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On bifurcation of solutions of the Yamabe problem in product manifolds ☆

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Abstract

We study local rigidity and multiplicity of constant scalar curvature metrics in arbitrary products of compact manifolds. Using (equivariant) bifurcation theory we determine the existence of infinitely many metrics that are accumulation points of pairwise non-homothetic solutions of the Yamabe problem. Using local rigidity and some compactness results for solutions of the Yamabe problem, we also exhibit new examples of conformal classes (with positive Yamabe constant) for which uniqueness holds.

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1. Introduction

The classical Yamabe problem asks for the existence of constant scalar curvature metrics in any given conformal class of Riemannian metrics on a compact manifold M. These metrics can be characterized variationally as critical points of the Hilbert–Einstein functional on conformal classes. The solution of Yamabe's problem, due to combined efforts of Yamabe [22], Trudinger [21], Aubin [2] and Schoen [18], provides *minimizers* of the Hilbert–Einstein functional in each conformal class. For instance, Einstein metrics are minima of the functional in their conformal class and in fact, except for round metrics on spheres, they are the unique metrics in their conformal class having constant scalar curvature, by a theorem of Obata [15]. It is also interesting to observe that, generically, minima of the Hilbert–Einstein functional in conformal classes are unique, see [1]. However, in many cases a rich variety of constant scalar curvature metrics arise as critical points that are not necessarily minimizers, and it is a very interesting question to classify all critical points. In this paper, we propose to use bifurcation theory to determine the existence of multiple constant scalar curvature metrics on products of compact manifolds. Multiplicity of solutions of the Yamabe problem

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in product manifolds has been studied in the literature, and several results have been obtained in the special case of products with round spheres, see for instance [6,9,16,19]. A somewhat different multiplicity result can be found in [17]; bifurcation theory is used in [7] to obtain a multiplicity result for the Yamabe equation on the sphere $\mathbb{S}^N$. In this paper we consider products of arbitrary factors with constant, but not necessarily positive, scalar curvature, and we prove a multiplicity result in an infinite number of conformal classes.

Let us describe our result more precisely. Given compact Riemannian manifolds $(M_0, \mathbf{g}^{(0)})$ and $(M_1, \mathbf{g}^{(1)})$, both having positive constant scalar curvature, one consider the *trivial* path $\mathbf{g}_\lambda$, $\lambda \in]0, +\infty[$, of constant scalar curvature metrics on the product $M = M_0 \times M_1$ defined by $\mathbf{g}_\lambda = \mathbf{g}^{(0)} \oplus \lambda\mathbf{g}^{(1)}$. The main results of the paper (Corollary 4.4, Theorem 4.5) state that there is a countable set $\Lambda \subset]0, +\infty[$ that accumulates (only) at 0 and at $+\infty$ such that:

- the family $(\mathbf{g}_\lambda)_\lambda$ is *locally rigid* at all points in $]0, +\infty[\setminus \Lambda$, i.e., for all $\lambda \in]0, +\infty[\setminus \Lambda$, any constant scalar curvature metric $\mathbf{g}$ on M which is sufficiently $\mathcal{C}^{2,\alpha}$-close to $\mathbf{g}_\lambda$ must be homothetic to some element of the trivial family;
- at all $\lambda_* \in \Lambda$, except for a finite subset, there is a bifurcating branch of constant scalar curvature metrics issuing from the trivial branch at $\mathbf{g}_{\lambda_*}$, and that consists of metrics that do not belong to the trivial family.

Rigidity and bifurcation results are also given when the scalar curvatures of $\mathbf{g}^{(0)}$ and of $\mathbf{g}^{(1)}$ are not both positive, see Theorem 4.13. Based on these results and other known facts about Yamabe metrics, one obtains some uniqueness and multiplicity results for constant scalar curvature metrics in fixed conformal classes, see Section 4.4. For instance, an interesting consequence of our bifurcation result yields the following: if $(M_1, \mathbf{g}^{(1)})$ has positive scalar curvature, then there is a subset $F \subset]0, 1]$ that has a countable number of accumulation points tending to 0 such that for all $\lambda \in F$, there are at least *three* distinct constant unit volume scalar curvature metrics in the conformal class of $\mathbf{g}_\lambda = \mathbf{g}^{(0)} \oplus \lambda\mathbf{g}^{(1)}$, see Proposition 4.14. Finally, using some recent compactness results for solutions of the Yamabe problem, see [8,12,13], we establish also the uniqueness of constant scalar curvature metrics in conformal classes in product of spheres or, more generally, in product of compact Einstein manifolds with positive scalar curvature, see Section 4.2.

The result is obtained as an application of a celebrated abstract bifurcation result of Smoller and Wasserman [20], which uses an assumption on the jump of the Morse index for a path of solutions of a family of variational problems. In the present paper, we consider the variational structure of the Yamabe problem given by the Hilbert–Einstein functional, defined on the set of metrics of volume 1 in a given conformal class of metrics. A very interesting observation on the Yamabe variational problem considered here is that the set Λ consisting of instants when the second variation of the Hilbert–Einstein functional degenerates do not correspond necessarily to jumps of the Morse index. Namely, the eigenvalues of the Jacobi operator for this functional are arranged into sequences of functions that are monotonic with respect to λ, but both increasing and decreasing functions appear, see Lemma 4.3. Thus, one can have a finite number of degeneracy instants $\lambda \in \Lambda$ where a compensation occurs, and no jump of the Morse index is produced by the passage through 0 of the eigenvalues. This raises an extremely interesting question on whether one can have local rigidity also at this sort of *neutral* degeneracy instants. In the last part of the paper we study this question, and we setup an equivariant framework to determine some sufficient conditions that guarantee bifurcation at *every* degeneracy instant. We define the notion of *harmonically freeness* for an isometric action of a Lie group G on a Riemannian manifold M, see Definition 4.6, which roughly speaking means that the corresponding isotropic representations of G on distinct eigenspaces of the Laplace–Beltrami operator of M should be direct sum of non-equivalent irreducible representations. The class of manifolds that admit a harmonically free isometric action of a Lie group includes, for instance, all compact symmetric spaces of rank 1, see Example 4.1. We obtain the result that, if one of the two factors M_0 or M_1 admits a harmonically free isometric action of some Lie group, then bifurcation of the family $(\mathbf{g}_\lambda)_\lambda$ must occur at every degeneracy instant (Proposition 4.7). This is obtained using the equivariant abstract bifurcation result of Smoller and Wasserman [20], by studying the representations of the Lie group G on the eigenspaces of the Jacobi operator of the Hilbert–Einstein functional.

The paper is organized as follows. Section 2 contains the essential facts on the variational framework of the constant scalar curvature problem in Riemannian manifolds; the basic references for details are [4,11,19]. Section 3 contains statements and proofs of a local rigidity theorem (implicit function theorem) and both the simple and the equivariant bifurcation result for the Yamabe variational problem. In Section 4 we study explicitly the case of product manifolds and prove our main results. Appendix A contains formal statements of an implicit function theorem and of two bifurcation theorems for variational problems defined on the total space of a fiber bundle, which are best suited for the theory developed in this paper.

2. The variational setting for the Yamabe problem

We will denote throughout by M a compact manifold without boundary, with $m = \dim(M) \geqslant 3$, and by $\mathbf{g}_\mathrm{R}$ an auxiliary Riemannian metric on M. The metric $\mathbf{g}_\mathrm{R}$ induces inner products and norms in all spaces of tensors on M, the Levi-Civita connection ∇_R of $\mathbf{g}_\mathrm{R}$ induces a connection in all vector spaces of tensors fields on M. Let $\mathcal{S}^k(M)$ be the space of all symmetric $(0,2)$-tensors of class $\mathcal{C}^k$ on M, with $k \geqslant 2$; this is a Banach space when endowed with the norm:

$$\|\tau\|_{\mathcal{C}^k} = \max_{j=0,\dots,k}\Big[\max_{p\in M}\big\|{\nabla_\mathrm{R}}^{(j)}\tau(p)\big\|_\mathrm{R}\Big].$$

Let $\mathcal{M}^k(M)$ denote the open cone of $\mathcal{S}^k(M)$ consisting of all Riemannian metrics on M; for all $\mathbf{g} \in \mathcal{M}^k(M)$, the tangent space $T_\mathbf{g}\mathcal{M}^k(M)$ is identified with the Banach space $\mathcal{S}^k(M)$. Given $\mathbf{g} \in \mathcal{M}^k(M)$, the *conformal class of* $\mathbf{g}$, denoted by $[\mathbf{g}]_k$ is the subset of $\mathcal{M}^k(M)$ consisting of metrics that are conformal to $\mathbf{g}$. For all $\mathbf{g}$, $[\mathbf{g}]_k$ is an open subset of a Banach subspace of $\mathcal{S}^k$, and thus it inherits a natural differential structure. As a matter of facts, in order to comply with certain Fredholmness assumptions in Bifurcation Theory, we need to introduce conformal classes of metrics having a Hölder type regularity $\mathcal{C}^{k,\alpha}$. To this aim, the most convenient definition is to consider a smooth[2] metric $\mathbf{g}$ on M, and setting:

$$[\mathbf{g}]_{k,\alpha} = \big\{\psi\cdot\mathbf{g}\colon \psi \in \mathcal{C}^{k,\alpha}(M),\ \psi > 0\big\};$$

thus, $[\mathbf{g}]_{k,\alpha}$ can be identified with the open subset of the Banach space $\mathcal{C}^{k,\alpha}(M)$ consisting of positive functions. The differential structure on $[\mathbf{g}]_{k,\alpha}$ is the one induced by $\mathcal{C}^{k,\alpha}(M)$.

For $\mathbf{g} \in \mathcal{M}^k(M)$, we will denote by $\nu_\mathbf{g}$ the volume form (or density, if M is not orientable) of $\mathbf{g}$, by $\mathrm{Ric}_\mathbf{g}$ the Ricci curvature of $\mathbf{g}$, and by $\kappa_\mathbf{g}$ its scalar curvature function, which is a function of class $\mathcal{C}^{k-2}$ on M.

The *volume function* $\mathcal{V}$ on $\mathcal{M}^k(M)$ is defined by:

$$\mathcal{V}(\mathbf{g}) = \int_M \nu_\mathbf{g}.$$

Observe that $\mathcal{V}(\mathbf{g})$ is smooth, and its differential is given by:

$$\mathrm{d}\mathcal{V}(\mathbf{g})\mathbf{h} = \frac{1}{2}\int_M \mathrm{tr}_\mathbf{g}(\mathbf{h})\nu_\mathbf{g}, \tag{1}$$

for all $\mathbf{h} \in \mathcal{S}^k(M)$. Let $\mathcal{M}_1^k(M)$ denote the subset of $\mathcal{M}^k(M)$ of those metrics $\mathbf{g}$ such that $\mathcal{V}(g) = 1$; let us also consider the scale-invariant *Hilbert–Einstein functional* on $\mathcal{M}^k(M)$, which is the function $\mathcal{A} : \mathcal{M}^k(M) \to \mathbb{R}$ defined by:

$$\mathcal{A}(\mathbf{g}) = \mathcal{V}(\mathbf{g})^{\frac{2-m}{m}}\int_M \kappa_\mathbf{g}\nu_\mathbf{g}.$$

We summarize here some well known facts about the critical points of $\mathcal{A}$:

Proposition 2.1.

