nabla \psi_0) - \nabla_Y \zeta_1{}_{|_{Y=\frac{X}{\gamma}}} \cdot \nabla \psi_0 + |D_Y| \tanh(h_0|D_Y|) \psi_1(X,Y)_{|_{Y=\frac{X}{\gamma}}} - \nabla \psi_0 \cdot \nabla_Y (\operatorname{sech}(h_0|D_Y|) b)_{|_{Y=\frac{X}{\gamma}}}.$$

The remainder $(G\psi)_{res}$ satisfies the estimate (r integer),

$$|(G\psi)_{res}|_{H^r} \leqslant M_r \mu^{-r/2-1/8} (|\nabla \psi_0|_{H^{r+3}} + \mu^{r/2} |\nabla_{X,Y} \psi_1|_{H^{r+1,r_0}} + |\nabla_{X,Y} \psi_1|_{H^{1,r_0+r}}). \tag{3.28}$$

Remark 3.11. The nonlocal operators in the expression for $(G\psi)_{eff}$ arise from the simultaneous homogenization process and shallow water limit. Homogenization analysis on a shallow water expansion would give a different result. The reason for this difference is that certain terms neglected in standard shallow water expansions are not negligible in the presence of rapidly varying bathymetry; their effects are described in the nonlocal terms of $(G\psi)_{eff}$.

Proof. Recalling that $P[\sigma] = P_0^{(0)} + \sqrt{\mu}(P_0^{(1)} + P_1)$, with $P_1 = P_1^{(0)} + \sqrt{\mu}P_1^{(1)}$, one has

$$\begin{aligned}G[\zeta, \beta b_\gamma]\psi &= \mathbf{e}_z \cdot P_0^{(0)}\nabla^\mu \phi_{\mathit{eff}\,|z=0} + \sqrt{\mu}\mathbf{e}_z \cdot \big(P_0^{(1)} + P_1\big)\nabla^\mu \phi_{\mathit{eff}\,|z=0} + \mu^{3/2}\mathbf{e}_z \cdot P[\sigma]\nabla^\mu \phi_{res|z=0} \\ &= \mathbf{e}_z \cdot P_0^{(0)}\nabla^\mu \phi_{\mathit{eff}\,|z=0} + \sqrt{\mu}\mathbf{e}_z \cdot \big(P_0^{(1)} + P_1^{(0)}\big)\nabla^\mu \psi_{0|z=0} + \mu\mathbf{e}_z \cdot P_1^{(1)}\nabla^\mu\psi_{0|z=0} \\ &\quad + \mu^{3/2}\mathbf{e}_z \cdot \big(P_0^{(1)} + P_1\big)\nabla^\mu\big(\phi_0^{(1)} + \phi_1^{(0)}\big)_{|z=0} + \mu^{3/2}\mathbf{e}_z \cdot P[\sigma]\nabla^\mu\phi_{res|z=0}.\end{aligned}$$

We now decompose

$$G[\zeta, \beta b_\gamma]\psi = (G\psi)_{\mathit{eff}} + \mu^{3/2}(G\psi)_{res},$$

with

$$\begin{aligned}(G\psi)_{\mathit{eff}} &= \mathbf{e}_z \cdot P_0^{(0)}\nabla^\mu \phi_{\mathit{eff}\,|z=0} + \sqrt{\mu}\mathbf{e}_z \cdot \big(P_0^{(1)} + P_1^{(0)}\big)\nabla^\mu \psi_{0|z=0}, \\ (G\psi)_{res} &= \frac{1}{\sqrt{\mu}}\mathbf{e}_z \cdot P_1^{(1)}\nabla^\mu \psi_{0|z=0} + \mathbf{e}_z \cdot \big(P_0^{(1)} + P_1\big)\nabla^\mu\big(\phi_0^{(1)} + \phi_1^{(0)}\big)_{|z=0} + \mathbf{e}_z \cdot P[\sigma]\nabla^\mu\phi_{res|z=0}.\end{aligned} \tag{3.29}$$

The two tasks of this proposition are to give an expression for $(G\psi)_{\mathit{eff}}$ and to prove an estimate for $(G\psi)_r$.

– *Explicit computation of* $(G\psi)_{\mathit{eff}}$. From the definition of ϕ_{eff}, we have

$$\begin{aligned}\frac{1}{\mu}(G\psi)_{\mathit{eff}} &= \frac{1}{\mu}\mathbf{e}_z \cdot \big(P_0 + \sqrt{\mu}P_1^{(0)}\big)\nabla^\mu\psi_{0|z=0} + \mathbf{e}_z \cdot P_0^{(0)}\nabla^\mu\phi_0^{(1)}{}_{|z=0} + \mathbf{e}_z \cdot P_0^{(0)}\nabla^\mu\phi_1^{(0)}{}_{|z=0} \\ &= -\nabla\cdot(h_0\nabla\psi_0) - \nabla_Y\zeta_1|_{Y=\frac{X}{\gamma}}\cdot\nabla\psi_0 + \frac{1}{h_0}\partial_z\phi_1^{(0)}\big|_{z=0}.\end{aligned} \tag{3.30}$$

Computing the last term in the RHS with the help of Proposition 3.5, we get

$$\frac{1}{h_0}\partial_z\phi_1^{(0)}|_{z=0} = |D_Y|\tanh\big(h_0|D_Y|\big)\psi_1(X,Y)_{|_{Y=\frac{X}{\gamma}}} + \nabla\psi_0\cdot\nabla_Y\big(\mathrm{sech}\big(h_0|D_Y|\big)b\big)_{|_{Y=\frac{X}{\gamma}}},$$

so that G_{eff} is indeed given by the expression stated in the proposition.

– *Control of the residual* $(G\psi)_{res}$. From the explicit expression (3.29) of $(G\psi)_{res}$ and Propositions 3.2 and 3.5 and Lemma 3.4, we get

$$\begin{aligned}\big|(G\psi)_{res}\big|_{H^r} &\leqslant \mu^{-r/2}M_r\big(|\nabla\psi_0|_{H^{r+1}} + \mu^{r/2}|\nabla_{X,Y}\psi_1|_{H^{r+1,r_0}} + |\nabla_{X,Y}\psi_1|_{H^{1,r_0+r}}\big) \\ &\quad + M_r\sum_{k=0}^{r}\mu^{-k/2}\big|\Lambda^{r-k}\nabla^\mu\phi_{res|z=0}\big|_{L^2},\end{aligned}$$

which motivates the following lemma.

