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On the kinetic energy profile of Hölder continuous Euler flows

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Abstract

In [8], the first author proposed a strengthening of Onsager's conjecture on the failure of energy conservation for incompressible Euler flows with Hölder regularity not exceeding $1/3$. This stronger form of the conjecture implies that anomalous dissipation will fail for a generic Euler flow with regularity below the Onsager critical space $L_t^\infty B_{3,\infty}^{1/3}$ due to low regularity of the energy profile.

The present paper is the second in a series of two papers whose results may be viewed as first steps towards establishing the conjectured failure of energy regularity for generic solutions with Hölder exponent less than $1/5$. The main result of this paper shows that any non-negative function with compact support and Hölder regularity $1/2$ can be prescribed as the energy profile of an Euler flow in the class $C_{t,x}^{1/5-\epsilon}$. The exponent $1/2$ is sharp in view of a regularity result of Isett [8]. The proof employs an improved greedy algorithm scheme that builds upon that in Buckmaster–De Lellis–Székelyhidi [1].

Keywords: Convex integration; Incompressible Euler equations; Weak solutions; Conservation of energy

1. Introduction

The present work concerns the construction of Hölder continuous solutions to the incompressible Euler equations on $\mathbb{R} \times \mathbb{R}^3$ and on $\mathbb{R} \times \mathbb{T}^3$

$$\begin{aligned} \partial_t v^l + \partial_j (v^j v^l) + \partial^l p &= 0 \\ \partial_j v^j &= 0 \end{aligned} \tag{E}$$

with a prescribed, possibly rough energy profile. As we consider solutions with fractional regularity, what we mean by a solution to (E) is a continuous velocity field $v : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}^3$ and pressure $p : \mathbb{R} \times \mathbb{T}^3 \to \mathbb{R}$ that together satisfy (E) in the sense of distributions.

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The main result of the paper is the following Theorem:

Theorem 1.1 *(Euler flows with prescribed energy profile). Let $\alpha < 1/5$, let $I \subseteq \mathbb{R}$ be a bounded open interval, and let $\bar{e}(t) \geq 0$ be any non-negative function with compact support in I which belongs to the class $\bar{e}(t) \in C_t^\gamma$ for some $\gamma > \frac{2\alpha}{1-\alpha}$. Then:*

1. *There exists a weak solution (v, p) to the incompressible Euler equations in the class $v \in C^\alpha_{t,x}(\mathbb{R} \times \mathbb{T}^3)$ with support contained in*

$$\operatorname{supp} v \cup \operatorname{supp} p \subseteq I \times \mathbb{T}^3$$

such that the energy profile of v is equal to $\int_{\mathbb{T}^3} |v|^2(t,x)dx = \bar{e}(t)$ for all $t \in \mathbb{R}$.

2. *Moreover, one may choose a one parameter family of solutions (v_A, p_A), $0 \leq A \leq 1$, with the above properties such that the energy profile of v_A is equal to $\int_{\mathbb{T}^3} |v_A|^2(t,x)dx = A\bar{e}(t)$ and such that $\|v_A\|_{C^\alpha_{t,x}} \to 0$ as $A \to 0$.*

The assumption that $e(t)$ is at least $2\alpha/(1-\alpha)$-Hölder is sharp in view of a regularity result in [8], which states that the energy profile of an Euler flow in the class $v \in L_t^\infty C_x^\alpha$, $0 < \alpha \leq 1/3$, belongs to the class $e(t) \in C_t^{2\alpha/(1-\alpha)}$. We remark that our arguments also allow one to achieve an energy profile that does not have compact support provided the norm $\|e\|_{C_t^\gamma} = \sup_t |e(t)| + \sup_t \sup_{|\Delta t| \neq 0} \frac{|e(t+\Delta t) - e(t)|}{|\Delta t|^\gamma}$ is finite. Furthermore, the proof of Theorem 1.1 extends easily to the nonperiodic setting (cf. Theorem 3.2 below).

Theorem 1.1 on solutions with prescribed rough energy profiles builds upon works [5,4,1], which exhibit solutions whose energy profiles can be any given smooth, strictly positive function on a closed interval $[0, T]$. These results show in particular that for any $\alpha < 1/5$ it is possible to construct solutions with $C^\alpha_{t,x}$ regularity whose energy profiles are strictly increasing or strictly decreasing (which we expect to be nongeneric solutions, as in Conjecture 1 of [6]). Theorem 1.1 improves on these results by obtaining sharp regularity for the energy profile, and by removing the restriction of having a strictly positive lower bound on the desired energy profile.

To achieve these improvements, we develop a more delicate greedy algorithm for choosing the energy increments at each stage of the iteration, and develop a sharper form of the Main Lemma in the iteration that allows us to execute this algorithm. A quadratic commutator estimate akin to the one used in the proof of energy conservation in [3,2] (as well as the proof of the $2\alpha/(1-\alpha)$-Hölder estimate for the energy profile in [8]) plays a key role in the proof. Our proof is also greatly simplified by the fact that we are able to achieve an exponential (rather than double-exponential) growth of frequencies in the iteration. This simplification is available thanks to improvements used to localize the construction in [6].

Our motivation for pursuing Theorem 1.1 extends from a strengthening of Onsager's conjecture proposed in [8], which states that a generic Euler flow in the class $C_t C_x^\alpha$, $0 < \alpha \leq 1/3$ will have an energy profile of the minimal regularity allowed by the equation. We refer to [6] for a thorough discussion.

2. The Main Lemma

In this section, we present the Main Lemma that is responsible for the proof of Theorem 1.1. The purpose of this lemma is to describe precisely the result of one step of the convex integration procedure. Theorem 1.1 follows from iteration of this Lemma as we will explain in Section 3.

We start by recalling the Euler–Reynolds system introduced in [5]. A vector field v^l, scalar field p and symmetric tensor field R^{jl} are said to be an Euler–Reynolds flow if (v, p, R) together satisfy the following PDE in the sense of distributions.

$$\begin{aligned} \partial_t v^l + \partial_j (v^j v^l) + \partial^l p &= \partial_j R^{jl} \\ \partial_j v^j &= 0 \end{aligned} \tag{1}$$

We will consider Euler–Reynolds flows on the domain $\mathbb{R} \times \mathcal{M}$ where $\mathcal{M}$ may be either a torus $\mathcal{M} = \mathbb{T}^3$ or $\mathcal{M} = \mathbb{R}^3$.

The Main Lemma of the iteration summarizes how, given an initial Euler–Reynolds flow (v, p, R) satisfying certain estimates, it is possible to perturb the velocity field and pressure to obtain a new Euler–Reynolds flow (v_1, p_1, R_1)

with stress tensor R_1^{jl} much smaller than the initial R^{jl}. To state the Main Lemma, we recall the notion of frequency and energy levels for Euler–Reynolds flows introduced in Sections 9 and 10 of [7].

Definition 2.1. Let $L \geq 1$ be a fixed integer. Let $\Xi \geq 2$, and let e_v and e_R be positive numbers with $e_R \leq e_v$. Let (v, p, R) be a solution to the Euler–Reynolds system on $\mathbb{R} \times \mathcal{M}$. We say that the frequency and energy levels of (v, p, R) are below (Ξ, e_v, e_R) (to order L in $C^0 = C^0_{t,x}(\mathbb{R} \times \mathcal{M})$) if the following estimates hold.

$$||\nabla^k v||_{C^0} \leq \Xi^k e_v^{1/2} \qquad k = 1, \dots, L \tag{2}$$

$$||\nabla^k p||_{C^0} \leq \Xi^k e_v \qquad k = 1, \dots, L \tag{3}$$

$$||\nabla^k R||_{C^0} \leq \Xi^k e_R \qquad k = 0, \dots, L \tag{4}$$

$$||\nabla^k (\partial_t + v \cdot \nabla) R||_{C^0} \leq \Xi^{k+1} e_v^{1/2} e_R \qquad k = 0, \dots, L-1 \tag{5}$$

Here ∇ refers only to derivatives in the spatial variables.

Our Main Lemma is based on the Main Lemma in Section 10 of [7] but also keeps track of how the support of the approximate solution enlarges after the addition of a correction. The following definition will be useful for keeping track of the support of the iteration, which is governed by the geometry of the flow map of the velocity field v.

Definition 2.2 *(v-Adapted Eulerian cylinder).* Let $\Phi_s = \Phi_s(t, x)$ be the flow map associated to a vector field v. Given $\tau, \rho > 0$ and a point (t_0, x_0) of the space–time $\mathbb{R} \times \mathcal{M}$, we define the *$v$-adapted Eulerian cylinder* $\hat{C}_v(\tau, \rho; t_0, x_0)$ centered at (t_0, x_0) with duration 2τ and base radius $\rho > 0$ to be

$$\hat{C}_v(\tau, \rho; t_0, x_0) := \left\{ \Phi_s(t_0, x_0) + (0, h) \ : \ |s| \leq \tau, |h| \leq \rho \right\} \tag{6}$$

In other words, $\hat{C}_v(\tau, \rho; t_0, x_0)$ is the union of spatial balls of radius ρ about the trajectory of (t_0, x_0) along the flow of v for $t \in [t_0 - \tau, t_0 + \tau]$.

Similarly, if $S \subseteq \mathbb{R} \times \mathcal{M}$ is a set, we define

$$\hat{C}_v(\tau, \rho; S) := \bigcup_{(t_0, x_0) \in S} \hat{C}_v(\tau, \rho; t_0, x_0) \tag{7}$$

With these definitions in hand, we can state the Main Lemma.

Lemma 2.1 *(The Main Lemma). Suppose that $L \geq 2$. Let K be the constant in Section 7.3 of [7], and let $M \geq 1$ be a constant. There exist constants $C_0, C > 1$, which depend only on M and L, such that following holds:*

Let (v, p, R) be any solution of the Euler–Reynolds system on $\mathbb{R} \times \mathcal{M}$ whose frequency and energy levels are below (Ξ, e_v, e_R) to order L in C^0.

Define the time-scale $\theta = \Xi^{-1} e_v^{-1/2}$, let N be any positive number obeying the bound $N \geq \left(\frac{e_v}{e_R}\right)^{3/2}$ and define the dimensionless parameter $\mathbf{b} = \left(\frac{e_v}{e_R N}\right)^{1/2}$.