(a) $\mathcal{M}_1^k(M)$ *is a smooth embedded codimension* 1 *submanifold of* $\mathcal{M}^k(M)$.
(b) $\mathcal{M}_1^{k,\alpha}(M,\mathbf{g}) = \mathcal{M}_1^k(M) \cap [\mathbf{g}]_{k,\alpha}$ *is a smooth embedded codimension* 1 *submanifold of* $[\mathbf{g}]_{k,\alpha}$. *For* $\mathbf{g}_0 \in \mathcal{M}_1^{k,\alpha}(M,\mathbf{g})$, *the tangent space* $T_{\mathbf{g}_0}\mathcal{M}_1^{k,\alpha}(M,\mathbf{g})$ *is identified with the closed subspace* $\mathcal{C}_*^{k,\alpha}(M,\mathbf{g}_0)$ *of* $\mathcal{C}^{k,\alpha}(M)$ *given by all functions* f *such that* $\int_M f\nu_{\mathbf{g}_0} = 0$.

[2] In fact, in most situations it will suffice to assume regularity $\mathcal{C}^{k+1}$ for $\mathbf{g}$.

(c) *$\mathcal{A}$ is a smooth functional on $\mathcal{M}^k(M)$ and on $[\mathbf{g}]_{k,\alpha}$.*
(d) *The critical points of $\mathcal{A}$ on $\mathcal{M}^k_1(M)$ are the* Einstein metrics *of volume* 1 *on M.*
(e) *The critical points of $\mathcal{A}$ on $\mathcal{M}^{k,\alpha}_1(M,\mathbf{g})$ are those metrics conformal to $\mathbf{g}$, having total volume* 1*, and that have constant scalar curvature.*
(f) *If $\mathbf{g}_0 \in \mathcal{M}^{k,\alpha}_1(M,\mathbf{g})$ is a critical point of $\mathcal{A}$ on $\mathcal{M}^{k,\alpha}_1(M,\mathbf{g})$, then the second variation $\mathrm{d}^2\mathcal{A}(\mathbf{g}_0)$ of $\mathcal{A}$ at $\mathbf{g}_0$ is identified with the quadratic form on $\mathcal{C}^{k,\alpha}_*(M,\mathbf{g}_0)$ defined by:*

$$\mathrm{d}^2\mathcal{A}(\mathbf{g}_0)(f,f) = \frac{m-2}{2}\int_M \big((m-1)\Delta_{\mathbf{g}_0} f - \kappa_{g_0} f\big) f\,\nu_{\mathbf{g}_0}. \tag{2}$$

Moreover, $\mathbf{g}_0$ is a nondegenerate[3] *critical point of $\mathcal{A}$ on $\mathcal{M}^{k,\alpha}_1(M,\mathbf{g})$ if either $\kappa_{\mathbf{g}_0}=0$ or if $\frac{\kappa_{g_0}}{m-1}$ is not an eigenvalue of $\Delta_{\mathbf{g}_0}$.*

Proof. For $\mathbf{g} \in \mathcal{M}^k_1(M)$, setting $\mathbf{h}=\mathbf{g}$ in (1) we get $\mathrm{d}\mathcal{V}(\mathbf{g})\mathbf{g} = \frac{1}{2}\int_M \mathrm{tr}_\mathbf{g}(\mathbf{g})\nu_\mathbf{g} = \frac{m}{2}\mathcal{V}(\mathbf{g}) > 0$. Thus, $\mathcal{M}^k_1(M)$ and $\mathcal{M}^{k,\alpha}_1(M,\mathbf{g})$ are the inverse image of a regular value of the volume function, which proves (a) and (b). For $\mathbf{g} \in \mathcal{M}^k_1(M)$, the tangent space $T_\mathbf{g}\mathcal{M}^k_1(M)$ is the kernel of $\mathrm{d}\mathcal{V}(\mathbf{g})$, i.e., the space of those $\mathbf{h} \in \mathcal{S}^k(M)$ such that $\int_M \mathrm{tr}_\mathbf{g}(\mathbf{h})\nu_\mathbf{g} = 0$, see (1). Setting $\mathbf{h} = f\cdot\mathbf{g}$, with $f \in \mathcal{C}^{k,\alpha}(M)$, we get $\int_M \mathrm{tr}_\mathbf{g}(\mathbf{h})\nu_\mathbf{g} = m\int_M f\nu_\mathbf{g}$; so, the tangent space[4] $T_\mathbf{g}\mathcal{M}^{k,\alpha}_1(M,\mathbf{g})$ is identified with $\mathcal{C}^{k,\alpha}_*(M,\mathbf{g})$.

The smoothness of $\mathcal{A}$ is clear, since it is the composition of an integral and a second order differential operator having smooth coefficients. The first variation formula for $\mathcal{A}$ is given by[5] (see for instance [19]):

$$\mathrm{d}\mathcal{A}(\mathbf{g})\mathbf{h} = -\int_M \left\langle \mathrm{Ric}_\mathbf{g} - \frac{1}{2}\kappa_\mathbf{g}\mathbf{g}, \mathbf{h}\right\rangle_\mathbf{g} \nu_\mathbf{g}, \tag{3}$$

$\mathbf{h} \in T_\mathbf{g}\mathcal{M}^k_1(M)$, from which it follows that $\mathbf{g} \in \mathcal{M}^k_1(M)$ is a critical point of $\mathcal{A}$ if and only if $\mathrm{Ric}_\mathbf{g} - \frac{1}{2}\kappa_\mathbf{g}\mathbf{g} = \lambda\cdot\mathbf{g}$ for some map λ, i.e., if and only if exists a function μ such that $\mathrm{Ric}_\mathbf{g} = \mu\cdot\mathbf{g}$. Taking traces, one sees that $\mu = \frac{1}{m}\kappa_\mathbf{g}$, i.e., $\mathbf{g}$ is Einstein. This proves (d). Setting $\mathbf{h} = f\cdot\mathbf{g}$ in (3), one obtains:

$$\mathrm{d}\mathcal{A}(\mathbf{g})(f\cdot\mathbf{g}) = \frac{m-2}{2}\int_M f\kappa_\mathbf{g}\nu_\mathbf{g}.$$

This is zero for all f with $\int_M f\nu_\mathbf{g} = 0$ iff and only if $\kappa_\mathbf{g}$ is constant, proving (e). Formula (2) can be found, for instance, in [10,19]. It is easy to see that the linear operator $(m-1)\Delta_\mathbf{g} - \kappa_g$ is (unbounded) self-adjoint on $L^2(M,\nu_\mathbf{g})$, that it leaves invariant the set of functions f such that $\int_M f\nu_\mathbf{g} = 0$, and that its restriction as a linear operator on $\mathcal{C}^{k,\alpha}_*(M,\mathbf{g})$ is Fredholm, and it has non-trivial kernel if and only if $\frac{\kappa_\mathbf{g}}{m-1}$ is a non-zero eigenvalue of $\Delta_\mathbf{g}$. □

Remark 2.2. An important observation for our theory is that, given $\lambda \in \mathbb{R}^+$, one has $\Delta_{\lambda\mathbf{g}} = \frac{1}{\lambda}\Delta_\mathbf{g}$ and $\kappa_{\lambda\mathbf{g}} = \frac{1}{\lambda}\kappa_\mathbf{g}$. This means that the spectrum of the operator $\Delta_\mathbf{g} - \frac{\kappa_\mathbf{g}}{m-1}$ is invariant by affine changes of the metric $\mathbf{g}$. On the other hand, $\nu_{\lambda\mathbf{g}} = \lambda^{\frac{m}{2}}\nu_\mathbf{g}$. When needed, we will normalize metrics to have volume 1, without changing the spectral theory of the operator $\Delta_\mathbf{g} - \frac{\kappa_\mathbf{g}}{m-1}$.

3. Bifurcation and local rigidity for the Yamabe problem

Let M be a fixed compact manifold without boundary, with $\dim(M) = m \geqslant 3$, and assume that $[a,b] \ni \lambda \mapsto \mathbf{g}_\lambda \in \mathcal{S}^k(M)$, $k \geqslant 2$, is a continuous path of Riemannian metrics on M having constant scalar curvature. An element $\lambda_* \in [a,b]$ is a *bifurcation instant* for the family $(\mathbf{g}_\lambda)_{\lambda\in[a,b]}$ if there exists a sequence $(\lambda_n)_{n\geqslant 1}$ in $[a,b]$ and a sequence $(\mathbf{g}_n)_{n\geqslant 1}$ in $\mathcal{S}^k(M)$ of Riemannian metrics on M satisfying:

[3] In the sense of Morse theory.
[4] If $\mathbf{g}_0 \in [\mathbf{g}]$, then clearly $[\mathbf{g}_0] = [\mathbf{g}]$ and $\mathcal{M}^{k,\alpha}_1(M,\mathbf{g}_0) = \mathcal{M}^{k,\alpha}_1(M,\mathbf{g})$. Thus, in this proof it will suffice to consider the case $\mathbf{g}_0 = \mathbf{g}$.
[5] The symbol $\langle\cdot,\cdot\rangle_\mathbf{g}$ in (3) denotes the inner product in the space of symmetric $(0,2)$ tensors induced by $\mathbf{g}$.

(a) for all $n \geqslant 1$, $\mathbf{g}_n$ belongs to the conformal class of $\mathbf{g}_{\lambda_n}$, but $\mathbf{g}_n \neq \mathbf{g}_{\lambda_n}$;
(b) for all $n \geqslant 1$, $\int_M \nu_{\mathbf{g}_n} = \int_M \nu_{\mathbf{g}_{\lambda_n}}$;
(c) for all $n \geqslant 1$, $\mathbf{g}_n$ has constant scalar curvature;
(d) $\lim_{n\to\infty} \lambda_n = \lambda_*$ and $\lim_{n\to\infty} \mathbf{g}_n = \mathbf{g}_{\lambda_*}$ in $\mathcal{S}^k(M)$.

If $\lambda_* \in [a, b]$ is not a bifurcation instant, then we say that the family $(\mathbf{g}_\lambda)_\lambda$ is *locally rigid* at λ_*. The implicit function theorem provides a sufficient condition for the local rigidity.