Lemma 3.12.

$$\big|\nabla^\mu\phi_{res|z=0}\big|_{L^2} \leqslant \mu^{-1/8}M_1\big(|\nabla\psi_0|_{H^2} + \mu^{1/2}|\nabla_{X,Y}\psi_1|_{H^{2,r_0}} + |\nabla_{X,Y}\psi_1|_{H^{1,1+r_0}}\big).$$

Proof. We write

$$\begin{aligned}\big|\nabla^\mu\phi_{res|z=0}\big|_{L^2} &\lesssim \mu^{1/8}\big|\nabla^\mu\phi_{res|z=0}\big|_{H^{1/2}} + \mu^{-1/8}\big|\nabla^\mu\phi_{res|z=0}\big|_{H^{-1/2}} \\ &\lesssim \mu^{1/8}\big(\mu^{1/4}\big\|\Lambda\nabla^\mu\phi_{res}\big\|_{L^2} + \mu^{-1/4}\big\|\partial_z\nabla^\mu\phi_{res}\big\|_{L^2}\big) \\ &\quad + \mu^{-1/8}\big(\big\|\nabla^\mu\phi_{res}\big\|_{L^2} + \big\|\Lambda^{-1}\partial_z\nabla^\mu\phi_{res}\big\|_{L^2}\big),\end{aligned}$$

where, for the second inequality, we have used two different version of the trace lemma, namely, $|F_{|z=0}|_{L^2} \lesssim \mu^{1/4}\|\Lambda^{1/2}F\|_{L^2} + \mu^{-1/4}\|\Lambda^{-1/2}\partial_z F\|_{L^2}$ and $|F_{|z=0}|_{L^2} \lesssim \|\Lambda^{1/2}F\|_{L^2} + \|\Lambda^{-1/2}\partial_z F\|_{L^2}$. The estimate follows therefore from the definition of ϕ_{res} and Proposition 3.6. □

Bound for $|\Lambda^{r-k}\nabla^\mu\phi_{res|z=0}|_{L^2}$ are obtained in the same manner, giving rise to a power of μ in the form $\mu^{-1/8-(r-k)/2}$. □

Later in the consistency analysis, we will need an estimate of $|(G\psi)_{res}|_{H^{1/2}}$. For this purpose, interpolating between H^{r-1} and H^r, we have ($r \geqslant 1$):

$$\begin{aligned}\left|(G\psi)_{res}\right|_{H^{r-\frac{1}{2}}} &\leqslant \mu^{1/4}\left|(G\psi)_{res}\right|_{H^r} + \mu^{-1/4}\left|(G\psi)_{res}\right|_{H^{r-1}} \\ &\leqslant M_r\mu^{-(r-\frac{1}{2})/2-1/8}\big(|\nabla\psi_0|_{H^{r+3}} + \mu^{r/2}|\nabla_{X,Y}\psi_1|_{H^{r+1,r_0}} + |\nabla_{X,Y}\psi_1|_{H^{1,r_0+r}}\big).\end{aligned} \tag{3.31}$$

4. Homogenization with estimates of the equations for water waves

Up to this point the canonical variables of the water wave problem (ζ, ψ) have been treated as data for an elliptic partial differential equation in a fixed domain. We now return to study the dynamics of the water waves system (2.3) as a time dependent problem. We first derive the effective PDEs satisfied by (ζ_0, ψ_0) and (ζ_1, ψ_1), after which we investigate the precision of our approximate solution (3.1) in satisfying the full water wave equation.

4.1. Effective equations

Consider the decomposition (1.7)–(1.8) to be an Ansatz for the full Euler equations (2.3), for which the slow and fast scale variables are identified through $Y = X/\sqrt{\mu}$ and $\tau = t/\sqrt{\mu}$. Substituting these expressions into the full equations and using the rigorous expansion (3.30), for the first component of equations of (2.3) we obtain an equation for the quantity $(\zeta_a, \psi_a) = (\zeta_0 + \sqrt{\mu}\zeta_1, \psi_0 + \mu\psi_1)$;

$$\begin{aligned}&\partial_t\zeta_0 + \partial_\tau\zeta_1 + h_0\Delta\psi_0 + \nabla\zeta_0\cdot\nabla\psi_0 + \nabla_Y\zeta_1\cdot\nabla\psi_0 \\ &\quad - |D_Y|\tanh\big(h_0|D_Y|\big)\psi_1(X,Y)_{|_{Y=\frac{X}{Y}}} - \nabla\psi_0\cdot\nabla_Y\big(\operatorname{sech}\big(h_0|D_Y|\big)\big)b_{|_{Y=\frac{X}{Y}}} = -\sqrt{\mu}\partial_t\zeta_1 + \sqrt{\mu}(G\psi_a)_{res}.\end{aligned} \tag{4.1}$$

At this point the fast and slow time scales are identified, $Y = X/\sqrt{\mu}$ and $\tau = t/\sqrt{\mu}$, and we have made no approximation. Similarly, for the second equation of (2.3) we obtain

$$\begin{aligned}&\partial_t\psi_0 + \sqrt{\mu}\partial_\tau\psi_1 + \mu\partial_t\psi_1 + \zeta_0 + \sqrt{\mu}\zeta_1 + \frac{1}{2}\left|\nabla\psi_0 + \sqrt{\mu}(\nabla_Y + \sqrt{\mu}\nabla_X)\psi_1\right|^2 \\ &\quad = \mu\frac{(\frac{1}{\mu}((G\psi_a)_{eff} + \mu^{3/2}(G\psi_a)_{res}) + (\nabla\zeta_0 + \nabla_Y\zeta_1 + \sqrt{\mu}\nabla_X\zeta_1)\cdot(\nabla\psi_0 + \sqrt{\mu}\nabla_Y\psi_1 + \mu\nabla_X\psi_1))^2}{2(1 + \mu|\nabla\zeta_0 + (\nabla_Y + \sqrt{\mu}\nabla_X)\zeta_1|^2)}.\end{aligned} \tag{4.2}$$