Let $e(t, x) : \mathbb{R} \times \mathcal{M} \to \mathbb{R}_{\geq 0}$ be any non-negative function which satisfies the lower bound

$$e(t, x) \geq K e_R \qquad \text{for all } (t, x) \in \hat{C}_v(\theta, \Xi^{-1}; \operatorname{supp} R) \tag{8}$$

(using the notation of Definition 2.2) and whose square root satisfies the estimates

$$||\nabla^k (\partial_t + v \cdot \nabla)^r e^{1/2}||_{C^0} \leq M \Xi^k (\mathbf{b}^{-1} \Xi e_v^{1/2})^r e_R^{1/2} \qquad 0 \leq r \leq 1, \ 0 \leq k + r \leq L \tag{9}$$

Then there exists a solution (v_1, p_1, R_1) of the Euler–Reynolds system of the form $v_1 = v + V$, $p_1 = p + P$ whose frequency and energy levels are below

$$(\Xi', e_v', e_R') = (C_0 N \Xi, e_R, \mathbf{b}^{-1} \frac{e_v^{1/2} e_R^{1/2}}{N}) \tag{10}$$

to order L in C^0, and such that the following are satisfied:

The correction $V = v_1 - v$ is of the form $V = \nabla \times W$ and can be guaranteed to obey the bounds

$$||V||_{C^0} \le C e_R^{1/2} \tag{11}$$
$$||\nabla V||_{C^0} \le C N \Xi e_R^{1/2} \tag{12}$$
$$||(\partial_t + v^j \partial_j) V||_{C^0} \le C \mathbf{b}^{-1} \Xi e_v^{1/2} e_R^{1/2} \tag{13}$$
$$||W||_{C^0} \le \Xi^{-1} N^{-1} e_R^{1/2} \tag{14}$$
$$\|\nabla W\|_{C^0} \le C e_R^{1/2} \tag{15}$$
$$||(\partial_t + v^j \partial_j) W||_{C^0} \le C \mathbf{b}^{-1} N^{-1} e_v^{1/2} e_R^{1/2} \tag{16}$$

The correction to the pressure $P = p_1 - p_0$ satisfies the estimates

$$||P||_{C^0} \le C e_R \tag{17}$$
$$||\nabla P||_{C^0} \le C N \Xi e_R \tag{18}$$
$$||(\partial_t + v \cdot \nabla) P||_{C^0} \le C \mathbf{b}^{-1} \Xi e_v^{1/2} e_R \tag{19}$$

The energy of the correction can be prescribed locally up to errors bounded uniformly in t by

$$\left| \int_{\mathcal{M}} |V|^2(t,x)\psi(x)dx - \int_{\mathcal{M}} e(t,x)\psi(x)dx \right| \le \mathbf{b}^{-1} \frac{e_v^{1/2} e_R^{1/2}}{N} \left(\|\psi\|_{L^1} + \Xi^{-1} \|\nabla \psi\|_{L^1} \right) \tag{20}$$

for any smooth test function $\psi(x) \in C_c^\infty(\mathcal{M})$. (In (20), we mean $L^1 = L^1(\mathcal{M})$.)

Finally, the space–time supports of V, P and R_1 are also contained in

$$\mathrm{supp}\, V \cup \mathrm{supp}\, P \cup \mathrm{supp}\, R_1 \subseteq \hat{C}_v(\mathbf{b}\theta, \Xi^{-1};\ \mathrm{supp}\, e) \tag{21}$$

Lemma 2.1 is very similar to the Main Lemma in [6], but with a few differences that will be important for the proof of Theorem 1.1.

1. The lemma allows for an additional loss of a factor $\mathbf{b}^{-1}$ in the cost of the advective derivative in the estimate (9).
2. There is a loss of a factor of $\mathbf{b}^{-1}$ in the estimate (20).
3. The implicit constants in the estimates (14) and (20) have been normalized to 1.
4. There is a gain of a factor of $\mathbf{b}$ in the time scale for the enlargement of support in (21).

In the following section, we explain the modifications of the proof of the Main Lemma in [6] that lead to Lemma 2.1. We note that the proof of the Main Lemma in [6], which considers only the case of $\mathcal{M} = \mathbb{R}^3$, extends easily to the case of $\mathcal{M} = \mathbb{T}^3$; the only alteration required is to use a partition of unity in space that is spatially periodic.

2.1. Modifying the proof of the Main Lemma in [6]

The proof of Lemma 2.1 is identical to the proof of the Main Lemma in [6] in the sense that every choice of parameter in the argument is left unchanged. The only differences in the proof are due to the inferior bound (9) on the advective derivative of the energy increment, which leads to worse estimates for a few terms in the argument that we will list here. Ultimately, the reason we are allowed to relax the bound on the advective derivative is that the cost of the advective derivative in (9) coincides with the inverse of the time scale in the construction:

$$\tau^{-1} \sim \mathbf{b}^{-1} \Xi e_v^{1/2} \tag{22}$$

In particular, there is no room here to allow for a bound which is any worse than (9) without losing regularity. It is therefore necessary to check a few of the estimates to make sure that the proof goes through with straightforward modifications. Here we list the necessary modifications in the proof of [6].

- All choices of parameters in the construction $(\epsilon_v, \epsilon_x, \epsilon_t, \tau, \rho, \lambda)$ are exactly the same. In particular, we have $\tau = a\mathbf{b}\Xi^{-1} e_v^{-1/2}$ for some constant a chosen sufficiently small.

- The fact that the bound (21) on the support of the iteration gains a factor of $\mathbf{b}$ can be observed from inspecting the bound [6, Equation 122] on the support of the stress. The time scale in this estimate is bounded by, say, 3τ, which is smaller than $\mathbf{b}\Xi^{-1}e_v^{-1/2}$ when the small constant a in the definition of τ is chosen appropriately.
- The constants in the estimates (14) and (20) can be made arbitrarily small by taking the constant B_λ in the construction to be sufficiently large when these terms are estimated. Here we have normalized these constants to be equal to 1.
- The choice of $\epsilon_t = cN^{-1}\Xi^{-1}e_R^{-1/2}$ for the time scale for the mollification along the flow made in [6, Equation 115] leads to a worse estimate on the error $\|e^{1/2} - \tilde{e}^{1/2}\|_{C^0}$ made in mollifying the energy increment. Namely, the bound [6, Equation 118] loses a factor of $\mathbf{b}^{-1}$, and is replaced instead by

$$\|e^{1/2} - \tilde{e}^{1/2}\|_{C^0} \le \mathbf{b}^{-1}\frac{e_v^{1/2}}{100N} \tag{23}$$

- The loss of $\mathbf{b}^{-1}$ in (23) ultimately leads to the loss of $\mathbf{b}^{-1}$ in (20) when bounding the error for prescribing the energy increment. Namely, this estimate introduces a $\mathbf{b}^{-1}$ in the estimate of line [6, Equation 161].
- The bound on the first advective derivative of $\tilde{e}^{1/2}$ also worsens by a factor of $\mathbf{b}^{-1}$. As a result, the estimates in [6, Proposition 8.3] incur a loss of $\mathbf{b}^{-1}$, and the estimate [6, Equation 153] must be replaced by

$$e_R^{1/2}\|D^{(a,r)}\tilde{e}^{1/2}\|_{C^0} + \|D^{(a,r)}R_\epsilon\|_{C^0} \le C_a\Xi^a e_R(\mathbf{b}^{-1}\Xi e_v^{1/2})^{(r\ge1)}(N\Xi e_R^{1/2})^{(r\ge2)}N^{(a+1-L)_+/L} \tag{24}$$

  Here we use the notation $(r \ge 1)$ and $(r \ge 2)$ to represent indicator functions. The reason that the second advective derivative incurs a cost of $\epsilon_t^{-1} = N\Xi e_R^{1/2}$, is that this estimate arises from differentiating the kernel used to mollify in time along the flow.
- The inferior bound in (24) affects the bounds for the advective derivatives of the amplitudes v_I stated in [6, Proposition 8.4]. These bounds take on the same pattern as the estimate (24), as now all of the worst terms occur when the advective derivatives fall on the factor of $\tilde{e}^{1/2}$. In particular, the first advective derivative incurs the same cost of $\tau^{-1} \sim \mathbf{b}^{-1}\Xi e_v^{1/2}$, but now the second advective derivative gives a greater cost of $(N\Xi e_R^{1/2})$. The estimates that replace [6, Equations 154–155] are:

$$\|D^{(a,r)}v_I\|_{C^0} \le C_a\Xi^a e_R^{1/2}\tau^{-(r\ge1)}(N\Xi e_R^{1/2})^{(r\ge2)}N^{(a+1-L)_+/L} \tag{25}$$

$$\|D^{(a,r)}\delta v_I\|_{C^0} \le C_a B_\lambda^{-1}N^{-1}\Xi^a e_R^{1/2}\tau^{-(r\ge1)}(N\Xi e_R^{1/2})^{(r\ge2)}N^{(a+2-L)_+/L} \tag{26}$$

  Note that the only difference compared to [6, Equations 154–155] lies in the bound on the second advective derivative.
- The only point in the argument at which the second advective derivative estimate is used comes in estimating the advective derivative of transport term Q_T^{jl}. The main term in Q_T^{jl} is given by the parametrix in [6, Section 9.1]

$$Q_T^{jl} = \sum_I \frac{1}{i\lambda}e^{i\lambda\xi_I}q(\nabla\xi_I)[(\partial_t + v_\epsilon^j\partial_j)v_I^l] + \text{ Lower order terms}$$

  As in [6, Section 9.1], we must estimate the cost of taking an advective derivative for this term. According to the estimate (25), the cost of taking a further advective derivative is no longer τ^{-1} as in [6], but rather is given by $\epsilon_t^{-1} = c^{-1}N\Xi e_R^{1/2}$. This cost is exactly the cost of $\left|\frac{\bar{D}}{\partial t}\right| \lesssim \Xi'(e_v')^{1/2} = CN\Xi e_R^{1/2}$ that must be verified for the advective derivative in order to conclude the proof of Lemma 2.1.

Having explained the modifications necessary to prove Lemma 2.1, we now explain how Lemma 2.1 can be applied to establish Theorem 1.1.

3. Prescribing the energy profile

In this Section, we show how our method can be applied to produce solutions with a prescribed energy profile, and we present a proof of Theorem 1.1. Here we outline our presentation of the proof.

To simplify our exposition, we start by proving a variant of Theorem 1.1 that illustrates the main ideas of our algorithm in the simplest case. This proof is carried out in Section 3.1. In Section 3.8, we then explain how the construction can be modified to handle the nonperiodic setting.

3.1. Prescribing the energy profile

In this Section, we establish a simplified version of Theorem 1.1 on prescribing the energy profile of solutions by repeated application of Lemma 2.1. Here we consider only the problem of prescribing the energy profile for a solution with regularity approaching $1/5$. In Section 3.5, we explain how to modify the argument to obtain solutions with regularity strictly below $1/5$, and to achieve solutions that are $C^\alpha_{t,x}$ perturbations of the 0 solution. In Section 3.8 below we explain how to modify the argument to prescribe the energy profile in the nonperiodic setting.

The construction explained in this section involves the introduction of several parameters which must be chosen in a particular logical order. We provide a summary of this construction and the logical structure of the choice of parameters in Section 3.6 below.

The theorem we establish in this Section is the following:

Theorem 3.1 *(Periodic Euler flows with prescribed energy profile). Let $\alpha < \alpha^* \leq 1/5$ and let $I \subseteq \mathbb{R}$ be a bounded open interval. Let $\bar{e}(t) \geq 0$ be any non-negative function with compact support in I which belongs to the class $\bar{e}(t) \in C^\gamma_t$ for $\gamma = \frac{2\alpha^*}{1-\alpha^*}$. Then there exists a weak solution (v, p) to the Euler equations in the class $v \in C^\alpha_{t,x}(\mathbb{R} \times \mathbb{T}^3)$ with compact support in $I \times \mathbb{T}^3$ such that the energy profile of v is given by*

$$\int_{\mathbb{T}^3} |v|^2(t,x)dx = \bar{e}(t), \qquad t \in \mathbb{R} \tag{27}$$

We will give the proof for the case $\alpha^* = 1/5$ (in which case $\gamma = 1/2$), since this case is most closely related to the proof of [6, Theorem 1.1], and the cases $\alpha^* < 1/5$ can be handled similarly. We will outline how to handle the more general case in Section 3.7 below, where we will also explain how to obtain a one parameter family of solutions tending to 0 in $C^\alpha_{t,x}$ as in the statement of Theorem 1.1.