3.1. A sufficient condition for local rigidity

Proposition 3.1. *Let $[a, b] \ni \lambda \mapsto \mathbf{g}_\lambda$ be a smooth path of Riemannian metrics of class $\mathcal{C}^k$, $k \geqslant 3$, having constant scalar curvature κ_λ for all λ, and let Δ_λ denote the Laplace–Beltrami operator of $\mathbf{g}_\lambda$. If $\kappa_{\lambda_*} = 0$ or if $\frac{\kappa_{\lambda_*}}{m-1}$ is not an eigenvalue of Δ_{λ_*} (i.e., if $\mathbf{g}_{\lambda_*}$ is a nondegenerate critical point of $\mathcal{A}$ in its conformal class), then the family $(\mathbf{g}_\lambda)_\lambda$ is locally rigid at λ_*.*

Proof. Up to a suitable normalization, we can assume $\int_M \nu_{\mathbf{g}_\lambda} = 1$ for all $\lambda \in [a, b]$, see Remark 2.2. Denote by $\mathcal{C}^{2,\alpha}_+(M)$ the open set of positive functions in $\mathcal{C}^{2,\alpha}(M)$, and by $\mathcal{D}$ the sub-bundle of the trivial fiber bundle $\mathcal{C}^{2,\alpha}_+(M) \times [a, b]$ over the interval $[a, b]$, defined by:

$$\mathcal{D} = \left\{ (\psi, \lambda) \in \mathcal{C}^{2,\alpha}_+(M) \times [a, b]\colon \int_M \psi^{\frac{m}{2}} \nu_{\mathbf{g}_\lambda} = 1 \right\}. \tag{4}$$

Also, let $\mathcal{E}$ the sub-bundle of $\mathcal{C}^{0,\alpha}(M) \times [a, b]$ defined by:

$$\mathcal{E} = \left\{ (\varphi, \lambda) \in \mathcal{C}^{0,\alpha}(M) \times [a, b]\colon \int_M \varphi \nu_{\mathbf{g}_\lambda = 0} \right\}. \tag{5}$$

Finally, consider the smooth map $F : \mathcal{D} \to \mathcal{E}$ given by:

$$F(\psi, \lambda) = \left(\kappa_{\psi\cdot\mathbf{g}_\lambda} - \int_M \kappa_{\psi\cdot\mathbf{g}_\lambda} \nu_{\mathbf{g}_\lambda}, \lambda \right) \in \mathcal{E}; \tag{6}$$

clearly, given $\psi \in \mathcal{C}^{2,\alpha}_+(M)$ and $\lambda \in [a, b]$, the metric $\psi \cdot \mathbf{g}_\lambda$ has volume equal to 1 and constant scalar curvature if and only if $(\psi, \lambda) \in \mathcal{D}$ and $F(\psi, \lambda) = (0, \lambda)$. This means that, in order to establish the desired result, we need to look at the structure of the inverse image $F^{-1}(\mathbf{0}_\mathcal{E})$ of the null section $\mathbf{0}_\mathcal{E}$ of the bundle $\mathcal{E}$. Note that F is a fiber bundle morphism, i.e., denoting by $\pi_\mathcal{D} : \mathcal{D} \to [a, b]$ and $\pi_\mathcal{E} : \mathcal{E} \to [a, b]$ the natural projections, one has $\pi_\mathcal{E} \circ F = \pi_\mathcal{D}$. The thesis will follows from the Implicit Function Theorem once we show that the *vertical derivative*[6] $\mathrm{d}_{\mathrm{ver}} F(\mathbf{1}, \lambda_*)$ of F at the point $(\mathbf{1}, \lambda_*)$ (here $\mathbf{1}$ is the constant function equal to 1 on M) is a (linear) isomorphism from the Banach space:

$$D_* = \left\{ \Psi \in \mathcal{C}^{2,\alpha}(M)\colon \int_M \Psi \nu_{\mathbf{g}_{\lambda_*}} = 0 \right\}$$

to the Banach space

$$E_* = \left\{ \Phi \in \mathcal{C}^{0,\alpha}(M)\colon \int_M \Phi \nu_{\mathbf{g}_{\lambda_*}} = 0 \right\}.$$

Observe that D_* is the tangent space at $\psi = \mathbf{1}$ of the fiber:

$$\mathcal{D}_{\lambda_*} = \left\{ \psi \in \mathcal{C}^{2,\alpha}_+(M)\colon \int_M \psi^{\frac{m}{2}} \nu_{\mathbf{g}_{\lambda_*}} = 1 \right\}.$$

[6] See Appendix A, Proposition A.1.

The vertical derivative $\mathrm{d_v} F(\mathbf{1}, \lambda_*)$ is easily computed as:

$$\frac{2}{m-2}\,\mathrm{d_{ver}} F(\mathbf{1}, \lambda_*)\Psi = (m-1)\Delta_{\lambda_*}\Psi - \kappa_{\lambda_*}\Psi - \int_M \big[(m-1)\Delta_{\lambda_*}\Psi - \kappa_{\lambda_*}\Psi\big]\nu_{\mathbf{g}_{\lambda_*}}$$
$$= (m-1)\Delta_{\lambda_*}\Psi - \kappa_{\lambda_*}\Psi. \tag{7}$$

For the second equality above, note that Δ_{λ_*} (as well as the operator given by multiplication by a constant) carries D_* to E_*. Under the assumption that $\kappa_{\lambda_*} = 0$ or that $\frac{\kappa_{\lambda_*}}{m-1}$ is not an eigenvalue of Δ_{λ_*}, $\mathrm{d_f} F(\mathbf{1}, \lambda_*)$ is injective on D_*. Moreover, the linear operator $\Delta_{\lambda_*} - \kappa_{\lambda_*}$ from $\mathcal{C}^{2,\alpha}(M)$ to $\mathcal{C}^{0,\alpha}(M)$ is Fredholm of index 0. Since the codimensions of D_* in $\mathcal{C}^{2,\alpha}(M)$ and of E_* in $\mathcal{C}^{0,\alpha}(M)$ are equal (both equal to 1), it follows that $\mathrm{d_f} F(\mathbf{1}, \lambda_*)$ is an isomorphism from D_* to E_*. This concludes the proof. □

Corollary 3.2. *If $\mathbf{g}_{\lambda_*}$ is an Einstein metric which is not the round metric on a sphere, then the family $(\mathbf{g}_\lambda)_\lambda$ is locally rigid at λ_*.*

Proof. By [10, Theorem 2.4], the positive eigenvalues of Δ_{λ_*} are strictly larger than κ_{λ_*} (i.e., $\mathbf{g}_{\lambda_*}$ is a strict local minimum of the Hilbert–Einstein functional in its conformal class). The conclusion follows from Proposition 3.1. □

By a result of Böhm, Wang and Ziller, see [5, Theorem C, p. 687], any metric with unit volume and constant scalar curvature which is $\mathcal{C}^{2,\alpha}$-close to an Einstein metric and which is not conformally equivalent to a round metric on the sphere must be a *Yamabe metric*, i.e., it realizes the minimum of the scalar curvature in its conformal class. Thus, in the situation of Corollary 3.2, $\mathbf{g}_\lambda$ is Yamabe for λ near λ_*. More generally, $\mathbf{g}_\lambda$ is a strict local minimum of the Hilbert–Einstein functional in its conformal class for λ in every interval $I \subset [a,b]$ containing λ_* such that either $\kappa_\lambda = 0$ or $\frac{\kappa_\lambda}{m-1}$ is not an eigenvalue of Δ_λ for all $\lambda \in I$. For instance, consider the manifold $\mathbb{S}^n$, $n \geqslant 2$, endowed with the standard round metric $\mathbf{g}$ (say, with normalized volume equal to 1); then, the (normalized) product metric $\mathbf{g}_\lambda = \mathbf{g} \oplus \lambda\mathbf{g}$ on $\mathbb{S}^n \times \mathbb{S}^n$ is a strict local minimum of the Hilbert–Einstein functional in its conformal class when $\lambda \in \left]\frac{n-1}{n}, \frac{n}{n-1}\right[$, see Section 4.2.

3.2. Bifurcation of solutions for the Yamabe problem

An instant $\lambda \in\,]0, +\infty[$ for which $\kappa_\lambda \neq 0$ and $\frac{\kappa_\lambda}{m-1}$ be an eigenvalue of Δ_λ will be called a *degeneracy instant* for the family $(\mathbf{g}_\lambda)_\lambda$. We will now establish some bifurcation results at the degeneracy instants of $(\mathbf{g}_\lambda)_\lambda$.

Theorem 3.3. *Let M be a compact manifold, with $\dim(M) = m \geqslant 3$, and let $[a,b] \ni \lambda \mapsto \mathbf{g}_\lambda \in \mathcal{S}^k(M)$, $k \geqslant 3$, is a $\mathcal{C}^1$-path of Riemannian metrics on M having constant scalar curvature. For all $\lambda \in [a,b]$, denote by κ_λ the scalar curvature of $\mathbf{g}_\lambda$, and by n_λ the number of eigenvalues of the Laplace–Beltrami operator Δ_λ (counted with multiplicity) that are less than $\frac{\kappa_\lambda}{m-1}$. Assume the following:*

(a) *$\frac{\kappa_a}{m-1}$ is either equal to 0, or it is not an eigenvalue of $\Delta_{\mathbf{g}_a}$;*
(b) *$\frac{\kappa_b}{m-1}$ is either equal to 0, or it is not an eigenvalue of $\Delta_{\mathbf{g}_b}$;*
(c) *$n_a \neq n_b$.*

Then, there exists a bifurcation instant $\lambda_* \in\,]a,b[$ for the family $(\mathbf{g}_\lambda)_\lambda$.

Proof. The result is obtained applying the non-equivariant bifurcation theorem [20, Theorem 2.1, p. 67] to the following setup. We will use a natural fiber bundle extension of this theorem, whose precise statement is given in Appendix A, Theorem A.2. Assume as in the proof of Proposition 3.1 that $\int_M \nu_{\mathbf{g}_\lambda} = 1$ for all λ, see Remark 2.2. Consider the fiber bundles $\mathcal{D}$ and $\mathcal{E}$, given respectively in (4) and (5), and let $F : \mathcal{D} \to \mathcal{E}$ be the map given in (6); the inverse image by F of the null section $\mathbf{0}_{\mathcal{D}}$ of $\mathcal{D}$ contains the constant section $\mathbf{1}_{\mathcal{E}} = \{\mathbf{1}\} \times [a,b]$, and the desired result is precisely a fiberwise bifurcation result for this setup. Let $H = L^2(M)$ denote the Hilbertable space of L^2-functions on M with respect to any of the measures induced by the volume forms $\nu_{\mathbf{g}_\lambda}$; for all λ, let H_λ be the closed subspace of H consisting of functions φ such that $\int_M \varphi \nu_{\mathbf{g}_\lambda} = 0$, endowed with the complete inner product $\langle \phi_1, \phi_2 \rangle_\lambda = \int_M \phi_1 \phi_2 \nu_{\mathbf{g}_\lambda}$. Note that $T_{\mathbf{1}}\mathcal{D}_\lambda$ is the Banach subspace of $\mathcal{C}^{2,\alpha}(M)$ consisting of maps Φ such that $\int_M \Phi \nu_{\mathbf{g}_\lambda} = 0$. The inclusion $\mathcal{C}^{k,\alpha}(M) \subset \mathcal{C}^{k-2,\alpha}(M) \subset L^2(M)$ induce inclusions $T_{\mathbf{1}}\mathcal{D}_\lambda \subset \mathcal{E}_\lambda \subset H_\lambda$ for all λ. The derivative $\mathrm{d}F(\cdot, \lambda)$ at $\mathbf{1}$ is identified with the vertical derivative $\mathrm{d}_{\mathrm{ver}} F(\mathbf{1}, \lambda)$ given in (7), which is a linear operator from $T_{\mathbf{1}}\mathcal{D}_\lambda$ to $\mathcal{E}_\lambda$ which is symmetric with respect to $\langle \cdot, \cdot \rangle_\lambda$. This is a Fredholm operator of index 0. Namely, recall that second order self-adjoint elliptic operators acting on sections of Euclidean vector bundles over compact manifolds are Fredholm maps of index zero from the space of $\mathcal{C}^{k,\alpha}$-sections to the space of $\mathcal{C}^{k-2,\alpha}$-sections, $k \geqslant 2$, see for instance [24, § 1.4] and [25, Theorem 1.1]. The spaces $T_{\mathbf{1}}\mathcal{D}_\lambda$ and $\mathcal{E}_\lambda$ are codimension 1 closed subspaces of $\mathcal{C}^{k,\alpha}(M)$ and of $\mathcal{C}^{k-2,\alpha}(M)$ respectively, and $\mathrm{d}_{\mathrm{f}} F(\mathbf{1}, \lambda)$ carries $T_{\mathbf{1}}\mathcal{D}_\lambda$ into $\mathcal{E}_\lambda$. This implies that the restriction of $\mathrm{d}_{\mathrm{f}} F(\mathbf{1}, \lambda)$ to $T_{\mathbf{1}}\mathcal{D}_\lambda$, with counterdomain $\mathcal{E}_\lambda$, is Fredholm of index 0.