Isolating the error terms in (4.2) onto the RHS, one obtains

$$\begin{aligned}&\partial_t\psi_0 + \sqrt{\mu}\partial_\tau\psi_1 + \zeta_0 + \sqrt{\mu}\zeta_1 + \frac{1}{2}|\nabla\psi_0|^2 + \sqrt{\mu}\nabla\psi_0\cdot\nabla_Y\psi_1 \\ &\quad = -\mu\partial_t\psi_1 - \mu\nabla\psi_0\cdot\nabla_X\psi_1 - \frac{1}{2}\mu|\nabla_Y\psi_1 + \sqrt{\mu}\nabla_X\psi_1|^2 \\ &\qquad + \mu\frac{(\frac{1}{\mu}((G\psi_a)_{eff} + \mu^{3/2}(G\psi_a)_{res}) + (\nabla\zeta_0 + \nabla_Y\zeta_1 + \sqrt{\mu}\nabla_X\zeta_1)\cdot(\nabla\psi_0 + \sqrt{\mu}\nabla_Y\psi_1 + \mu\nabla_X\psi_1))^2}{2(1 + \mu|\nabla\zeta_0 + (\nabla_Y + \sqrt{\mu}\nabla_X)\zeta_1|^2)}.\end{aligned} \tag{4.3}$$

The bottom profile b is function of the fast variables $Y = X/\sqrt{\mu}$.

Now adopt the point of view that we seek multi-scale approximations to the system of Eqs. (4.1)–(4.3). To impose this scaling regime, make the assumption that the variables t, X and τ, Y are independent, so that $(\zeta_0 + \sqrt{\mu}\zeta_1, \psi_0 + \mu\psi_1)$ are multiscale functions of the variables (t, X, τ, Y). In these equations, ζ_0 and ψ_0 are functions of X, t only, while ζ_1 and ψ_1 are multi-scale functions of both time and space. The original variables will be re-imposed when we return to the identification $Y = X/\sqrt{\mu}$ and $\tau = t/\sqrt{\mu}$. In order to justify this otherwise formal separation of slow and fast scales, we will use results on scale separation that appear in [3,10].

Eqs. (4.1)–(4.2) are two equations for the four unknown quantities $(\zeta_0, \zeta_1, \psi_0, \psi_1)$. In order to obtain well-defined evolution equations for them, one must identify dynamics that take place on the slow and fast space and time scales. This is performed using the following scale separation lemmas, in which the underlying periodic nature of the cell problem plays a rôle.

Proposition 4.1. *Let g be a continuous function on $\mathbb{R}^d$ which is periodic over $\mathbb{T}^d$, and denote $\bar{g} = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} g(Y)\,dY$ its average value on $\mathbb{T}^d$. For any function $f(X)$ in the Schwarz space $\mathcal{S}(\mathbb{R}^d)$, we have*

$$\int g(X/\gamma)f(X)\,dX = \bar{g}\int f(X)\,dX + O\big(\gamma^N\big), \tag{4.4}$$

for any N.

Proposition 4.2. *Let $g(X,Y)$ be a continuous function on $\mathbb{R}^d\times\mathbb{R}^d$ which is periodic in $Y\in\mathbb{T}^d$, and denote $\bar{g}(X) = \frac{1}{(2\pi)^d}\int_{\mathbb{T}^d} g(X,Y)\,dY$ its average value over $\mathbb{T}^d$. For any function $f(X)$ in the Schwarz space $\mathcal{S}(\mathbb{R}^d)$, we have*

$$\int g(X, X/\gamma)f(X)\,dX = \int \bar{g}(X)f(X)\,dX + O\big(\gamma^N\big), \tag{4.5}$$

for any N.

The formal derivation of the effective equations satisfied by (ζ_0,ψ_0) is to write (4.1)–(4.2) in the sense of distributions, using test functions f that depend only on the large scale variable X, averaging over the variables Y, and neglecting all the terms that are understood to be of lower order (a rigorous justification of this process will be the object of the next section). Denote the mean value over the Y variables by $(\bar{\zeta}_a,\overline{\psi}_a) = (\frac{1}{|\Gamma|}\int_\Gamma \zeta_a(X,Y)\,dY, \frac{1}{|\Gamma|}\int_\Gamma \psi_a(X,Y)\,dY)$. Then

$$\partial_t\bar{\zeta}_a = \partial_t\zeta_0 + \partial_\tau\overline{\zeta_1} + \sqrt{\mu}\partial_t\overline{\zeta_1}, \tag{4.6}$$

and similarly

$$\partial_t\overline{\psi}_a = \partial_t\psi_0 + \sqrt{\mu}\partial_\tau\overline{\psi_1} + \mu\partial_t\overline{\psi_1}. \tag{4.7}$$

We assume that $\bar{\zeta}_1 = 0$ and $\overline{\psi}_1 = 0$, an assumption that will be shown to be consistent with Eqs. (4.11) and (4.12) derived below. We obtain therefore $\bar{\zeta}_a \simeq \zeta_0$ and $\overline{\psi}_a \simeq \psi_0$ at lowest order. For Eq. (4.1), using that the Y-derivative of a periodic function of Y has mean value zero, we find that

$$\partial_t\zeta_0 = -h_0\Delta\psi_0 - \nabla\zeta_0\cdot\nabla\psi_0. \tag{4.8}$$

From Eq. (4.2), we find

$$\partial_t\psi_0 + (\zeta_0 + \sqrt{\mu}\bar{\zeta}_1) + \frac{1}{2}|\nabla\psi_0|^2 = 0. \tag{4.9}$$

The lowest order approximation to the system (4.8)–(4.9) takes the form of the classical shallow water system (with $V_0 = \nabla\psi_0$ and $h_0 = 1+\zeta_0$), namely

$$\begin{cases} \partial_t\zeta_0 = -h_0\nabla\cdot V_0 - \nabla\zeta_0\cdot V_0, \\ \partial_t V_0 + \nabla\zeta_0 + V_0\cdot\nabla V_0 = 0. \end{cases} \tag{4.10}$$

Returning to (4.1)–(4.2) and using Eqs. (4.10) satisfied by (ζ_0,ψ_0), we obtain at the next order of approximation the equation

$$\partial_\tau\zeta_1 + V_0\cdot\nabla_Y\zeta_1 = |D_Y|\tanh\big(h_0|D_Y|\big)\psi_1 + V_0\cdot\nabla_Y\operatorname{sech}\big(h_0|D_Y|\big)b. \tag{4.11}$$

If $\bar{\zeta}_1 = 0$ at time $\tau = 0$, it remains so for all times. For the evolution equation for ψ_1, we find

$$\partial_\tau\psi_1 + V_0\cdot\nabla_Y\psi_1 + \zeta_1 = 0. \tag{4.12}$$

Again, if $\overline{\psi}_1 = 0$ at time $\tau = 0$, it remains so for all times.