Theorem 3.1 is proved by iterating Lemma 2.1. The solution (v, p) stated in Theorem 3.1 will be obtained as a uniform limit of a sequence of solutions $(v_{(k)}, p_{(k)}, R_{(k)})$ to the Euler–Reynolds equations, beginning with the trivial solution $(0, 0, 0)$.

The sequence of Euler–Reynolds flows $(v_{(k)}, p_{(k)}, R_{(k)})$ will have frequency-energy levels below certain values $(\Xi_{(k)}, e_{v,(k)}, e_{R,(k)})$ which are chosen to satisfy iteration rules of the form

$$\Xi_{(k+1)} = C_0 Z^{5/2} \Xi_{(k)} \tag{28}$$

$$e_{v,(k+1)} = e_{R,(k)} \tag{29}$$

$$e_{R,(k+1)} = \frac{e_{R,(k)}}{Z} \tag{30}$$

These solutions are obtained by repeatedly applying Lemma 2.1 with a choice of $N = Z^{5/2}$ and a parameter M which will be specified later. Recall from [6, Section 11.4] that, for $\alpha < 1/5$ and Z sufficiently large depending on α and the constants in the statement of Lemma 2.1, this choice of parameters leads to convergence of $(v_{(k)}, p_{(k)})$ in the $C^\alpha_{t,x} \times C^{2\alpha}_{t,x}$ norm to a weak solution to the Euler equations. The choice of a large parameter Z corresponds in the context of the construction to a choice of a rapid frequency and time scale in the first stage of the iteration (see Equation (42) below), and also to a large ratio between consecutive frequencies in the whole iteration. We remark that the power $Z^{5/2}$ appearing in the iteration rules (28)–(30) corresponds to taking $\alpha^* = 1/5$.

In specifying the construction, it will be helpful to introduce the parameters

$$\begin{aligned} \mathbf{b} &= \left(\frac{e_v^{1/2}}{e_R^{1/2} N}\right)^{1/2}_{(k)} = Z^{-1} \\ \theta_{(k)} &= \Xi^{-1}_{(k)} e^{-1/2}_{v,(k)}, \qquad \hat{\tau}_k = \mathbf{b}\,\Xi^{-1}_{(k)} e^{-1/2}_{v,(k)} \end{aligned} \tag{31}$$

in order to distinguish the important time scales in the iteration. The time scale $\theta_{(k)}$ corresponds to the natural time scale of motion for the flow of the velocity field $v_{(k)}$, whereas the time scale $\hat{\tau}_k$ corresponds up to a constant to the more rapid time scale employed in the time cutoffs of the construction.

To keep a perspective on the relative magnitude of the parameters, we point out that the iteration rules (28)–(30) and Equation (31) imply (neglecting powers of C_0)

$$\Xi_{(k)} \sim Z^{5k/2}\Xi_{(0)}, \quad \frac{e_{v,(k)}}{Z} = e_{R,(k)} = \frac{e_{R,(0)}}{Z^k},$$
$$\theta_{(k)} \sim Z^{-2k}\theta_{(0)}, \quad \frac{\theta_{(k)}}{Z} = \hat{\tau}_k \sim Z^{-2k}\hat{\tau}_0.$$

Later on it will turn out that $e_{R,(0)} = \sup_t \bar{e}(t)$ is bounded independent of Z, while we will choose $\Xi_{(0)} \geq (Z^{1-\frac{3\gamma}{2}} e_{R,(0)}^{-1-\frac{\gamma}{2}})^{1/\gamma}$ according to equation (42) below. These choices and (31) dictate the small initial time scales $\theta_{(0)}$ and $\hat{\tau}_0$.

Along the way, we will keep track of the time supports of the solutions and errors, by defining a sequence of sets $I_{(k)} \subseteq I$ such that the following claims hold

Claim 1 *(Growing supports). For all $k \geq 0$, we have*

$$\operatorname{supp} v_{(k)} \cup \operatorname{supp} p_{(k)} \cup \operatorname{supp} R_{(k)} \subseteq I_{(k)} \times \mathbb{T}^3. \tag{32}$$
$$I_{(k)} \subseteq I_{(k+1)} \tag{33}$$

To fully specify the iteration, we must construct the functions $e_{(k)}(t,x)$ which prescribe the energy increment at each stage of the iteration, and specify the parameters of the construction, including the initial frequency energy levels $(\Xi_{(0)}, e_{v,(0)}, e_{R,(0)})$ for the base case of the iteration and the parameter Z. Since we are considering the case of spatially periodic solutions, we may consider energy increments $e_{(k)}(t)$ which depend only on the variable t.

Our goal in choosing the energy increments is to ensure that the solution (v, p) constructed in the limit has energy profile given by $\bar{e}(t)$. Since the approximate solutions $v_{(k)}(t)$ converge uniformly to the limiting solution, it suffices to show that the energy profiles of the approximate solutions

$$E_k(t) := \int_{\mathbb{T}^3} |v_{(k)}|^2(t,x)dx \tag{34}$$

converge pointwise to the desired energy profile $\bar{e}(t)$ as $k \to \infty$. This convergence will be obtained by ensuring that the inductive Claims 2 and 3 below hold throughout the iteration. In order to state these claims, we introduce the following notation, which will be used in the remainder of the paper.

Definition 3.1. Given any set $J \subseteq \mathbb{R}$ and any $\overline{\tau} \in \mathbb{R}_{\geq 0}$, we define

$$I(\overline{\tau}; J) := \{t + \Delta t \in \mathbb{R} : t \in J, \ |\Delta t| \leq \overline{\tau}\}. \tag{35}$$

Claim 2 *(There is always room to add more energy where the error is supported). For every $k \geq 0$ and $t \in \mathbb{R}$, we have*

$$\bar{e}(t) - E_k(t) \geq 0. \tag{36}$$

Moreover, for all $t \in I(2\theta_{(k)}; I_{(k)})$, we have

$$\bar{e}(t) - E_k(t) \geq 3K e_{R,(k)} \tag{37}$$

where K is the constant in the lower bound (8) of Lemma 2.1.

Claim 3 *(The energy threshold is nearly saturated). There is an absolute constant $\overline{M}$ such that the upper bound*

$$\sup_t |\bar{e}(t) - E_k(t)| \leq \overline{M} e_{R,(k)} \tag{38}$$

holds uniformly.

Note that Claim 3 implies the uniform convergence of $E_k(t) \to \bar{e}(t)$ as $k \to \infty$. The condition (37) is required for continuing the iteration; this condition is present to ensure that one can construct an energy increment compatible with the conditions (8) and (9) while condition (38) is maintained. Thus, the proof of Theorem 3.1 reduces to specifying an iteration in which Claims 2–3 remain satisfied.[3]

We now explain our rule for specifying the iteration. Our construction will involve choosing two large constants (Y and Z), which will be chosen in alphabetical order during the course of the proof. First, we define a sequence of "gaps"

$$\Delta E_k = Y e_{R,(k+1)} = Y \frac{e_{R,(k)}}{Z} \tag{39}$$

with Y some constant to be chosen later on. Our goal is to choose an energy increment $e_{(k)}(t)$ which ensures that the number ΔE_k is a lower bound for the gap in the energy profile $(\bar{e}(t) - E_{k+1}(t)) \geq \Delta E_k$ which remains after stage k of the iteration on the support of the error.

According to conditions (8), (9) and (38), we should choose at each stage an energy increment $e_{(k)}(t)$ which will nearly saturate the energy threshold, but we must leave a gap of size $(\bar{e} - e_{(k)} - E_k) \sim e_{R,(k+1)}$ on the support of the error to ensure we can continue the iteration in the next stage. A sensible first guess for the energy increment we desire would be the function

$$\hat{e}_{(k)}(t) = (\bar{e}(t) - E_k(t) - \Delta E_k)_+ \tag{40}$$

(Recall that we use the notation $y_+ = \max\{y, 0\}$.) The key calculation that motivates this choice is given in Section 3.4.2 below.

The problem with this guess is that the function $\hat{e}_{(k)}$ is only Lipschitz, whereas Lemma 2.1 requires control over derivatives of the square root $e_{(k)}^{1/2}(t)$. We therefore modify the function $\hat{e}_{(k)}^{1/2}$ by prescribing an energy profile of the form

$$e_{(k)}^{1/2}(t) = (\bar{e} - E_k - \Delta E_k)_+^{1/2} * \eta_{\hat{\tau}_k} = \hat{e}_{(k)}^{1/2} * \eta_{\hat{\tau}_k} \tag{41}$$

The function $\eta_{\hat{\tau}_k}$ here denotes a standard, non-negative mollifying kernel in the time variable with support in the interval $\operatorname{supp} \eta_{\hat{\tau}_k} \subseteq \{|t| \leq \hat{\tau}_k\}$. The number $\hat{\tau}_k$ is the timescale of the construction defined in (31). For intuition, one can picture the formula (41) graphically in the case $E_k = 0$ as shifting the graph of $\bar{e}(t)$ downwards by an amount ΔE_k, taking a square root and then averaging over translates in t by a width less than $\hat{\tau}_k$.

We will see during the course of the proof that the C_t^γ regularity of $\bar{e}$ will be essential for verifying that the assumptions (8), (9) can be carried on during the iteration. Without sufficient regularity for the function $\bar{e}$, the regularized function $e_{(k)}$ may be a poor approximation to the desired energy increment (40). One must also worry that the time mollification in (41) may cause the energy profile of the approximate solutions to exceed the energy threshold if the regularity of $\bar{e}$ is too low.

3.2. Prescribing the energy increment: the base case

We initialize the construction by taking our Euler–Reynolds flow to be $(v_{(0)}, p_{(0)}, R_{(0)}) = (0, 0, 0)$. For the initial set of times containing the support of the iteration, we take $I_{(0)} = \emptyset$ to be the empty set. We must now choose the initial frequency energy levels $(\Xi_{(0)}, e_{v,(0)}, e_{R,(0)})$. For the initial energy level $e_{R,(0)}$ we take $e_{R,(0)} = \sup_{t\in\mathbb{R}} \bar{e}(t)$. This choice and the Ansatz (29)–(30) dictate our choice of $e_{v,(0)} = Z e_{R,(0)}$. Observe that these choices guarantee that Claims 2–3 and the containment (32) hold at the stage $k = 0$.

