

# Global existence for reaction–diffusion systems with nonlinear diffusion and control of mass

El Haj Laamri <sup>a</sup>, Michel Pierre <sup>b,\*</sup>

<sup>a</sup> Institut Elie Cartan, Université de Lorraine, B.P 239, 54506 Vandœuvre-lès-Nancy, France

<sup>b</sup> Ecole Normale Supérieure de Rennes, IRMAR, UEB, Campus de Ker Lann, 35170 Bruz, France

Received 7 January 2015; received in revised form 20 November 2015; accepted 1 March 2016

Available online 17 March 2016

## Abstract

We prove here global existence in time of weak solutions for some reaction–diffusion systems with natural structure conditions on the nonlinear reactive terms which provide positivity of the solutions and uniform control of the total mass. The diffusion operators are nonlinear, in particular operators of the porous media type  $u_i \mapsto -d_i \Delta u_i^{m_i}$ . Global existence is proved under the assumption that the reactive terms are bounded in  $L^1$ . This extends previous similar results obtained in the semilinear case when the diffusion operators are linear of type  $u_i \mapsto -d_i \Delta u_i$ .

© 2016 L'Association Publications de l'Institut Henri Poincaré. Published by Elsevier B.V. All rights reserved.

MSC: 35K10; 35K40; 35K57

Keywords: Reaction–diffusion systems; Nonlinear diffusion; Porous media equation; Global existence

## 1. Introduction

The goal of this paper is the study of global existence in time of solutions to reaction–diffusion systems of the following type

$$\begin{cases} \text{for all } i = 1, \dots, m, \\ \partial_t u_i - \Delta \varphi_i(u_i) &= f_i(u_1, u_2, \dots, u_m) & \text{in } ]0, +\infty[ \times \Omega \\ u_i(t, \cdot) &= 0, & \text{on } ]0, +\infty[ \times \partial\Omega, \\ u_i(0, \cdot) &= u_{i0} \geq 0 & \text{in } \Omega. \end{cases} \quad (1)$$

Here  $\Omega$  is a bounded open subset of  $\mathbb{R}^N$  with a regular boundary,  $\varphi_i, i = 1, \dots, m$  are continuous increasing functions from  $[0, +\infty)$  into  $[0, +\infty)$  with  $\varphi_i(0) = 0$  and the  $f_i$  are regular functions such that the two following main properties occur:

\* Corresponding author.

E-mail addresses: [El-Haj.Laamri@univ-lorraine.fr](mailto:El-Haj.Laamri@univ-lorraine.fr) (E.H. Laamri), [michel.pierre@ens-rennes.fr](mailto:michel.pierre@ens-rennes.fr) (M. Pierre).

- $(P)$ : the nonnegativity of the solutions is preserved for all time;
- $(M)$ : the total mass of the components is controlled for all time (sometimes even exactly preserved).

Properties  $(P)$  and  $(M)$  are natural in applications: these systems are mathematical models for evolution phenomena undergoing at the same time spatial diffusion and (bio-)chemical type of reactions. The unknown functions are generally densities, concentrations, temperature so that their nonnegativity is required. Moreover, often a control of the total mass, sometimes even preservation of the total mass, is naturally guaranteed by the model. Interest has increased recently for these models in particular for applications in biology, ecology, environment and population dynamics.

Mathematically speaking,  $(P)$  is satisfied (like for systems of ordinary differential equations) if and only if  $f = (f_i)_{1 \leq i \leq m}$  is *quasipositive* whose meaning is recalled in (5).

Condition  $(M)$  is satisfied if for instance

$$\sum_{1 \leq i \leq m} f_i \leq 0 \quad (2)$$

or, more generally, if this sum is reasonably controlled (see (6) for precise assumption).

These two conditions imply  $L^1(\Omega)$ -bounds on the solutions which are uniform on each finite time interval (see Lemma 2.3):

$$\forall i = 1, \dots, m, \forall T > 0, \sup_{t \in [0, T]} \|u_i(t)\|_{L^1(\Omega)} < +\infty.$$

Unlike uniform  $L^\infty(\Omega)$ -estimates on each finite interval, such  $L^1$ -estimates are not enough to imply existence of global solution on  $(0, +\infty)$ . More structure is needed for global existence.

Actually, many results of global existence are known for these systems in the *semilinear case* when the diffusions are linear and given for instance by  $\varphi_i(u_i) = d_i u_i$ ,  $d_i \in (0, +\infty)$ . Existence of regular bounded solutions on  $(0, +\infty)$  may be found for example in [25,16,27,26,19,18,36,10,9,22,39,14,17,3,4] and in several other articles whose references may be found in the survey [31] or in the book [37]. However, it is well-known that the solutions may blow up in  $L^\infty(\Omega)$  in finite time as proved in [33,34] where explicit finite time blow up in  $L^\infty(\Omega)$  are given. Thus, even in the semilinear case, it is necessary to deal with *weak solutions* if one expects global existence.

Our main goal here is to exploit the good “ $L^1$ -framework” provided by the two conditions  $(P)$ ,  $(M)$  and to see how the main results of global existence of weak solutions extend from the semilinear case to the case when the  $\varphi_i$  are nonlinear, in particular of the *porous media type*, namely  $\varphi_i(u_i) = d_i u_i^{m_i}$ ,  $m_i \geq 1$ . In this case, degeneracy of the diffusion occurs at the same time for small  $u_i$  and for large  $u_i$ .

We are interested in looking for extensions to these nonlinear diffusions of the two following main results proved in the semilinear case:

- first the global existence result of weak solutions for (1) when  $(P)$ ,  $(M)$  hold and when moreover an *a priori*  $L^1$ -estimate holds for the nonlinear reactive part, namely (see [29,30] and the survey [31]).

$$\forall i = 1, \dots, m, \forall T > 0, \int_{(0, T) \times \Omega} |f_i(u_1, \dots, u_m)| < +\infty. \quad (3)$$

- next the fact that global existence of weak solutions hold for *quadratic nonlinearities*  $f_i$  satisfying only  $(P)$ ,  $(M)$ . This is a consequence of the latter result and of a main *a priori*  $L^2((0, T) \times \Omega)$ -estimate on the solutions implied by  $(P) + (M)$  and which is interesting for itself. This estimate was noticed in [12,28,33,34] and then widely exploited, see for example [11,6,7,35,32,8,5,9,31].

We will first see that these two results extend to the case when the  $\varphi_i$  are nonlinear but nondegenerate (that is when  $\varphi_i'$  is bounded from below and from above, see Proposition 2.4). But, the situation is more complicated and not so clear in the degenerate case  $\varphi_i(u_i) = d_i (u_i)^{m_i}$ . More precisely:

1. We are able to prove global existence of solutions under the a priori estimate (3) if  $m_i \in \left(\frac{(N-2)^+}{N}, 2\right)$  for all  $i$ . We do not know whether the restriction  $m_i < 2$  is only technical or due to deeper phenomena. But, at least, it appears

as being necessary to extend the approach of the semilinear case as such. This is explained in more details next (see [Theorem 2.6](#), [Corollary 2.11](#) and their proofs).

2. On the other hand, we can prove that the a priori  $L^2$ -estimate of the semilinear case has a natural extension to the degenerate case, this for any  $m_i \geq 1$ . Indeed, under the only assumptions  $(P)$ ,  $(M)$ , the solutions  $u_i$  are a priori bounded in  $L^{m_i+1}((0, T) \times \Omega)$  for all  $T > 0$ . This allows in particular to prove global existence for System (1) with quadratic reactive terms or with growth less than  $m_i + 1$  and some other classical reactive terms (see [Theorem 2.7](#), [Corollary 2.8](#) and [Corollary 2.9](#)).

A main reason to try to exploit the “ $L^1$ -framework” provided by  $(P)$ ,  $(M)$  for System (1) is that, like in the semilinear case, the operator  $u_i \rightarrow \partial_t u_i - d_i \Delta u_i^{m_i}$  has good  $L^1$ -compactness properties in the sense that the following mapping is compact when  $m > (N - 2)^+ / N$ :

$$(w_0, F) \in L^1(\Omega) \times L^1((0, T) \times \Omega) \mapsto w \in L^1((0, T) \times \Omega),$$

where  $w$  is the solution of

$$\partial_t w - d \Delta w^m = F \text{ in } (0, T) \times \Omega, \quad w = 0 \text{ on } (0, T) \times \partial\Omega, \quad w(0) = w_0.$$

This provides compactness for the solutions of the adequate approximations of System (1). Next, the main difficulty—which is actually serious—is to show that the limit of these approximate solutions is indeed solution of the limit system.

Note that, besides the semilinear case, this kind of  $L^1$ -approach was also used with success in [20] for such systems with nonlinear diffusions of the  $p$ -Laplacian type  $\partial_t u_i - \nabla \cdot (|\nabla u_i|^{p-2} \nabla u_i)$ . Let us also mention some global existence and finite time blow up in [15,27] and [21] for  $2 \times 2$  systems with nonlinear diffusion  $\varphi_i(u_i) = u_i^{m_i}$ ,  $i = 1, 2$  and with growth conditions on the reactive terms like

$$f_1(u_1, u_2) = u_1^\alpha + u_2^\beta + C_1, \quad f_2(u_1, u_2) = u_1^\delta + u_2^\gamma, \quad 1 \leq \alpha, \delta \leq m_1, \quad 1 \leq \beta, \gamma \leq m_2.$$

A particular example of System (1) with

$$m = 2, \quad \varphi_1(u_1) = u_1^{m_1}, \quad \varphi_2(u_2) = d_2 u_2, \quad f_1 \leq 0, \quad f_2 = -f_1$$

was also shown in [20] to have weak solutions for  $m_1 \in [1, 2)$  and initial data  $(u_{10}, u_{20}) \in L^{m_1+1}(\Omega) \times L^2(\Omega)$  and as well strong bounded global solutions for bounded initial data and polynomial growth of  $f_1$  (even for general  $\varphi_1$  in this case). Nondegenerate nonlinear diffusions were also considered in [12] and [32] with quadratic reactive terms.

## 2. Main results

Throughout this paper, we denote  $Q := (0, +\infty) \times \Omega$ ,  $Q_T := (0, T) \times \Omega$ ,  $\Sigma := (0, +\infty) \times \partial\Omega$ ,  $\Sigma_T := (0, T) \times \partial\Omega$  and, for  $p \in [1, +\infty)$

$$\|u(t)\|_{L^p(\Omega)} = \left( \int_{\Omega} |u(t, x)|^p dx \right)^{1/p}, \quad \|u\|_{L^p(Q_T)} = \left( \int_0^T \int_{\Omega} |u(t, x)|^p dt dx \right)^{1/p},$$

$$\|u(t)\|_{L^\infty(\Omega)} = \text{ess sup}_{x \in \Omega} |u(t, x)|, \quad \|u\|_{L^\infty(Q_T)} = \text{ess sup}_{(t,x) \in Q_T} |u(t, x)|.$$

For  $i = 1, \dots, m$ , let  $f_i : Q \times [0, +\infty)^m \rightarrow \mathbb{R}$  be such that

$$\text{Regularity: } \begin{cases} f_i \text{ is measurable,} \\ \forall T > 0, f(\cdot, \cdot, 0), g(\cdot, \cdot, 0) \in L^1(Q_T), \\ \exists K : [0, +\infty) \rightarrow [0, +\infty) \text{ nondecreasing such that:} \\ |f_i(t, x, r) - f_i(t, x, \hat{r})| \leq K(M) \|r - \hat{r}\|, \\ \text{for all } M > 0, \text{ for all } r, \hat{r} \in (0, M)^m \text{ and a.e. } (t, x) \in Q, \end{cases} \tag{4}$$

where  $\|r\| = \sum_{1 \leq i \leq m} |r_i|$  is the norm chosen in  $\mathbb{R}^m$ .