The result is that (4.10) is the shallow water system (1.9) for (ζ_0, V_0), with $V_0 = \nabla\psi_0$, and the dispersive corrections are given by (4.11)–(4.12). This derivation, and a rigorous justification of it, are the principal subject of this paper.

4.2. Regularity of the approximate solution

The approximate solution is constructed from the solution of a system of simpler model equations (4.10) and (4.11)–(4.12) where the first is a version of the classical shallow water equations. The second is the system for linear water waves in the rapid variables (Y,τ), with a forcing term due to the presence of bottom variations, whose coefficients depend upon $(\zeta_0(t,X),\psi_0(t,X))$, for which the slow variables are considered as being fixed.

Theorem 4.3. *For $r>d/2+1$, given initial data $(\zeta_0(\cdot,0),V_0(\cdot,0))$ in $H^r(\mathbb{R}^d)\times H^r(\mathbb{R}^d)^d$ such that* (3.4) *is satisfied. Then there exist $T>0$ and a smooth solution (ζ_0,V_0) in $C([-T,T];H^r(\mathbb{R}^d)\times H^r(\mathbb{R}^d)^d)$ to* (4.10) *with this initial data.*

Proof. The shallow water system can be written as a symmetric hyperbolic system for the vector function $(\zeta_0, V_0=\nabla\psi_0)$. For Sobolev index $r>d/2+1$, these equations are locally well posed in time for $(\zeta_0(\cdot,0),V_0(\cdot,0))\in H^r\times H^r$ satisfying the condition (3.4) (see for example [16]). □

The components of the corrector $(\zeta_1,\psi_1)=(\zeta_1(\tau,Y;t,X),\psi_1(\tau,Y;t,X))$ are multiple scale functions, satisfying a system of the form

$$\partial_\tau\begin{pmatrix}\zeta_1\\ \psi_1\end{pmatrix}+V_0(t,X)\cdot\nabla_Y\begin{pmatrix}\zeta_1\\ \psi_1\end{pmatrix}+\begin{pmatrix}0 & -|D_Y|\tanh(h_0(t,X)|D_Y|)\\ I & 0\end{pmatrix}\begin{pmatrix}\zeta_1\\ \psi_1\end{pmatrix}=\begin{pmatrix}f\\ g\end{pmatrix}, \tag{4.13}$$

for which the large scale variables (t,X) enter as parameters. In the case of Eqs. (4.11)–(4.12) the inhomogeneous forcing functions are given by

$$f(Y;t,X)=V_0(t,X)\cdot\nabla_Y\operatorname{sech}\big(h_0(t,X)|D_Y|\big)b(Y),\qquad g=0 \tag{4.14}$$

(in particular, it is autonomous, namely independent of τ, and its zero Fourier mode $\overline{f}$ vanishes).

Initial data for this system is given in the form $(\zeta_1(0,\cdot;t,X),\psi_1(0,\cdot;t,X))$ in $H^r_Y(\mathbb{T}^d)\times H^{r+1/2}_Y(\mathbb{T}^d)$ with zero mean on $\mathbb{T}^d$. The dependence of these solutions on the variables (t,X) will be quantified in a paragraph below.

For the sake of clarity, we omit the dependence on the variables (t,X) in the statement below, since they only act as parameters in (4.13). It is also convenient to introduce the energy or order r ($r\in\mathbb{R}$) defined for all couple of function (u,v) on $\mathbb{T}^d$ with zero mean by

$$\big\|(u,v)\big\|^2_{E^r}=\sum_{k\neq 0}\big(1+k^2\big)^r\big(|\hat u_k|^2+|k|\tanh\big(h_0(t,X)|k|\big)|\hat v_k|^2\big), \tag{4.15}$$

where $\hat u_k$ and $\hat v_k$ stand for the Fourier components of u and v.

Theorem 4.4. *Let $r\in\mathbb{R}$. For all $(f,g)\in C(\mathbb{R}:H^r_Y\times H^{r+1/2}_Y(\mathbb{T}^d))$ with zero mean and all $(\zeta_1(0,\cdot),\psi_1(0,\cdot))\in H^r_Y(\mathbb{T}^d)\times H^{r+1/2}_Y(\mathbb{T}^d)$ with zero mean, there exists a unique solution (ζ_1,ψ_1) of* (4.13) *in $C(\mathbb{R}:H^r_Y(\mathbb{T}^d)\times H^{r+1/2}_Y(\mathbb{T}^d))$ with initial values given by $(\zeta_1(0,\cdot),\psi_1(0,\cdot))$. Moreover, this solution has zero mean and one has*

$$\forall\tau\in\mathbb{R},\quad \big\|\big(\zeta_1(\tau),\psi_1(\tau)\big)\big\|^2_{E^r}\leqslant\big\|\big(\zeta_1(0),\psi_1(0)\big)\big\|^2_{E^r}+\tau\sup_{0\leqslant\tau'\leqslant\tau}\big\|\big(f(\tau'),g(\tau')\big)\big\|^2_{E^r}.$$

This theorem implies that (ζ_1,ψ_1) is bounded in $H^r\times H^{r+1/2}$ over any time interval $\tau\in[-T_1,T_1]$. However, this bound may grow as $T_1\to\infty$ due to the possible presence of secular terms. Furthermore, when considering the dependence of this solution on the parameters (t,X), secular growth of the quantities $\partial_X(\zeta_1,\psi_1)$, $\partial_t(\zeta_1,\psi_1)$ is quite possible, and would affect the validity of the solution decomposition (1.7)–(1.8) over long time intervals $\tau\in[-T/\gamma,T/\gamma]$. In Theorem 4.5, we show that such effects do not occur, at least in the absence of Bragg resonances, and for initial data $(\zeta_1(0,\cdot),\psi_1(0,\cdot))$ chosen to be stationary in the local environment defined by (ζ_0,ψ_0).

