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Multiple positive solutions for a critical quasilinear equation via Morse theory ☆

Existence de plusieurs solutions positives d'une équation quasi-linéaire avec exposant critique par la théorie de Morse

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Abstract

We deal with the existence of solutions for the quasilinear problem

$$(P_\lambda)\qquad \begin{cases} -\Delta_p u = \lambda u^{q-1} + u^{p^*-1} & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $N \geqslant p^2$, $1 < p \leqslant q < p^*$, $p^* = Np/(N-p)$, $\lambda > 0$ is a parameter. Using Morse techniques in a Banach setting, we prove that there exists $\lambda^* > 0$ such that, for any $\lambda \in (0, \lambda^*)$, (P_λ) has at least $\mathcal{P}_1(\Omega)$ solutions, possibly counted with their multiplicities, where $\mathcal{P}_t(\Omega)$ is the Poincaré polynomial of Ω. Moreover for $p \geqslant 2$ we prove that, for each $\lambda \in (0, \lambda^*)$, there exists a sequence of quasilinear problems, approximating (P_λ), each of them having at least $\mathcal{P}_1(\Omega)$ distinct positive solutions.

Résumé

On s'interesse à l'existence de solutions pour l'équation quasi-linéaire

$$(P_\lambda)\qquad \begin{cases} -\Delta_p u = \lambda u^{q-1} + u^{p^*-1} & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

où Ω est un domaine de $\mathbb{R}^N$ avec frontière régulière, $N \geqslant p^2$, $1 < p \leqslant q < p^*$, $p^* = Np/(N-p)$, $\lambda > 0$ est un paramètre. Par des techniques de la théorie de Morse dans le cadre des espaces de Banach, un démontre l'existence de $\lambda^* > 0$ tel que, pour tout $\lambda \in (0, \lambda^*)$, (P_λ) possède au moins $\mathcal{P}_1(\Omega)$ solutions, considerées avec leur multiplicité, où $\mathcal{P}_1(\Omega)$ est le polynôme de

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Poincaré de Ω. En outre, pour $p \geqslant 2$, on démontre que, pour tout $\lambda \in (0, \lambda^*)$, il existe une suite de problèmes quasi-linéaires qui approchent (P_λ), chacun desquels a au moins $\mathcal{P}_1(\Omega)$ solutions positives différentes.

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1. Introduction

Let us consider the quasilinear elliptic problem

$$(P_\lambda) \qquad \begin{cases} -\Delta_p u = \lambda u^{q-1} + u^{p^*-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $N \geqslant p^2$, $1 < p \leqslant q < p^*$, $p^* = Np/(N-p)$, $\lambda > 0$ is a parameter.

The first striking result, in the case $p = 2$, is due to Pohozaev [30] who proved that if Ω is starshaped with respect to some point and $\lambda = 0$, then (P_λ) has no solution. Some years later, in the celebrated paper [9], Brezis and Nirenberg showed that if $N \geqslant 4$ and $p = q = 2$, problem (P_λ) has a solution for any $\lambda \in (0, \lambda_1)$ where λ_1 is the first eigenvalue of $-\Delta$ with Dirichlet boundary condition on Ω (cf. [2]) and no solution when $\lambda \geqslant \lambda_1$ or $\lambda \leqslant 0$ and Ω is starshaped. Moreover they proved that when $N = 3$, $p = q = 2$, Ω is the ball in $\mathbb{R}^N$, then problem (P_λ) has a solution if and only if $\lambda \in (\frac{\lambda_1}{4}, \lambda_1)$. The paper [9] stimulated a vast amount of research on this subject.

The quasilinear critical problem (P_λ) was considered by Azorero and Peral in [3]. They showed that if $N \geqslant p^2$, and $p = q$, then (P_λ) has a solution for any $\lambda \in (0, \lambda_1)$, where λ_1 denotes the first eigenvalue of p-Laplace operator $-\Delta_p$ with Dirichlet boundary condition on Ω, no solutions when $\lambda \geqslant \lambda_1$ or $\lambda \leqslant 0$ and Ω is starshaped. The same results are obtained by Guedda and Veron in [23] with a different approach.

Moreover in [3] (see also [4]) Azorero and Peral proved that when $q \in (p, p^*)$, $N \geqslant p^2$, (P_λ) has a solution for any $\lambda > 0$.