Since Δ_λ is a positive discrete operator, it follows that $\Delta_\lambda - \frac{\kappa_\lambda}{m-1}$ has spectrum which consists of a sequence of finite multiplicity eigenvalues, and only a finite number of them is negative. Note that $T_{\mathbf{1}}\mathcal{D}_\lambda$ is a codimension 1 closed subspace of $\mathcal{C}^{k,\alpha}(M)$ that is orthogonal relatively to $\langle \cdot, \cdot \rangle_\lambda$ to the eigenspace of the first eigenvalue of $\Delta_\lambda - \frac{\kappa_\lambda}{m-1}$, which consists of constant functions. This implies that the restriction of $\Delta_\lambda - \frac{\kappa_\lambda}{m-1}$ to $T_{\mathbf{1}}\mathcal{D}_\lambda$ has the same eigenvalues of $\Delta_\lambda - \frac{\kappa_\lambda}{m-1}$, except for the first one (given exactly by $-\frac{\kappa_\lambda}{m-1}$), each of them with the same eigenspace. In particular, jumps of the dimension of the negative eigenspace of $\mathrm{d}_{\mathrm{f}} F(\mathbf{1}, \lambda)$ occur precisely when jumps of the dimension of the negative eigenspace of $\Delta_\lambda - \frac{\kappa_\lambda}{m-1}$ occur.

In conclusion, assumptions (a) and (b) imply that $\mathrm{d}_{\mathrm{f}} F(\mathbf{1}, \lambda)$ is an isomorphism at $\lambda = a$ and at $\lambda = b$, respectively. Assumption (c) implies that there is a jump in the dimension of the negative eigenspace of $\mathrm{d}_{\mathrm{f}} F(\mathbf{1}, \lambda)$, as λ runs from a to b. The discreteness of the spectrum implies the existence of an isolated instant $\lambda_* \in \,]a, b[$ where $\mathrm{d}_{\mathrm{f}} F(\mathbf{1}, \lambda_*)$ is singular, and where a jump of the dimension of the negative eigenspace of $\mathrm{d}_{\mathrm{f}} F(\mathbf{1}, \lambda)$ occurs. Bifurcation must then occur at λ_*, see Theorem A.2. □

One can give a more general bifurcation result using an equivariant setup. Assume in the above situation that there exists a (finite dimensional) *nice* (in the sense of [20][7]) Lie group G of diffeomorphisms of M that preserves all the metrics $\mathbf{g}_\lambda$. This means that, denoting by I_λ the isometry group of $(M, \mathbf{g}_\lambda)$, G is contained in the intersection $\bigcap_{\lambda \in [a,b]} I_\lambda$. It is easy to see that for every λ and every eigenvalue ρ of Δ_λ, one has a linear (anti-)representation[8] of $\pi_{\lambda,\rho} : G \to \mathrm{GL}(V_{\lambda,\rho})$, where $V_{\lambda,\rho}$ is the ρ-eigenspace of $\Delta_{\mathbf{g}_\lambda}$. Such a representation is defined by:

$$\pi_{\lambda,\rho}(\phi) f = f \circ \phi,$$

for all $\phi \in G$ and all $f \in V_{\lambda,\rho}$. For all λ, let us denote by π_λ^- the direct sum representation:

$$\pi_\lambda^- = \bigoplus_{\rho \leqslant \frac{\kappa_\lambda}{m-1}} \pi_{\lambda,\rho}$$

of G on the vector space V_λ^- given by the direct sum:

$$V_\lambda^- = \bigoplus_{\rho \leqslant \frac{\kappa_\lambda}{m-1}} V_{\lambda,\rho}.$$

Recall that two linear representations $\pi_i : G \to \mathrm{GL}(V_i)$, $i = 1, 2$, of the group G on the vector space V_i are *equivalent* if there exists an isomorphism $T : V_1 \to V_2$ such that $\pi_2(g) \circ T = T \circ \pi_1(g)$ for all $g \in G$.

We then have the following extension of Theorem 3.3:

Theorem 3.4. *In the above situation, assume that*:

- $\frac{\kappa_a}{m-1}$ *is either equal to* 0*, or it is not an eigenvalue of* $\Delta_{\mathbf{g}_a}$;

[7] A group G is nice if, given unitary representations of G on the finite dimensional inner product spaces V and W, assuming that the quotient spaces $D(V)/S(V)$ and $D(W)/S(W)$ have the same equivariant homotopy type as G-spaces (D is the unit disk and S is the unit sphere), then the two representations are equivalent. For instance, denoting by G_0 the connected component of the identity of G, G is nice if either $G/G_0 = \{1\}$ or if G/G_0 is the product of a finite number of copies of $\mathbb{Z}_2$, or of a finite number of copies of $\mathbb{Z}_3$.

[8] Note that the action of G on $\mathcal{M}^k(M)$ by pull-back is on the right.

- $\frac{\kappa_b}{m-1}$ *is either equal to* 0*, or it is not an eigenvalue of* $\Delta_{\mathbf{g}_b}$;
- π_a^- *and* π_b^- *are not equivalent.*

Then, there exists a bifurcation instant $\lambda_* \in\,]a, b[$ *for the family* $(\mathbf{g}_\lambda)_\lambda$.

Proof. This uses the equivariant bifurcation result of [20, Theorem 3.1], applied to the setup described in the proof of Theorem A.2. See Theorem A.3 for the precise statement needed for our purposes. Note that the (right) action of G on $\mathcal{D}$ is given by $(\psi, \lambda) \cdot \phi = (\psi \circ \phi, \lambda)$, for all $(\psi, \lambda) \in \mathcal{D}$ and all $\phi \in G$, similarly for the action of G on $\mathcal{E}$, and the function F is equivariant with respect to this action. Clearly, constant functions are fixed by this action, and the remaining assumptions of Theorem A.3 are easily checked, as in the proof of Theorem 3.3. □

4. Bifurcation in product manifolds

Let $(M_0, \mathbf{g}^{(0)})$, $(M_1, \mathbf{g}^{(1)})$ be compact Riemannian manifolds with constant scalar curvature denoted by $\kappa^{(0)}$ and $\kappa^{(1)}$ respectively. Let m_0 (resp. m_1) be the dimension of M_0 (resp. M_1), and assume $m_0 + m_1 \geqslant 3$. For all $\lambda \in\,]0, +\infty[$ denote by $\mathbf{g}_\lambda = \mathbf{g}^{(0)} \oplus \lambda \cdot \mathbf{g}^{(1)}$ the metric on $M = M_0 \times M_1$. Clearly, $\mathbf{g}_\lambda$ has constant scalar curvature

$$\kappa_\lambda = \kappa^{(0)} + \frac{1}{\lambda}\kappa^{(1)}. \tag{8}$$

Observe that, as to degeneracy instants and bifurcation, the role played by the manifolds $(M_0, \mathbf{g}^{(0)})$ and $(M_1, \mathbf{g}^{(1)})$ is symmetric. Namely, degeneracy instants and bifurcation instants for the family $(\mathbf{g}_\lambda)_\lambda$ coincide respectively with degeneracy instants and bifurcation instants for the family of metrics $\mathbf{h}_\lambda = \frac{1}{\lambda}\mathbf{g}^{(0)} \oplus \mathbf{g}^{(1)}$ on $M = M_0 \times M_1$.

Set $m = m_0 + m_1 = \dim(M)$, and let $\mathcal{J}_\lambda$ be the Jacobi operator of the Hilbert–Einstein functional along $\mathbf{g}_\lambda$, given by:

$$\mathcal{J}_\lambda = \Delta_\lambda - \frac{\kappa_\lambda}{m-1},$$

defined on the space:

$$\left\{\Psi \in \mathcal{C}^{2,\alpha}(M)\colon \int_M \Psi \nu_{\mathbf{g}_\lambda} = 0\right\}$$

and taking values in the space:

$$\left\{\Phi \in \mathcal{C}^{0,\alpha}(M)\colon \int_M \Phi \nu_{\mathbf{g}_\lambda} = 0\right\};$$

let $\Sigma(\mathcal{J}_\lambda)$ be its spectrum. This spectrum coincides with the spectrum of $\Delta_\lambda - \frac{\kappa_\lambda}{m-1}$ as an operator from $\mathcal{C}^{2,\alpha}(M)$ to $\mathcal{C}^{0,\alpha}(M)$, with the point $-\frac{\kappa_\lambda}{m-1}$ removed.

Denote by $0 = \rho_1^{(i)} < \rho_2^{(i)} < \rho_3^{(i)} < \cdots$ the sequence of eigenvalues of $\Delta_{\mathbf{g}^{(i)}}$, $i = 0, 1$, and denote by $\mu_j^{(i)}$ the multiplicity of $\rho_j^{(i)}$; Then:

$$\Sigma(\mathcal{J}_\lambda) = \left\{\sigma_{i,j}(\lambda)\colon i, j \geqslant 0,\ i + j > 0\right\},$$

where:

$$\sigma_{i,j}(\lambda) = \rho_i^{(0)} + \frac{1}{\lambda}\rho_j^{(1)} - \frac{1}{m-1}\left(\kappa^{(0)} + \frac{1}{\lambda}\kappa^{(1)}\right). \tag{9}$$

The multiplicity of $\sigma_{i,j}(\lambda)$ in $\Sigma(\mathcal{J}_\lambda)$ is equal to the product $\mu_i^{(0)}\mu_j^{(1)}$, note however that the $\sigma_{i,j}$'s need not be all distinct. Our interest is to determine the distribution of zeros of the functions $\lambda \mapsto \sigma_{i,j}(\lambda)$ as i and j vary; such zeros correspond to degeneracy instants of the Jacobi operator $\mathcal{J}_\lambda$. Towards this goal, we make a preliminary observation.

Remark 4.1. Each function $\sigma_{i,j}$ which is not identically zero has at most one zero in $]0, +\infty[$. Moreover, for any fixed i and $\bar{\lambda} \in\,]0, +\infty[$, there is at most one j for which $\sigma_{i,j}(\bar{\lambda}) = 0$. This depends on the fact that the sequence

$j \mapsto \rho_j^{(1)}$ is strictly increasing. Similarly, for each j and $\bar\lambda \in \,]0,+\infty[$, there is at most one value of i for which $\sigma_{i,j}(\bar\lambda) = 0$.