Proof. For the sake of simplicity, we take $g=0$ in the proof below; the adaptation to the general case is straightforward. Considered independently of the large scale variables (t,X), the system of Eqs. (4.13) over $Y\in\mathbb{T}^d$ has constant coefficients, and the solution operator can be conveniently expressed in Fourier transform. The evolution of the individual Fourier modes is described by the system

$$\partial_\tau \begin{pmatrix} \hat{\zeta}_{1k} \\ \hat{\psi}_{1k} \end{pmatrix} + \begin{pmatrix} ik\cdot V_0 & 0 \\ 0 & ik\cdot V_0 \end{pmatrix} \begin{pmatrix} \hat{\zeta}_{1k} \\ \hat{\psi}_{1k} \end{pmatrix} + \begin{pmatrix} 0 & -\omega_k^2 \\ I & 0 \end{pmatrix} \begin{pmatrix} \hat{\zeta}_{1k} \\ \hat{\psi}_{1k} \end{pmatrix} = \begin{pmatrix} \hat{f}_k \\ 0 \end{pmatrix}, \tag{4.16}$$

where the local velocity is $V_0(t,X)$, and the frequency is given by $\omega_k = (|k|\tanh(h_0(t,X)|k|))^{1/2}$, both of which depend parametrically upon the long scale spatial variable (t,X).

Defining new coordinates $u_k := \omega_k^{-1/2}\hat{\zeta}_{1,k}$ and $v_k := \omega_k^{+1/2}\hat{\psi}_{1,k}$, the propagator for (4.13)–(4.16) is given by

$$\exp\left[\tau \begin{pmatrix} -ik\cdot V_0 & 0 \\ 0 & -ik\cdot V_0 \end{pmatrix} + \tau \begin{pmatrix} 0 & \omega_k \\ -\omega_k & 0 \end{pmatrix}\right] = e^{-ik\cdot V_0\tau} \begin{pmatrix} \cos(\omega_k\tau) & \sin(\omega_k\tau) \\ -\sin(\omega_k\tau) & \cos(\omega_k\tau) \end{pmatrix}.$$

Using complex notation for this system, define

$$Z_k := u_k + iv_k, \qquad W_k := u_k - iv_k,$$

with which we express the general solution to (4.16);

$$Z_k(\tau) = e^{-i\tau[\omega_k + k\cdot V_0]} Z_k(0) + \int_0^\tau e^{-i(\tau-s)[\omega_k + k\cdot V_0]} \omega_k^{-1/2} \hat{f}_k(s)\,ds,$$

$$W_k(\tau) = e^{+i\tau[\omega_k - k\cdot V_0]} W_k(0) + \int_0^\tau e^{+i(\tau-s)[\omega_k - k\cdot V_0]} \omega_k^{-1/2} \hat{f}_k(s)\,ds. \tag{4.17}$$

Standard use of the Plancherel identity implies that

$$\|Z(\tau,\cdot)\|^2_{H_y^{r+1/4}} \leqslant \|Z(0,\cdot)\|^2_{H_y^{r+1/4}} + C_0|\tau| \|f(s,\cdot)\|^2_{L_s^\infty([-\tau,\tau]:H_y^r)},$$

$$\|W(\tau,\cdot)\|^2_{H_y^{r+1/4}} \leqslant \|W(0,\cdot)\|^2_{H_y^{r+1/4}} + C_0|\tau| \|f(s,\cdot)\|^2_{L_s^\infty([-\tau,\tau]:H_y^r)},$$

where we have used that $\omega_k \simeq \langle k\rangle^{1/2}$. Recovering our original variables

$$\frac{1}{2}(Z_k + W_k) = u_k = \omega_k^{-1/2}\hat{\zeta}_{1k}, \qquad \frac{1}{2i}(Z_k - W_k) = v_k = \omega_k^{+1/2}\hat{\psi}_{1k},$$

the result is as stated in Theorem 4.4, with in addition a quantitative estimate on the growth in the fast time variable τ. □

4.3. The Bragg resonance condition

Solutions to the linear equation (4.13) exist for all $\tau \in \mathbb{R}$, however $(\zeta_1(\cdot,\tau), \psi_1(\cdot,\tau))$ may exhibit secular growth in time; more precisely, it may grow linearly with respect to τ. This is a concern for our model system because $\tau = t/\gamma$ is the rapid timescale, therefore over physically relevant time intervals of $\mathcal{O}(1)$ in the slow time variable t, solutions of (4.13) may grow from $\mathcal{O}(1)$ quantities to $\mathcal{O}(1/\gamma)$ quantities, thus leaving the range of validity of our assumption regarding the asymptotic scaling regime. This secular growth for Fourier modes $(\hat{\zeta}_{1,k}, \hat{\psi}_{1,k})(\tau)$ is due to the presence of Bragg resonances of the kth Fourier mode of the corrector solution with the periodic variations of the bottom topography defined by $b(Y)$ for which $\hat{b}_k \neq 0$. Note that these resonances differ from the classical Bragg resonances which are obtained with surface waves and bottoms of comparable wavelength [19]; to our knowledge they had not been exhibited before. Such a resonance occurs at time t and in X if for some $k \neq 0$ such that $\hat{b}_k \neq 0$,

$$\omega_k(X,t)^2 = \big(k\cdot V_0(X,t)\big)^2, \tag{4.18}$$

where $V_0 = \nabla\psi_0$, $\omega_k = (|k|\tanh(h_0|k|))^2$.