The first multiplicity result for (P_λ) has been achieved by Rey in [31] in the semilinear case. Precisely Rey proved that if $N \geqslant 5$, $p = q = 2$, for λ small enough (P_λ) has at least $cat(\Omega)$ solutions, where $cat(\Omega)$ denotes the Ljusternik–Schnirelmann category of Ω in itself. This result, as Rey wrote in the introduction of the paper [31], was suggested by the fact that the number of solutions to (P_λ) is related to the properties of the Green's function of Ω. Precisely in [32], he has showed that if $N \geqslant 4$, $p = q = 2$, and u_λ is a solution of (P_λ), which concentrates around a point x_0 as $\lambda \to 0$, then x_0 is a critical point of the Robin's function, the regular part of the Green's function. Conversely if $N \geqslant 5$, $p = q = 2$, any nondegenerate critical point x_0 of the Robin's function generates a family of solutions of (P_λ), concentrating around x_0 as $\lambda \to 0$.

Throughout a different approach, based on some ideas introduced by Benci and Cerami in [6], Lazzo obtained in [24] the same result of Rey [31], under the weaker condition $N \geqslant 4$. Really, for $p = q = 2$, in [28], Passaseo improved the results in [31,24] proving that if Ω is not contractible, (P_λ) has at least $cat(\Omega) + 1$ solutions. Furthermore Passaseo showed that the number of solutions of (P_λ) is not related to the topology of Ω, but to the topology of a domain $\tilde{\Omega}$ which differs from Ω by a set of small capacity. For instance if Ω is obtained by $\tilde{\Omega}$ cutting off a set of small capacity, then problem (P_λ) has at least $cat(\tilde{\Omega}) + 1$ distinct solutions, for λ small, even if the domain Ω is contractible in itself.

Recently in [1], Alves and Ding have proved a multiplicity result for the quasilinear problem (P_λ), in the spirit of the papers [31,24]. They showed that if $N \geqslant p^2$ and $2 \leqslant p \leqslant q < p^*$, then there exists $\lambda^* > 0$ such that for each $\lambda \in (0, \lambda^*)$ problem (P_λ) has at least $cat(\Omega)$ solutions.

In this work we aim to obtain a better information on the number of solutions of problem (P_λ), for small value of parameter λ, via the Morse theory and the domain topology. We recall the following definitions.

Definition 1.1. Let $\mathbb{K}$ be a field. For any $B \subset A \subset \mathbb{R}^n$, we denote $\mathcal{P}_t(A, B)$ the Poincaré polynomial of the topological pair (A, B), defined by

$$\mathcal{P}_t(A, B) = \sum_{k=0}^{+\infty} \dim H^k(A, B) t^k,$$

where $H^k(A, B)$ stands for the kth Alexander–Spanier relative cohomology group of (A, B), with coefficient in $\mathbb{K}$; we also set $H^k(A) = H^k(A, \emptyset)$ and $\mathcal{P}_t(A) = \mathcal{P}_t(A, \emptyset)$ is called the Poincaré polynomial of A.

Definition 1.2. Let X be a Banach space and f be a C^1 functional on X. Let $\mathbb{K}$ be a field. Let u be a critical point of f, $c = f(u)$, and U be a neighborhood of u. We call

$$C_q(f, u) = H^q\big(f^c \cap U, \big(f^c \setminus \{u\}\big) \cap U\big)$$

the qth critical group of f at u, $q = 0, 1, 2, \ldots$, where $f^c = \{v \in X\colon\ f(v) \leqslant c\}$, $H^q(A, B)$ stands for the qth Alexander–Spanier cohomology group of the pair (A, B) with coefficients in $\mathbb{K}$. By the excision property of the singular cohomology theory the critical groups do not depend on a special choice of the neighborhood U.

Definition 1.3. We denote $\mathcal{P}_t(f, u)$ the Morse polynomial of f in u, defined by

$$\mathcal{P}_t(f, u) = \sum_{k=0}^{+\infty} \dim C_k(f, u) t^k.$$

We call the *multiplicity* of u the number $\mathcal{P}_1(f, u) \in \mathbb{N} \cup \{+\infty\}$.