During the course of the iteration (see Line (60) below), we will have to show that the quotient

$$Q_{(k)} = \frac{|\hat{\tau}_k|^\gamma}{e_{R,(k+1)}}$$

remains uniformly bounded, independent of the choices of Y and Z. We remark that this point is the reason for the numerology $\gamma = \frac{2\alpha^*}{1-\alpha^*}$. With this motivation, we choose $\Xi_{(0)}$ to achieve the inequality $Q_{(0)} \leq 1$. Recall from (31) that $\hat{\tau}_0 = \mathbf{b}\,\Xi_{(0)}^{-1} e_{v,(0)}^{-1/2} = Z^{-3/2} \Xi_{(0)}^{-1} e_{R,(0)}^{-1/2}$ and $e_{R,(1)} = \frac{1}{Z} e_{R,(0)}$. The goal $Q_{(0)} \leq 1$ is therefore accomplished by choosing a value $\Xi_{(0)}$ such that

[3] We point out that the Claims 2–3 are also carried along the iteration in the schemes for prescribing energy in [5,4,1]. The difference in this respect is that these papers assume a strictly positive lower bound on $\bar{e}(t)$, and the lower bound (37) is assumed to hold everywhere.

$$\Xi_{(0)} \geq \left(Z^{1-\frac{3\gamma}{2}} e_{R,(0)}^{-1-\frac{\gamma}{2}} \right)^{1/\gamma} \tag{42}$$

We have now specified the initial frequency energy levels (up to the specification of Z), but we are not quite ready to proceed with the iteration by applying Lemma 2.1. Namely, we want to apply Lemma 2.1 with the choice of energy increment $e_{(k)}(t)$ defined by (41). However, in order to apply Lemma 2.1, we are required to specify the constant M in the upper bounds (9) for the energy profile, which turns out to depend on the choice of Y. Once we have determined the value of Y (which is accomplished in Section 3.4 below), we will be able to apply Lemma 2.1 for a specified value of M. In particular, the constant C_0 in the iteration rule (28) comes from Lemma 2.1 and depends on the value of Y.

In Section 3.3 below, we continue the proof of Theorem 3.1 by verifying that our choice of energy increment $e_{(k)}(t)$ defined by (41) remains for all indices k an admissible choice of energy function in Lemma 2.1 for the sequence of frequency energy levels dictated by (28)–(30). In the process, we specify the sequence $I_{(k)}$ and verify that Claims 2–3 hold with this choice of energy increment $e_{(k)}(t)$, provided the function $\bar{e}(t)$ has sufficient regularity.

3.3. *Prescribing the energy increment: admissibility of the energy function*

In this Section, we define the sequence $I_{(k)}$, and we verify that the energy function defined in (41) is always an admissible choice of energy function for applying Lemma 2.1.

For $k \geq 0$, we define $I_{(k+1)}$ to be[4]

$$I_{(k+1)} := I(2\hat{\tau}_k; \{t \in \mathbb{R} : \bar{e}(t) - E_k(t) \geq \Delta E_k\}). \tag{43}$$

In what follows, we will assume that the constant Y has already been chosen so that Claim 2 holds, and also that Claim 3 is satisfied for a specified constant $\overline{M}$.

With the assumptions that Y has already been chosen and that Claim 3 is satisfied for a specified constant $\overline{M}$, we obtain the following bounds on the square root of the energy increment:

$$\left\| \left(\frac{d}{dt} \right)^r e_{(k)}^{1/2} \right\|_{C_t^0} \leq A(\mathbf{b}^{-1} \Xi e_v^{1/2})^r [\overline{M} e_{R,(k)}]^{1/2}, \qquad 0 \leq r \leq 2 \tag{44}$$

Indeed, at the level $r = 0$, we have

$$\|e_{(k)}^{1/2}\|_{C^0} \leq \|(\bar{e} - E_k - \Delta E_k)_+^{1/2}\|_{C^0} \leq \|\bar{e} - E_k\|_{C^0}^{1/2} \leq \overline{M}^{1/2} e_{R,(k)}^{1/2}$$

using our induction hypothesis Equation (38). The estimates for higher derivatives follow by differentiating the mollifier in the definition (41) of $e_{(k)}^{1/2}$.

Here A is some absolute constant, but we have not yet specified $\overline{M}$, which will turn out to depend on our choice of Y. We postpone these choices for Section 3.4.

The estimate (44) specifies the value of $M = A\overline{M}^{1/2}$ with which we may apply Lemma 2.1. To conclude that the function $e_{(k)}^{1/2}$ is admissible, we must also verify the lower bound (8) on the set $t \in I(\theta_{(k)}; I_{(k)})$.

For stage $k = 0$, the lower bound (8) is vacuous. To see that the lower bound holds at later stages, we first establish a lower bound for the function $(\bar{e} - E_k - \Delta E_k)_+$ on a slightly larger set of times. Namely, for any $t' \in I(2\theta_{(k)}; I_{(k)})$, we have a lower bound

$$\bar{e}(t') - E_k(t') \geq 3K e_{R,(k)}$$

by Claim 2. We now impose the requirement

$$Z \geq \frac{2}{K} Y \tag{45}$$

to ensure the bound $\Delta E_k = \frac{Y}{Z} e_{R,(k)} \leq \frac{1}{2} K e_{R,(k)}$ for ΔE_k defined in (39). We now have that

$$(e(t') - E_k(t') - \Delta E_k)_+ = (e(t') - E_k(t') - \Delta E_k) \geq 2K e_{R,(k)} \tag{46}$$

[4] We remark that in principle the set $I_{(k+1)}$ is allowed to be empty.

for all $t' \in I(2\theta_{(k)}; I_{(k)})$. As a consequence, we have for $t' \in I(2\theta_{(k)}; I_{(k)})$

$$(e(t') - E_k(t') - \Delta E_k)_+^{1/2} \geq (2K e_{R,(k)})^{1/2}. \tag{47}$$

From this lower bound, we obtain the desired lower bound

$$e_{(k)}^{1/2}(t) \geq (2K e_{R,(k)})^{1/2}, \qquad t \in I(\theta_{(k)}; I_{(k)}) \tag{48}$$

for the function $e_{(k)}^{1/2} = (e - E_k - \Delta E_k)_+^{1/2} * \eta_{\hat{\tau}_k}$ because the time scale $\hat{\tau}_k$ in the mollification is smaller than $\theta_{(k)}$, and because the kernel in the mollification is non-negative with integral equal to 1. Since $\operatorname{supp} R_{(k)} \subseteq I_{(k)} \times \mathbb{T}^3$ by Claim 2, it follows that our choice of $e_{(k)}(t)$ is admissible for Lemma 2.1.

To conclude the proof, we now verify Claims 1, 2 and 3. In the process, we will specify the constants Y and $\overline{M}$.

3.4. *Verification of Claims 1–3*

In the following Section, we verify that Claims 1–3 hold during the iteration given the choice of energy function in (41) and the choice of $I_{(k)}$ defined in (43). In the process, we explain the choice of the constants Y and $\overline{M}$ (which are required to be independent of the constants in Lemma 2.1, and also must be independent of the parameter Z). In this proof, we will therefore say that a constant C is *universal* if it is independent of Y, $\overline{M}$ and Z, and we will use the letter $\hat{C}$ to denote constants which are universal. Some of the constants here will depend (in an increasing manner) on the homogeneous Hölder seminorm of $\bar{e}$, which we denote by

$$\|\bar{e}\|_{\dot{C}_t^\gamma} = \sup_t \sup_{\Delta t \neq 0} \frac{|\bar{e}(t + \Delta t) - \bar{e}(t)|}{|\Delta t|^\gamma}$$

Our starting point is to prove an estimate on the control of the energy profile that will be used at several points in the verification of Claims 1–3.

The main estimate on the energy gap Claims 2–3 require us to control the difference $\bar{e}(t) - E_{k+1}(t)$. Our main tool for estimating this difference is the following Lemma.

Lemma 3.1. *We have an approximation*

$$\bar{e}(t) - E_{k+1}(t) = \bar{e}(t) - E_k(t) - (\bar{e}(t) - E_k(t) - \Delta E_k)_+ + O((1 + \|\bar{e}\|_{\dot{C}_t^\gamma}) e_{R,(k+1)}) \tag{49}$$

where the constant in the $O(\,)$ is universal.

Proof. The starting point for establishing this control is the following calculation:

$$E_{k+1}(t) = E_k(t) + \int_{\mathbb{T}^3} (|v_{(k)} + V_{(k)}|^2(t,x) - |v_{(k)}|^2(t,x))dx \tag{50}$$

$$= E_k(t) + \int_{\mathbb{T}^3} |V_{(k)}|^2(t,x)dx + 2\int_{\mathbb{T}^3} v_{(k)} \cdot V_{(k)}(t,x)dx \tag{51}$$

For the last term, Lemma 2.1 gives an estimate

$$\left|\int_{\mathbb{T}^3} v_{(k)} \cdot V_{(k)} dx\right| \leq \left|\int_{\mathbb{T}^3} \nabla \times v_{(k)} \cdot W_{(k)} dx\right| \tag{52}$$

$$\leq \hat{C}(\Xi_{(k)} e_{v,(k)}^{1/2})(N_{(k)}^{-1} \Xi_{(k)}^{-1} e_{R,(k)}^{1/2}) \tag{53}$$

$$\leq \hat{C}\mathbf{b}^2 e_{R,(k)} = \hat{C}\mathbf{b} e_{R,(k+1)} = \frac{\hat{C}}{Z} e_{R,(k+1)} \tag{54}$$

Note that the constant here can be made smaller than 1 if Z is larger than some universal constant.

For the second term, Lemma 2.1 gives a bound

$$\left| \int_{\mathbb{T}^3} |V_{(k)}|^2(t,x)dx - \int_{\mathbb{T}^3} e_{(k)}(t)dx \right| \le \mathbf{b}^{-1} \frac{e_{v,(k)}^{1/2} e_{R,(k)}^{1/2}}{N_{(k)}} \tag{55}$$

$$\le e_{R,(k+1)} \tag{56}$$

We assume here for simplicity that our torus $\mathbb{T}^3$ has unit volume. Note that both the estimates above use the remark in Lemma 2.1 on the universality of the constants in (14) and (20).

Combining these estimates, (51) gives

$$E_{k+1} = E_k + e_{(k)}(t) + O(e_{R,(k+1)}) \tag{57}$$

where the constant in the $O(\cdot)$ notation is universal.

The proof of Lemma 3.1 concludes by applying the following Lemma, which gives an estimate for how well the smoothed out energy increment

$$e_{(k)}(t) = [(\bar{e} - E_k - \Delta E_k)_+^{1/2} * \eta_{\hat{\tau}_k}]^2$$

approximates the desired energy increment $\tilde{e}_{(k)}(t)$.

Lemma 3.2. *There is a universal constant $\hat{C}$ such that*

$$\|e_{(k)}(t) - (\bar{e} - E_k - \Delta E_k)_+\|_{C_t^0} \le \hat{C} \left(\|\bar{e}\|_{\dot{C}_t^\gamma} |\hat{\tau}_k|^\gamma + \Xi_{(k)} e_{v,(k)}^{1/2} e_{R,(k)} |\hat{\tau}_k| \right) \tag{58}$$

For now we postpone the proof of Lemma 3.2, which is based on the commutator estimate of [3], and the following bound on the derivative of the energy profile of the approximate solution

$$\|\frac{d}{dt} E_k\|_{C_t^0} \le \hat{C}\, \Xi_{(k)} e_{v,(k)}^{1/2} e_{R,(k)} \tag{59}$$

We will return to the proof of Lemma 3.2 in Section 3.5.