In absence of such resonances, it is quite easy to check that there is no secular growth of the first corrector: $(\zeta_1(\tau,\cdot;t,X), \psi_1(\tau,\cdot;t,X))$ remains bounded with respect to τ in the energy norm (4.15). This easily follows from the fact that (ζ_1, ψ_1) solves (4.13) with a forcing function f given by (4.14) which is independent of τ. The time integral in (4.17), which is at the origin of the secular growth, can then be explicitly computed, and it is obviously bounded in absence of Bragg resonance.

Controlling the error corresponding to our approximation requires however some bound on the parametric derivatives of ζ_1 and ψ_1 (i.e. their derivatives with respect to t and X). These parametric derivatives also solve a problem of the form (4.13), albeit with a different forcing term (f, g). Theorem 4.4 can therefore be used to give some control on the energy norm of these parametric derivatives; however the forcing term now depends on τ and the linear secular growth in τ that appears in the estimate of Theorem 4.4 cannot be removed as above.

As previously explained, this secular growth is destructive for our approximation. While it cannot be avoided in general for the parametric derivatives of (ζ_1, ψ_1), we still have some freedom to eliminate it. Indeed, the choice for the initial condition associated to (4.13) is so far completely arbitrary. It turns out that, in absence of Bragg resonance, there is *one* choice of initial data that removes the secular growth. This removal is quite spectacular since the corresponding solutions are *independent* of τ (and therefore bounded together with all their parametric derivatives). These constant solutions are found by removing the τ-derivative in (4.13)–(4.14) which leads to solving the following problem,

$$\begin{pmatrix} V_0 \cdot \nabla_Y & -|D_Y|\tanh(h_0|D_Y|) \\ I & V_0 \cdot \nabla_Y \end{pmatrix} \begin{pmatrix} \zeta_1 \\ \psi_1 \end{pmatrix} = \begin{pmatrix} V_0 \cdot \nabla_Y \operatorname{sech}(h_0|D_Y|)b \\ 0 \end{pmatrix}.$$

In absence of Bragg resonance, this yields, for all $k \in \mathbb{Z}$,

$$\hat{\zeta}_{1,k} = -\frac{(V_0 \cdot k)^2 \operatorname{sech}(h_0|k|)}{-(V_0 \cdot k)^2 + |k|\tanh(h_0|k|)}\hat{b}_k, \qquad \hat{\psi}_{1,k} = -i\frac{(V_0 \cdot k)\operatorname{sech}(h_0|k|)}{-(V_0 \cdot k)^2 + |k|\tanh(h_0|k|)}\hat{b}_k. \tag{4.19}$$

A quantitative measure of nonresonance with respect to a sequence $\{0 < B_k < +\infty\colon k \in \mathbb{Z}^d\}$ is necessary for the analysis that follows; the kth Fourier modes $(\hat{\zeta}_{1,k}, \hat{\psi}_{1,k})$ are *nonresonant* at (X, t) with respect to the homogenized solution $(\zeta_0(\cdot, t), \psi_0(\cdot, t))$ and the bottom topography $b(Y)$ if $\hat{b}_k \neq 0$ and one has

$$\left|\omega_k(X,t)^2 - \big(k \cdot V_0(X,t)\big)^2\right| > \frac{1}{B_k}. \tag{4.20}$$

The sequence $\{B_k\}$ is effectively a bound on the small divisor condition governing Bragg resonances, locally in (X, t). Given a sequence $\{B_k\colon k \in \mathbb{Z}^d\}$ such that $B_k < e^{\bar{h}|k|/2}/\delta$, if (4.20) holds for all $k \neq 0$, a local stationary solution exists, and furthermore the secular growth of local solutions can be controlled locally in (X, t). When (4.20) holds uniformly in (X, t) for all $\hat{b}_k \neq 0$, it is a nonresonant situation (relative to the small divisor conditions $\{B_k\}$) and solutions can be controlled globally.

Theorem 4.5. *Let $r \in \mathbb{N}$, $r' > d/2 + r + 1$, $T > 0$ and $(\zeta_0, V_0) \in C^r([-T, T]; H^{r'-r}(\mathbb{R}^d)^{1+d})$ be a solution of the shallow water equations* (4.10)*, such that*

$$\exists \alpha_0 > 0,\ \forall (X,t) \in \mathbb{R}^d \times [-T,T], \quad 1 + \zeta_0(X,t) \geqslant \alpha_0.$$

Assume also that the nonresonance condition (4.20) *holds with $B_k < e^{\bar{h}|k|}/\delta$ (for some $\delta > 0$ and $0 < \bar{h} < \alpha_0$). Then there exists a unique locally stationary solution of* (4.13)*, which is given by* (4.19)*. In particular, one has, for all $0 \leqslant s \leqslant r' - r$ and all $s' > 0$,*

$$\begin{aligned} &|\zeta_1|_{C^r([-T,T];H^s_X \times H^{s'}_Y)} + \left|\psi_1(\cdot)\right|_{C^r([-T,T];H^s_X \times H^{s'}_Y)} \\ &\quad \leqslant C_{rss'}\big(|\zeta_0|_{C([-T,T];C^r_X)}, |\psi_0|_{C([-T,T];C^r_X)}\big)\big(|\zeta_0|_{C^r([-T,T];H^s_X)} + |\psi_0|_{C^r([-T,T];H^s_X)}\big)|b|_{L^2_Y}. \end{aligned} \tag{4.21}$$

Proof. Let $F_k := \mathbb{R} \times \mathbb{R}^d \mapsto \mathbb{R}$ defined as

$$F_k(V,\zeta) = -\frac{(V \cdot k)^2 \operatorname{sech}((1+\zeta)|k|)}{-(V \cdot k)^2 + |k|\tanh((1+\zeta)|k|)},$$

so that $\hat{\zeta}_{a,k} = F(V_0, \zeta_0)\hat{b}_k$. The mapping F_k is smooth, vanishes at the origin, and is rapidly decaying together with all its derivatives as a consequence of the nonresonance assumption (4.20). It follows therefore from Moser's estimate that

$$|k|^{s'}|\hat{\zeta}_{1,k}|_{C([-T,T];H^s_X)} \leqslant C\big(s, s', |\zeta_0|_{C([-T,T];L^\infty_X)}, |V_0|_{C([-T,T];L^\infty_X)}\big)\big(|\zeta_0|_{C([-T,T];H^s_X)} + |V_0|_{C([-T,T];H^s_X)}\big)|\hat{b}_k|,$$

so that the bound given in the lemma stems from Plancherel's inequality in the case $r = 0$. Bounds for $r > 0$ and on ψ_1 are obtained in the same way. □