In the past years the relations between the topological properties of the domain and the multiplicity of solutions to semilinear elliptic problems have been largely investigated. We mention the celebrated paper [7] where Benci and Cerami estimated the number of solutions of the semilinear elliptic problem

$$(S_\lambda)\quad \begin{cases} -\lambda\Delta u + u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with smooth boundary, $N \geqslant 3$, $\lambda > 0$ is a parameter, $f : \mathbb{R}^+ \to \mathbb{R}$ is a $C^{1,1}$ function with $f(0) = f'(0) = 0$, having a subcritical growth at infinity. By means of Morse techniques, they showed that the number of positive solutions of problem (S_λ), counted with their multiplicities, depends on the topology of Ω, actually on $\mathcal{P}_t(\Omega)$, the Poincaré polynomial of Ω.

The functional analytic setting, when (S_λ) is set up, is a Hilbert space. The multiplicity is exactly one, if the solution is nondegenerate in the classical sense given in a Hilbert space, namely the second derivative of the associated Euler functional in the solution is an isomorphism between the Hilbert space and its dual. The nondegeneracy condition is generally verified via the perturbation results in [26], which guarantee that each isolated critical point can be resolved in a finite number of nondegenerate critical points of a C^1 locally approximating functional. Let us emphasize that the perturbation results in [26] rely on a infinite dimensional version of Sard's Theorem, due to Smale [34] so that they need the Fredholm properties of the second derivates in the critical points.

In our work, we obtain a first result, which correlates the topological properties of the domain and the number of solutions of (P_λ), counted with their multiplicities.

Theorem 1.4. *Assume that $N \geqslant p^2$, $1 < p \leqslant q < p^*$, $p^* = Np/(N-p)$. There exists $\lambda^* > 0$ such that, for any $\lambda \in (0, \lambda^*)$, (P_λ) has at least $\mathcal{P}_1(\Omega)$ solutions, possibly counted with their multiplicities.*

As showed in [7], the application of the Morse theory yields better results than the application of Ljusternik–Schnirelman theory for topologically rich domain. For example if Ω is obtained by an open contractible domain cutting off k holes, we derive that the number of solutions of (P_λ) is affected by k, even if the category of Ω is 2 (see Remark 3.11 and also [7]).

Theorem 1.4 assures $\mathcal{P}_1(\Omega)$ distinct solutions, if one is able to interpret the *multiplicity* for a solution of (P_λ). For this reason, we need a deep insight into this notion. In the present work we consider the case $p \geqslant 2$. In this context, the variational setting of the quasilinear problem (P_λ) is the Banach space $W_0^{1,p}(\Omega)$, which is not Hilbert for $p \neq 2$. The Euler functionals associated to problem (P_λ) are C^2, but several conceptual difficulties arise in order to perform a local Morse theory and perturbation results like in [26]. Firstly all the solutions of (P_λ) are degenerate in the classic sense, as $W_0^{1,p}(\Omega)$ is not isomorphic to its dual space (cf. [19]). Moreover, denoting by I_λ the Euler functional associated to (P_λ), we lack the Fredholm properties of I''_λ in its critical points, so that the perturbation results in [26] cannot be applied. As far as we know, there are no kind of results in the spirit of the paper by Rey [32].

As in [15], we introduce the following *weaker* notion of nondegeneracy.

Definition 1.5. Let A be an open subset of $W_0^{1,p}(\Omega)$ and $g : A \to \mathbb{R}$ be a C^2 functional. We say that a critical point u of g is nondegenerate if $g''(u)$ is injective from $W_0^{1,p}(\Omega)$ to its dual $W^{-1,p'}(\Omega)$.

We remark that the above notion of nondegeneracy coincides with the usual one if the space is Hilbert and the operator is Fredholm. This is not our case, when $p > 2$. We emphasize that in 1969 Smale, as written by Uhlenbeck in [38], conjectured that injectivity is enough for developing Morse theory in some Banach settings.

Using the above notion of nondegeneracy, we give a sharp interpretation of the multiplicity of a critical point of (P_λ) in terms of approximating elliptic problems. This result is contained in Theorem 5.1. We remark that this approach is new also for the case $p = 2$. Indeed the perturbation results by Marino and Prodi [26] furnish an interpretation of the multiplicity in terms of C^1 locally approximating functional, which cannot be, in general, the Euler functional of some semilinear problem.