Lemma 3.1 now follows from Lemma 3.2 if we can estimate the right hand side of (58) by $O(e_{R,(k+1)})$. For the second term in (58), recalling $|\hat{\tau}_k| = \mathbf{b}\Xi_{(k)}^{-1} e_{v,(k)}^{-1/2}$ and $e_{R,(k+1)} = Z^{-1} e_{R,(k)} = \mathbf{b} e_{R,(k)}$ gives

$$\Xi_{(k)} e_{v,(k)}^{1/2} e_{R,(k)} |\hat{\tau}_k| = e_{R,(k+1)}$$

For the first term in (58), we want to estimate

$$|\hat{\tau}_k|^\gamma = Q_{(k)} e_{R,(k+1)}$$

$$Q_{(k)} = \frac{|\hat{\tau}_k|^\gamma}{e_{R,(k+1)}} \tag{60}$$

For $k = 0$, we established the inequality $Q_{(0)} \le 1$ in line (42). For larger values of k we can decide whether $Q_{(k)}$ increases in size by calculating

$$Q_{(k+1)} = (C_0^{-\gamma} Z^{-2\gamma} Z) Q_{(k)} \tag{61}$$

Here $C_0 > 1$ is the large constant in Lemma 3.1 for the gain in the frequency level. Recalling now that $\gamma = \frac{1}{2}$, we see that $Q_{(k+1)} \le Q_{(k)} \le 1$ for all k, which concludes the proof of Lemma 3.1. □

Given Lemma 3.1, we are now in a position to verify Claims 1, 2 and 3. We start by verifying Claims 1 and 2, which will require us to fix our choice of Y.

3.4.1. Verifying Claims 1 and 2

We now check that Claims 1 and 2 hold provided the constants Y and Z are chosen appropriately. Our proof of Claims 1 and 2 will proceed by induction, and it will be necessary to couple these claims together in the argument in order to close the induction. For simplicity, we will suppress the dependence of constants on the norm $\|\bar{e}\|_{\dot{C}_t^\gamma}$ in what follows.

Let $k \geq 0$. To confirm Claims 1 and 2, we must verify that

$$\bar{e}(t) - E_{k+1}(t) \geq 0 \qquad \text{for } t \in \mathbb{R}, \tag{62}$$

$$\bar{e}(t) - E_{k+1}(t) \geq 3K e_{R,(k+1)} \qquad \text{for } t \in I(2\theta_{(k+1)}; I_{(k+1)}), \tag{63}$$

$$\operatorname{supp} v_{(k+1)} \cup \operatorname{supp} p_{(k+1)} \cup \operatorname{supp} R_{(k+1)} \subseteq I_{(k+1)} \times \mathbb{T}^3, \tag{64}$$

$$I_{(k)} \subseteq I_{(k+1)} \tag{65}$$

hold under the induction hypothesis that (62)–(65) hold for k replacing $k+1$. In the base case of the iteration (i.e. $k+1=0$), the requirements (62)–(64) are satisfied trivially, as $I_{(0)} = \emptyset$ and $E_0(t) = 0$, while the containment (65) imposes no restriction. We now consider the case $k+1 > 0$.

Recall that in (43), we defined

$$I_{(k+1)} = I(2\hat{\tau}_k; \{t \in \mathbb{R} : \bar{e}(t) - E_k(t) \geq \Delta E_k\}).$$

We begin by checking the containment (65). Let t_0 be an element of $I_{(k)}$. By inequality (63) for k, we have that $\bar{e}(t_0) - E_k(t_0) \geq 3K e_{R,(k)}$. Recalling that $\Delta E_k = Y e_{R,(k+1)} = Y \frac{e_{R,(k)}}{Z}$, we have that $t_0 \in I_{(k+1)}$ as long as we impose the condition (45) on our choice of Z, which implies $Z \geq 2K^{-1}Y \geq (3K)^{-1}Y$. Assuming this restriction, we have that $I_{(k)} \subseteq I_{(k+1)}$.

We next verify the containment (64) for $k+1$. Observe that the definition of $I_{(k+1)}$ implies

$$I(\hat{\tau}_k; \operatorname{supp} e_{(k)}) \times \mathbb{T}^3 \subseteq I_{(k+1)} \times \mathbb{T}^3, \tag{66}$$

where $e_{(k)}$ is defined by $e_{(k)}^{1/2}(t) := (\bar{e}(t) - E_k(t) - \Delta E_k)^{1/2} * \eta_{\hat{\tau}_k}$. By the containment (21) of Lemma 2.1, the containment (66), and recalling our choice of $\hat{\tau}_k$, we obtain

$$\operatorname{supp} R_{(k)} \subseteq I_{(k+1)} \times \mathbb{T}^3, \quad \operatorname{supp} V_{(k)} \cup \operatorname{supp} P_{(k)} \subseteq I_{(k+1)} \times \mathbb{T}^3. \tag{67}$$

From the definition of $(v_{(k+1)}, p_{(k+1)}) = (v_{(k)} + V_{(k)}, p_{(k)} + P_{(k)})$, it follows that (64) holds for $k+1$.

We now prove that (62) holds for $k+1$ under the assumption that (63) holds for $k+1$. From the definition of $v_{(k+1)} = v_{(k)} + V_{(k)}$ and the containment (67), it follows that

$$\bar{e}(t) - E_{k+1}(t) = \bar{e}(t) - E_k(t) \geq 0 \qquad \text{for } t \notin I_{(k+1)}, \tag{68}$$

since $t \notin I_{(k+1)}$ implies that $E_{k+1}(t) = \int |v_{(k)} + V_{(k)}|^2(t,x)\,dx = \int |v_{(k)}|^2(t,x)dx = E_k(t)$. Under the assumption that (63) holds for $k+1$, we also have the inequality (62) for $t \in I_{(k+1)}$, which confirms (62) for $k+1$.

To complete proof of Claims 1 and 2, it only remains to establish (63) for $k+1$, for an appropriate choice of Y (independent of k, of course). First observe that for $t \in I_{(k+1)}$, we have by Lemma 3.1

$$\bar{e}(t) - E_{k+1}(t) = \Delta E_k + O(e_{R,(k+1)}) \geq (Y - C)e_{R,(k+1)}, \tag{69}$$

where $C \geq 0$ is a constant independent of k. Now let $t \in I(2\theta_{(k+1)}; I_{(k+1)})$. Writing t in the form $t = t' + \Delta t$ where $t' \in I_{(k+1)}$ and $|\Delta t| \leq 2\theta_{(k+1)}$, we have

$$\begin{aligned}\bar{e}(t) - E_{k+1}(t) &= \bar{e}(t') - E_{k+1}(t') + (\bar{e}(t) - \bar{e}(t')) - (E_{k+1}(t) - E_{k+1}(t')) \\ &\geq \Big(Y - C - 2^\gamma \|\bar{e}\|_{\dot{C}_t^\gamma} \frac{\theta_{(k+1)}^\gamma}{e_{R,(k+1)}} - \hat{C}\,\Xi_{(k+1)} e_{v,(k+1)}^{1/2} 2\theta_{(k+1)}\Big) e_{R,(k+1)}\end{aligned}$$

where we used (69), Hölder continuity of $\bar{e}$ and (59) on the last line. We bound the last term from below by $-2\hat{C}e_{R,(k+1)}$, recalling the identity $\theta_{(k+1)} = \Xi_{(k+1)}^{-1} e_{v,(k+1)}^{-1/2}$. The third term is bounded by

$$\theta_{(k+1)}^\gamma / e_{R,(k+1)} = Z^{\gamma-1} Q_{(k+1)} \leq 1,$$

which follows from the iteration rules, the definitions (31) and (60) of $\theta_{(k)}$ and $Q_{(k)}$, the choices of $Z \geq 1$, $Q_{(0)} \leq 1$, and the fact that $\gamma - 1 = -\frac{1}{2} < 0$. From these estimates we arrive at

$$\bar{e}(t) - E_{k+1}(t) \geq (Y - C - 2^{\gamma}\|\bar{e}\|_{\dot{C}_t^{\gamma}} - 2\hat{C})e_{R,(k+1)} \quad \text{for } t \in I(2\theta_{(k+1)}; I_{(k+1)}) \tag{70}$$

Choosing $Y \geq C + 2^{\gamma}\|\bar{e}\|_{\dot{C}_t^{\gamma}} + 2\hat{C} + 3K$, the desired statement (63) follows.

3.4.2. Verifying the upper bound Claim 3

We now verify the upper bound in Claim 3 for stage $k+1$, and in the process we specify the constant $\overline{M}$ for this upper bound. This estimate follows quickly from Lemma 3.1 now that the constant Y has already been chosen and we have the lower bound $\bar{e}(t) - E_{k+1}(t) \geq 0$ from Claim 2. The proof splits into two cases. In the first case, we consider t for which $\bar{e}(t) - E_k - \Delta E_k \geq 0$. In this case, Lemma 3.1 gives

$$\bar{e}(t) - E_{k+1}(t) = \bar{e} - E_k - (\bar{e} - E_k - \Delta E_k)_+ + O(e_{R,(k+1)}) \tag{71}$$

$$= \Delta E_k + O(e_{R,(k+1)}) = Y e_{R,(k+1)} + O(e_{R,(k+1)}) \tag{72}$$

where the constant in the $O(\,)$ notation is universal. In particular, we have (38) on this set.

For other values of t, we have an upper bound $\bar{e}(t) - E_k < \Delta E_k$. In this case, Lemma 3.1 gives the same upper bound

$$\bar{e}(t) - E_{k+1}(t) = \bar{e} - E_k - (\bar{e} - E_k - \Delta E_k)_+ + O(e_{R,(k+1)}) \tag{73}$$

$$\leq \Delta E_k + O(e_{R,(k+1)}) = Y e_{R,(k+1)} + O(e_{R,(k+1)}) \tag{74}$$

Recalling that the constant in the $O(\,)$ notation is universal, the above bound depends only on Y.

We now choose the constant $\overline{M}$ in Claim 3 depending on Y such that the estimates (72)–(74) hold. This choice of $\overline{M}$ together with the estimate (44) now determine the constant M in our applications of Lemma 2.1.

With this bound, we have established Claim 3, which concludes our proof of Theorem 3.1. The last remaining detail is to establish the Lemma 3.2, which had been used in the proof of Lemma 3.1. We accomplish this step in Section 3.5 below.

3.5. Proof of Lemma 3.2

In this Section, we complete the proof of Theorem 3.1 by establishing Lemma 3.2.

The main idea is to decompose the difference into two terms

$$e_{(k)}(t) - (\bar{e} - E_k - \Delta E_k)_+ = T_I + T_{II} \tag{75}$$

$$T_I = [(\bar{e} - E_k - \Delta E_k)_+^{1/2} * \eta_{\hat{\tau}_k}]^2 - (\bar{e} - E_k - \Delta E_k)_+ * \eta_{\hat{\tau}_k} \tag{76}$$

$$T_{II} = (\bar{e} - E_k - \Delta E_k)_+ * \eta_{\hat{\tau}_k} - (\bar{e} - E_k - \Delta E_k)_+ \tag{77}$$

We can then establish the estimate of Lemma 3.2 for each term individually using the estimates

$$\begin{aligned} |\bar{e}(t + \Delta t) - \bar{e}(t)| &\leq \|\bar{e}\|_{\dot{C}_t^{\gamma}} |\Delta t|^{\gamma} \\ |E_k(t + \Delta t) - E_k(t)| &\leq \hat{C}\,\Xi_{(k)} e_{v,(k)}^{1/2} e_{R,(k)} |\Delta t| \end{aligned} \tag{78}$$

To obtain the second estimate involving E_k in (78), we apply an observation in [1], which is that this bound can be obtained from the Euler–Reynolds equations in the same way that one usually proves conservation of energy for Euler.

$$\frac{d}{dt}E_k = \frac{d}{dt}\int_{\mathbb{T}^3} |v_{(k)}|^2(t,x)dx = \int_{\mathbb{T}^3} v_{(k),l}\partial_j R_{(k)}^{jl} dx \tag{79}$$

$$= -\int_{\mathbb{T}^3} \partial_j v_{(k),l} R_{(k)}^{jl} dx \tag{80}$$

The desired bound for the term T_{II}

$$\|T_{II}\|_{C^0_t} \le \hat{C}\left(\|\bar{e}\|_{\dot{C}^\gamma_t}|\hat{\tau}_k|^\gamma + \Xi_{(k)}e^{1/2}_{v,(k)}e_{R,(k)}|\hat{\tau}_k|\right) \tag{81}$$

now follows easily from (78), where $\hat{C}$ is some constant depending on the volume of the torus $\mathbb{T}^3$.