The question as to how often Bragg resonances occur merits a discussion. For $k \neq 0$ fixed, (4.20) is an open condition on the state parameters $(\zeta_0, V_0) \in C^0$. In two dimensions (that is, when $d = 1$), it is related to the local Froude number of the flow, defined by

$$Fr^2(X,t) := \frac{V_0^2(X,t)}{h_0(X,t)};$$

indeed, the nonresonance condition (4.20) can be stated equivalently as

$$Fr^2(X,t) = \frac{\tanh(h_0(X,t)|k|)}{h_0(X,t)|k|}$$

and in particular for supercritical flows $Fr^2(X,t) \geqslant 1$, Bragg resonances are absent. However, the Froude number is an indication of criticality which is local in X, and because $V_0 \in H^r$, solutions can be supercritical only on compact sets. For subcritical flows, and for (X,t) fixed, at most one Fourier mode can be in resonance.[2] If $b(Y)$ is a trigonometric polynomial, then there are a finite number of resonances. Any 2-resonances are separated by a region of nonresonance, and for $V_0 \to 0$ as $X \to \pm\infty$ further resonances are avoided. In particular, if k_{max} denotes the highest nonzero Fourier mode of b and k_{min} the lowest one, then Bragg resonances are possible only if

$$Fr^2_{min} \leqslant Fr^2 \leqslant Fr^2_{max}, \quad \text{with} \quad Fr^2_{min} = \frac{\tanh(h_0(X,t)|k_{min}|)}{h_0(X,t)|k_{min}|}, \qquad Fr^2_{max} = \frac{\tanh(h_0(X,t)|k_{max}|)}{h_0(X,t)|k_{max}|}.$$

For general $b(Y)$ with infinitely many nonzero Fourier coefficients, any zero of velocity V_0 is a point of accumulation of resonances and in particular, small resonant patches will appear because $V_0 \to 0$ as $X \to \pm\infty$. Their asymptotic strength is related to the large k – behavior of $|\hat{b}_k|$.

The character of resonance for $d \geqslant 2$ is different. For $b(Y)$ given by a trigonometric polynomial, resonances are isolated for the same reason as for $d = 1$. In the case of a general $b(Y)$, there is the potential for a dense set of resonances in the state space (ζ_0, V_0) and not just at $V_0 = 0$. This can be seen through the parametric dependence on (ζ_0, V_0) of the resonant condition (4.18) in wavenumber space. Given (ζ_0, V_0), this condition defines a hypersurface E_k in $k \in \mathbb{Z}^d$, which, if it passes through a lattice point $k \in \mathbb{Z}^d$ with $\hat{b}_k \neq 0$, gives rises to a resonance. Even if it does not intersect a lattice point, under arbitrarily small perturbations at (ζ_0, V_0), it will. Hence the set of resonant states (ζ_0, V_0) is dense.

Nonetheless, in the measure theoretic sense, Bragg resonances are relatively rare. That is, there is a set of states (ζ_0, V_0) for which (4.20) is satisfied for all $k \neq 0$, such that its complement has measure less than $C\delta$. Indeed, fix $k \neq 0$. The gradient of the resonance condition with respect to (ζ_0, V_0) on the curve E_k is non-vanishing, and is of amplitude of order $\mathcal{O}(|k|)$. Thus state variables (ζ_0, V_0) of distance $(B_k|k|)^{-1}$ from E_k will satisfy the nonresonance condition (4.20). The union over $k \neq 0$ of $(B_k|k|)^{-1}$-tubular neighborhoods of the sets E_k consists of the 'bad' states, for which there exists at least one near resonance as in (4.20). This union has relative measure bounded above by

$$\sum_{k\neq 0} \frac{1}{|k|B_k} < C\delta.$$

Therefore the resonant set is dense, but has relatively small measure in the space of states (ζ_0, V_0). Moreover, we see that the set for which (4.20) is satisfied has the character of a Cantor set.

[2] This follows from the fact that $\tanh(x)/x$ is strictly decaying on $\mathbb{R}^+$.

4.4. Consistency analysis

The purpose of this section is to evaluate the error that is made when approximating the solution (ζ, ψ) of the full water wave problem by the functions (3.1), where the components (ζ_0, ψ_0) and (ζ_1, ψ_1) satisfy the effective system of Eqs. (4.9) and (4.11) respectively.

Write the full water wave problem (2.3) as

$$E_1(\zeta, \psi) = 0, \qquad E_2(\zeta, \psi) = 0, \tag{4.22}$$

where E_1 and E_2 identify with the LHS of Eqs. (2.3). We denote our construction of an approximate solution by $E_a := (E_1(\zeta_a, \psi_a), E_2(\zeta_a, \psi_a))$, where (ζ_a, ψ_a) is defined in (1.7)–(1.8). For this approximate solution, the error is given by the expression

$$E_1(\zeta_a, \psi_a) = -\sqrt{\mu}(G\psi_a)_{res} - \sqrt{\mu}\partial_t\zeta_1, \tag{4.23}$$

$$\begin{aligned} &E_2(\zeta_a, \psi_a) \\ &\quad = \mu\nabla\psi_0 \cdot \nabla_X\psi_1 + \frac{\mu}{2}|\nabla_Y\psi_1 + \sqrt{\mu}\nabla_X\psi_1|^2 - \mu\partial_t\psi_1 \\ &\qquad - \mu\frac{(\frac{1}{\mu}((G\psi_a)_{eff} + \mu^{3/2}(G\psi_a)_{res}) + (\nabla\zeta_0 + \nabla_Y\zeta_1 + \sqrt{\mu}\nabla_X\zeta_1) \cdot (\nabla\psi_0 + \sqrt{\mu}\nabla_Y\psi_1 + \mu\nabla_X\psi_1))^2}{2(1 + \mu|\nabla\zeta_0 + (\nabla_Y + \sqrt{\mu}\nabla_X)\zeta_1|^2)}. \end{aligned} \tag{4.24}$$

The statement that the expression (ζ_a, ψ_a) is a good approximation for Eqs. (2.3) is that the error is small, in an appropriate norm, for small μ. Theorem 4.6 is a result of this form, implying the consistency of the approximate solution. We recall that the leading term (ζ_0, ψ_0) of the approximation solves the nonlinear shallow water equations (1.9) while, in absence of Bragg resonances, the correctors (ζ_1, ψ_1) are explicitly given by (4.19).