Using the result in Theorem 5.1, we prove the following multiplicity results. In what follows, we say that $\partial\Omega$ satisfies the interior sphere condition if for each $x_0 \in \partial\Omega$ there exists a ball $B_R(x_1) \subset \Omega$ such that $\overline{B_R(x_1)} \cap \partial\Omega = \{x_0\}$.

Theorem 1.6. *Assume that $\partial\Omega$ satisfies the interior sphere condition and that $N \geqslant p^2$, $2 < p \leqslant q < p^*$, $p^* = Np/(N-p)$. There exists $\lambda^* > 0$ such that, for any $\lambda \in (0, \lambda^*)$, either (P_λ) has at least $\mathcal{P}_1(\Omega)$ distinct solutions or, if not, for any sequence (α_n), with $\alpha_n > 0$, $\alpha_n \to 0$, there exists a sequence (f_n) with $f_n \in C^1(\overline{\Omega})$, $\|f_n\|_{C^1} \to 0$ such that problem*

$$(P_n)\quad \begin{cases} -\operatorname{div}\big((|\nabla u|^2 + \alpha_n)^{(p-2)/2}\nabla u\big) = \lambda u^{q-1} + u^{p^*-1} + f_n & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least $\mathcal{P}_1(\Omega)$ distinct solutions, for n large enough.

Theorem 1.7. *Assume that $\partial\Omega$ satisfies the interior sphere condition and that $N \geqslant 4$, $2 \leqslant q < 2^*$, $2^* = 2N/(N-2)$. Then there exists $\lambda^* > 0$ such that, for any $\lambda \in (0, \lambda^*)$, either (P_λ) has at least $\mathcal{P}_1(\Omega)$ distinct solutions or, if not, there exists a sequence (f_n) with $f_n \in C^1(\overline{\Omega})$, $\|f_n\|_{C^1} \to 0$ such that problem*

$$(L_n)\quad \begin{cases} -\Delta u = \lambda u^{q-1} + u^{p^*-1} + f_n & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least $\mathcal{P}_1(\Omega)$ distinct solutions, for n large enough.

Theorems 1.6 and 1.7 are quantitative results which give an interpretation to the number of solutions of (P_λ). The proofs of these theorems rely on the construction of an approximating functional to I_λ, having only *nondegenerate* critical points in the sense introduced in Definition 1.5. For nondegenerate critical points of the approximating functional, we are able to compute the critical groups, which are topological objects, in terms of differential notions, like the Morse index (see Theorem 4.2), so that the multiplicity of a nondegenerate critical point is exactly one. By Theorem 2.4 it follows that the Morse polynomial $\mathcal{P}_t(I_\lambda, u_0)$ (see Definition 1.3) can be computed in terms of the sum of the Morse polynomials of the approximating functional in each critical point and a partially controlled remainder term.

We remark that the idea of combining the Splitting Theorem and Sard's Lemma, in the finite dimensional case, can be traced back to Chang [10] in the special case of a C^2 functional, defined on a Hilbert space, having an isolated critical point.

Perturbations results in Morse theory for quasilinear problem having a right-hand side subcritically at infinity are obtained in [17,14] (see also [13,20]).

Concerning multiplicity results of nontrivial solutions (not necessarily positive) for some critical quasilinear problem, we quote [33,29]. Finally we mention a recent result by Degiovanni and Lancelotti [21], where the existence of a nontrivial solution (not necessarily positive) for the critical problem (P_λ) is proved for any $\lambda > \lambda_1$, $\lambda \neq \lambda_m$, where (λ_m) is a suitable sequence of eigenvalues of $-\Delta_p$.

Throughout the paper we use the following notations:

(1) $\|\cdot\|$ denotes the usual norm both in $W_0^{1,p}(\Omega)$ and in $W^{-1,p}(\Omega)$;
(2) $\|\cdot\|_{1,2}$ denotes the usual norm in $W_0^{1,2}(\Omega)$;
(3) $\|\cdot\|_\infty$ denotes the usual norm in $L^\infty(\Omega)$;
(4) $|\cdot|_r$ denotes the usual norm in $L^r(\Omega)$;
(5) $\|\cdot\|_{C^1}$ and $\|\cdot\|_{C^2}$ denote the usual norms in $C^1(W_0^{1,p}(\Omega))$ and $C^2(W_0^{1,p}(\Omega))$;
(6) $\langle\cdot,\cdot\rangle : W^{-1,p'}(\Omega) \times W_0^{1,p}(\Omega) \to \mathbb{R}$ denotes the duality pairing;
(7) $d(\cdot,\cdot)$ denotes the distance function in each metric space;
(8) M^r denotes $\{v \in W_0^{1,p}(\Omega)\colon d(v,M) < r\}$, where $M \subset W_0^{1,p}(\Omega)$ and $r > 0$;
(9) $f^c = \{v \in W_0^{1,p}(\Omega)\colon f(v) \leqslant c\}$, $f_a^b = \{v \in W_0^{1,p}(\Omega)\colon a \leqslant f(v) \leqslant b\}$, $\mathrm{int}(f_a^b) = \{v \in W_0^{1,p}(\Omega)\colon a < f(v) < b\}$.

2. Some abstract recalls in Morse theory

We need to recall some useful definitions and results (cf. [11,12,35]).

Definition 2.1. Let X be a Banach space and f be a C^1 functional on X. Let C be a closed subset of X. A sequence (u_n) in C is a Palais–Smale sequence for f if $\|f(u_n)\| \leqslant M$ uniformly in n, while $f'(u_n) \to 0$ as $n \to +\infty$.

We say that f satisfies $(P.S.)$ on C if any Palais–Smale sequence in C has a strongly convergent subsequence.

Let $c \in \mathbb{R}$. We say that f satisfies $(P.S.)_c$ if any sequence (u_n) in X, such that $f(u_n) \to c$ and $f'(u_n) \to 0$ as $n \to +\infty$, has a strongly convergent subsequence.

Definition 2.2. Let X be a Banach space and f be a C^2 functional on X. If u is a critical point of f, the Morse index of f in u is the supremum of the dimensions of the subspaces of X on which $f''(u)$ is negative definite. It is denoted by $m(f,u)$. Moreover, the large Morse index of f in u is the sum of $m(f,u)$ and the dimension of the kernel of $f''(u)$. It is denoted by $m^*(f,u)$.

Next theorem is a topological version of the classical Morse relation (cf. Theorem 4.3 in [11]).

Theorem 2.3. *Let X be a Banach space and f be a C^1 functional on X. Let $a, b \in \mathbb{R}$ be two regular values for f, with $a < b$. If f satisfies the $(P.S.)_c$ condition for all $c \in (a,b)$ and $u_1, \ldots, u_l$ are the critical points of f in $f^{-1}(a,b)$, then*

$$\sum_{q=0}^{+\infty}\left(\sum_{j=1}^{l} \dim C_q(f,u_j)\right)t^q = \mathcal{P}_t(f^b, f^a) + (1+t)Q(t), \tag{2.1}$$

where $Q(t)$ is a formal series with coefficients in $\mathbb{N} \cup \{+\infty\}$.

We point out that the above series are formal, as (2.1) means that the coefficients (possibly $+\infty$) of each t^q are the same on both sides of the equality.

In order to obtain a multiplicity result of solutions to problem (P_λ) via Morse relations, we recall an abstract theorem, proved in [14] (see also [5] and [11]).

Theorem 2.4. *Let A be a open subset of a Banach space X. Let f be a C^1 functional on A and $u \in A$ be an isolated critical point of f. Assume that there exists an open neighborhood U of u such that $\overline{U} \subset A$, u is the only critical point of f in $\overline{U}$ and f satisfies the Palais–Smale condition in $\overline{U}$.*
Then there exists $\bar{\mu} > 0$ such that, for any $g \in C^1(A, \mathbb{R})$ such that

- $\|f - g\|_{C^1(A)} < \bar{\mu}$,
- *g satisfies the Palais–Smale condition in $\overline{U}$,*
- *g has a finite number $\{u_1, u_2, \dots, u_m\}$ of critical points in U,*

we have

$$\sum_{j=1}^{m} \mathcal{P}_t(g, u_j) = \mathcal{P}_t(f, u) + (1+t)Q(t),$$

where $Q(t)$ is a formal series with coefficients in $\mathbb{N} \cup \{+\infty\}$.