**Quasipositivity:** (P)  $\begin{cases} f_i(t, x, r_1, \dots, r_{i-1}, 0, r_{i+1}, \dots, r_m) \geq 0, \\ \text{for all } r = (r_i)_{1 \leq i \leq m} \geq 0, \text{ a.e. } (t, x) \in Q. \end{cases}$  (5)

**Control of mass:** (M)  $\begin{cases} \forall r \in [0, +\infty)^m, \text{ for a.e. } (t, x), \sum_i f_i(t, x, r) \leq \sigma \|r\| + h(t, x) \\ \text{for some } \sigma \in [0, +\infty), h \in L^1(Q_T)^+ \text{ for all } T > 0. \end{cases}$  (6)

... These three above properties will be assumed throughout the paper...

**Remark 2.1.** Note that all results given in this paper immediately extend if (M) is replaced by the existence of  $\alpha_i \in (0, +\infty)$  such that

$$\forall r \in [0, +\infty)^m, \text{ a.e. } (t, x), \sum_i \alpha_i f_i(t, x, r) \leq \sigma \|r\| + h(t, x).$$

Indeed we may multiply each  $i$ -th equation by  $\alpha_i$  and changing  $u_i$  into  $v_i := \alpha_i u_i$ . For simplicity, and without loss of generality, we will work here with (M) as above.

For  $i = 1, \dots, m$ , let  $\varphi_i : [0, +\infty) \rightarrow [0, +\infty)$  be increasing, continuously differentiable on  $(0, +\infty)$  with  $\varphi_i(0) = 0$ . We will mainly consider two situations:

*The nondegenerate case:*

$$\exists a_i, b_i \in (0, +\infty), \forall s \in (0, +\infty), 0 < a_i \leq \varphi_i'(s) \leq b_i < +\infty. \tag{7}$$

*The possibly degenerate case:*

$$\forall s \in [0, +\infty), \varphi_i(s) = d_i s^{m_i}, m_i \in (0, +\infty), d_i \in (0, +\infty). \tag{8}$$

We consider the associated System (1) where the *weak solution* of each equation is understood in the sense of nonlinear semigroups in  $L^1(\Omega)$  (see [40] for various definitions of solutions). More precisely, if  $\varphi$  denotes one of the  $\varphi_i$  and if  $(w_0, F) \in L^\infty(\Omega) \times L^\infty(Q_T)$ , we will use, especially in the approximation processes, the following notion of *bounded solutions*:

$$\begin{cases} w \in \mathcal{C}([0, T]; L^1(\Omega)) \cap L^\infty(Q_T), \varphi(w) \in L^2(0, T; H_0^1(\Omega)), \\ \partial_t w - \Delta \varphi(w) = F \text{ in the sense of distributions in } Q_T, \\ w(0) = w_0. \end{cases} \tag{9}$$

If  $\varphi$  satisfies one of the conditions (7) or (8), then for  $(w_0, F)$  given in  $L^\infty(\Omega) \times L^\infty(Q_T)$ , such a solution exists and is unique (see e.g. [40, Chapters 5 and 6]). Moreover, if  $\hat{w}$  is the solution associated with  $(\hat{w}_0, \hat{F}) \in L^\infty(\Omega) \times L^\infty(Q_T)$ , we have

$$\|w(t) - \hat{w}(t)\|_{L^1(\Omega)} \leq \|w_0 - \hat{w}_0\|_{L^1(\Omega)} + \int_0^t \|F(s) - \hat{F}(s)\|_{L^1(\Omega)} ds, \tag{10}$$

so that

$$(w_0, F) \in L^1(\Omega) \times L^1(Q_T) \mapsto w \in \mathcal{C}([0, T]; L^1(\Omega))$$

is a contraction. This allows to extend by density, in a unique way, the notion of solution to any  $(F, w_0) \in L^1(Q_T) \times L^1(\Omega)$  and we will denote it by

$$w := S_\varphi(w_0, F). \tag{11}$$

This is the notion of solution that will mainly be used in this paper. Note that it satisfies

$$\begin{cases} w \in \mathcal{C}([0, T]; L^1(\Omega)), \varphi(w) \in L^1(Q_T) \text{ and } \forall \psi \in \mathcal{C}_T, \\ - \int_\Omega \psi(0) w_0 - \int_{Q_T} \partial_t \psi w + \varphi(w) \Delta \psi = \int_{Q_T} \psi F, \end{cases} \tag{12}$$

where

$$\mathcal{C}_T = \{\psi : [0, T] \times \overline{\Omega} \rightarrow \mathbb{R}; \psi, \partial_t \psi, \partial_{x_i x_j}^2 \psi \text{ are continuous, } \psi = 0 \text{ on } \Sigma_T, \psi(T) = 0\}. \tag{13}$$

The latter property (12) corresponds to the notion of *very weak solution* in Definition 6.2 of [40].

Solutions in the sense of (11) satisfy the maximum principle and order properties:

$$[w_0 \geq 0, F \geq 0] \Rightarrow [S_\varphi(w_0, F) \geq 0], [w_1 \geq w_2, F_1 \geq F_2] \Rightarrow [S_\varphi(w_1, F_1) \geq S_\varphi(w_2, F_2)].$$

Recall also that  $w := S_\varphi(w_0, F)$  satisfies (see e.g. [40])

$$\forall p \in [1, +\infty], \forall t \in [0, T], \|w(t)\|_{L^p(\Omega)} \leq \|w_0\|_{L^p(\Omega)} + \int_0^t \|F(s)\|_{L^p(\Omega)} ds. \tag{14}$$

We now define what we mean by a solution to our System (1).

**Definition 2.2.** Given  $u_{i0} \in L^1(\Omega)$ ,  $u_{i0} \geq 0$ ,  $i = 1, \dots, m$ , by **global weak solution** to System (1), we mean  $u = (u_1, u_2, \dots, u_m) : (0, +\infty) \times \Omega \rightarrow [0, +\infty)^m$  such that, for all  $i = 1, \dots, m$  and for all  $T > 0$

$$\begin{cases} u_i \in \mathcal{C}([0, +\infty); L^1(\Omega)), \varphi_i(u_i) \in L^1(0, T; W_0^{1,1}(\Omega)), \\ u_i = S_{\varphi_i}(u_{i0}, f_i(u)). \end{cases} \tag{15}$$

Note that we only deal with nonnegative solutions.

*The approximate reaction–diffusion system* Next we consider the following approximation of System (1) with solution  $u^n := (u_1^n, \dots, u_m^n)$  in the sense of (9) for each equation, that is

$$\begin{cases} \text{for all } i = 1, \dots, m, \\ \text{for all } T > 0, u_i^n \in L^\infty(Q_T) \ u_i \geq 0, \ \varphi_i(u_i^n) \in L^2(0, T; H_0^1(\Omega)), \\ \partial_t u_i^n - \Delta \varphi_i(u_i^n) = f_i^n(u^n) \text{ in } Q, \\ u_i^n(t, \cdot) = 0 \text{ on } \Sigma, \\ u_i^n(0, \cdot) = u_{i0}^n \geq 0 \text{ in } \Omega, \end{cases} \tag{16}$$

where  $u_{i0}^n \in L^\infty(\Omega)^+$  converges to  $u_{i0}$  in  $L^1(\Omega)$  and the approximate nonlinearities  $f_i^n$  satisfy (4) with  $K(\cdot)$  independent of  $n$ , (5), (6) with  $\sigma, h$  independent of  $n$  and are in  $L^\infty(Q_T \times \mathbb{R}^m)$  for each  $n$ . The convergence of  $f_i^n$  toward  $f_i$  is defined as follows. Let us denote

$$\epsilon_M^n := \max_{1 \leq i \leq m} \sup_{0 \leq \|r\| \leq M} |f_i^n(t, x, r) - f_i(t, x, r)|. \tag{17}$$

We will assume that

$$\epsilon_M^n \rightarrow 0 \text{ in } L^1(Q_T) \text{ and a.e. as } n \rightarrow +\infty. \tag{18}$$

As a typical example, we may choose

$$f_i^n := \frac{f_i}{1 + \frac{1}{n} \sum_{1 \leq j \leq m} |f_j|}. \tag{19}$$

Note that, with this choice,  $\|f_i^n\|_{L^\infty(Q)} \leq n$  and the other properties may easily be checked (in (4),  $K(M)$  has to be replaced by  $(2 + m)K(M)$ ).

**Lemma 2.3.** Assume that, for  $1 \leq i \leq m$ ,  $\varphi_i$  satisfies either (7) or (8) with  $m_i > 0$ . Then the approximate system (16) has a (global and regular) solution  $u^n$  and there exists  $C : [0, +\infty) \rightarrow [0, +\infty)$ , independent of  $n$  such that

$$\forall n, \forall T > 0, \sup_{t \in [0, T]} \sum_{1 \leq i \leq m} \|u_i^n(t)\|_{L^1(\Omega)} \leq C(T) \left[ 1 + \sum_{1 \leq i \leq m} \|u_{i0}\|_{L^1(\Omega)} \right].$$

Now, let us assume, like in the semilinear case that, for whatever reason, an a priori  $L^1$ -estimate holds for the solution  $u^n$  of the approximate System (16), namely

$$\forall T > 0, \sup_n \sum_{1 \leq i \leq m} \|f_i^n(u^n)\|_{L^1(Q_T)} < +\infty. \tag{20}$$

Examples of such situations and applications will be given later (see also the survey [31]). The question is to decide whether, like in the semilinear case,  $u^n$  converges to a global weak solution of (1).

A first—not surprising—result is that, when the nonlinearities  $\varphi_i$  are nondegenerate, then this convergence property does hold. Moreover, the a priori  $L^2$ -estimate holds as well. Indeed we have the following proposition (and this is a particular case of Theorem 2.6 and Theorem 2.7 below):

**Proposition 2.4.** *Assume all functions  $\varphi_i$  are nondegenerate in the sense of (7). Then, up to a subsequence,  $u^n$  converges in  $[L^1(Q_T)]^m$  for all  $T > 0$  to a global weak solution of (1) in the sense of Definition 2.2. If moreover  $u_0 \in L^2(\Omega)^m$  and  $h \in L^1_{loc}([0, +\infty); L^2(\Omega))$  in the assumption (6), then there exists  $C : [0, +\infty) \rightarrow [0, +\infty)$  such that*

$$\sup_n \sum_{1 \leq i \leq n} \|u_i^n\|_{L^2(Q_T)} \leq C(T) \left[ 1 + \sum_{1 \leq i \leq m} \|u_{i0}\|_{L^2(\Omega)} \right].$$

**Remark 2.5.** Versions of the above  $L^2$ -estimate may also be found in [12,28,32] where they were used to prove global existence results for systems of type (1) with nondegenerate  $\varphi_i$  and quadratic reactive terms. Global existence with general right-hand side bounded in  $L^1$  seems however to be new.

Now the question is to decide what happens in the degenerate case. We can prove the following.

**Theorem 2.6.** *Assume that, for  $1 \leq i \leq m$ ,  $\varphi_i$  satisfies either (7) or (8) with  $m_i \in ((N-2)^+/N, 2)$ . Assume  $L^1$ -estimate (20) holds. Then, up to a subsequence,  $u^n$  converges in  $[L^1(Q_T)]^m$  for all  $T > 0$  to a global weak solution of (1) in the sense of Definition 2.2.*

As commented in the introduction, we do not know whether the restriction  $m_i < 2$  is necessary or not. We will explain where it naturally appears in the proof and suggest some possible reasons. We will deduce a global existence result for System (1) below in Corollary 2.11.

On the other hand, it turns out that the a priori  $L^2$ -estimate does have a natural extension no matter the value of the  $m_i$ .

**Theorem 2.7.** *Assume that, for  $1 \leq i \leq m$ ,  $\varphi_i$  satisfies either (7) or (8) with  $m_i > 0$ . If moreover  $h \in L^1_{loc}([0, +\infty); L^2(\Omega))$  in the assumption (6), then there exists  $C : [0, +\infty) \rightarrow [0, +\infty)$  such that*

$$\sup_n \sum_{1 \leq i \leq m} \|u_i^n\|_{L^{m_i+1}(Q_T)} \leq C(T) \left[ 1 + \sum_{1 \leq i \leq m} \|u_{i0}^n\|_{L^2(\Omega)} \right],$$

where we set  $m_i := 1$  in case (7).