We now show that the bounds in (78) also imply the same estimate for the term T_I. The main idea is to view the difference T_I as a quadratic commutator term as in the well-known commutator estimate of [3] (i.e. the term can be written in the form $f * \eta_\epsilon g * \eta_\epsilon - (fg) * \eta_\epsilon$ for the appropriate functions f and g and the appropriate mollifying kernel η_ϵ). Setting $\hat{e}^{1/2}_{(k)}(t) = (\bar{e}(t) - E_k(t) - \Delta E_k)^{1/2}_+$, this commutator structure allows us to write the term T_I as

$$T_I = \int_{\mathbb{R}} \left(\hat{e}^{1/2}_{(k)}(t+\tau) - \eta_{\hat{\tau}_k} * \hat{e}^{1/2}_{(k)}(t)\right)^2 \eta_{\hat{\tau}_k}(\tau)d\tau \tag{82}$$

We first estimate the integrand of (82) pointwise at each fixed value of $\tau \in \mathbb{R}$. We begin with the elementary inequality

$$|(y+\Delta y)^{1/2}_+ - (y)^{1/2}_+| \le |\Delta y|^{1/2} \qquad \text{for all } y, \Delta y \in \mathbb{R}.$$

Taking $y = \bar{e}(t) + E_k(t) - \Delta E_k$ and $y + \Delta y = \bar{e}(t+\tau) + E_k(t+\tau) - \Delta E_k$ in the above inequality, we apply the bounds in (78) to obtain the estimate

$$|\hat{e}^{1/2}_{(k)}(t+\tau) - \hat{e}^{1/2}_{(k)}(t)| \le \hat{C}^{1/2}\left(\|\bar{e}\|_{\dot{C}^\gamma_t}|\tau|^\gamma + \Xi_{(k)}e^{1/2}_{v,(k)}e_{R,(k)}|\tau|\right)^{1/2}$$

for all $\tau \in \mathbb{R}$. From the above estimate on the modulus of continuity of $\hat{e}^{1/2}_{(k)}$ and the containment $\operatorname{supp}\eta_{\hat{\tau}_k} \subseteq \{\tau \in \mathbb{R} \,:\, |\tau| \le \hat{\tau}_k\}$, it is now straightforward to estimate the integrand of (82) pointwise, and to obtain the desired bound (81) for the term T_I. This bound concludes the proof of Lemma 3.2.

3.6. *Proof of Theorem 3.1: summary*

In this Section, we summarize the proof of Theorem 3.1, and clarify the logical order in which the parameters involved in proving these claims are chosen. What we have shown in Sections 3.2–3.5 above is the following statement:

Proposition 3.1 *(Summary of the iteration). Given a positive number $\alpha < \alpha^* = 1/5$, an open interval $I \subseteq \mathbb{R}$, a compactly supported, non-negative function $\bar{e}(t) \in C^\gamma_t(I)$, $\gamma = \frac{2\alpha^*}{1-\alpha^*} = \frac{1}{2}$, and a real number $B > 0$ that bounds the Hölder semi-norm $\|\bar{e}\|_{\dot{C}^\gamma_t}$ from above, there exist:*

- *Non-negative constants $\overline{M}$, C_0 and Z;*
- *A sequence of parameters $(\Xi_{(k)}, e_{v,(k)}, e_{R,(k)})$, $\Xi_{(k)} \ge 2$, $e_{v,(k)}, e_{R,(k)} \ge 0$;*
- *A sequence of Euler–Reynolds flows $(v_{(k)}, p_{(k)}, R_{(k)})$;*
- *A sequence of subsets $I_{(k)} \subseteq I$,*

such that

- *The iteration rules (28)–(30) relating $(\Xi_{(k)}, e_{v,(k)}, e_{R,(k)})$, C_0 and Z hold for all $k \ge 0$.*
- *The frequency energy levels of $(v_{(k)}, p_{(k)}, R_{(k)})$ are below $(\Xi_{(k)}, e_{v,(k)}, e_{R,(k)})$ to order 2 in C^0.*
- *The sequence $(v_{(k)}, p_{(k)})$ converges in $C^\alpha_{t,x} \times C^{2\alpha}_{t,x}(I \times \mathbb{T}^3)$ to a solution of incompressible Euler.*
- *The containment $\operatorname{supp} v_{(k)} \cup \operatorname{supp} p_{(k)} \cup \operatorname{supp} R_{(k)} \subseteq I_{(k)} \times \mathbb{T}^3$ holds as stated in (32).*
- *The containment $I_{(k)} \subseteq I_{(k+1)}$ holds as stated in (33) for all $k \ge 0$.*
- *For $E_k(t) = \frac{1}{2}\int_{\mathbb{T}^3}|v_{(k)}|^2(t,x)dx$, the inequalities (36), (37) and (38) which relate the functions $\bar{e}(t)$, $E_k(t)$ to the sets $I_{(k)}$ and the parameters $\overline{M}$, Z, and $(\Xi_{(k)}, e_{v,(k)}, e_{R,(k)})$ hold for all $k \ge 0$.*

The proof of Proposition 3.1 involves the introduction of a parameter Y which is used to define the energy increments $e_{(k)}(t)$ during the iteration. We also define a parameter $\mathbf{b} = Z^{-1}$ and time scales $\theta_{(k)} = \Xi^{-1}_{(k)}e^{-1/2}_{v,(k)}$ and $\hat{\tau}_k = Z^{-1}\Xi^{-1}_{(k)}e^{-1/2}_{v,(k)}$ for ease of notation.

In the base case of the iteration (Section 3.2), we choose the initial Euler–Reynolds flow to be $(v_{(0)}, p_{(0)}, R_{(0)}) = (0, 0, 0)$, and $I_{(0)} = \emptyset$, while the initial energy level $e_{R,(0)}$ is chosen to depend only on the norm $\|\bar{e}(t)\|_{C^0_t}$. At this point the parameters $(\Xi_{(0)}, e_{v,(0)})$ remain unspecified as they will depend on the choice of Z. The reason for this dependence is that we desire a sharp time scale in the first stage of the iteration, and this goal is accomplished by taking a large value of Z.

The parameter Y appearing in (39) is the next parameter specified. This parameter is chosen at the end of Section 3.4.1. The choice of Y depends only on: certain universal constants C and $\hat{C}$ appearing in Lemma 3.1 and Section 3.5 where Lemma 3.2 is proven; the universal constant K from the construction; and the upper bound (B above) for the $\dot{C}^\gamma_t$ Hölder semi-norm of $\bar{e}$.

The constant $\overline{M}$ in Claim 3 is the next parameter to be specified. This constant depends on the parameter Y and the other absolute constants from the Lemmas in Section 3.4. The choice of $\overline{M}$ is made in Section 3.4.2. With the constant $\overline{M}$ determined, the sequence of upper bounds

$$\left\| \left(\frac{d}{dt}\right)^r e_{(k)}^{1/2} \right\|_{C^0_t} \leq A(Z\Xi_{(k)} e_{v,(k)}^{1/2})^r [\overline{M} e_{R,(k)}]^{1/2}, \qquad 0 \leq r \leq 2 \tag{83}$$

stated on the right hand side of (44) are fully determined up to the choice of Z and the determination of C_0. (In this equation, we have substituted Z for $\mathbf{b}^{-1}$ in order to distinguish the $\mathbf{b}$ in the definition of $\hat{\tau}_k$ and the parameter $\left(\frac{e_v^{1/2}}{e_R^{1/2} N}\right)$ appearing in Lemma 2.1.)

We choose C_0 to be the constant whose existence is asserted by Lemma 2.1 with $L = 2$ and $M = A\overline{M}^{1/2}$.

With C_0 chosen, it is possible to determine the appropriate choice of Z subject to some requirements. First, Z is sufficiently large depending on α, C_0 and other absolute constants to ensure $C^\alpha_{t,x} \times C^{2\alpha}_{t,x}$ convergence of the sequence $(v_{(k)}, p_{(k)})$. More precisely, Z is chosen sufficiently large so that the sequence of bounds on the $C^\alpha_{t,x} \times C^{2\alpha}_{t,x}$ norms of the corrections which result from the iteration will be summable. Furthermore, Z satisfies the requirements $Z \geq 2K^{-1}Y$ coming from (45).

With the constants C_0 and Z determined, the full sequence of parameters $(\Xi_{(k)}, e_{v,(k)}, e_{R,(k)})$ along with the time scales $\theta_{(k)}, \hat{\tau}_k$ are determined by induction according to the iteration rules (28)–(30), and the initial choice of $e_{v,(0)} = Ze_{R,(0)}$ and $\Xi_{(0)}$ made in (42). The energy increment $e_{(0)}^{1/2}(t) = (\bar{e}(t) - Ye_{R,(0)})_+^{1/2} * \eta_{\hat{\tau}_0}$ for initializing the iteration has also been determined (it is possible that $e_{(0)}(t) = 0$). The set $I_{(1)} := I(2\hat{\tau}_0; \{t \in \mathbb{R} : \bar{e}(t) \geq \frac{Y}{Z} e_{R,(0)}\})$ has also been determined according to (43) (it is possible that $I_{(1)}$ is empty).

With these parameters, we generate a sequence of Euler–Reynolds flows $(v_{(k)}, p_{(k)}, R_{(k)})$ by repeated application of Lemma 2.1. This iteration simultaneously generates a function $e_{(k)}(t)$ and a set $I_{(k)} \subseteq \mathbb{R}$ associated to each Euler Reynolds flow $(v_{(k)}, p_{(k)}, R_{(k)})$ according to the formulas (41) and (43). The assumption that $\bar{e}$ has compact support in I together with inequality (37) imply by induction that $I_{(k)} \subseteq I$ for all $k \geq 0$. Lemma 2.1 is applied in each stage choosing $L = 2$ and the parameter M to be the constant $A\overline{M}$. In each stage, we take[5] $N = Z^{5/2}$, and define the energy increment $e_{(k)}(t)$ according to (41).