3. The topological result

Assume that $N \geqslant p^2$ and $1 < p \leqslant q < p^*$, $p^* = pN/(N-p)$. Standard arguments prove that the solutions of (P_λ) correspond to the critical points of the C^1 functional $I_\lambda : W_0^{1,p}(\Omega) \to \mathbb{R}$ defined by setting

$$I_\lambda(u) = \frac{1}{p} \int_\Omega |\nabla u|^p \, dx - \frac{\lambda}{q} \int_\Omega (u^+)^q \, dx - \frac{1}{p^*} \int_\Omega (u^+)^{p^*} \, dx. \tag{3.1}$$

We introduce the Nehari manifolds

$$\Sigma_\lambda = \left\{u \in W_0^{1,p}(\Omega) \setminus \{0\}\colon \langle I'_\lambda(u), u\rangle = 0\right\}.$$

Suitably modifying the proof of [7, Lemma 2.2], it is easy to show that, for any $\lambda > 0$, Σ_λ is a 1-codimensional submanifold of $W_0^{1,p}(\Omega)$, as it is C^1-diffeomorphic to

$$\left\{u \in W_0^{1,p}(\Omega)\colon \|u\| = 1\right\} \setminus \left\{u \in W_0^{1,p}(\Omega)\colon u \leqslant 0 \text{ a.e.}\right\}.$$

Moreover, each nontrivial critical point of I_λ is a nonnegative function which belongs to Σ_λ.

We state some results, which are proved in [3,4,1].

As usually, we denote by S the best Sobolev constant of the embedding $W_0^{1,p}(\Omega) \hookrightarrow L^{p^*}(\Omega)$ given by $S = \inf\{\|u\|^p\colon u \in W_0^{1,p}(\Omega),\ |u|_{p^*} = 1\}$.

Lemma 3.1. *Let $N \geqslant p^2$. Then I_λ satisfies the $(P.S.)_c$ condition for all $c \in (0, \frac{S^{N/p}}{N})$.*

Theorem 3.2. *Let $N \geqslant p^2$. I_λ possesses the mountain-pass geometry (M.P., for short), is bounded from below on Σ_λ and $\inf I_\lambda(\Sigma_\lambda)$, which we denote by c_λ, is the M.P. level, i.e.*

$$c_\lambda \stackrel{def}{=} \inf I_\lambda(\Sigma_\lambda) = \inf_{v \neq 0} \max_{t \geqslant 0} I_\lambda(tv).$$

Moreover c_λ is decreasing in λ and $\lim_{\lambda \to 0} c_\lambda = \frac{S^{N/p}}{N}$.

Up to translations, we may assume that $0 \in \Omega$. Moreover, in what follows, we fix $r > 0$ such that $\mathrm{B}_r(0) = \{x \in \mathbb{R}^n\colon d(x, 0) < r\} \subset \Omega$ and the sets

$$\Omega_r^+ = \left\{x \in \mathbb{R}^n\colon d(x, \Omega) < r\right\}, \qquad \Omega_r^- = \left\{x \in \Omega\colon d(x, \partial\Omega) > r\right\}$$

are both homotopically equivalent to Ω.

Further, we consider the space

$$W_{0,\mathrm{rad}}^{1,p}(B_r) = \left\{u \in W_0^{1,p}\left(\mathrm{B}_r(0)\right)\colon u(x) = u\left(|x|\right)\right\}$$

and set

$$I_{\lambda,\mathrm{rad}}(u)=\frac{1}{p}\int\limits_{B_r(0)}|\nabla u|^p\,dx-\frac{\lambda}{q}\int\limits_{B_r(0)}(u^+)^q\,dx-\frac{1}{p^*}\int\limits_{B_r(0)}(u^+)^{p^*}\,dx\quad\forall u\in W^{1,p}_{0,\mathrm{rad}}(B_r),$$

$$\Sigma_{\mathrm{rad}}=\big\{u\in W^{1,p}_{0,\mathrm{rad}}(B_r)\setminus\{0\}\colon\ \big\langle I'_{\mathrm{rad}}(u),u\big\rangle=0\big\},$$

$$m_r(\lambda)=\inf I_{\mathrm{rad}}(\Sigma_{\mathrm{rad}}).$$

Theorem 3.3. *Using the previous notations, $m_r(\lambda)$ is the M.P. level of I_{rad}, i.e.*

$$m_r(\lambda)=\inf\Big\{\max_{t\geqslant 0}I_{\mathrm{rad}}(tu)\colon\ u\in W^{1,p}_{0,\mathrm{rad}}(B_r),\ u\neq 0\Big\}.$$

Moreover $m_r(\lambda)$ is decreasing in λ and $\lim_{\lambda\to 0}m_r(\lambda)=\frac{S^{N/p}}{N}$.