We deduce the following global existence result.

**Corollary 2.8.** *Assume that, for  $1 \leq i \leq m$ ,  $\varphi_i$  satisfies either (7) or (8) with  $m_i \geq 1$ . Assume there exists  $\epsilon > 0$  such that*

$$\sum_{1 \leq i \leq m} |f_i(u)| \leq C \left[ 1 + \sum_{1 \leq i \leq m} u_i^{m_i+1-\epsilon} \right]. \quad (21)$$

Then, for all  $u_0 \in L^2(\Omega)^m$ ,  $u_0 \geq 0$ , the system (1) has a global weak solution in the sense of Definition 2.2.

As we will see in the proof, the main point of the “ $-\epsilon$ ” in the above assumption is that it makes the nonlinearities  $f_i^n(u^n)$  not only bounded in  $L^1(Q_T)$ , but *uniformly integrable*. This is the main tool to pass to the limit in the reactive terms. Actually, any other assumption guaranteeing this uniform integrability of the  $f_i^n(u^n)$  will lead to global existence. For instance, it follows from this theorem that global existence holds for the typical system modeling the chemical reaction

$$U_1 + U_3 \rightleftharpoons U_2 + U_4.$$

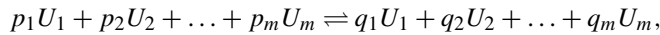
Indeed, applying the mass action law for the reactive terms and a Darcy’s law for the diffusion lead to the following  $4 \times 4$  system for the concentrations  $u_i = u_i(t, x)$  of the components  $U_i$ ,  $1 \leq i \leq 4$ :

$$\begin{cases} \text{for } 1 \leq i \leq 4, \\ \partial_t u_i - d_i \Delta \varphi_i(u_i) = (-1)^i [u_1 u_3 - u_2 u_4] \text{ in } Q, \\ u_i = 0 \text{ on } \Sigma, \quad u_i(0) = u_{i0} \geq 0. \end{cases} \tag{22}$$

The following result is a direct consequence of [Corollary 2.8](#) when  $m_i > 1$  for all  $1 \leq i \leq 4$ . If some of the  $m_i$  are equal to 1, then an extra argument is needed to prove that the reactive terms are not only bounded in  $L^1$ , but uniformly integrable. This is coming from the entropy inequality and from an  $L^1$ -estimate that it provides on  $u_i^{m_i+1} (\log u_i)^2$  (extending the  $L^2$ -techniques of [\[11\]](#)).

**Corollary 2.9.** *Assume that, for  $1 \leq i \leq 4$ ,  $\varphi_i$  satisfies either (7) or (8) with  $m_i \geq 1$ . Then, for all  $u_0 = (u_{i0})_{1 \leq i \leq 4}$  with  $u_{i0} \geq 0$  and  $u_{i0} \log u_{i0} \in L^2(\Omega)$  for  $1 \leq i \leq 4$ , System (22) has a global weak solution in the sense of [Definition 2.2](#).*

**Remark 2.10.** We may also consider more general reversible chemical reactions of the form



where  $p_i, q_i$  are nonnegative integers. According to the usual mass action kinetics and with Darcy’s laws for the diffusion, the evolution of the concentrations  $u_i$  of  $U_i$  may be modeled by the following system

$$\partial_t u_i - d_i \Delta u_i^{m_i} = (p_i - q_i) \left( k_2 \prod_{j=1}^m u_j^{q_j} - k_1 \prod_{j=1}^m u_j^{p_j} \right), \quad i = 1 \dots m, \tag{23}$$

where  $k_1, k_2$  are positive diffusion coefficients and where a stoichiometric law holds like  $\sum_i \alpha_i p_i = \sum_i \alpha_i q_i$  for some  $\alpha_i \in (0, +\infty)$ . Similarly to [Corollary 2.9](#), global existence of weak solutions may be proven when

$$\sum_i \frac{p_i}{m_i + 1} \leq 1, \quad \sum_i \frac{q_i}{m_i + 1} \leq 1. \tag{24}$$

Indeed, together with [Theorem 2.7](#), this guarantees that the reactive terms are bounded in  $L^1(Q_T)$ . The entropy inequality, which is valid for this system as well as for (22), allows to prove their uniform integrability in all cases when (24) holds. This approach is the same as for [Corollary 2.9](#) and relies also on an  $L^1$ -estimate on  $u_i^{m_i+1} (\log u_i)^2$ . For completeness, we also give the main steps of the proof of this remark after the proof of [Corollary 2.9](#).

It is known that, for instance for a  $2 \times 2$  system, an a priori  $L^1$  bound of type (20) holds as soon as two linear relations between  $f_1, f_2$  hold rather than only one, like

$$f_1 + f_2 \leq 0, \quad f_1 + \lambda f_2 \leq 0, \quad \lambda \in [0, 1).$$

More generally, if there are  $m$  linearly independent similar inequalities in an  $m \times m$  system, then estimate (20) holds. Actually, by coupling [Theorem 2.6](#) and [Theorem 2.7](#), we may even prove the following.

**Corollary 2.11.** *Assume that for all  $1 \leq i \leq m$ ,  $\varphi_i$  satisfies either (7) or (8) with  $m_i \in ((N - 2)^+ / N, 2)$ . Assume moreover that there exists an invertible  $m \times m$  matrix  $P$  with nonnegative entries and  $\mathbf{b} \in \mathbb{R}^m$  such that*

$$\forall r \in [0, +\infty)^m, \quad Pf(r) \leq \mathbf{b} \left[ 1 + \sum_i r_i^{1+m_i} \right], \tag{25}$$

where again  $m_i := 1$  in case (7). Then, for all  $u_0 \in L^1(\Omega)^m, u_0 \geq 0$ , the System (1) has a global weak solution in the sense of [Definition 2.2](#).

**Remark 2.12.** We emphasize the fact that any  $L^1(\Omega)$ -initial data is allowed in this result. As particular standard situations covered by [Corollary 2.11](#), we have the  $2 \times 2$  systems where the nonlinearities are as in the two following examples:

- 1)  $f_1 \geq 0, f_2 = -f_1.$
- 2)  $f_1(u_1, u_2) = \lambda u_1^p u_2^q - u_1^\alpha u_2^\beta, f_2(u_1, u_2) = -u_1^p u_2^q + u_1^\alpha u_2^\beta, \lambda \in [0, 1), p, q, \alpha, \beta \geq 1.$

Indeed, (25) is satisfied with  $\mathbf{b} = (0, 0)$  and successively

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, P = \begin{pmatrix} 1 & 1 \\ 1 & \lambda \end{pmatrix}.$$

Note that in these two examples, there is no restriction on the growth of  $f_1, f_2$ , but as stated in Corollary 2.11, it is required that  $m_i < 2$ . On the other hand, when applying Corollary 2.8 to this system (see also Remark 2.10), we obtain global existence of weak solution, no matter the values of the  $m_1, m_2$ , but with the growth conditions

$$\frac{p}{m_1 + 1} + \frac{q}{m_2 + 1} < 1, \frac{\alpha}{m_1 + 1} + \frac{\beta}{m_2 + 1} < 1.$$

The case  $m_1 = m_2 = 3, p = \beta = 5, q = \alpha = 2$  is for instance not covered (except may be in small space dimensions) by any of the above results although the reactive terms are a priori bounded in  $L^1(Q_T)$  (see the proof of Corollary 2.11) and even if  $\lambda = 0$ . *This is an interesting open problem.*

### 3. The proofs

**Proof of Lemma 2.3.** Since the  $f_i^n$  are bounded for each  $n$ , existence of a (unique) bounded global solution  $u^n$  is classical. Let us recall a procedure without too many details. Given  $T \in (0, +\infty)$ , we consider the set

$$\mathcal{W} := \{v \in \mathcal{C}([0, T]; L^1(\Omega)^m); \forall i = 1, \dots, m, v_i(0) = u_{i0}^n, \|v_i\|_{L^\infty(Q_T)} \leq R\},$$

where  $R = \|u_{i0}^n\|_{L^\infty(\Omega)} + nT$  (recall that  $f_i^n$  is uniformly bounded by  $n$ ). We equip  $\mathcal{W}$  with the norm  $\|v\| := \max_i \sup_{t \in [0, T]} \|v_i(t)\|_{L^1(\Omega)}$ . Then, we consider the mapping  $\mathcal{F}$  which to  $v^n = (v_1^n, \dots, v_m^n) \in \mathcal{W}$  associates the solution  $u^n = (u_1^n, \dots, u_m^n) \in \mathcal{W}, u_i^n = S_{\varphi_i}(u_{i0}^n, f_i^n(\pi(v^n)))$  where  $\pi : \mathbb{R}^m \rightarrow [0, +\infty)^m$  is the projection onto the positive cone, that is  $\pi(r_1, \dots, r_m) = (r_1^+, \dots, r_m^+)$ . Using the estimates (10)–(14), it is easy to prove that  $\mathcal{F}$  send  $\mathcal{W}$  into itself and that some iterate of  $\mathcal{F}$  is a strict contraction. Whence the existence of a fixed point  $u^n$ . The  $L^2(0, T; H_0^1(\Omega))$ -regularity holds by construction for these bounded solutions (see [40]). Next, multiplying each equation by  $(u_i^n)^- = -\inf\{u_i^n, 0\}$ , integrating on  $Q_T$  and summing over  $i$ , thanks to the quasipositivity of  $f^n$  we deduce that  $(u_i^n)^- \equiv 0$ , whence the nonnegativity of  $u^n$ . We refer e.g. to [20] for more details.

Next, summing all the  $m$  equations of (16) and integrating on  $\Omega$  gives, using (M):

$$\partial_t \int_{\Omega} \sum_{1 \leq i \leq m} u_i^n(t) \leq \int_{\Omega} \sum_{1 \leq i \leq m} f_i^n(u^n) \leq \sigma \sum_{1 \leq i \leq m} \|u_i^n(t)\|_{L^1(\Omega)} + h = \sigma \int_{\Omega} \sum_{1 \leq i \leq m} u_i^n(t) + h.$$

Integrating this Gronwall’s inequality gives for all  $t \in [0, T]$

$$\sum_{1 \leq i \leq m} \|u_i^n(t)\|_{L^1(\Omega)} \leq e^{\sigma T} \left[ \sum_{1 \leq i \leq m} \|u_{i0}\|_{L^1(\Omega)} + \|h\|_{L^1(Q_T)} \right].$$

Whence the estimate of Lemma 2.3.  $\square$

Let us now recall the main compactness properties of the solutions of (11). Here  $\varphi : [0, +\infty) \rightarrow [0, +\infty)$  denotes one of the functions  $\varphi_i$ .

**Lemma 3.1.** *Assume  $\varphi$  satisfies (7) or (8) with  $m_i > \frac{(N-2)^+}{N}$ . Then the mapping*

$$(w_0, F) \in L^1(\Omega) \times L^1(Q_T) \mapsto S_\varphi(w_0, F) \in L^1(Q_T)$$

*is compact.*

**Proof.** For a proof, see [2].  $\square$



**Lemma 3.2.** Let  $\varphi(w) = d w^q, d \in (0, +\infty), q > (N - 2)^+ / N$ . Then, for  $(w_0, F) \in L^1(\Omega) \times L^1(Q_T), w = S_\varphi(w_0, F)$  of (11) satisfies

$$\int_{Q_T} |w|^{q\alpha} \leq C \text{ for } 0 < \alpha < 1 + \frac{2}{qN}, \tag{26}$$

$$\int_{Q_T} |\nabla w^q|^\beta \leq C_2 \text{ for } 1 \leq \beta < 1 + \frac{1}{1 + qN}, \tag{27}$$

where  $C = C(T, \alpha, \beta, q, \|w_0\|_{L^1(\Omega)}, \|F\|_{L^1(Q_T)})$ .