Our choices of parameters have been made such that both $N = Z^{5/2}$ and $e_{(k)}(t)$ defined in (43) are admissible according to the requirements $N \geq (e_{v,(k)}/e_{R,(k)})^{3/2}$, (8) and (9). The requirement $N \geq (e_{v,(k)}/e_{R,(k)})^{3/2}$ follows by induction from the parameter evolution rules. To verify the estimates in (9), we must check that the factor of Z appearing in the right hand side of inequality (83) is no larger than the loss of the factor $\mathbf{b}^{-1} = \left(\frac{e_v^{1/2}}{e_R^{1/2} N}\right)_{(k)}^{-1/2}$ allowed by the Lemma. From the parameter evolution rules, it follows by induction that this factor is equal to Z for all k. It follows by induction that the admissibility condition (9) is satisfied for all indices k under the assumption that inequality (83) holds. In Section 3.3, we verify that the inequality (83) holds for the sequence of functions $e_{(k)}(t)$ using the inductive Claim 3. In Section 3.3, we also verify the required lower bound (8) of Lemma 2.1 using the inductive Claims 1–2.

[5] We remark that in principle N could be allowed to depend on the stage k, as was the case in [7].

3.7. Extending Theorem 3.1 to Theorem 1.1

Having concluded the proof of Theorem 3.1, we outline how our argument above extends to establish the additional statements in Theorem 1.1. We address the additional technical issues involved in the nonperiodic case in Section 3.8.

We first observe that the ideas of our proof of the case $\gamma = \frac{1}{2}$ in Theorem 3.1 can be extended to give solutions $v \in C^\alpha_{t,x}$ with prescribed energy profiles $\bar{e} \in C^\gamma_t$ having lower regularity $0 < \alpha < \alpha^* \leq 1/5$, $\gamma = \frac{2\alpha^*}{1-\alpha^*}$. To achieve this result, one can state a variant of Lemma 2.1 where the stress is reduced at an inferior rate of $e'_R = \mathbf{b}^{-\beta}\frac{e_v^{1/2} e_R^{1/2}}{N}$ for some $\beta \geq 1$, and the number $\mathbf{b}^{-1}$ is replaced by $\mathbf{b}^{-\beta}$ in all of the estimates. One must also replace the smallness factor $\mathbf{b}$ in the enlargement of the time support by the smaller factor $\mathbf{b}^\beta$. In this case, the same argument establishes Theorem 3.1 with lesser regularity. The crucial point at which the exponent $\gamma = \frac{2\alpha^*}{1-\alpha^*}$ comes into play is in the estimate (61), where we estimate the difference between the smoothed out energy increment and the desired energy increment. As β tends to infinity, the threshold α^* for the Hölder regularity tends to 0, as does $\gamma = \frac{2\alpha^*}{1-\alpha^*}$. In the opposite direction, if one assumes the same lemma but with $0 \leq \beta \leq 1$, then the threshold α^* would tend to $1/3$ while the required regularity $\gamma = \frac{2\alpha^*}{1-\alpha^*}$ would tend to 1.

Next, we observe that our solutions with prescribed energy profiles in the class C^γ_t become arbitrarily small in the $C^\alpha_{t,x}$ topology if $\alpha < \alpha^*$ and $\gamma = \frac{2\alpha^*}{1-\alpha^*}$ when we consider a one-parameter family of energy profiles tending to 0 as in the statement of Theorem 1.1. To check this observation, consider the algorithm in the proof of Theorem 3.1, and apply this algorithm to an energy profile

$$\bar{e}_A(t) = A\bar{e}(t) \tag{84}$$

where $A \leq 1$ is some constant. Note that the choice of the constants Y, $\overline{M}$ and Z in our algorithm do not depend on A, but rather depend only on the desired regularity α and possibly on an upper bound for the $\dot{C}^\gamma_t$ semi-norm of $\bar{e}$ (see Section 3.1). Recall also that the bounds on the $C^\alpha_{t,x}$ norms of the corrections $V_{(k)}$ decrease exponentially by a certain factor for all indices $k \geq 0$ thanks to the choice of the parameter Z. Thus, to check that the algorithm produces a solution that is arbitrarily small in $C^\alpha_{t,x}$ as A tends to 0, the key point is to check that the size of the initial correction $V_{(0)}$ becomes arbitrarily small in $C^\alpha_{t,x}$ as the parameter A tends to 0.

To check that this smallness holds, recall from [6, Section 11.4] that the correction obeys the estimate

$$\|V_{(0)}\|_{C^\alpha_{t,x}} \leq C(N_{(0)}\Xi_{(0)})^\alpha e_{R,(0)}^{1/2} \tag{85}$$

The constant C here is universal. The parameters $N_{(0)}$ and $\Xi_{(0)}$ both depend on Z, but we can ignore this dependence since Z is fixed. What we consider here is the dependence on A. Recall from Section 3.2 that $e_{R,(0)}$ is proportional to $\|A\bar{e}\|_{C^0}$, and is therefore proportional to A. The initial frequency level is chosen in Section 3.2 to have size $\Xi_{(0)} \sim e_{R,(0)}^{-\frac{1}{\gamma}-\frac{1}{2}}$. Our estimate (85) therefore scales as

$$\|V_{(0)}\|_{C^\alpha_{t,x}} \lesssim A^{\frac{1}{2}-\alpha(\frac{1}{\gamma}+\frac{1}{2})}$$

with an implied constant depending on the choices of Y, $\overline{M}$ and Z. The above bound tends to 0 as $A \to 0$ provided $\gamma > \frac{2\alpha}{1-\alpha}$. Thus, the initial correction $V_{(0)}$, and furthermore the sum of all corrections $\sum_{k=0}^\infty V_{(k)}$, may be made arbitrarily small in $C^\alpha_{t,x}$ by applying our algorithm to the energy profile $A\bar{e}(t)$ with A small and $\bar{e} \in C^\gamma_t$.

Remark. From this calculation, we can view our Theorem 1.1 as providing some evidence for the conjecture in [8] that irregularity of the energy profile is characteristic of generic solutions to Euler with Hölder regularity strictly below $1/3$. Here we have shown that, within the range of exponents $\alpha < 1/5$ and $\gamma > \frac{2\alpha}{1-\alpha}$, Euler flows which are arbitrarily small perturbations of 0 in $C^\alpha_{t,x}$ can have energy profiles that fail to have C^γ regularity in time. In view of [6, Theorem 1.1], it is very likely that our methods show that solutions with such irregular energy profiles can approximate any sufficiently smooth solution to Euler in $C^\alpha_{t,x}$. We are optimistic that the method of convex integration can be extended to make statements about generic Euler flows, at least those with regularity below $1/5$.

3.8. Prescribing the energy profile: the nonperiodic setting

In this Section, we describe how to modify the proof of Theorem 1.1 to construct a Hölder continuous weak solution to the Euler equations with a prescribed energy profile in the non-periodic setting. Our main result in this setting is as follows.

Theorem 3.2 *(Nonperiodic Euler flows with prescribed energy profile). Let $\alpha < \alpha^* \leq 1/5$ and let $\mathcal{U}$ be a non-empty open subset of $\mathbb{R}^3$. Let $I \subseteq \mathbb{R}$ be a bounded open interval and let $\bar{e}(t) \geq 0$ be any non-negative function with compact support in I which belongs to the class $\bar{e}(t) \in C_t^\gamma$ for $\gamma = \frac{2\alpha^*}{1-\alpha^*}$. Then:*

1. *There exists a weak solution (v, p) to the Euler equations in the class $v \in C^\alpha_{t,x}(\mathbb{R} \times \mathbb{R}^3)$ with support contained in*
$$\operatorname{supp} v \cup \operatorname{supp} p \subseteq I \times \mathcal{U} \tag{86}$$
such that the energy profile of v is given by
$$\int_{\mathbb{R}^3} |v|^2(t, x)dx = \bar{e}(t), \qquad t \in \mathbb{R} \tag{87}$$
2. *Moreover, one may choose a one parameter family of solutions (v_A, p_A), $0 \leq A \leq 1$ with the above properties such that the energy profile of v_A is equal to $\int_{\mathbb{R}^3} |v_A|^2(t, x)dx = A\bar{e}(t)$ and such that $\|v_A\|_{C^\alpha_{t,x}} \to 0$ as $A \to 0$.*

Proof. Unless otherwise stated, we employ the same notation as in the proof of Theorem 3.1.

We begin with some initial reductions of the theorem. As before, we will only consider the case $\alpha^* = 1/5$ and $\gamma = 1/2$ of the first statement. Extension of this special case to the full statement of Theorem 3.2 proceeds exactly as in Section 3.7.

Furthermore, in order to simplify the proof, we will produce a weak solution (v, p) to the Euler equations in $C^\alpha_{t,x}(\mathbb{R} \times \mathbb{R}^3)$ supported in the space–time cylinder
$$\operatorname{supp}(v, p) \subseteq \operatorname{supp} \bar{e} \times B(2; 0), \tag{88}$$
where $B(2; 0) \subseteq \mathbb{R}^3$ is simply the closed ball of radius 2 centered at the origin. By making straightforward modifications to the argument below, one can also arrange for the solutions of Theorem 3.2 (v, p) to have spatial supports contained in an arbitrarily small open subset $\mathcal{U} \subseteq \mathbb{R}^3$, as asserted in (86).

Beginning with the trivial solution $(v_{(0)}, p_{(0)}, R_{(0)}) = (0, 0, 0)$, we will construct a sequence of solutions $(v_{(k)}, p_{(k)}, R_{(k)})$ to the Euler–Reynolds equations that obeys the following properties:

1. Each solution $(v_{(k)}, p_{(k)}, R_{(k)})$ has frequency-energy levels below $(\Xi_{(k)}, e_{v,(k)}, e_{R,(k)})$, which evolves under the iteration rules (28)–(30); recall that these rules ensure that $(v_{(k)}, p_{(k)})$ converges to a weak solution to the Euler equations in $C^\alpha_{t,x} \times C^{2\alpha}_{t,x}$ for $0 < \alpha < \alpha^* = 1/5$.
2. In addition to keeping track of the time support of the Euler–Reynolds flows $(v_{(k)}, p_{(k)}, R_{(k)})$, we now need to take into account their supports in space. We will construct sets $I_{(k)} \subseteq \mathbb{R}$, $B_{(k)} \subseteq \mathbb{R}^3$ so that for each k, we have the space–time support condition
$$\operatorname{supp} v_{(k)} \cup \operatorname{supp} p_{(k)} \cup \operatorname{supp} R_{(k)} \subseteq I_{(k)} \times B_{(k)} \tag{89}$$
and $B_{(k)} \subseteq \mathbb{R}^3$ is an open ball satisfying
$$B_{(k)} \subseteq B_{(k+1)} \subseteq B(2; 0). \tag{90}$$
The last property ensures that the limiting Euler flow (v, p) obeys (88).
3. Finally, the energy profiles $E_k(t)$ converge pointwise to $\bar{e}(t)$ as $k \to \infty$, i.e.,
$$E_k(t) := \int_{\mathbb{R}^3} |v_{(k)}|^2(t, x)\, dx \to \bar{e}(t) \quad \text{as } k \to \infty. \tag{91}$$
This property achieves the desired energy prescription (87).

As before, we construct the sequence $(v_{(k)}, p_{(k)}, R_{(k)})$ via iteration of Lemma 2.1. To this end, we need to choose:

- The initial space–time set $I_{(0)} \times B_{(0)}$.
- The initial frequency-energy levels $(\Xi_{(0)}, e_{v,(0)}, e_{R,(0)})$.
- The iteration factor Z.
- The energy increment $\widetilde{e}_{(k)}(t,x)$ and the space–time set $I_{(k)} \times B_{(k)}$ for each $k \geq 0$.