Remark 3.4. Once fixed $r>0$, we can repeat the same construction for any $\varrho\in(0,r)$, defining the levels $m_\varrho(\lambda)$. If $m_\varrho(\lambda)$ is a critical level for any $\varrho\in(0,r)$, then Theorem 1.4 is proved. So we can suppose that $m_r(\lambda)$ is not a critical level for I_λ.

We define the continuous map $\beta:\Sigma_\lambda\to\mathbb{R}^N$ by setting

$$\beta(u)=\frac{\int_\Omega x(u^+(x))^{p^*}\,dx}{\int_\Omega (u^+(x))^{p^*}\,dx}.$$

Lemma 3.5. *There exists $\lambda^*>0$ such that if $\lambda\in(0,\lambda^*)$, $u\in\Sigma_\lambda$ and $I_\lambda(u)\leqslant m_r(\lambda)$, then $\beta(u)\in\Omega_r^+$.*

Proof. By way of contradiction, let $\{\lambda_n\}$ and $\{u_n\}$ be such that $\lambda_n\to 0$, $u_n\in\Sigma_{\lambda_n}$, $I_{\lambda_n}(u_n)\leqslant m_r(\lambda_n)$ and $\beta(u_n)\notin\Omega_r^+$. Since $\lim_{\lambda\to 0}m_r(\lambda)=\lim_{\lambda\to 0}c_\lambda=\frac{S^{N/p}}{N}$,

$$\lim_{n\to+\infty}I_{\lambda_n}(u_n)=\frac{S^{N/p}}{N}. \tag{3.2}$$

From $u_n\in\Sigma_{\lambda_n}$ and $I_{\lambda_n}(u_n)\leqslant m_r(\lambda_n)$, we have that $\|u_n\|$ is bounded and hence $\lambda_n|u_n^+|_q\to 0$.
Consequently, as $u_n\in\Sigma_{\lambda_n}$ and (3.2) holds, we get

$$\lim_{n\to+\infty}\|u_n\|^p=\lim_{n\to+\infty}|u_n^+|_{p^*}^{p^*}=S^{N/p}. \tag{3.3}$$

Defining $w_n=u_n/|u_n^+|_{p^*}$, we see that $|w_n^+|_{p^*}=1$ and, by (3.3), $\lim_{n\to+\infty}\|w_n\|^p=S$.
Furthermore, the functions $\tilde w_n(x)=w_n^+(x)$ satisfy

$$|\tilde w_n|_{p^*}=1\quad\text{and}\quad\|\tilde w_n\|^p\to S.$$

Let us introduce the following notation

$$\mathcal{D}^{1,p}(\mathbb{R}^n)=\left\{u\in L^{p^*}(\mathbb{R}^n)\colon\ \frac{\partial u}{\partial x_i}\in L^p(\mathbb{R}^n)\ \text{for}\ i=1,\dots,N\right\}.$$

By Lemma 3.1 in [1], there is $\{\varepsilon_n\}$ in $\mathbb{R}^+$ and $\{y_n\}$ in $\mathbb{R}^N$, such that $\varepsilon_n\to 0$, $y_n\to y\in\overline{\Omega}$ and $v_n(x)=\varepsilon_n^{(N-p)/p}\tilde w_n(\varepsilon_n x+y_n)\to v$ in $\mathcal{D}^{1,p}(\mathbb{R}^N)$, with $v(x)>0$.
Considering $\phi\in C_0^\infty(\mathbb{R}^N)$ such that $\phi(x)=x$ in Ω, we infer

$$\beta(u_n)=\frac{\int_\Omega x(u_n^+(x))^{p^*}\,dx}{\int_\Omega (u_n^+(x))^{p^*}\,dx}=\int\limits_{\mathbb{R}^N}\phi(x)\big(\tilde w_n(x)\big)^{p^*}\,dx=\int\limits_{\mathbb{R}^N}\phi(\varepsilon_n z+y_n)\big(v_n(z)\big)^{p^*}\,dz.$$