If  $\varphi$  is nondegenerate in the sense of (7), then the estimates (26) and (27) are valid with  $q = 1$ .

**Proof.** For a proof, see Lukkari [23, Lemma 4.7] for the case  $q > 1$  and Lukkari [24, Lemma 3.5] for the case  $\frac{(N-2)^+}{N} < q < 1$ . In these two references, the proof is given with zero initial data, but with right-hand side a bounded measure. We may use the measure  $\delta_{t=0} \otimes w_0 dx$  to include the case of initial data  $w_0$ . We may also use the results in [1, Theorem 2.9]). The estimate in the nondegenerate case may be obtained in a similar way.  $\square$

In several of the proofs below, we will use the famous Vitali’s Lemma (see e.g. [13, theorem 2.24, page 150], [38, chapter 16]).

**Lemma 3.3 (Vitali).** Let  $(E, \mu)$  be a measured space such that  $\mu(E) < +\infty$ , let  $1 \leq p < +\infty$  and let  $\{f_n\}_n \subset L^p(E)$  such that  $f_n \rightarrow f$  a.e. If  $\{f_n^p\}_n$  is uniformly integrable over  $E$ , then  $f \in L^p(E)$  and  $f_n \rightarrow f$  in  $L^p(E)$ .

We now deduce various compactness properties of the approximate solution  $u^n$  of (16).

**Lemma 3.4.** Assume that  $\varphi_i$  satisfy (7) or (8) with  $m_i > (N - 2)^+ / N$  and that the  $L^1$ -estimate (20) holds for the solution  $u^n$  of (16). Then, up to a subsequence, and for all  $T > 0$  and  $1 \leq i \leq m$ ,

- $u_i^n$  converge in  $L^1(Q_T)$  and a.e. to some  $u_i \in L^1(Q_T)$ ,
- $\varphi_i(u_i^n)$  converge in  $L^\alpha(Q_T)$  and a.e. to  $\varphi_i(u_i)$  for all  $\alpha \in [1, 1 + 2/(m_i N)]$  in case (8) and all  $\alpha \in [1, 1 + 2/N]$  in case (7),
- $\varphi_i(u_i) \in L^\beta(0, T; W_0^{1,\beta}(\Omega))$  for all  $\beta \in [1, 1 + 1/(1 + m_i N)]$  in case (8) and for all  $\beta \in [1, 1 + 1/(1 + N)]$  in case (7),
- $f_i^n(u^n)$  converges a.e. to  $f_i(u) \in L^1(Q_T)$ .

**Proof of Lemma 3.4.** By the estimate (20),  $f_i^n(u^n)$  is bounded in  $L^1(Q_T)$ . According to Lemma 3.1,  $u_i^n$  is relatively compact in  $L^1(Q_T)$  for all  $T > 0$ . Therefore, up to a subsequence, we may assume that  $u_i^n$  converge in  $L^1(Q_T)$  for all  $T > 0$  and a.e. in  $Q$  as well to some limit  $u_i \in L^1(Q_T)$ .

Next, by Lemma 3.2,  $\varphi_i(u_i^n)$  is bounded in  $L^\alpha(Q_T)$  for  $\alpha \in [1, 1 + 2/(m_i N)]$  [even for  $\alpha \in [1, 1 + 2/N]$  in the nondegenerate case] and for all  $T > 0$ . By arbitrariness of  $\alpha$  in this interval open to the right,  $\varphi_i(u_i^n)^\alpha$  is even uniformly integrable. Since it also converges a.e. to  $\varphi_i(u_i)$ , by the Vitali’s Lemma 3.3, the convergence holds strongly in  $L^\alpha(Q_T)$  to  $\varphi_i(u_i)$ .

Next, thanks to the estimate of the gradient in Lemma 3.2,  $\varphi_i(u_i^n)$  stays bounded in the space  $L^\beta(0, T; W_0^{1,\beta}(\Omega))$  for all  $\beta \in [1, 1 + 1/(1 + m_i N)]$  [even all  $\beta \in [1, 1 + 1/(1 + N)]$  in the nondegenerate case]. These spaces being reflexive (for  $\beta > 1$ ), it follows that  $\varphi_i(u_i)$  also belongs to these same spaces.

Finally, due to the definition of the  $f_i^n$  and to the a.e. convergence of  $u^n$  to  $u = (u_i)_{1 \leq i \leq m}$ , it is clear that  $f_i^n(u^n)$  converges a.e. to  $f_i(u)$ . By Fatou’s Lemma,  $f_i(u) \in L^1(Q_T)$ .  $\square$

**Remark 3.5.** To prove that the limit  $u$  is solution of the limit problem, we would “only need” to prove that the convergence of  $f_i^n(u^n)$  to  $f_i(u)$  holds in the sense of distributions and not only a.e. But this is where the main difficulty of the proof lies. Indeed,  $f_i^n(u^n)$  is bounded in  $L^1(Q_T)$ . Therefore it converges in the sense of measures to  $f_i(u) + \mu$  where  $\mu$  is a bounded measure. The point is to prove that this measure is equal to zero.

An easy situation is when the  $f_i^n(u^n)$  are uniformly integrable and not only bounded in  $L^1(Q_T)$ . Then, using Vitali’s Lemma 3.3, we deduce that the convergence of  $f_i^n(u^n)$  to  $f_i(u)$  holds in  $L^1(Q_T)$  and therefore in the sense of distributions. It follows that  $u$  is solution of the limit problem.

Actually, our method here, similar to the one in [30], will be to first prove that  $u$  is a supersolution of the limit system. This is where the main difficulty is concentrated. The main result is stated in the next proposition. It is interesting to emphasize that the conclusion of this proposition is valid without the structure property  $(M)$ . This property  $(M)$  will only be used later to prove the reverse inequality.

**Proposition 3.6.** *Under the assumptions of Lemma 3.4, the limit  $u$  is a supersolution of the limit system, which means that, for all  $\psi \in C_T$  as defined in (13),  $\psi \geq 0$  and for  $1 \leq i \leq m$ :*

$$-\int_{\Omega} \psi(0)u_{i0} + \int_{Q_T} -\partial_t \psi u_i + \nabla \psi \nabla \varphi_i(u_i) \geq \int_{Q_T} \psi f_i(u),$$

where  $u_i \in L^\infty((0, T); L^1(\Omega))$ ,  $\varphi_i(u_i) \in L^1((0, T); W_0^{1,1}(\Omega))$ .

**Preliminary remark about Proposition 3.6 and its proof.** Note that the result of this Proposition is interesting for itself and, as we already remark, is valid without the structure assumption  $(M)$  on the nonlinearities  $f_i^n$ . The ideas of the proof of Proposition 3.6 are taken from [30]. A first idea is that, if  $w$  is a solution of the heat equation, then  $T_k(w)$  is a supersolution of the heat equation where  $T_k$  is a regular approximation of the truncation function  $r \in [0, +\infty) \mapsto \inf\{r, k\}$  as defined below. Here, we first prove that  $T_k(u_i)$  is indeed a supersolution for all  $k$ : by letting  $k$  go to  $+\infty$ , it will follow that  $u_i$  itself is a supersolution, whence Proposition 3.6.

In order to obtain that  $T_k(u_i)$  is a supersolution, we pass to the limit as  $n \rightarrow +\infty$  in the inequation satisfied by an adequate approximation of  $T_k(u_i^n)$ . But to pass to the limit in the sense of distributions in the nonlinear reaction terms (which a priori converge only a.e.), each truncation of the  $i$ -th equation must also involve all the  $u_j^n$ ,  $j \neq i$ : more precisely, in the semilinear case, the method was to write for each  $i$ , the inequation satisfied by  $T_k(u_i^n + \eta \sum_{j \neq i} u_j^n)$  with  $\eta > 0$ , then first to let  $n \rightarrow +\infty$  for  $\eta, k$  fixed, and next to let  $\eta \rightarrow 0$ , then  $k \rightarrow +\infty$  (see [30]). The main work was to justify the step  $\eta \rightarrow 0$  which involves estimates on the gradient of the solutions.

Here, the ideas are the same, but we have to adapt them to nonlinear diffusions. Besides and because of the degeneracy due this nonlinearity, gradient estimates are not as good as for linear diffusions, especially near  $u_i = 0$ . Moreover, the nonlinear diffusion requires more complex truncations than in the linear case. This is why we consider the truncating process (29) below.

To prepare the proof of Proposition 3.6, let us introduce the truncating functions  $T_k : [0, +\infty) \rightarrow [0, +\infty)$  of class  $C^3$  which satisfy the following for all  $k \geq 1$ :

$$\begin{cases} T_k(r) = r \text{ if } r \in [0, k - 1], \\ T_k(r) \leq k; \\ T_k'(r) = 0 \text{ if } r \geq k \\ 0 \leq T_k'(r) \leq 1, -1 \leq T_k''(r) \leq 0 \text{ for all } r \geq 0. \end{cases} \tag{28}$$

Next, for all  $i = 1, \dots, m$  and for  $(n, \eta, k) \in \mathbb{N}^* \times (0, 1) \times [1, +\infty[$ , we introduce

$$A_{i,\eta,k}^n = \partial_t (T_k(u_i^n)T_k'(\eta V_i^n)) - \nabla \cdot (T_k'(u_i^n)T_k''(\eta V_i^n)\nabla \varphi_i(u_i^n)), \quad V_i^n = \sum_{j \neq i} u_j^n. \tag{29}$$

**Remark 3.7.** To give some light on the choice of the above expression, note that, when  $\eta \rightarrow 0$ , then  $T_k'(\eta V_i^n) \rightarrow 1$  and when  $k \rightarrow +\infty$ , then  $T_k$  tends to the identity so that this expression approximates  $\partial_t u_i^n - \nabla \cdot (\nabla \varphi_i(u_i^n)) = \partial_t u_i^n - \Delta \varphi_i(u_i^n)$ .

We check that

$$\begin{cases} A_{i,\eta,k}^n = T_k'(u_i^n)T_k''(\eta V_i^n)f_i^n(u^n) + A_i^n + B_i^n \text{ where} \\ A_i^n = \eta T_k(u_i^n)T_k''(\eta V_i^n)(V_i^n)_t = \eta T_k(u_i^n)T_k''(\eta V_i^n) \sum_{j \neq i} [\Delta \varphi_j(u_j^n) + f_j^n(u^n)], \\ \text{that we write as an obvious sum: } A_i^n =: \sum_{j \neq i} X_j^n + Y_j^n, \\ B_i^n = -\nabla \varphi_i(u_i^n)\nabla [T_k'(u_i^n)T_k''(\eta V_i^n)]. \end{cases} \tag{30}$$

The proof of Proposition 3.6 will mainly rely on the following estimate.

**Lemma 3.8.** *There exist  $\delta > 0, C > 0$  independent of  $n$  and  $\eta$  such that, for all  $i = 1, \dots, m$  and for all  $\psi \in \mathcal{C}_T, \psi \geq 0$ :*

$$\int_{Q_T} A_{i,\eta,k}^n \psi \geq \int_{Q_T} T'_k(u_i^n) T'_k(\eta V_i^n) f_i^n(u_n) \psi - C D(\psi) \eta^\delta, \tag{31}$$

where  $D(\psi) = \|\psi\|_{L^\infty(Q_T)} + \|\nabla\psi\|_{L^\infty(Q_T)}$ .