Let i_* and j_* be the smallest nonnegative integers with the property that:

$$\rho_{i_*}^{(0)} \geqslant \frac{\kappa^{(0)}}{m-1}, \qquad \rho_{j_*}^{(1)} \geqslant \frac{\kappa^{(1)}}{m-1}. \tag{10}$$

Let us say that the pair of metrics $(\mathbf{g}^{(0)}, \mathbf{g}^{(1)})$ is *degenerate* if equalities hold in both inequalities of (10). In this situation, the Jacobi operator $\mathcal{J}_\lambda$ is degenerate for all $\lambda > 0$, namely, $\sigma_{i_*,j_*}(\lambda) = 0$ for all λ.

Remark 4.2. Clearly, if either $\kappa^{(0)} < 0$ or $\kappa^{(1)} < 0$, then $(\mathbf{g}^{(0)}, \mathbf{g}^{(1)})$ is not degenerate. We observe also that if either one of the two metrics $\mathbf{g}^{(0)}$ or $\mathbf{g}^{(1)}$ is Einstein with positive scalar curvature, then the pair $(\mathbf{g}^{(0)}, \mathbf{g}^{(1)})$ is never degenerate. Namely, if say $\mathbf{g}^{(0)}$ is Einstein and $\kappa^{(0)} > 0$, then $\kappa^{(0)} = m_0 \mathrm{Ric}_{\mathbf{g}^{(0)}}$; using Lichnerowicz–Obata theorem (see for instance [3, Ch. 3, §D], or [14]) one gets:

$$\rho_1^{(0)} \geqslant \frac{m_0}{m_0-1}\mathrm{Ric}_{\mathbf{g}^{(0)}} = \frac{\kappa^{(0)}}{m_0-1} > \frac{\kappa^{(0)}}{m-1}.$$

This says that $i_* = 1$, and that equality does not hold in the first inequality of (10). We note however that when the metrics $\mathbf{g}^{(0)}$ and $\mathbf{g}^{(1)}$ are not Einstein, then the integers i_* and j_* defined above can be arbitrarily large. For instance, given any manifold $(\bar M, \bar{\mathbf{g}})$ with positive scalar curvature $\bar\kappa$, then the product Riemannian manifold $M_0 = \bar M \times \mathbb{S}^1(r)$, where $\mathbb{S}^1(r)$ is the circle of radius $r > 0$, has constant scalar curvature larger than $\bar\kappa$, and every eigenvalue of its Laplace–Beltrami operator goes to 0 as $r \to +\infty$. This implies that i_* becomes arbitrarily large as $r \to +\infty$.

Except for case of degenerate pairs, the operator $\mathcal{J}_\lambda$ is singular only at a discrete countable set of instants λ in $]0,+\infty[$. We consider separately the (most interesting) case that both scalar curvatures $\kappa^{(0)}$ and $\kappa^{(1)}$ are positive.

4.1. The case of positive scalar curvatures

Lemma 4.3. *Assume $(\mathbf{g}^{(0)}, \mathbf{g}^{(1)})$ nondegenerate, and that $\kappa^{(0)}, \kappa^{(1)} > 0$. The functions $\sigma_{i,j}(\lambda)$ satisfy the following properties.*

(a) *For all $i, j \geqslant 0$, the map $\lambda \mapsto \sigma_{i,j}(\lambda)$ is strictly monotone in $]0,+\infty[$, except possibly the maps σ_{i,j_*}, that are constant equal to $\rho_i^{(0)} - \frac{\kappa^{(0)}}{m-1}$ when $\rho_{j_*}^{(1)} = \frac{\kappa^{(1)}}{m-1}$.*
(b) *For $i \neq i_*$ and $j \neq j_*$, the map $\sigma_{i,j}(\lambda)$ admits a zero if and only if:*
  - *either $j < j_*$ and $i > i_*$, in which case $\sigma_{i,j}$ is strictly increasing,*
  - *or if $j > j_*$ and $i < i_*$, in which case $\sigma_{i,j}$ is strictly decreasing.*

(c) *If $\rho_{i_*}^{(0)} = \frac{\kappa^{(0)}}{m-1}$, then $\sigma_{i_*,j}$ does not have zeros for any j. If $\rho_{i_*}^{(0)} > \frac{\kappa^{(0)}}{m-1}$, then $\sigma_{i_*,j}$ has a zero if and only if $j < j_*$.*
(d) *If $\rho_{j_*}^{(1)} = \frac{\kappa^{(1)}}{m-1}$, then σ_{i,j_*} does not have zeros for any i. If $\rho_{j_*}^{(1)} > \frac{\kappa^{(1)}}{m-1}$, then σ_{i,j_*} has a zero if and only if $i < i_*$.*

Proof. The entire statement follows readily from a straightforward analysis of (9), writing $\sigma_{i,j}(\lambda) = A_i + \frac{1}{\lambda}B_j$, with $A_i = \rho_i^{(0)} - \frac{\kappa^{(0)}}{m-1}$, and $B_j = \rho_j^{(1)} - \frac{\kappa^{(1)}}{m-1}$. □

Corollary 4.4. *If $(\mathbf{g}^{(0)}, \mathbf{g}^{(1)})$ is nondegenerate, then the set of instants λ in the open half line $]0,+\infty[$ at which the Jacobi operator is singular is countable and discrete; it consists of a strictly increasing unbounded sequence and a strictly decreasing sequence tending to 0. For all other values of λ, $\mathcal{J}_\lambda$ is an isomorphism, and in particular, the family $(\mathbf{g}_\lambda)_\lambda$ is locally rigid at these instants.*

Proof. By Lemma 4.3, each function $\sigma_{i,j}$ has at most one zero, thus there is only a countable numbers of degeneracy instants for $\mathcal{J}_\lambda$. For $j > j_*$ and $i < i_*$, the zero $\lambda_{i,j}$ of $\sigma_{i,j}$ satisfies:

$$\lambda_{i,j} = \left|\frac{B_j}{A_i}\right| \geqslant B_j \cdot \left[\frac{\kappa^{(0)}}{m-1} - \rho_{i_*-1}^{(0)}\right]^{-1} \longrightarrow +\infty, \quad \text{as } j \to +\infty.$$

Similarly, for $i > i_*$ and $j < j_*$, the zero $\lambda_{i,j}$ of $\sigma_{i,j}$ satisfies:

$$0 < \lambda_{i,j} = \left|\frac{B_j}{A_i}\right| \leqslant A_i^{-1} \cdot \frac{\kappa^{(1)}}{m-1} \longrightarrow 0, \quad \text{as } i \to +\infty.$$

The conclusion follows. □

Theorem 4.5. *Let $(M_0, \mathbf{g}^{(0)})$ and $(M_1, \mathbf{g}^{(1)})$ be compact Riemannian manifolds with positive constant scalar curvature; assume that the pair $(\mathbf{g}^{(0)}, \mathbf{g}^{(1)})$ is nondegenerate. For $\lambda \in \,]0, +\infty[$, let $\mathbf{g}_\lambda$ denote the metric $\mathbf{g}^{(0)} \oplus \lambda \mathbf{g}^{(1)}$ on the product $M_0 \times M_1$. Then, there exists a sequence $(\lambda_n^{(1)})_n$ tending to 0 as $n \to \infty$ and a sequence $(\lambda_n^{(2)})_n$ tending to $+\infty$ as $n \to \infty$ consisting of bifurcation instants for the family $(\mathbf{g}_\lambda)_\lambda$.*

Proof. By Corollary 4.4, there are two sequences of instants λ at which the Jacobi operator $\mathcal{J}_\lambda$ is singular; these instants are our candidates to be bifurcation instants. In principle one cannot guarantee that at each of these instants there is a jump in the dimension of the negative eigenspace of $\mathcal{J}_\lambda$; namely, the eigenvalues $\sigma_{i,j}(\lambda)$ described in Lemma 4.3 can be either increasing or decreasing. Nevertheless, the zeroes of those eigenvalues that are increasing functions accumulate (only) at zero, while the zeroes of those eigenvalues that are decreasing functions accumulate (only) at $+\infty$. This implies that at all but a finite number of degeneracy instants there is jump of dimension in the negative eigenspace of $\mathcal{J}_\lambda$. The conclusion follows then from Theorem 3.3. □

Note that the case of degenerate pairs cannot be treated with Theorem 3.3, because $\mathcal{J}_\lambda$ is degenerate for all λ, and thus assumptions (a) and (b) are never satisfied in this case.

Theorem 4.5 leaves an open question on whether there may be some degeneracy instants for the Jacobi operator $\mathcal{J}_\lambda$ at which bifurcation *does not* occur. In principle, this situation might occur at those instants λ at which two or more eigenvalue functions $\sigma_{i,j}$ vanish, compensating the positive and the negative contributions to the dimension of the negative eigenspace. Let us call *neutral* a degeneracy instant of this type. It is quite intuitive that existence of neutral degeneracy instants should not occur generically, although a formal proof of this fact might be quite awkward.

There is an interesting case in which one can establish bifurcation also at neutral degeneracy instants, using the equivariant result of Theorem 3.4. This case is studied in the sequel. Let us give the following definition:

Definition 4.6. Two representations π_i, $i = 1, 2$ of a group G are said to be *essentially equivalent* if one of the two is equivalent to the direct sum of the other with a number of copies of the trivial representation of G. Let G be a group acting by isometries on a Riemannian manifold $(N, \mathbf{h})$. The action will be called *harmonically free* if, given an arbitrary family $V_1, \dots, V_r, V_1', \dots, V_s'$ of pairwise distinct eigenspaces of the Laplacian $\Delta_{\mathbf{h}}$, then the corresponding representations of G on the direct sums $V = \bigoplus_{i=1}^r V_i$ and $V' = \bigoplus_{j=1}^s V_j'$ are not essentially equivalent.

For instance, the natural action of the orthogonal group $\mathrm{O}(n)$ on the round sphere $\mathbb{S}^{n+1}$ is harmonically free. Namely, the representation of $\mathrm{O}(n)$ on each eigenspace of the Laplacian of $\mathbb{S}^{n+1}$ is irreducible. Moreover, the dimension of the eigenspaces of the Laplacian of $\mathbb{S}^{n+1}$ form a strictly increasing sequence, from which it follows that the representations of $\mathrm{O}(n)$ on the eigenspaces of the Laplacian of $\mathbb{S}^{n+1}$ are pairwise non-equivalent. This in particular implies that direct sum of any two distinct families of eigenspaces of the Laplacian are never essentially equivalent.

Example 4.1. More generally, the action of the isometry group of a compact manifold is harmonically free when the eigenspaces of the Laplacian are irreducible and pairwise non-equivalent. An important class of examples of this situation (see [3, Ch. III, §C]) is given by the compact symmetric spaces of rank one, which consists of the following homogeneous spaces G/H with a G-invariant metric:

- the real projective spaces $\mathbb{R}P^k$, with $G = \mathrm{O}(k+1)$ and $H = \mathrm{O}(k) \times \{-1, 1\}$;
- the complex projective spaces $\mathbb{C}P^k$, with $G = \mathrm{U}(k+1)$ and $H = \mathrm{U}(k) \times \mathrm{U}(1)$;
- the quaternionic projective spaces $\mathbb{H}P^k$, with $G = \mathrm{Sp}(k+1)$ and $H = \mathrm{Sp}(k) \times \mathrm{Sp}(1)$;
- the Cayley plane $\mathbb{P}^2(\mathrm{Ca})$, with $G = F_4$ and $H = \mathrm{Spin}(9)$.