Theorem 4.6. *Under the assumptions of Theorem* 4.5*, the approximate solution* (ζ_a, ψ_a) *given by expression* (1.7)–(1.8) *satisfies the following consistency estimates*

$$\big|E_1(\zeta_a, \psi_a)\big|_{L^2} \leqslant C_a\mu^{3/8}, \qquad \big|E_2(\zeta_a, \psi_a)\big|_{H^{1/2}} \leqslant C_a\mu^{3/4}, \tag{4.25}$$

for the error term for the water wave system (2.3)*. The constant* C_a *is of the form*

$$C_a = C\left(\frac{1}{\alpha_0}, |\zeta_0|_{C^4}, |V_0|_{H^4}, |b|_{L^2}\right).$$

Remark 4.7. The quantities $(\zeta_0, \psi_0, \zeta_1, \psi_1)$ are solutions of the model equations and following Theorems 4.3, 4.4 and 4.5, are bounded along with their derivatives in terms of the initial data. The norm in which the error is measured is relatively weak, the reason being that we are dealing with a problem with fast oscillating functions. It is however a natural norm for this problem since it coincides with the norm of the energy functional associated to the water waves equations.

Proof. The first component E_1 satisfies

$$\big|E_1(\zeta_a, \psi_a)\big|_{L^2} \leqslant \sqrt{\mu}\big|(G\psi)_{res}\big|_{L^2}\sqrt{\mu}|\partial_t\zeta_1|_{L^2}.$$

By Proposition 3.10, the norm $|(G\psi)_{res}|_{L^2}$ is bounded by $\mu^{-1/8}$. This estimate involves norms of ζ_1, ψ_1 and their derivatives that can becontrolled using Theorem 4.5 in terms of norms of the leading term ζ_0 and ψ_0. Thus the estimate of the first component of E_a is shown to be as stated in the theorem.

The second component of the error E_2 is given in (4.24). It is made up of a complicated nonlinear expression. This nonlinear quantity consists of several types of terms, distinguished by whether they depend upon surface variables alone, or whether they depend upon the Dirichlet–Neumann operator and thus on the solution of an elliptic boundary value problem with oscillatory coefficients. Further terms of the RHS depend only upon surface variables, as products of functions of X and/or multiscale functions. We will use the interpolation estimates in the form (for the sake of clarity, we omit the dependence on time here)

$$|f|_{H^{1/2}} \leqslant \mu^{-1/4}|f|_{L^2} + \mu^{1/4}|f|_{H^1}. \tag{4.26}$$

Products such as $|\nabla\psi_0(X)\cdot\nabla_Y\psi_1(X, X/\sqrt{\mu})|_{H^{1/2}}$ are controlled by

$$\mu|\nabla\psi_0\cdot\nabla_Y\psi_1|_{H^{1/2}} \leqslant \mu^{3/4}|\nabla\psi_0|_{C^1}|\nabla_Y\psi_1|_{H^{1,r_0+1}}. \tag{4.27}$$

Products of multiscale functions are bounded by

$$\mu\big||\nabla_Y\psi_1|^2\big|_{H^{1/2}} \leqslant \mu^{3/4}|\nabla_Y\psi_1|_{C^0_{XY}}|\nabla_Y\psi_1|_{H^{0,r_0+1}}. \tag{4.28}$$

Other terms in the first line of the RHS of (4.24) are bounded similarly. We now turn to the second line of the RHS of (4.24). It has the form $\mu\frac{A}{B}$ that we need to bound in $H^{1/2}$ norm. The denominator B satisfies $B = 2(1 + |\nabla\zeta_0 + \mu(\nabla_Y + \sqrt{\mu}\nabla_X)\zeta_1|^2) \geqslant 2$ and we can therefore write

$$\mu\left|\frac{A}{B}\right|_{H^{1/2}} \leqslant \mu\left(\mu^{-1/4}\left|\frac{A}{B}\right|_{L^2} + \mu^{1/4}\left|\frac{A}{B}\right|_{H^1}\right) \leqslant \mu^{3/4}|A|_{L^2} + \mu^{5/4}|\nabla A|_{L^2} + \mu^{5/4}|A\nabla B|_{L^2}. \tag{4.29}$$

The numerator A contains many terms. To bound its L^2 norm, we have for example terms of the form

$$\left|\left(\frac{1}{\mu}(G\psi)_{eff}\right)^2\right|_{L^2} \leqslant C \tag{4.30}$$

where C depends on $|\nabla\zeta_0|_{C^2}$, $|\nabla\psi_0|_{C^2}$, $|\nabla_{X,Y}\zeta_1|_{H^{3,r_0+1}}$, $|\nabla_{X,Y}\psi_1|_{H^{3,r_0+1}}$. Here again, Theorem 4.5 is used to control the last two quantities in terms of norms of ζ_0 and ψ_0. Estimates of terms of the numerator A which depend upon the quantity $(G\psi)_{res}$ from the Dirichlet–Neumann operator use the results of Section 3 on the boundary value problem with periodic oscillatory coefficients. For example,

$$\big|\big(\mu^{1/2}(G\psi)_{res}\big)^2\big|_{L^2} \leqslant C\mu^{3/4}. \tag{4.31}$$

Examination of all terms leads to $|A|_{L^2} \leqslant C$. Noting that the computation of ∇B gives one factor μ and that each derivation costs a factor $\mu^{1/2}$, we find

$$|\nabla B|_{L^4} \leqslant C\mu^{1/4}. \tag{4.32}$$

Considering all terms similarly, we arrive to the conclusion of Theorem 4.6. □

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