**Proof of Lemma 3.8.** It is a direct consequence of formula (30) and of Lemmas 3.12 and 3.13 below.  $\square$

The proof of Lemmas 3.12 and 3.13 below will require the following preliminary estimate:

**Lemma 3.9.** *Let  $F \in L^1(Q_T)^+, w_0 \in L^1(\Omega)^+$ . Then  $w = S_\varphi(w_0, F)$  as defined in (11) satisfies the following: there exists  $C = C\left(\int_{Q_T} F, \int_\Omega w_0\right)$  such that, for all nondecreasing  $\theta : (0, +\infty) \rightarrow (0, +\infty)$  of class  $\mathcal{C}^1$  and with  $\theta(0^+) = 0$*

$$\int_{[\theta(w) \leq k]} |\nabla\theta(w)| |\nabla\varphi(w)| = \int_{[\theta(w) \leq k]} \nabla\theta(w) \nabla\varphi(w) \leq C k. \tag{32}$$

In particular,

$$\int_{[\varphi(w) \leq k]} |\nabla\varphi(w)|^2 \leq C k, \quad \int_{[w \leq k]} |\nabla w|^2 \leq C k^{2-\bar{m}} \tag{33}$$

with  $\bar{m} = 1$  in case (7) and with  $\bar{m} = m_i$  in case (8) assuming  $m_i < 2$ .

**Remark 3.10.** The main restriction  $m_i < 2$  discussed in the introduction appears in the above statement. The proof of Theorem 2.6 requires to control the  $L^2$ -norm of  $\nabla u_i^n$  on the level sets  $[u_i^n \leq k]$ . This  $L^2$ -norm is not bounded if  $m_i \geq 2$  because of the degeneracy around the points where  $u_i^n = 0$ . It is however valid for the large values of  $u_i^n$ . But this does not seem to be sufficient for the proof.

**Proof of Lemma 3.9.** As usual, we make the computations for regular enough solutions and they are preserved by approximation for all semigroup solutions.

Multiply equation  $\partial_t w - \Delta\varphi(w) = F$  by  $T_{k+1}(\theta(w))$ . We obtain

$$\int_\Omega J_k(w)(T) + \int_{Q_T} T'_{k+1}(\theta(w)) \nabla\theta(w) \nabla\varphi(w) = \int_{Q_T} T_{k+1}(\theta(w)) F + \int_\Omega J_k(w_0),$$

where  $J'_k(r) = T_{k+1}(\theta(r)), J_k(0) = 0$ . Since  $T_{k+1} \leq k + 1$ , we have  $J_k(r) \leq (k + 1)r$  so that

$$\int_{[\theta(w) \leq k]} |\nabla\theta(w)| |\nabla\varphi(w)| \leq (k + 1) \left( \int_{Q_T} F + \int_\Omega w_0 \right) \leq C k.$$

Choosing  $\theta := \varphi$  gives the first estimate of (33). The second one is clear in the nondegenerate case (7). If  $\varphi_i(r) = d_i r^{m_i}$  with  $m_i < 2$ , we choose  $\theta(r) := r^{2-m_i}$  to obtain

$$d_i(2 - m_i)m_i \int_{[w^{2-m_i} \leq k]} |\nabla w|^2 \leq C k,$$

which gives the second estimate of (33) by changing  $k$  into  $k^{2-m_i}$ .  $\square$

**Remark 3.11.** The two next lemmas provide the expected estimates for  $B_i^n$ , then  $A_i^n$ . We will often use that, for some  $C$  independent of  $n$  and  $\eta$ , it follows from (32) that, for  $i, j = 1, \dots, m$

$$\int_{[\eta \varphi_j(u_j^n) \leq k]} |\nabla \varphi_j(u_j^n)|^2 \leq Ck/\eta, \quad \int_{[\eta V_i^n \leq k]} |\nabla V_i^n|^2 \leq C[k/\eta]^{2-M}, \quad M = \max\{1, \max_i m_i\}, \tag{34}$$

where we used the inclusion:  $\forall j \neq i, [\eta V_i^n \leq k] \subset [\eta u_j^n \leq k]$ .

**Lemma 3.12.** *There exist  $C \geq 0, \delta > 0$  independent of  $n$  and  $\eta$  such that, for all  $i = 1, \dots, m$  and for all  $\psi \in \mathcal{C}_T, \psi \geq 0$*

$$\int_{Q_T} \psi B_i^n \geq -\eta^\delta C \|\psi\|_{L^\infty(Q_T)}. \tag{35}$$

**Proof of Lemma 3.12.** We have

$$\begin{aligned} \int_{Q_T} \psi B_i^n &= - \int_{Q_T} \psi \nabla \varphi_i(u_i^n) \nabla [T_k'(u_i^n) T_k''(\eta V_i^n)] \\ &= - \int_{Q_T} \psi \nabla \varphi_i(u_i^n) [\nabla u_i^n T_k''(u_i^n) T_k'(\eta V_i^n) + T_k'(u_i^n) T_k''(\eta V_i^n) \eta \nabla V_i^n] \\ &\geq -\eta \int_{Q_T} \psi T_k'(u_i^n) T_k''(\eta V_i^n) \nabla \varphi_i(u_i^n) \nabla V_i^n, \end{aligned}$$

the last inequality coming from  $T_k'' \leq 0, \varphi_i' \geq 0, \psi \geq 0$ . By Schwarz’s inequality and for some  $C = C(k)$

$$\begin{aligned} \int_{Q_T} |\psi T_k'(u_i^n) T_k''(\eta V_i^n) \nabla \varphi_i(u_i^n) \nabla V_i^n| &\leq C \|\psi\|_{L^\infty(Q_T)} \left( \int_{[u_i^n \leq k]} |\nabla \varphi_i(u_i^n)|^2 \right)^{1/2} \left( \int_{[\eta V_i^n \leq k]} |\nabla V_i^n|^2 \right)^{1/2} \\ &\leq C \|\psi\|_{L^\infty(Q_T)} \sqrt{\varphi_i(k)} [k/\eta]^{1-M/2}, \quad M := \max\{1, \max_i m_i\}, \end{aligned}$$

where the last inequality is obtained through (32) and (34). Thus,  $\int_{Q_T} \psi B_i^n \geq -CD(\psi)\eta^{M/2}$  for some  $C = C(k)$ . Whence (35) with  $\delta = M/2$ .  $\square$

**Lemma 3.13.** *There exist  $\delta > 0, C \geq 0$  independent of  $n$  and  $\eta$  such that, for all  $i = 1, \dots, m$  and for all  $\psi \in \mathcal{C}_T, \psi \geq 0$ :*

$$\int_{Q_T} \psi A_i^n \geq -\eta^\delta CD(\psi). \tag{36}$$

**Proof of Lemma 3.13.** We will need several steps. Recall that  $A_i^n = X_i^n + Y_i^n$ .

- Let us bound  $\int_{Q_T} Y_j^n \psi$ . We have

$$\int_{Q_T} Y_j^n \psi = \eta \int_{Q_T} \psi T_k(u_i^n) T_k''(\eta V_i^n) f_j^n(u^n),$$

so that, using the  $L^1$ -bound on  $f_j^n$ , we obtain

$$\int_{Q_T} Y_j^n \cdot \psi \geq -\eta C(k) \|\psi\|_{L^\infty(Q_T)}. \tag{37}$$

- Let us bound  $\int_{Q_T} X_j^n \psi$ . We have

$$\int_{Q_T} X_j^n \psi = \int_{Q_T} \eta \psi T_k(u_i^n) T_k''(\eta V_i^n) \Delta \varphi_j(u_j^n) = -I^n - J^n$$

where

$$I^n = \eta \int_{[\eta V_i^n \leq k]} \nabla \varphi_j(u_j^n) \nabla \psi T_k(u_i^n) T_k''(\eta V_i^n) \quad \text{and} \quad J^n = K_{1,n} + K_{2,n}$$

with

$$K_{1,n} = \eta \int_{[\eta V_i^n \leq k] \cap [u_i^n \leq k]} \psi \nabla \varphi_j(u_j^n) T_k'(u_i^n) T_k''(\eta V_i^n) \nabla u_i^n,$$

$$K_{2,n} = \eta^2 \int_{[\eta V_i^n \leq k]} \psi \nabla \varphi_j(u_j^n) T_k(u_i^n) T_k'''(\eta V_i^n) \nabla V_i^n,$$

- Let us bound  $I^n$ . By (27):

$$|I^n| \leq C(k) D(\psi) \eta \int_{Q_T} |\nabla \varphi_j(u_j^n)| \leq C \eta.$$

- Let us bound  $K_{1,n}$ . By Schwarz's inequality, (32)–(34) and  $[\eta V_i^n \leq k] \subset [\eta u_j^n \leq k]$

$$|K_{1,n}| \leq \eta \int_{[\eta u_j^n \leq k] \cap [u_i^n \leq k]} \psi |\nabla \varphi_j(u_j^n)| T_k'(u_i^n) |T_k''(\eta V_i^n)| |\nabla u_i^n|$$

$$\leq C(k) \eta \|\psi\|_{L^\infty(Q_T)} \left( \int_{[\eta u_j^n \leq k]} |\nabla \varphi_j(u_j^n)|^2 \right)^{1/2} \left( \int_{[u_i^n \leq k]} |\nabla u_i^n|^2 \right)^{1/2}$$

$$\leq CD(\psi) \eta \sqrt{\varphi_j(k/\eta)} k^{1-m_i/2} \leq CD(\psi) \eta^{1-m_j/2},$$

where we used  $\varphi_j(r) \leq Cr^{m_j}$  for  $r \geq 1$ .

- Let us bound  $K_{2,n}$ . Using again Schwarz's inequality, (32)–(34) and  $[\eta V_i^n \leq k] \subset [\eta u_j^n \leq k]$ , we obtain:

$$|K_{2,n}| \leq \eta^2 \int_{[\eta V_i^n \leq k]} \psi |\nabla \varphi_j(u_j^n)| |\nabla V_i^n| |T_k(u_i^n)| |T_k'''(\eta V_i^n)|$$

$$\leq C \eta^2 \|\psi\|_{L^\infty(Q_T)} \left[ \int_{[\eta u_j^n \leq k]} |\nabla \varphi_j(u_j^n)|^2 \right]^{1/2} \left[ \int_{[\eta V_i^n \leq k]} |\nabla V_i^n|^2 \right]^{1/2},$$

$$\leq C \eta^2 D(\psi) \sqrt{\varphi_j(k/\eta)} [k/\eta]^{1-M/2}$$

$$\leq C \eta^2 D(\psi) [k/\eta]^{1-m_j/2} [k/\eta]^{1-M/2}$$

$$\leq CD(\psi) \eta^{(m_j+M)/2}. \quad \square$$

**Proof of Proposition 3.6.** Recall that, by Lemma 3.8, we have for all  $\psi \in C_T$ ,  $\psi \geq 0$

$$\int_{Q_T} A_{i,\eta,k}^n \psi \geq \int_{Q_T} T'_k(u_i^n) T'_k(\eta V_i^n) f_i^n(u_n) \psi - C D(\psi) \eta^\delta, \tag{38}$$

where

$$A_{i,\eta,k}^n = \partial_t (T_k(u_i^n) T'_k(\eta V_i^n)) - \nabla \cdot (T'_k(u_i^n) T'_k(\eta V_i^n) \nabla \varphi_i(u_i^n)), \quad V_i^n = \sum_{j \neq i} u_j^n.$$

Note also that

$$\begin{cases} \int_{Q_T} A_{i,\eta,k}^n \psi = - \int_{\Omega} T_k(u_{i0}^n) T'_k(\eta V_i^n(0)) \psi(0) \\ + \int_{Q_T} -T_k(u_i^n) T'_k(\eta V_i^n) \partial_t \psi + T'_k(u_i^n) T'_k(\eta V_i^n) \nabla \varphi_i(u_i^n) \nabla \psi. \end{cases} \tag{39}$$

The main point is to pass to the limit in (38) and (39). We do it in the following order: first  $n \rightarrow +\infty$ , then  $\eta \rightarrow 0$ , finally  $k \rightarrow +\infty$ .