In these examples, the eigenspaces of the Laplacian are irreducible by the natural action of G, see [3, Proposition C.I.8], and the dimension of these eigenspaces form a strictly increasing sequence. In particular, they are pairwise non-equivalent. Observe also that all these examples have constant scalar curvature, by homogeneity. In fact, all these examples are *two point homogeneous*, which implies that they are Einstein.

Proposition 4.7. *Under the hypothesis of Theorem* 4.5*, assume in addition that there exists a nice Lie group G with an isometric and harmonically free action on either $(M_0, \mathbf{g}^{(0)})$ or on $(M_1, \mathbf{g}^{(1)})$. Then, every degeneracy instant for the Jacobi operator $\mathcal{J}_\lambda$ is a bifurcation instant for the family $(\mathbf{g}_\lambda)_\lambda$.*

Proof. We can assume that G acts on $(M_0, \mathbf{g}^{(0)})$. For all $\lambda \in]0, +\infty[$, one obtains a non-trivial isometric action of G on $(M, \mathbf{g}_\lambda)$ by setting $g \cdot (x_0, x_1) = (g \cdot x_0, x_1)$, $g \in G$, $x_0 \in M_0$ and $x_1 \in M_1$. Let $\bar{\lambda}$ be a neutral degeneracy instants for the family $\mathbf{g}_\lambda$, and let $\sigma_{i,j}$ be one of the eigenvalue functions that vanish at $\bar{\lambda}$. For all λ, the eigenspace of $\sigma_{i,j}(\lambda)$ is the direct sum of the i-th eigenspace V_i of $\Delta_{\mathbf{g}^{(0)}}$ and the j-th eigenspaces W_j of $\Delta_{\mathbf{g}^{(1)}}$. There is a representation of G on this direct sum, given by the direct sum of the natural representation of G on the eigenspace V_i of $\Delta_{\mathbf{g}^{(0)}}$ and the trivial representation of G on W_j. As λ increases and crosses $\bar{\lambda}$, the space $V_i \oplus W_j$ is added or removed from the negative eigenspace of $\mathcal{J}_\lambda$, according to whether $\sigma_{i,j}$ is decreasing or increasing.

Denote by $\mathcal{H}_0$ the direct sum of eigenspaces of those eigenvalues $\sigma_{i,j}$ that are negative on the interval $[\bar{\lambda} - \varepsilon, \bar{\lambda} + \varepsilon]$. Then, for $\varepsilon > 0$ small enough, the negative eigenspace of $\mathcal{J}_{\bar{\lambda}-\varepsilon}$ is a direct sum of the form:

$$\mathcal{H}_0 \oplus \bigoplus_{k=1}^{r} V_{i_k} \oplus W_{j_k},$$

and the negative eigenspace of $\mathcal{J}_{\bar{\lambda}+\varepsilon}$ is the direct sum

$$\mathcal{H}_0 \oplus \bigoplus_{l=r+1}^{r+s} V_{i_l} \oplus W_{j_l},$$

where the family $V_{i_1}, \dots, V_{i_r}, V_{i_{r+1}}, \dots, V_{i_s}$ consists of pairwise distinct eigenspaces of $\Delta_{\mathbf{g}^{(0)}}$. This follows from the fact that if $(i, j) \neq (i', j')$ and $\sigma_{i,j}(\bar{\lambda}) = \sigma_{i',j'}(\bar{\lambda}) = 0$, then necessarily $i \neq i'$ and $j \neq j'$, see Remark 4.1. The representation $\pi^-_{\bar{\lambda}-\varepsilon}$ is the direct sum of the representations of G on $\mathcal{H}_0$, on $V = \bigoplus_{k=1}^{r} V_{i_k}$, plus a number of copies of the trivial representation of G, while he representation $\pi^-_{\bar{\lambda}+\varepsilon}$ is the direct sum of the representations of G on $\mathcal{H}_0$, on $V' = \bigoplus_{l=r+1}^{r+s} V_{i_l}$ plus a number of copies of the trivial representation of G. Hence, $\pi^-_{\bar{\lambda}-\varepsilon}$ and $\pi^-_{\bar{\lambda}+\varepsilon}$ are not equivalent, because the action of G on $(M_0, \mathbf{g}^{(0)})$ is harmonically free. The result follows then from Theorem 3.4. □

Corollary 4.8. *Let $(M_1, \mathbf{g}^{(1)})$ be a compact symmetric space of rank* 1*. Given any compact Riemannian manifold $(M_0, \mathbf{g}^{(0)})$ with positive constant scalar curvature, then the family $\mathbf{g}_\lambda = \mathbf{g}^{(0)} \oplus \lambda \mathbf{g}^{(1)}$ on $M_0 \times M_1$ has a countable number of degeneracy instants that accumulate at* 0 *and at $+\infty$. There is bifurcation at every degeneracy instant.*

Proof. Set $m_0 = \dim(M_0) \geqslant 2$, write $M_1 = G/H$, and consider the isometric action of G by left multiplication. Since compact symmetric spaces of rank 1 are Einstein and have positive scalar curvature, then the pair $(\mathbf{g}^{(0)}, \mathbf{g}^{(1)})$ is nondegenerate, see Remark 4.2. Finally, observe that all the groups G, except for $G = \mathrm{O}(k+1)$, that appear in Example 4.1, are connected, hence they are nice. Also the orthogonal group $\mathrm{O}(k+1)$ is nice, as $\mathrm{O}(k+1)/\mathrm{SO}(k+1) \cong \mathbb{Z}_2$. The result now follows from Corollary 4.4 and Proposition 4.7, keeping in mind that the action of G on M is harmonically free, see Example 4.1. □

4.2. Product of spheres

Consider the case when M is the product of two spheres $\mathbb{S}^n \times \mathbb{S}^n$ of same dimension n, endowed with the metric $\mathbf{g}_\lambda = \mathbf{g} \oplus \lambda \mathbf{g}$, where $\mathbf{g}$ is the standard round metric on $\mathbb{S}^n$. Since $\mathbf{g}_\lambda$ and $\mathbf{g}_{\frac{1}{\lambda}}$ belong to the same conformal class, it suffices to consider the case $\lambda \in]0, 1]$.

The j-th eigenvalue of $\Delta_{\mathbf{g}}$ is $\rho_j = j(j+n-1)$, which gives

$$\sigma_{i,j}(\lambda) = \frac{1}{\lambda}\left[j(j+n-1) - \frac{n(n-1)}{2n-1}\right] + i(i+n-1) - \frac{n(n-1)}{2n-1};$$

by Corollary 4.8, every zero of σ_{ij} is a bifurcation instant. One computes easily that $\sigma_{i,j}$ has a zero in the interval $]0,1]$ only if $j=0$; the zero of $\sigma_{i,0}$ in $]0,1]$ is given by:

$$\lambda_i(n) = \frac{n(n-1)}{i(i+n-1)(2n-1) - n(n-1)}, \quad i > 0;$$

this forms a strictly decreasing sequence tending to 0 as $i \to +\infty$, and its maximum is $\lambda_1(n) = \frac{n-1}{n}$. By Proposition 3.1, the family $\mathbf{g}_\lambda$ is locally rigid in the interval $]\frac{n-1}{n}, \frac{n}{n-1}[$.

Since for $\lambda = 1$ the metric $\mathbf{g}_\lambda$ on $\mathbb{S}^n \times \mathbb{S}^n$ is Einstein, we know that $\mathbf{g}_1$ is the unique metric in its conformal class with given volume and constant scalar curvature. It is an interesting open question if the same is true for the metric $\mathbf{g}_\lambda$, for $\lambda \in]\frac{n-1}{n}, \frac{n}{n-1}[$. Our local rigidity result gives a partial answer to this question, in that it excludes the existence of other constant scalar curvature metrics with given volume *near* $\mathbf{g}_\lambda$ for $\lambda \in]\frac{n-1}{n}, \frac{n}{n-1}[$. This result can be improved as follows:

Proposition 4.9. *Consider the product manifold $M = \mathbb{S}^n \times \mathbb{S}^n$ endowed with the metric $\mathbf{g}_\lambda = \mathbf{g} \oplus \lambda \cdot \mathbf{g}$, where $\mathbf{g}$ is the round metric on $\mathbb{S}^n$. Consider the set*:

$$\mathcal{A} = \left\{\lambda \in \left]\frac{n-1}{n}, \frac{n}{n-1}\right[: \text{the conformal class of } \mathbf{g}_\lambda \text{ contains only one metric with constant scalar curvature and volume } v_\lambda\right\}; \tag{11}$$

Then, $\mathcal{A}$ is an open subset of $]\frac{n-1}{n}, \frac{n}{n-1}[$ containing 1.

If $\bar\lambda$ is an accumulation point of $\mathcal{A}$, then every constant curvature metric in the conformal class of $\mathbf{g}_{\bar\lambda}$ which is not homothetic to $\mathbf{g}_{\bar\lambda}$ is degenerate.[9]

Proof. Clearly $1 \in \mathcal{A}$, as we observed above. By taking homotheties, we can assume that the volume of each $\mathbf{g}_\lambda$ is equal to 1. Assume $\lambda_* \in \mathcal{A}$ and, by absurd, that there exists a sequence $\lambda_k \in]\frac{n-1}{n}, \frac{n}{n-1}[\setminus \mathcal{A}$ with $\lim_{k\to\infty} \lambda_k = \lambda_*$. Let $\mathbf{g}_k$ be a constant scalar curvature metric in the conformal class of $\mathbf{g}_{\lambda_k}$ and of volume 1 which is different from $\mathbf{g}_{\lambda_k}$. By the local rigidity around λ_*, for k large $\mathbf{g}_{\lambda_k}$ cannot enter in some neighborhood of $\mathbf{g}_{\lambda_*}$. The set of unit volume constant scalar curvature metrics on $\mathbb{S}^n \times \mathbb{S}^n$ that belong to the conformal class of some $\mathbf{g}_\lambda$, with $\lambda \in [\frac{n-1}{n}, \frac{n}{n-1}]$ is compact in the $\mathcal{C}^2$-topology; this follows easily from [8,12,13], see Proposition 4.10 below. Hence, the sequence $\mathbf{g}_k$ must have a subsequence converging in the $\mathcal{C}^2$-topology to a metric $\mathbf{g}_\infty$ which belongs to the conformal class of $\mathbf{g}_{\lambda_*}$. By continuity, $\mathrm{vol}(M, \mathbf{g}_\infty) = 1$ and $\mathbf{g}_\infty$ has constant scalar curvature. This gives a contradiction, because it must be $\mathbf{g}_\infty \neq \mathbf{g}_{\lambda_*}$, but $\lambda_* \in \mathcal{A}$. This shows that $\mathcal{A}$ is open.