We set

$$I_{(0)} = \emptyset, \quad B_{(0)} = B(1;0) \tag{92}$$

Similarly as before, we then take

$$e_{R,(0)} = \frac{A'}{|B(1;0)|} \max_{t\in I} \bar{e}(t), \quad e_{v,(0)} = Z e_{R,(0)}, \tag{93}$$

where $A' > 0$ is an absolute constant that will be specified in (103) below. The iteration factor Z will be chosen so that

$$Z \geq C_0^{\frac{2\alpha}{5\epsilon}}, \quad Z \geq (2C_0^{-1})^{2/5}, \tag{94}$$

where $C_0 > 1$ is the constant arising in the iteration rule (28). The first condition, which coincides with [6, Section 11.4], ensures that the resulting Euler flow (v,p) belongs to $C_{t,x}^{1/5-\epsilon} \times C_{t,x}^{2/5-2\epsilon}$. The second condition will be used to control the growth of $B_{(k)}$ defined in (97). We emphasize, however, that the value of Z will be fixed only later. Similarly, for $\Xi_{(0)}$, we require

$$\Xi_{(0)} \geq \max\{100, \ (Z^{1-\frac{3\gamma}{2}} e_{R,(0)}^{-1-\frac{\gamma}{2}})^{1/\gamma}\}, \tag{95}$$

but its actual value will be fixed after Z has been chosen. The frequency-energy levels $(\Xi_{(k)}, e_{v,(k)}, e_{R,(k)})$ for $k \geq 1$ are determined by the iteration rules (28)–(30). Note that by (94), (95) and the iteration rules (see also the proof of (61) before), we have

$$Q_{(k)} := \frac{|\hat{\tau}_k|^\gamma}{e_{R,(k+1)}} \leq Q_{(0)} \leq 1. \tag{96}$$

Now it only remains to specify the space–time sets $I_{(k)} \times B_{(k)}$ and the energy density $\widetilde{e}_{(k)}(t,x)$. We need $\widetilde{e}_{(k)}(t,x)$ to be admissible in the sense that (8), (9) are satisfied with $(\Xi_{(k)}, e_{v,(k)}, e_{R,(k)})$ as specified above. On the other hand, we need to ensure that the desired properties (89), (90) and (91) hold with our choice of $I_{(k)} \times B_{(k)}$ and $\widetilde{e}_{(k)}(t,x)$.

Here, our strategy is to essentially reduce the proof to that of Theorem 3.1. We proceed recursively: Under the assumption that $I_{(k)} \times B_{(k)}, v_{(k)}, p_{(k)}, R_{(k)}$ have been constructed so that $(v_{(k)}, p_{(k)}, R_{(k)})$ has frequency-energy level below $(\Xi_{(k)}, e_{v,(k)}, e_{R,(k)})$ and (89) is satisfied, we construct appropriate $I_{(k+1)} \times B_{(k+1)}$ and $\widetilde{e}_{(k)}$. First, we define $B_{(k+1)}$ as

$$B_{(k+1)} := B(10\Xi_{(k)}^{-1}; B_{(k)}) \quad \text{for } k \geq 0 \tag{97}$$

where $B(\overline{\rho}; S) := \{x + \Delta x : x \in S, \ |\Delta x| \leq \overline{\rho}\}$. Since $B_{(0)} = B(1;0)$, note that each $B_{(j)}$ is a closed ball centered at the origin as well; we will denote the radius of $B_{(j)}$ by $r_{(j)}$. By the second condition in (94), (95) and the iteration rules, (90) follows. Next, as in the proof of Theorem 3.1, we set

$$e_{(k)}^{1/2}(t) = (\bar{e}(t) - E_k(t) - \Delta E_k)_+^{1/2} * \eta_{\hat{\tau}_k} \tag{98}$$

where the gap ΔE_k now takes the form[6]

$$\Delta E_k = Y e_{R,(k+1)} |B_{(k)}|, \tag{99}$$

and Y is a constant to be chosen later. We then take the energy density $\widetilde{e}_{(k)}^{1/2}(t,x)$ to be of the form

6 This definition is exactly analogous to (39), as we assumed $|\mathbb{T}^3| = 1$ before.

$$\widetilde{e}_{(k)}^{1/2}(t,x) = e_{(k)}^{1/2}(t)\chi_{(k)}(x) \tag{100}$$

where $\chi_{(k)}(x)$ is a smooth non-negative function on $\mathbb{R}^3$ that obeys

$$\operatorname{supp} \chi_{(k)} \subseteq B(5\Xi_{(k)}^{-1}; B_{(k)}), \tag{101}$$

$$B(2\Xi_{(k)}^{-1}; B_{(k)}) \subseteq \{x \in \mathbb{R}^3 : \chi_{(k)}(x) = \chi_{(k)}(0)\}, \tag{102}$$

$$\int \chi_{(k)}(x) = 1, \quad |\nabla^m \chi_{(k)}| \leq A' \frac{\Xi_{(k)}^m}{|B_{(k)}|} \quad \text{for } 0 \leq m \leq 2, \tag{103}$$

for some absolute constant $A' > 0$. To construct such a function $\chi_{(k)}(x)$, consider a smooth radial function $\eta(r)$ which equals 1 on $\{r \leq r_{(k)} + 2\Xi_{(k)}^{-1}\}$ and vanishes on $\{r \geq r_{(k)} + 5\Xi_{(k)}^{-1}\}$, and then normalize $\chi_{(k)}(x) = c\eta(|x|)$ so that $\int \chi_{(k)} = 1$. Finally, the set $I_{(k+1)}$ is defined as

$$I_{(k+1)} = I(2\hat{\tau}_k; \{t \in \mathbb{R} : \bar{e}(t) - E_k(t) \geq \Delta E_k\}) \quad \text{for } k \geq 0, \tag{104}$$

where we recall the notation $I(\overline{\tau}; J) = \{t + \Delta t : t \in J, \ |\Delta t| \leq \overline{\tau}\}$.

The Ansatz (100) reduces the question of admissibility of the energy density $\widetilde{e}_{(k)}(t,x)$ to that of the energy profile $e_{(k)}(t)$, which we have dealt with in the proof of Theorem 3.1. Indeed, note that

$$\hat{C}_{v_{(k)}}(\theta_{(k)}, \Xi_{(k)}^{-1}; \operatorname{supp} R_{(k)}) \subseteq I(\theta_{(k)}; I_{(k)}) \times B(\Xi_{(k)}^{-1}; B_{(k)})$$

which follows from $\operatorname{supp} v_{(k)} \cup \operatorname{supp} R_{(k)} \subseteq I_{(k)} \times B_{(k)}$ (i.e., (89) for k) and by the definition of Eulerian cylinders (see Definition 2.2). Hence, the desired lower bound (8) follows (using (102)) once we prove the bound $e_{(k)}(t)\chi_{(k)}(0) \geq 2K e_{R,(k)}$ for $t \in I(\theta_{(k)}; I_{(k)})$. Using (103), we can further reduce (8) to the following lower bound on $e_{(k)}(t)$:

$$e_{(k)}(t) \geq \frac{2K|B_{(k)}|}{A'} e_{R,(k)}, \quad t \in I(\theta_{(k)}; I_{(k)}). \tag{105}$$

Next, by (101) and (102), we have

$$\operatorname{supp} v_{(k)} \cap \operatorname{supp} \nabla \chi_{(k)} = \emptyset, \tag{106}$$

which implies

$$\nabla^m (\partial_t + v_{(k)} \cdot \nabla)^r \widetilde{e}_{(k)}^{1/2}(t,x) = (\frac{d}{dt})^r e_{(k)}^{1/2}(t) \nabla^m \chi_{(k)}(x). \tag{107}$$

Hence, by (103), the desired upper bound (9) for $\widetilde{e}_{(k)}^{1/2}(t,x)$ with $L = 2$ follows once M is chosen to be such that

$$\|(\frac{d}{dt})^r e_{(k)}^{1/2}\|_{C_t^0} \leq \frac{M}{A'} (\mathbf{b}^{-1} \Xi_{(k)} e_{v,(k)})^r e_{R,(k)}^{1/2}, \quad 0 \leq r \leq 1. \tag{108}$$

Repeating the arguments in the proof of Theorem 3.1, the following analogues of Claims 1, 2 and 3 can be established using induction (note that (93) ensures that these claims hold for $k = 0$):

Claim 4 *(Growing supports). For all $k \geq 0$, we have*

$$\operatorname{supp} v_{(k)} \cup \operatorname{supp} p_{(k)} \cup \operatorname{supp} R_{(k)} \subseteq I_{(k)} \times B_{(k)} \tag{109}$$

$$I_{(k)} \subseteq I_{(k+1)} \tag{110}$$

$$B(1,0) \subseteq B_{(k)} \subseteq B_{(k+1)} \subseteq B(2,0) \tag{111}$$

Claim 5 *(There is always room to add more energy where the error is supported). For every $t \in \mathbb{R}$, we have*

$$\bar{e}(t) - E_k(t) \geq 0. \tag{112}$$

Moreover, for $t \in I(2\theta_{(k)}; I_{(k)})$, we have

$$\bar{e}(t) - E_k(t) \geq \frac{3K}{A'} e_{R,(k)} |B_{(k)}|, \tag{113}$$

where K is the constant in the lower bound (8) of Lemma 2.1, A' is the constant in (103).

Claim 6 *(The energy threshold is nearly saturated). There is an absolute constant* $\overline{M}$ *such that the upper bound*

$$\sup_t |\bar{e}(t) - E_k(t)| \le \overline{M} e_{R,(k)} |B_{(k)}| \tag{114}$$

holds uniformly.

We remark that the factors of $|B_{(k)}|$ on the right-hand sides of (113) and (114) ensure that these estimates are dimensionally correct; note that in (37) and (38), we had $|\mathbb{T}^3| = 1$. The presence of the factor $|B_{(k)}|$ does not cause any significant modification of the proof, as $|B_{(k)}|$ is bounded from below and above by absolute constants by construction, i.e., $|B(1;0)| \le |B_{(k)}| \le |B(2;0)|$. The absolute constant $A' > 0$ in (113) does not introduce any difficulty as well. It is in the proof of these claims that the constant $Y > 0$ in (99) is fixed, and the iteration constant Z is required to be even larger depending on Y. We omit the routine modifications.

We are now ready to conclude the proof of Theorem 3.2. Arguing as in the proofs of (48) and (44), the desired estimates (105) and (108) (with a constant $M > 0$ independent of k) for $e_{(k)}(t)$ follow from Claims 4, 5 and 6 once Z is chosen to be sufficiently large. Hence Lemma 2.1 (with $L = 2$) can be applied to $(v_{(k)}, p_{(k)}, R_{(k)})$ to produce $(v_{(k+1)}, p_{(k+1)}, R_{(k+1)})$ with frequency-energy levels below $(\Xi_{(k+1)}, e_{v,(k+1)}, e_{R,(k+1)})$. The support property (89) for $(v_{(k+1)}, p_{(k+1)}, R_{(k+1)})$ follows from (21), and (91) is a quick consequence of (114). □

Conflict of interest statement

No conflict of interest.

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