- Let  $n \rightarrow +\infty$  along the subsequence introduced in Lemma 3.4 ( $\eta$  and  $k$  are fixed). Since  $u_{i0}^n \rightarrow u_{i0}$  in  $L^1(\Omega)$  and since  $T_k, T'_k$  are Lipschitz continuous

$$\int_{\Omega} T_k(u_{i0}^n) T'_k(\eta V_i^n(0)) \psi(0) \rightarrow \int_{\Omega} T_k(u_{i0}) T'_k(\eta V_i(0)) \psi(0).$$

For the last integral in (39), since, for all  $j = 1, \dots, m$ ,  $u_j^n$  converges in  $L^1(Q_T)$  and a.e. to  $u_j$ , it follows that  $T_k(u_i^n) T'_k(\eta V_i^n) \rightarrow T_k(u_i) T'_k(\eta V_i)$  in  $L^1(Q_T)$  where we set  $V_i := \sum_{j \neq i} u_j$ . It also follows that  $T'_k(\eta V_i^n)$  converges in  $L^2(Q_T)$  to  $T'_k(\eta V_i)$ . Next,  $T'_k(u_i^n) \nabla \varphi(u_i^n)$  is bounded in  $L^2(Q_T)$  by (33) in Lemma 3.9. Therefore it converges weakly in  $L^2(Q_T)$ . Its limit is necessarily  $T'_k(u_i) \nabla \varphi(u_i)$ . Indeed,  $T'_k(u_i^n) \nabla \varphi(u_i^n) = \nabla S_k(u_i^n)$  where we set  $S_k(r) := \int_0^r T'_k(s) \varphi'(s) ds$ . Since  $S_k(u_i^n)$  converges a.e. to  $S_k(u_i)$  and is bounded, the convergence holds in the sense of distributions. Therefore the distribution limit of  $\nabla S_k(u_i^n)$  is  $\nabla S_k(u_i) = T'_k(u_i) \nabla \varphi_i(u_i)$ . This ends the proof of the passing to the limit in (39).

Now, to pass to the limit in the right-hand side of (38), let us denote

$$W_n := T'_k(u_i^n) T'_k(\eta V_i^n) f_i^n(u^n), \quad W := T'_k(u_i) T'_k(\eta V_i) f_i(u)$$

and let us show that  $W_n$  converges to  $W$  in  $L^1(Q_T)$ . Since  $W_n = 0$  outside the set  $[u_i^n \leq k] \cup [V_i^n \leq k/\eta]$ , if  $M := \max\{k, k/\eta\}$ , we may write (see the definition (17) and property (4) and recall that  $|T'_k| \leq 1$ )

$$|W_n| \leq |f_i^n(t, x, u^n)| \leq |f_i(t, x, 0)| + \epsilon_0^n + K(M) \|u^n(t, x)\|.$$

By assumption (see (18)), as  $n \rightarrow +\infty$ ,  $\epsilon_0^n$  tends to 0 in  $L^1(Q_T)$ . Moreover,  $u^n$  converges in  $L^1(Q_T)^m$  to  $u$ . Therefore, to prove the convergence of  $W_n$  in  $L^1(Q_T)$ , it is sufficient to prove that it converges a.e. We know that, for all  $j$ ,  $u_j^n$  converges a.e. to  $u_j$ . Therefore,  $T'_k(u_i^n) T'_k(\eta V_i^n)$  converges a.e. to  $T'_k(u_i) T'_k(\eta V_i)$ . It remains to check that

$$f_i^n(t, x, u^n(t, x)) \text{ converges a.e. } (t, x) \text{ to } f_i(t, x, u(t, x)). \tag{40}$$

Let  $D$  be the subset of  $(t, x) \in Q_T$  such that, at the same time,  $u^n(t, x)$  converges to  $u(t, x)$  with  $\|u(t, x)\| < +\infty$  and  $\epsilon_p^n(t, x)$  converges to 0 for all positive integer  $p$  as  $n \rightarrow +\infty$  along the subsequence introduced in Lemma 3.4. We know that  $Q_T \setminus D$  is of zero Lebesgue measure. Now let  $(t, x) \in D$  and let  $p > \|u(t, x)\|$ . For  $n$  large enough,  $\|u^n(t, x)\| < p$  and we may write for all  $i = 1, \dots, m$  (using the definition (17) and property (4)):

$$\begin{cases} |f_i^n(t, x, u^n(t, x)) - f_i(t, x, u(t, x))| \leq \epsilon_p(t, x) + |f_i(t, x, u^n(t, x)) - f_i(t, x, u(t, x))| \\ \leq \epsilon_p(t, x) + K(p) \|u^n(t, x) - u(t, x)\|. \end{cases}$$

The right-hand side of this inequality tends to 0 by definition of  $D$ .

According to the above analysis, we can pass to the limit as  $n \rightarrow +\infty$  in (38) and (39) and we obtain that

$$\begin{cases} - \int_{\Omega} T_k(u_{i0}) T'_k(\eta V_i(0)) \psi(0) + \int_{Q_T} -T_k(u_i) T'_k(\eta V_i) \partial_t \psi + T'_k(u_i) T'_k(\eta V_i) \nabla \varphi_i(u_i) \nabla \psi \\ \geq \int_{Q_T} T'_k(u_i) T'_k(\eta V_i) f_i(u) \psi - C D(\psi) \eta^\delta. \end{cases} \tag{41}$$

- We now let  $\eta \rightarrow 0$  for fixed  $k$  in (41). Since  $f_i^n(u^n)$  converges a.e. to  $f_i(u)$  (see (40)) and is bounded in  $L^1(Q_T)$ , Fatou’s lemma implies that  $f_i(u) \in L^1(Q_T)$ . As  $\eta \rightarrow 0$ ,  $T_k'(\eta v_i) \rightarrow 1$  a.e. and stays bounded by 1, then by dominated convergence, we can replace at the limit  $T_k'(\eta V_i)$  in all integrals of (41). Thanks to  $\delta > 0$ , we then obtain

$$-\int_{\Omega} T_k(u_{i0})\psi(0) + \int_{Q_T} -T_k(u_i) \partial_t \psi + T_k'(u_i) \nabla \varphi_i(u_i) \nabla \psi \geq \int_{Q_T} T_k'(u_i) f_i(u) \psi. \tag{42}$$

- Finally, we let  $k \rightarrow +\infty$  in this inequality (42). Then  $T_k(u_i)$  increases to  $u_i$  and  $T_k'(u_i)$  increases to 1,  $\nabla \varphi_i(u_i)$  is at least in  $L^1(Q_T)$  (see (27)) and  $f_i(u) \in L^1(Q_T)$ . Therefore, we easily pass to the limit in (42) to obtain

$$-\int_{\Omega} u_{i0}\psi(0) + \int_{Q_T} -u_i \partial_t \psi + \nabla \varphi_i(u_i) \nabla \psi \geq \int_{Q_T} f_i(u) \psi. \tag{43}$$

And this ends the proof of Proposition 3.6.  $\square$

**Proof of Theorem 2.6.** By Proposition 3.6, we already know that the limit  $u$  is a supersolution in the sense that (43) is satisfied for all  $\psi \in C_T$ ,  $\psi \geq 0$  and for all  $i = 1, \dots, m$ . We will show with the help of the (M) structure property (6) that the inverse inequality is satisfied for the sum of these  $m$  expressions, namely

$$-\int_{\Omega} [\sum_i u_{i0}] \psi(0) + \int_{Q_T} -[\sum_i u_i] \partial_t \psi + [\sum_i \nabla \varphi_i(u_i)] \nabla \psi \leq \int_{Q_T} [\sum_i f_i(u)] \psi. \tag{44}$$

This will imply that equality holds in each of the inequalities (43).

Going back to the approximate system (16) and adding the  $m$  equations lead to the fact that, for all  $\psi$  as above,

$$-\int_{\Omega} [\sum_i u_{i0}^n] \psi(0) + \int_{Q_T} -\left[ \sum_i u_i^n \right] \partial_t \psi + \left[ \sum_i \nabla \varphi_i(u_i^n) \right] \nabla \psi = \int_{Q_T} \left[ \sum_i f_i^n(u^n) \right] \psi.$$

We already know that, along an adequate subsequence of  $n \rightarrow +\infty$ ,  $u_i^n$  converges in  $L^1(Q_T)$  to  $u_i$  and that  $\nabla \varphi(u_i^n)$  converges weakly in  $L^2(Q_T)$  to  $\nabla \varphi_i(u_i)$  (see the proof of Proposition 3.6). Hence, the left-hand side of this equality converges to the expected limit as  $n \rightarrow +\infty$ .

For the right-hand side, the assumption (6) on the  $f_i^n$  says that

$$\sigma \|u^n\| + h - \sum_i f_i^n(u^n) \geq 0.$$

We know that  $u^n$  converges in  $L^1(Q_T)$  to  $u$  and, according to (40), that  $f_i^n(u^n)$  converges a.e. to  $f_i(u)$ . By Fatou’s Lemma

$$\int_{Q_T} \left[ \sigma \|u\| + h - \sum_i f_i(u) \right] \psi \leq \int_{Q_T} (\sigma \|u\| + h) \psi + \liminf_{n \rightarrow +\infty} \int_{Q_T} -\left[ \sum_i f_i^n(u^n) \right] \psi.$$

Therefore

$$\limsup_{n \rightarrow \infty} \int_{Q_T} \left[ \sum_i f_i^n(u^n) \right] \psi \leq \int_{Q_T} \left[ \sum_i f_i(u) \right] \psi,$$

whence (44). And as explained above, this implies that equality holds in (43). We will use below the version obtained after integration by parts, namely that, for all  $\psi \in C_T$

$$-\int_{\Omega} u_{i0}\psi(0) + \int_{Q_T} -u_i \partial_t \psi - \varphi_i(u_i) \Delta \psi = \int_{Q_T} f_i(u) \psi. \tag{45}$$

We know that, at least  $\varphi_i(u_i) \in L^1(0, T; W_0^{1,1}(\Omega))$  by Lemma 3.4. To conclude the proof of Theorem 2.6, it remains to show that we exactly have

$$u_i = S_{\varphi_i}(u_{i0}, f_i(u)), \tag{46}$$

in the sense of (11).

To this end, we first go back to (42). Note that by approximation, this inequality remains valid if one replaces  $T_k$  by the “exact” truncation function

$$\forall r \in [0, k], \Sigma_k(r) = r, \quad \forall r \in [k, +\infty), \Sigma_k(r) = k,$$

and since  $\Sigma'_k(u_i) \nabla \varphi_i(u_i) = \nabla \varphi_i(\Sigma_k(u_i))$  a.e., then (42) may be written

$$-\int_{\Omega} \Sigma_k(u_{i0}) \psi(0) + \int_{Q_T} -\Sigma_k(u_i) \partial_t \psi + \nabla \varphi_i(\Sigma_k(u_i)) \nabla \psi \geq \int_{Q_T} \Sigma'_k(u_i) f_i(u) \psi. \tag{47}$$

Inequality (47) says in some sense that  $\Sigma_k(u_i)$  is a bounded “supersolution” of  $\partial_t \Sigma_k(u_i) - \Delta \varphi_i(\Sigma_k(u_i)) \geq \Sigma'_k(u_i) f_i(u)$ . By the comparison Theorem 6.5 in [40] (see also Proposition 6.4 in [40]), we may deduce that

$$\Sigma_k(u_i) \geq S_{\varphi_i}(\Sigma_k(u_{i0}), \Sigma'_k(u_i) f_i(u)).$$

Now, passing to the limit as  $k \rightarrow +\infty$  and using the contraction property (10), we deduce (note that  $(\Sigma_k(u_{i0}), \Sigma_k(u_i) f_i(u))$  converges to  $(u_{i0}, f_i(u))$  in  $L^1(\Omega) \times L^1(Q_T)$ ):

$$u_i \geq S_{\varphi_i}(u_{i0}, f_i(u)) =: U_i.$$

But, since  $u_i$  satisfies (45) and since so does  $U_i$  by (12), we have

$$\int_{Q_T} (u_i - U_i) \partial_t \psi + [\varphi_i(u_i) - \varphi_i(U_i)] \Delta \psi = 0.$$