Let $\bar\lambda$ be an accumulation point of $\mathcal{A}$ that does not belong to $\mathcal{A}$, and let $\bar{\mathbf{g}} \neq \mathbf{g}_{\bar\lambda}$ be a constant scalar curvature metric in the conformal class of $\mathbf{g}_{\bar\lambda}$ having volume equal to $v_{\bar\lambda}$. If $g_{\bar\lambda}$ were nondegenerate, then by the implicit function theorem (see Proposition 3.1) one could construct a differentiable path of constant scalar curvature metrics $\lambda \mapsto \mathbf{h}_\lambda$, $\lambda \in]\bar\lambda - \varepsilon, \bar\lambda + \varepsilon[$, with $\mathbf{h}_{\bar\lambda} = \bar{\mathbf{g}}$, with $\mathbf{h}_\lambda \neq \mathbf{g}_\lambda$ in the conformal class of $\mathbf{g}_\lambda$ and of volume equal to v_λ for all λ. This contradicts the fact that for $\lambda \in \mathcal{A}$ near $\bar\lambda$, $\mathbf{g}_\lambda$ is the unique such a metric in its conformal class. $\square$

We have used a compactness result for solutions of the Yamabe problem:

Proposition 4.10. *Let M be a compact manifold and let $\mathcal{K}$ be a set of smooth Riemannian metrics on M which is compact in the $\mathcal{C}^{k,\alpha}$-topology with k* sufficiently large,[10] *and such that one of the following assumptions is satisfied*:

[9] I.e., a degenerate critical point of the Hilbert–Einstein functional $\mathcal{A}$ in $\mathcal{M}_1^{2,\alpha}(M, \mathbf{g}_{\bar\lambda})$, see item (f) in Proposition 2.1.

[10] Sufficiently large depending only on $\dim(M)$, see [8, Lemma 10.1, p. 172] for details.

(a) $\dim(M) \leqslant 7$;
(b) *for all* $\mathbf{g} \in \mathcal{K}$, *then the Weyl tensor* $W_{\mathbf{g}}$ *of* $\mathbf{g}$ *satisfies*

$$\left|W_{\mathbf{g}}(p)\right| + \left|\nabla W_{\mathbf{g}}(p)\right| > 0$$

at every point $p \in M$;
(c) $\dim(M) \leqslant 24$ *and* M *is spin.*

Then, the set of unit volume constant scalar curvature metrics that belong to the conformal class of some $\mathbf{g} \in \mathcal{K}$ *is compact in the* $\mathcal{C}^2$*-topology. In particular, the conclusion holds for the family of metrics* $\mathcal{K}_n = \{\mathbf{g}_\lambda\colon \lambda \in [\frac{n-1}{n}, \frac{n}{n-1}]\}$ *in the product* $M = \mathbb{S}^n \times \mathbb{S}^n$.

Proof. The result follows from the arguments in [8,12,13], see in particular [8, Lemma 10.1]. For the second statement, observe that the manifolds $(\mathbb{S}^n \times \mathbb{S}^n, \mathbf{g}_\lambda)$ satisfy assumption (b). Namely, the Weyl tensor of $\mathbf{g}_\lambda$ is never vanishing in $\mathbb{S}^n \times \mathbb{S}^n$, since this is a homogeneous metric which is not locally conformally flat for every λ. The given set $\mathcal{K}_n$ is compact in the $\mathcal{C}^{k,\alpha}$-topology for all k. □

In fact, the result of Proposition 4.9 extends immediately to the case of products of arbitrary Einstein manifolds of positive scalar curvature. We need an elementary result first:

Lemma 4.11. *Let* $W^{(0)}$, $W^{(1)}$ *and* W *be the Weyl tensors of* $(M_0, \mathbf{g}^{(0)})$, $(M_1, \mathbf{g}^{(1)})$ *and* $(M_0 \times M_1, \mathbf{g}^{(0)} \oplus \mathbf{g}^{(1)})$ *respectively. Assume that* M_0 *is Einstein at* p *and* M_1 *is Einstein at* q. *Then* W *vanishes at a point* $(p, q) \in M_0 \times M_1$ *if and only if the following hold*:

(a) $W^{(0)}(p) = 0$, $W^{(1)}(q) = 0$,
(b) $m_1(m_1-1)\kappa^{(0)} + m_0(m_0-1)\kappa^{(1)} = 0$, *where* $m_j = \dim(M_j) \geqslant 2$ *and* $\kappa^{(j)}$ *is the scalar curvature of* M_j, $j = 0, 1$.

In particular, if both $\kappa^{(0)}$ *and* $\kappa^{(1)}$ *are positive, then* (b) *is not satisfied and therefore* $W(p, q) \neq 0$.

Proof. A direct elementary computation using the standard decomposition of a curvature tensor into its irreducible components, see for instance [4]. □

A more general result that characterizes conformally flat product manifolds can be found in [23, Theorem 4].

Proposition 4.12. *Let* $(M_0^{m_0}, \mathbf{g}^{(0)})$ *and* $(M_1^{m_1}, \mathbf{g}^{(1)})$ *be compact Einstein manifolds of positive scalar curvature* $\kappa^{(0)}$ *and* $\kappa^{(1)}$ *respectively. Denote by* $\mathbf{g}_\lambda$, $\lambda \in]0, +\infty[$, *the metric* $\mathbf{g}^{(0)} \oplus \lambda \mathbf{g}^{(1)}$ *on the product manifold* $M = M_0 \times M_1$. *Then, there exists an open subset* $\mathcal{A}$ *of* $]0, +\infty[$ *containing* $\lambda_* = \frac{m_0 \kappa^{(1)}}{m_1 \kappa^{(0)}}$ *such that for all* $\lambda \in \mathcal{A}$, $\mathbf{g}_\lambda$ *is the unique constant scalar curvature metric in its conformal class, up to homotheties.*

If $\bar{\lambda}$ *is an accumulation point of* $\mathcal{A}$, *then every constant curvature metric in the conformal class of* $\mathbf{g}_{\bar{\lambda}}$ *which is not homothetic to* $\mathbf{g}_{\bar{\lambda}}$ *is degenerate.*

Proof. The proof of Proposition 4.9 can be repeated *verbatim* here, observing that the value $\lambda_* = \frac{m_0 \kappa^{(1)}}{m_1 \kappa^{(0)}}$ corresponds to the unique Einstein metric of the family $\mathbf{g}_\lambda$. As to the compactness, note that assumption (b) of Proposition 4.10 is always satisfied in products of Einstein manifolds with positive scalar curvature, by Lemma 4.11. □

4.3. The case of non-positive scalar curvature

Let us now study the bifurcation problem for the family $\mathbf{g}_\lambda$ of metrics on the product $M_0 \times M_1$ under the assumption that either $\kappa^{(0)}$ or $\kappa^{(1)}$ are non-positive. First, we observe that if both $\kappa^{(0)}$ and $\kappa^{(1)}$ are non-positive, then the pair $(\mathbf{g}^{(0)}, \mathbf{g}^{(1)})$ is nondegenerate. If $\kappa^{(0)} \leqslant 0$ and $\kappa^{(1)} > 0$, then the pair $(\mathbf{g}^{(0)}, \mathbf{g}^{(1)})$ is degenerate if and only if $\kappa^{(0)} = 0$ and $\rho_{j_*}^{(1)} = \frac{\kappa^{(1)}}{m-1}$.

Theorem 4.13. *If $\kappa^{(0)} \leqslant 0$ and $\kappa^{(1)} \leqslant 0$, then the family $\mathbf{g}_\lambda$ has no degeneracy instants, and thus it is locally rigid at every $\lambda \in]0, +\infty[$.*

If $\kappa^{(0)} \leqslant 0$, $\kappa^{(1)} > 0$ and the pair $(\mathbf{g}^{(0)}, \mathbf{g}^{(1)})$ is nondegenerate, then the set of degeneracy instants for the Jacobi operator $\mathcal{J}_\lambda$ is a strictly decreasing sequence λ_n that converges to 0 as $n \to \infty$. Moreover, every degeneracy instant is a bifurcation instant for the family $(\mathbf{g}_\lambda)_\lambda$.

Symmetrically, if $\kappa^{(0)} > 0$, $\kappa^{(1)} \leqslant 0$ and the pair $(\mathbf{g}^{(0)}, \mathbf{g}^{(1)})$ is nondegenerate, then the set of degeneracy instants for the Jacobi operator $\mathcal{J}_\lambda$ is a strictly increasing unbounded sequence λ_n, and every degeneracy instant is a bifurcation instant for the family $(\mathbf{g}_\lambda)_\lambda$.

Proof. Follows from an elementary analysis of the zeroes of the functions $\sigma_{i,j}(\lambda)$ given in (9). In the first case $\sigma_{i,j}(\lambda) > 0$ for all $i, j = 0, 1, \ldots, i + j \neq 0$. In the second (resp. in the third) one, the function $\sigma_{i,j}(\lambda)$ admits a zero for all $i \geqslant 0$, and for $j \in \{0, 1, \ldots, j_* - 1\}$ (resp., for all $j \geqslant 0$, and for $i \in \{0, 1, \ldots, i_* - 1\}$) . Then we have a sequence of instants $(\lambda_n)_n$, that converges to 0 (resp. to $+\infty$) as $n \to \infty$ (see the proof of Corollary 4.4), at each of which there is a jump in the dimension of the negative eigenspace of $\mathcal{J}_\lambda$. The conclusion follows from Theorem 3.3. $\square$

4.4. A multiplicity result in conformal classes of the bifurcating branches

Let us consider the case of constant scalar curvature manifolds $(M_0, \mathbf{g}^{(0)})$ and $(M_1, \mathbf{g}^{(1)})$, with $\kappa^{(1)} > 0$, and consider the product manifold $M = M_0 \times M_1$ endowed with the family of metrics $\mathbf{g}_\lambda = \mathbf{g}^{(0)} \oplus \lambda \mathbf{g}^{(1)}$. Let us recall the following terminology. A unit volume metric $\mathbf{g}$ on M is a *Yamabe* metric if it has constant scalar curvature, and it realizes the minimum of all the scalar curvature among the unit volume constant scalar curvature in its conformal class. Let $\mathcal{Y}(M)$ denote the *Yamabe invariant* of M; recall that this is the supremum of the scalar curvature of all Yamabe metrics of M. It is well known that $\mathcal{Y}(M) \leqslant \mathcal{Y}(\mathbb{S}^m)$.

Proposition 4.14. *Let λ_n be the decreasing sequence of bifurcation instants for the family $\mathbf{g}_\lambda$, with* $\lim_{n\to\infty} \lambda_n = 0$. *Then, for n sufficiently large, the conformal class of each metric in the branch bifurcating from $\mathbf{g}_{\lambda_n}$ contains at least* three *distinct unit volume constant scalar curvature metrics.*

Proof. Since $\kappa^{(1)} > 0$, one has $\lim_{\lambda\to 0^+} \kappa_\lambda = +\infty$, see (8). Thus, for $\lambda > 0$ sufficiently small, $\kappa_\lambda > \mathcal{Y}(\mathbb{S}^m) \geqslant \mathcal{Y}(M)$, which implies that for λ small enough, $\mathbf{g}_\lambda$ is not a Yamabe metric. Thus, for n large, $\mathbf{g}_{\lambda_n}$ is not a Yamabe metric, and by continuity also nearby metrics are not Yamabe. Hence, each conformal class of the bifurcating branch issuing from $\mathbf{g}_{\lambda_n}$ contains a constant scalar curvature of the family, another distinct constant scalar curvature near by, and a Yamabe metric. $\square$

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Appendix A. Fiberwise implicit function theorem and bifurcation

In this appendix we give a formal statement of an implicit function theorem and two bifurcation results for functions defined on the total space of a fiber bundle. Their proof is obtained readily from standard results, and they will be omitted.