Choosing  $\psi(t, x) = (T - t)\zeta(x)$  where  $-\Delta \zeta = 1$  in  $\Omega$ ,  $\zeta = 0$  on  $\partial\Omega$ , we obtain

$$\int_{Q_T} (u_i - U_i) \zeta + [\varphi_i(u_i) - \varphi_i(U_i)](T - t) = 0.$$

Since  $\zeta > 0$ ,  $u_i \geq U_i$ ,  $\varphi_i$  increasing, we deduce  $u_i \equiv U_i$  whence (46).  $\square$

**Proof of Theorem 2.7.** We add the  $m$  equations of System (16) to obtain

$$\partial_t \left( \sum_i u_i^n \right) - \Delta \left( \sum_i \varphi_i(u_i^n) \right) = \sum_i f_i^n(u_i^n) \leq \sigma \|u^n\| + h = \sigma \sum_i u_i^n + h.$$

We rewrite this as

$$\partial_t \left( e^{-\sigma t} \sum_i u_i^n \right) - \Delta \left( e^{-\sigma t} \sum_i \varphi_i(u_i^n) \right) \leq e^{-\sigma t} h \leq h.$$

Let us set  $W(t) := e^{-\sigma t} \sum_i u_i^n$ ,  $Z(t) := \int_0^t e^{-\sigma s} \sum_i \varphi_i(u_i^n(s)) ds$ . Integrating the last inequality in time leads to

$$W(t) - \Delta Z(t) \leq W(0) + \int_0^t h(s) ds. \tag{48}$$

We now multiply this inequality by  $\partial_t Z (\geq 0)$  and we integrate over  $Q_T$ :



$$\int_{Q_T} (\partial_t Z) W + \int_{Q_T} \nabla \partial_t Z \cdot \nabla Z \leq \int_{Q_T} \partial_t Z \left[ W(0) + \int_0^t h \right] \leq \int_{\Omega} \left[ W(0) + \int_0^T h \right] Z(T).$$

We have  $\int_{Q_T} \nabla \partial_t Z \cdot \nabla Z = \frac{1}{2} \int_{\Omega} |\nabla Z(T)|^2 \geq 0$ . Moreover, the above right hand-side is bounded for all  $T > 0$ . To see it, we may introduce the solution of

$$-\Delta \theta_0 = W(0) + \int_0^T h \text{ in } \Omega, \quad \theta_0 = 0 \text{ on } \partial\Omega, \quad \theta_0 \geq 0.$$

And we multiply the equation (48) at time  $t = T$  by  $\theta_0$  to find, after integration by parts

$$\int_{\Omega} W(T)\theta_0(T) + \int_{\Omega} \left[ W(0) + \int_0^T h \right] Z(T) \leq \int_{\Omega} |\nabla \theta_0|^2 \leq C \|W(0) + \int_0^T h\|_{L^2(\Omega)}.$$

Finally, for all  $T > 0$ , we obtained  $C(T) \in (0, +\infty)$  such that

$$\int_{Q_T} e^{-2\sigma t} \left[ \sum_i u_i^n \right] \left[ \sum_i \varphi_i(u_i^n) \right] \leq C(T).$$

In particular, if  $\varphi_i(u_i) = d_i u_i^{m_i}$ , we obtain

$$d_i \int_{Q_T} u_i^{m_i+1} \leq e^{2\sigma T} C(T).$$

And if  $\varphi_i$  is nondegenerate as in (7), this estimate is also valid with  $m_i = 1$ .  $\square$

**Proof of Corollary 2.8.** For all  $i = 1, \dots, m$ , we set

$$u_{i0}^n := \inf\{u_{i0}, n\}, \quad \forall r \in [0, +\infty)^m, \text{ a.e. } (t, x) \in Q, \quad f_i^n(t, x, r) = \frac{f_i(t, x, r)}{1 + \frac{1}{n} \sum_j |f_j(t, x, r)|}.$$

As already stated (see the comments following (19)), these approximations  $f_i^n$  satisfy (4), (5), (6) with values independent of  $n$ . Thus, we may consider the solutions of the approximate system (16) and apply Theorem 2.7 which implies that, for all  $i = 1, \dots, m$ ,  $u_i^n$  is bounded in  $L^{m_i+1}(Q_T)$ . Together with the assumption (21), it follows that  $f_i^n(u^n)$  is uniformly integrable on  $Q_T$ . Indeed, for all measurable set  $K \subset Q_T$  with Lebesgue measure denoted by  $|K|$ , we have (recall that  $|f_i^n| \leq |f_i|$ )

$$\int_K \sum_i |f_i^n(u^n)| \leq C \left[ |K| + \sum_i \int_K (u_i^n)^{m_i+1-\epsilon} \right] \leq C \left[ |K| + \sum_i \left( \int_{Q_T} (u_i^n)^{m_i+1} \right)^{\frac{m_i+1-\epsilon}{m_i+1}} |K|^{\frac{\epsilon}{m_i+1}} \right].$$

Since  $\sup_n \int_{Q_T} (u_i^n)^{m_i+1} < +\infty$ , this implies that  $\int_K \sum_i |f_i^n(u^n)|$  may be made uniformly small by taking  $|K|$  small enough. This is exactly the uniform integrability of the  $f_i^n(u^n)$ .

Moreover,  $f_i^n(u^n)$  converges a.e. to  $f_i(u)$ . Therefore, at least up to a subsequence, by Vitali’s Lemma 3.3, we may deduce that  $f_i^n(u^n)$  converges in  $L^1(Q_T)$  for all  $T < +\infty$  to  $f_i(u_i)$ . This implies that  $u_i^n = S_{\varphi_i}(u_{i0}^n, f_i^n(u^n))$  converges to  $u_i = S_{\varphi_i}(u_{i0}, f_i(u))$ .

Finally, by the estimate (27) in Lemma 3.2 of  $\nabla \varphi_i(u_i^n)$  in  $L^\beta(Q_T)$  with  $\beta > 1$ , it follows that  $\varphi_i(u_i)$  is (at least) in  $L^1(0, T; W_0^{1,1}(\Omega))$ . This ends the proof of Corollary 2.8.  $\square$

**Proof of Corollary 2.9.** Note first that the reactive terms in System (22) satisfy the three assumptions (4), (5) and (6) with  $\sigma = 0, h = 0$ . If  $\varphi_i(u_i) = d_i u_i^{m_i}$  for at least one odd and one even value of  $i \in \{1, \dots, 4\}$ , then the assumptions of Corollary 2.8 are satisfied: indeed, if for instance  $m_1 > 1$ , we may write Young’s inequality

$$u_1 u_3 \leq \frac{1}{p} u_1^p + \frac{1}{q} u_3^q, \quad p = (m_1 + 3)/2 < m_1 + 1, \quad q = (m_1 + 3)/(m_1 + 1) < 2,$$

and similarly for  $u_2 u_4$ . Whence global existence of weak solutions. With  $m_i \geq 1, i = 1, \dots, 4$ , the strict condition (21) is not necessarily satisfied. We need an extra argument to obtain strong compactness in  $L^1(Q_T)$  of the reactive terms. We could use the  $L^2$ -compactness approach used in [31] and [8, Lemma 5].

Here, as in [11], we can more easily use the entropy structure of the system which would apply as well to general reversible reactions (see Remark 2.10). This will provide uniform integrability of the approximate reactive terms and, together with the a.e. convergence and Vitali’s Lemma 3.3, strong  $L^1(Q_T)$  compactness as well, whence the result of Corollary 2.9.

We use the same approximation as in the proof of Corollary 2.8. For  $i = 1, \dots, 4$ , let us set

$$w_i^n := u_i^n \log u_i^n - u_i^n + 1 (\geq 0), \quad z_i^n = \int_1^{u_i^n} \log r \varphi_i'(r) dr \geq 0. \tag{49}$$

We have

$$\partial_t w_i^n - \Delta z_i^n = \log u_i^n f_i^n(u^n) - \frac{\varphi_i'(u_i^n)}{u_i^n} |\nabla u_i^n|^2.$$

The main point is that

$$\sum_{1 \leq i \leq 4} \log u_i^n f_i^n(u^n) = -(u_1^n u_3^n - u_2^n u_4^n) (\log(u_1^n u_3^n) - \log(u_2^n u_4^n)) / [1 + \frac{1}{n} \sum_{1 \leq i \leq 4} |f_i(u^n)|] \leq 0.$$

We deduce that

$$\partial_t (\sum_{1 \leq i \leq 4} w_i^n) - \Delta \sum_{1 \leq i \leq 4} z_i^n \leq 0. \tag{50}$$

We now make the same computation as in the proof of Theorem 2.7. We integrate this inequality in time, we multiply by  $\sum_i z_i^n (\geq 0)$  and we integrate over  $Q_T$ . We obtain

$$\int_{Q_T} (\sum_{1 \leq i \leq 4} w_i^n) (\sum_{1 \leq i \leq 4} z_i^n) + \frac{1}{2} \int_{\Omega} |\nabla \int_0^T \sum_{1 \leq i \leq 4} z_i^n|^2 \leq \int_{\Omega} \sum_{1 \leq i \leq 4} w_i^n(0) \int_0^T \sum_{1 \leq i \leq 4} z_i^n. \tag{51}$$

From

$$-\Delta \int_0^T \sum_{1 \leq i \leq 4} z_i^n \leq \sum_{1 \leq i \leq 4} w_i^n(0) \leq \sum_{1 \leq i \leq 4} u_{i0} |\log u_{i0}| + 4 \in L^2(\Omega), \quad \sum_{1 \leq i \leq 4} z_i^n = 0 \text{ on } \partial\Omega,$$

we deduce that  $\int_0^T \sum_i z_i^n$  is bounded in  $L^2(\Omega)$  independently of  $n$ . Thus, it follows from (51) that for some  $C(T) \in (0, +\infty)$

$$\int_{Q_T} (\sum_{1 \leq i \leq 4} w_i^n) (\sum_{1 \leq i \leq 4} z_i^n) \leq C(T).$$

Now, in the nondegenerate case,  $\varphi_i'(u_i^n) \geq a_i$  for some  $a_i > 0$  so that  $z_i^n \geq a_i w_i^n$  and the above last estimate implies  $\int_{Q_T} a_i (\log u_i^n)^2 (u_i^n)^2 \leq C(T)$ . If  $\varphi_i(u_i^n) = d_i (u_i^n)^{m_i}$ , we have  $z_i^n = d_i \log u_i^n (u_i^n)^{m_i} - (m_i)^{-1} [(u_i^n)^{m_i} - 1]$ . From the same estimate above, we deduce

$$\int_{Q_T} (\log u_i^n)^2 (u_i^n)^{m_i+1} \leq C(T).$$

In all cases, we obtain that  $(u_i^n)^2$  are uniformly integrable on  $Q_T$ . Thus we can pass to the limit in  $L^1(Q_T)$  in the quadratic terms  $f_i^n(u^n)$ .  $\square$

**Proof of Remark 2.10.** Let us first assume  $k_1 = k_2 =: k$  in (23). With the same notation as in the just above proof of Corollary 2.9, we have

$$\begin{aligned} \sum_{i=1}^m \log u_i^n f_i^n(u^n) &= -k \left[ \prod_i u_i^{q_i} - \prod_i u_i^{p_i} \right] \log \frac{\prod_i u_i^{q_i}}{\prod_i u_i^{p_i}} \leq 0, \\ \partial_t \left( \sum_i w_i^n \right) - \Delta \sum_i z_i^n &\leq 0. \end{aligned}$$

We now multiply this last inequality by  $\sum_i z_i^n$  and, by the same computation as in (51) and in the lines which follow (51), we deduce as well that  $\int_{Q_T} (\log u_i^n)^2 (u_i^n)^{m_i+1} \leq C(T)$  for all  $i = 1, \dots, m$ . Therefore  $(u_i^n)^{m_i+1}$  is uniformly integrable.