A.1. Implicit function theorem

Given fiber bundles $\pi_i : E_i \to B_i$, $i = 1, 2$, and a $\mathcal{C}^1$-morphism of fiber bundles $\mathrm{M} : E_1 \to E_2$, the *vertical derivative of* M *at* $e \in E_1$ is the linear map

$$\mathrm{d}_{\mathrm{ver}}\mathrm{M}(e) : T_e\mathcal{F}(e) \to T_{\mathrm{M}(e)}\mathcal{F}\big(\mathrm{M}(e)\big)$$

given by the differential of the restriction $\mathrm{M}|_{\mathcal{F}(e)} : \mathcal{F}(e) \to \mathcal{F}(\mathrm{M}(e))$, where $\mathcal{F}(e) = \pi_1^{-1}(\pi_1(e)) \subset E_1$ is the fiber of E_1 through the point e, and $\mathcal{F}(\mathrm{M}(e)) = \pi_2^{-1}(\pi_2(M(e))) \subset E_2$ is the fiber of E_2 through $\mathrm{M}(e)$.

We have used in the proof of Proposition 3.1 a sort of *fiber bundle implicit function theorem*, whose statement is as follows:

Proposition A.1. *Let $\pi_i : E_i \to B$, $i = 1, 2$, be fiber bundles, let $\mathrm{M} : E_1 \to E_2$ be a fiber bundle morphism of class $\mathcal{C}^k$, $k \geqslant 1$, let $s : U \subset B \to E_2$ be a local section of E_2 of class $\mathcal{C}^k$, with U open subset of B containing x_0, $s(x_0) = e_2$, and let $e_1 \in \mathrm{M}^{-1}(e_2)$. Assume that the vertical derivative $\mathrm{d}_{\mathrm{ver}}\mathrm{M}(e_1)$ is an isomorphism. Then, there exists an open neighborhood V of e_1 in E_1, with $U' = \pi_1(V) \subset U$, and a $\mathcal{C}^k$-section $\tilde{s} : U' \to E_1$ with $\tilde{s}(x_0) = e_1$, such that $e \in V \cap \mathrm{M}^{-1}(s(U))$ if and only if $e \in \tilde{s}(U')$.* $\square$

A.2. Fiberwise bifurcation

We propose a slightly more general statement of a celebrated bifurcation result by Smoller and Wasserman, see [20]. Recall that the basic setup of [20] consists of a path $\lambda \mapsto \mathrm{M}_\lambda$ of *gradient operators* from a *fixed* Banach space B_2 to another *fixed* Banach space B_0, with $B_2 \subset B_0$, and a path $\lambda \to u_\lambda \in B_2$ satisfying $M_\lambda(u_\lambda) = 0$ for all λ. The main results in [20] give sufficient conditions for the existence of bifurcation branch of solutions of the equation $F(u, \lambda) = \mathrm{M}_\lambda(u) = 0$ issuing from some point of the path u_λ, both in the general and in the equivariant case. These results are used in the present paper in a slightly different context, in that our setup consists of a gradient operators F_λ defined on a smoothly *varying* Banach submanifold $\mathcal{D}_\lambda$ of a fixed Banach space, and taking values also in a smoothly *varying* family $\mathcal{E}_\lambda$ of closed subspaces of a Banach space. An extension of the results in [20] to this situation is quite straightforward, using local charts and projections, nevertheless it may be interesting to provide a precise statement of the result which is employed in the present paper.

Let us give a few definitions. Given a Banach space B, a family $[a, b] \ni \lambda \mapsto B_\lambda$ of Banach submanifolds of B is said to be a $\mathcal{C}^1$-family of submanifolds of B if the set $\mathcal{B} = \{(x, \lambda) \in B \times [a, b]\colon x \in B_\lambda\}$ has the structure of a $\mathcal{C}^1$-sub-bundle of the trivial bundle $B \times [a, b]$ over $[a, b]$. For instance, given a $\mathcal{C}^1$-function $f : B \times [a, b] \to \mathbb{R}$ such that $\frac{\partial f}{\partial x} \neq 0$ at all points in $f^{-1}(0)$, then the family $B_\lambda = \{(x, \lambda)\colon f(x, \lambda) = 0\}$ is a $\mathcal{C}^1$-family of submanifolds of B. Similarly, by a $\mathcal{C}^1$-family of closed subspaces of the Banach space B we mean a family $[a, b] \ni \lambda \mapsto S_\lambda$ of Banach subspaces of B such that the set $\mathcal{S} = \{(x, \lambda)\colon \lambda \in [a, b],\ x \in S_\lambda\}$ is a sub-bundle of the trivial Banach space bundle $B \times [a, b]$ over $[a, b]$. If $\lambda \mapsto x_\lambda \in B$ is a $\mathcal{C}^1$-path, $\mathcal{B} = \bigcup_\lambda (B_\lambda \times \{\lambda\})$ is a $\mathcal{C}^1$-family of submanifolds of B, with $x_\lambda \in B_\lambda$ for all λ, then the path $\lambda \mapsto T_{x_\lambda} S_\lambda$ is a $\mathcal{C}^1$-family of closed subspaces of B.

Theorem A.2. *Let B_0, B_2 be Banach spaces, H a Hilbertable space. Let $[a, b] \ni \lambda \mapsto \mathcal{D}_\lambda \subset B_2$ be a $\mathcal{C}^1$-family of submanifolds of B_2, and let $[a, b] \ni \lambda \mapsto \mathcal{E}_\lambda \subset B_0$ and $[a, b] \ni \lambda \mapsto H_\lambda \subset H$ be $\mathcal{C}^1$-families of closed subspaces of B_0 and of H respectively. Let $F : \mathcal{D} \to \mathcal{E}$ be a $\mathcal{C}^1$ bundle morphism, and assume that the following are satisfied*:

(a) *$\lambda \mapsto e_\lambda \in \mathcal{E}_\lambda$ is a $\mathcal{C}^1$-section of the bundle $\mathcal{E}$*;
(b) *$\lambda \mapsto d_\lambda \in \mathcal{D}_\lambda$ is a $\mathcal{C}^1$-section of the bundle $\mathcal{D}$, with*

$$F(d_\lambda, \lambda) = (e_\lambda, \lambda)$$

for all λ;
(c) *it is given a $\mathcal{C}^1$-family of complete inner products $\lambda \mapsto \langle\cdot,\cdot\rangle_\lambda$ in H_λ*;
(d) *there are continuous inclusions $B_2 \subset B_0 \subset H$ that induce inclusions $T_{d_\lambda}\mathcal{D}_\lambda \subset \mathcal{E}_\lambda \subset H_\lambda$ for all λ*;
(e) *for all λ, the map $F_\lambda = F(\cdot, \lambda) : \mathcal{D}_\lambda \to \mathcal{E}_\lambda$ is a gradient operator at d_λ, i.e., the differential $\mathrm{d}F(\cdot, \lambda) : T_{d_\lambda}\mathcal{D}_\lambda \to \mathcal{E}_\lambda$ is symmetric relatively to the inner product $\langle\cdot,\cdot\rangle_\lambda$*;
(f) *$\mathrm{d}F(\cdot, \lambda) : T_{d_\lambda}\mathcal{D}_\lambda \to \mathcal{E}_\lambda$ is Fredholm of index 0 for all λ*;
(g) *for all λ, there exists an $\langle\cdot,\cdot\rangle_\lambda$-orthonormal basis $e_1^\lambda, e_2^\lambda, \ldots$ of H_λ consisting of eigenvectors of $\mathrm{d}F(\cdot, \lambda)$*;
(h) *the corresponding eigenvectors have finite multiplicities, and for all λ the number n_λ of eigenvalues* (*counted with multiplicities*) *of $\mathrm{d}F(\cdot, \lambda)$ that are negative is finite*;
(i) *there exists $\lambda_* \in]a, b[$ such that, for $\varepsilon > 0$ sufficiently small*:
   – *$\mathrm{d}F(\cdot, \lambda_* - \varepsilon)$ and $\mathrm{d}F(\cdot, \lambda_* + \varepsilon)$ are non-singular*;
   – *$n_{\lambda_* - \varepsilon} \neq n_{\lambda_* + \varepsilon}$.*

Then, λ_* is a bifurcation instant *for the equation*

$$F(\cdot,\lambda)=(e_\lambda,\lambda),$$

i.e., there exists a sequence $d_n \in B_2$, and a sequence λ_n in $[a,b]$, with $d_n \in \mathcal{D}_{\lambda_n}$ for all n, $\lim_{n\to\infty}\lambda_n=\lambda_*$, $\lim_{n\to\infty} d_n = d_{\lambda_*}$, $d_n \neq d_{\lambda_n}$ for all n, and such that

$$F(d_n,\lambda_n)=(e_{\lambda_n},\lambda_n)$$

for all n.

Proof. Sufficiently small neighborhoods of $(d_{\lambda_*},\lambda_*)$ in $\mathcal{D}$ and of $(e_{\lambda_*},\lambda_*)$ in $\mathcal{E}$ are identified respectively with open subsets of products $T_{d_{\lambda_*}}\mathcal{D}_{\lambda_*}\times[\lambda_*-\varepsilon,\lambda_*+\varepsilon]$ and $\mathcal{E}_{\lambda_*}\times[\lambda_*-\varepsilon,\lambda_*+\varepsilon]$. Using these identifications, the bundle morphism F is given by a $\mathcal{C}^1$-path of gradient operators F_λ between open subsets of the Banach spaces $T_{d_{\lambda_*}}D_{\lambda_*}$ and $\mathcal{E}_{\lambda_*}$. The result is then obtained as a straightforward application of [20, Theorem 2.1]. □

In the situation described by items (a)–(h) in Theorem A.2, assume that G is a connected (or more generally, a *nice* in the sense of [20]) Lie group, and that B_0, B_2 and H are G-spaces. Assume that $\mathcal{D}_\lambda$, $\mathcal{E}_\lambda$ and H_λ are G-invariant for all λ, and that F is G-equivariant, i.e.:

$$F(g\cdot d,\lambda)=g\cdot F(d,\lambda)$$

for all $(d,\lambda)\in\mathcal{D}$ and all $g\in G$. Assume further that $g\cdot d_\lambda=d_\lambda$ and $g\cdot e_\lambda=e_\lambda$ for all $g\in G$ and all λ. It is easy to see that every eigenspace of $\mathrm{d}F(\cdot,\lambda)$ is G-invariant for all λ. Denote by π_λ^- the representation of G on the finite dimensional space given by the direct sum of all eigenspaces of $\mathrm{d}F(\cdot,\lambda)$ corresponding to negative eigenvalues.

Theorem A.3. *Let $\lambda_*\in\,]a,b[$ be such that, for $\varepsilon>0$ sufficiently small*:

- $\mathrm{d}F(\cdot,\lambda_*-\varepsilon)$ *and* $\mathrm{d}F(\cdot,\lambda_*+\varepsilon)$ *are non-singular*;
- $\pi^-_{\lambda_*-\varepsilon}$ *and* $\pi^-_{\lambda_*+\varepsilon}$ *are not equivalent.*

Then, λ_* is a bifurcation instant for the equation $F(\cdot,\lambda)=(e_\lambda,\lambda)$.

Proof. The result is an application of [20, Theorem 3.1], using a local product structure of $\mathcal{D}$ and $\mathcal{E}$ around the points $(d_{\lambda_*},\lambda_*)$ and $(e_{\lambda_*},\lambda_*)$. □

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