Now let  $r_i := (m_i + 1)/q_i$  for  $i = 1, \dots, m$  and  $s := 1 - \sum_{i=1}^m q_i/(m_i + 1)$ , this last number being nonnegative by assumption (24). Then, using  $\sum_i (r_i)^{-1} + s = 1$ , by Young’s inequality we have

$$\prod_{i=1}^m (u_i^n)^{q_i} \leq \sum_{i=1}^m r_i^{-1} (u_i^n)^{m_i+1} + s.$$

This implies that the product  $\prod_i (u_i^n)^{q_i}$  is itself uniformly integrable and similarly for  $\prod_i (u_i^n)^{p_i}$ . Therefore, as in Corollary 2.9, we can pass to the limit in  $L^1(Q_T)$  for the nonlinear reaction terms of the approximate problem to System (23).

Finally, to treat the case  $k_1 \neq k_2$ , if for instance  $p_1 - q_1 \neq 0$ , we may just change the definition of the functions  $w_1^n, z_1^n$  as

$$w_1^n := u_1^n \log(\lambda u_1^n) - u_1^n + 1/\lambda, \quad z_1^n = \int_1^{u_1^n} \log(\lambda r) \varphi_1'(r) dr,$$

with  $\lambda^{p_1-q_1} := k_1/k_2$ . The rest is unchanged.  $\square$

**Proof of Corollary 2.11.** Again, we consider the same approximation as in the proof of Corollary 2.8. By Theorem 2.6, it is sufficient to prove that the  $L^1(Q_T)$ -estimate (20) holds. Let us denote  $P = (p_{ij})_{1 \leq i, j \leq m}$ . By Assumption (25), and using (19), we have

$$\forall i = 1, \dots, m, \quad \sum_j p_{ij} f_j^n(u^n) = \frac{\sum_j p_{ij} f_j(u^n)}{1 + \frac{1}{n} \sum_p |f_p(u^n)|} \leq b_i [1 + \sum_j (u_j^n)^{m_j+1}].$$

Since the right-hand side is nonnegative, we can even write

$$\forall i = 1, \dots, m, \quad \left[ \sum_j p_{ij} f_j^n(u^n) \right]^+ \leq b_i [1 + \sum_j (u_j^n)^{m_j+1}].$$

But, by Theorem 2.7,  $u_j^n$  is bounded in  $L^{m_j+1}(Q_T)$  independently of  $n$ . Therefore, for some  $C(T) \in (0, +\infty)$

$$\int_{Q_T} \left[ \sum_j p_{ij} f_j^n(u^n) \right]^+ \leq C(T).$$

Now multiplying each  $j$ -th equation of the approximate System (16) by  $p_{ij}$  and summing over  $j$  leads, for all  $i = 1, \dots, m$ , to

$$\sum_j p_{ij} [\partial_t u_j^n - \Delta \varphi_j(u_j^n)] + \left[ \sum_j p_{ij} f_j^n(u^n) \right]^- = \left[ \sum_j p_{ij} f_j^n(u^n) \right]^+.$$

Integrating over  $Q_T$  and using positivity of the various terms gives

$$\int_{Q_T} \left[ \sum_j p_{ij} f_j^n(u^n) \right]^- \leq \int_{Q_T} \left[ \sum_j p_{ij} f_j^n(u^n) \right]^+ + \sum_j p_{ij} u_{j0}.$$

We deduce that for some  $C(T) \in (0, +\infty)$

$$\sum_i \int_{Q_T} \left| \sum_j p_{ij} f_j^n(u^n) \right| \leq C(T) \text{ or } \int_{Q_T} \|P f^n(u^n)\| \leq C(T),$$

where  $\forall r \in \mathbb{R}^m$ ,  $\|r\| = \sum_i |r_i|$ . If we denote also by  $\|\cdot\|$  the induced norm on  $m \times m$  matrices, then we have

$$\int_{Q_T} \|f^n(u^n)\| = \int_{Q_T} \|P^{-1} P f^n(u^n)\| \leq \|P^{-1}\| \int_{Q_T} \|P f^n(u^n)\| \leq C(T). \quad \square$$

### Conflict of interest statement

No conflict of interest.

### References

- [1] B. Abdellaoui, I. Peral, M. Walias, Some existence and regularity results for porous medium and fast equations with a gradient term, *Trans. Am. Math. Soc.* (2016), in press.
- [2] P. Baras, Compacité de l'opérateur  $f \mapsto u$  solution d'une équation non linéaire  $\frac{du}{dt} + Au \ni f$ , *C. R. Acad. Sci., Sér. A* 286 (1978) 1113–1116.
- [3] A. Barabanova, On the global existence of solutions of a reaction–diffusion system with exponential nonlinearity, *Proc. Am. Math. Soc.* 122 (1994) 827–831.
- [4] S. Benachour, B. Rebiai, Global classical solutions for reaction–diffusion systems with nonlinearities of exponential growth, *J. Evol. Equ.* 10 (2010) 511–527.
- [5] M. Bendahmane, T. Lepoutre, A. Marrocco, B. Perthame, Conservative cross diffusions and pattern formation through relaxation, *J. Math. Pures Appl.* (9) 92 (6) (2009) 651–667.
- [6] D. Bothe, M. Pierre, Quasi-steady-state approximation for a reaction–diffusion system with fast intermediate, *J. Math. Anal. Appl.* 368 (1) (2010) 120–132.
- [7] D. Bothe, M. Pierre, The instantaneous limit for reaction–diffusion systems with a fast irreversible reaction, *Discrete Contin. Dyn. Syst., Ser. B* 5 (1) (2012) 49–59.
- [8] D. Bothe, M. Pierre, G. Rolland, Cross-diffusion limit for a reaction–diffusion system with fast reversible reaction, *Commun. Partial Differ. Equ.* 37 (2012) 1940–1966.
- [9] J. Cañizo, L. Desvillettes, K. Fellner, Improved duality estimates and applications to reaction–diffusion equations, *Commun. Partial Differ. Equ.* 39 (6) (2014) 1185–1204.
- [10] L. Desvillettes, K. Fellner, Global existence for quadratic systems of reaction–diffusion, *Adv. Nonlinear Stud.* 7 (2007) 491–511.
- [11] L. Desvillettes, K. Fellner, M. Pierre, J. Vovelle, Global existence for quadratic systems of reaction–diffusion, *Adv. Nonlinear Stud.* 7 (2007) 491–511.
- [12] W.E. Fitzgibbon, J.J. Morgan, S.J. Waggoner, Generalized Lyapunov methods for interactive systems in biology, in: Eds O. Arino, D.E. Axelrod, M. Kimmel (Eds.), *Mathematical Population Dynamics, Proceedings of the 2nd International Conference*, Marcel Dekker, Inc., 1991, pp. 177–188.
- [13] I. Fonseca, G. Leoni, *Modern Methods in the Calculus of Variations:  $L^p$  Spaces*, Springer-Verlag, 2007.
- [14] T. Goudon, A. Vasseur, Regularity analysis for systems or reaction–diffusion equations, *Ann. Sci. Éc. Norm. Supér.* (4) 43 (2010) 117–142.
- [15] V. Galaktionov, S. Kurdyumov, A. Samarskii, A parabolic system of quasilinear equations I, *Differ. Uravn.* 19 (12) (1983) 2123–2140 (in Russian).
- [16] S.L. Hollis, R.H. Martin, M. Pierre, Global existence and boundedness in reaction–diffusion systems, *SIAM J. Math. Anal.* 18 (3) (1987) 744–761.
- [17] A. Haraux, A. Youkana, On a result of K. Masuda concerning reaction–diffusion equations, *Tôhoku Math. J.* 40 (1988) 159–163.
- [18] J.I. Kanel, M. Kirane, Global solutions of reaction–diffusion systems with a balance law and nonlinearities of exponential growth, *J. Differ. Equ.* 165 (2000) 24–41.
- [19] S. Kouachi, Existence of global solutions to reaction–diffusion equations via a Lyapounov functional, *Electron. J. Differ. Equ.* 68 (2001) 1–10.
- [20] E.-H. Laamri, Existence globale pour des systèmes de réaction-diffusion dans  $L^1$ , Thèse Université de Nancy I (1988).
- [21] E.-H. Laamri, Etude de l'existence de solutions globales d'un système de réaction-diffusion parabolique fortement non linéaire (Study of the existence of global solutions of a strongly nonlinear parabolic reaction–diffusion system), *Ann. Fac. Sci. Toulouse Math.* (5) 12 (3) (1991) 373–390 (in French).

- [22] E.-H. Laamri, Global existence of classical solutions for a class of reaction–diffusion systems, *Acta Appl. Math.* 115 (2) (2011) 153–165.
- [23] T. Lukkari, The porous medium equation with measure data, *J. Evol. Equ.* 10 (2010) 711–729.
- [24] T. Lukkari, The fast diffusion equation with measure data, *Nonlinear Differ. Equ.* 19 (2012) 329–343.
- [25] K. Masuda, On the global existence and asymptotic behavior of reaction–diffusion equations, *Hokkaido Math. J.* 12 (1983) 360–370.
- [26] R.H. Martin, M. Pierre, Nonlinear reaction–diffusion systems, in: W.F. Ames, C. Rogers (Eds.), *Nonlinear Equations in the Applied Science*, in: *Math. Sci. Eng.*, vol. 185, Ac. Press, New York, 1991.
- [27] J. Morgan, Global existence for semilinear parabolic systems, *SIAM J. Math. Anal.* 20 (5) (1989) 1128–1144.
- [28] J. Morgan, S. Waggoner, Global existence for a class of quasilinear reaction–diffusion systems, *Commun. Appl. Anal.* 8 (2004) 153–166.
- [29] M. Pierre, An  $L^1$ -method to prove global existence in some reaction–diffusion systems, in: J.I. Diaz, P.L. Lions (Eds.), *Contributions to Nonlinear Partial Differential Equations*, in: *Pitman Res. Notes in Math. Séries*, 1987, pp. 220–231.
- [30] M. Pierre, Weak solutions and supersolutions in  $L^1$  for reaction–diffusion systems, *J. Evol. Equ.* 3 (2003) 153–168.
- [31] M. Pierre, Global existence in reaction–diffusion systems with dissipation of mass: a survey, *Milan J. Math.* 78 (2) (2010) 417–455.
- [32] M. Pierre, G. Rolland, Global existence for a class of quadratic reaction–diffusion systems with nonlinear diffusions and  $L^1$  initial data, *Nonlinear Anal. Theory Methods Appl.*, <http://dx.doi.org/10.1016/j.na.2015.11.025>.
- [33] M. Pierre, D. Schmitt, Blow-up in reaction–diffusion systems with dissipation of mass, *SIAM J. Math. Anal.* 18 (2) (1997) 169–259.
- [34] M. Pierre, D. Schmitt, Blow-up in reaction–diffusion systems with dissipation of mass, *SIAM Rev.* 42 (1) (2000) 93–106.
- [35] M. Pierre, R. Texier-Picard, Global existence for degenerate quadratic reaction–diffusion systems, *Ann. Inst. Henri Poincaré, Anal. Non Linéaire* 26 (2009) 1553–1568.
- [36] J. Prüss, Maximal regularity for evolution equations in  $L_p$ -spaces, *Conf. Sem. Mat. Univ. Bari* 285 (2003) 1–39.
- [37] P. Quittner, Ph. Souplet, *Superlinear Parabolic Problems: Blow-Up, Global Existence, Steady States*, Advanced Texts, Birkhäuser, 2007.
- [38] R.L. Schilling, *Measures, Integrals and Martingales*, Cambridge University Press, 2005.
- [39] D. Schmitt, Existence globale ou explosion pour les systèmes de réaction-diffusion avec contrôle de masse, Thèse Université Henri Poincaré, 1995.
- [40] J.L. Vazquez, *The Porous Medium Equation Mathematical Theory*, Oxford Mathematical Monographs, 2006.