Moreover, by Lebesgue Theorem,

$$\int\limits_{\mathbb{R}^N}\phi(\varepsilon_n x+y_n)\big(v_n(x)\big)^{p^*}\,dx\to y\in\overline{\Omega},$$

so that $\lim_{n\to\infty}\beta(u_n)=y\in\overline{\Omega}$, in contradiction with $\beta(u_n)\notin\Omega_r^+$. □

By Theorem 3.3, for each $\lambda>0$ we can consider $v_\lambda\in\Sigma_{\lambda,\mathrm{rad}}$, such that $I_{\lambda,\mathrm{rad}}(v_\lambda)=m_r(\lambda)$. Let us introduce $\gamma:\Omega_r^-\to I_\lambda^{m_r(\lambda)}\cap\Sigma_\lambda$ defined by

$$\gamma(y)(x)=\begin{cases}v_\lambda(x-y) & \text{if } x\in B_r(y),\\ 0 & \text{otherwise.}\end{cases}$$

Note that γ is continuous and, as v_λ is radial,

$$\beta\circ\gamma(y)=y\quad\forall y\in\Omega_r^-. \tag{3.4}$$

So we get the following result.

Lemma 3.6. *There exists $\lambda^*>0$ such that if $\lambda\in(0,\lambda^*)$, then*

$$\dim H^k\big(I_\lambda^{m_r(\lambda)}\cap\Sigma_\lambda\big)\geqslant\dim H^k(\Omega).$$

Proof. Let λ^* be as in Lemma 3.5. We denote by γ^k and β^k the homomorphisms induced by γ and β respectively between the kth cohomology groups, i.e.

$$H^k(\Omega_r^+)\xrightarrow{\beta^k}H^k\big(I_\lambda^{m_r(\lambda)}\cap\Sigma_\lambda\big)\xrightarrow{\gamma^k}H^k(\Omega_r^-).$$

Since, from (3.4), $\gamma^k\circ\beta^k=\mathrm{id}^k$ and Ω_r^+ is homotopically equivalent to Ω, the assert follows. □

From now on, for any $b\geqslant a$, we will denote $(I_\lambda)_a^b$ simply by I_a^b.

Lemma 3.7. *$I_\lambda^{-1}\{a\}$ is a deformation retract of $I_a^b\setminus\Sigma_\lambda$, for any $a\in(0,c_\lambda)$ and $b\geqslant a$.*

Proof. Let $C=W_0^{1,p}(\Omega)\setminus(\Sigma_\lambda\cup\{0\})$. For any $u\in C$, the function

$$t\in[0,+\infty)\longrightarrow I_\lambda(tu)$$

has one maximum point θ_u, and $\theta_u\neq1$ since $tu\in\Sigma_\lambda$ if and only if $t=\theta_u$.

Adapting the proof of [7, Lemma 2.2], we infer that $u\to\theta_u$ is continuous, so that $A=\{u\in C:\ \theta_u<1\}$ and $B=\{u\in C:\ \theta_u>1\}$ are open sets and $I_a^b\setminus\Sigma_\lambda\subset A\cup B$.

Let $u\in I_a^b\setminus\Sigma_\lambda$. If $u\in A$, let $\delta(u)$ be the only value $t\geqslant1$ such that $I_\lambda(tu)=a$.

The function $\delta:I_a^b\cap A\to\mathbb{R}$ is continuous. In fact, let $F:(0,+\infty)\times A\to\mathbb{R}$ be defined by $F(t,u)=I_\lambda(tu)-a$ and $u_0\in I_a^b\cap A$. Let $t_0\geqslant1$ be such that $F(t_0,u_0)=0$. Since $\theta_{u_0}<1$ while $t_0\geqslant1$, we get that $t_0u_0\notin\Sigma_\lambda$ and

$$\frac{\partial F}{\partial t}(t_0,u_0)=\big\langle I'_\lambda(t_0u_0),u_0\big\rangle\neq0,$$

so, by the Implicit Function Theorem, δ is continuous.

Analogously, if $u\in B$, let $\delta(u)$ be defined as the only $t\in(0,1]$ such that $I_\lambda(tu)=a$, so that the function $\delta:I_a^b\cap B\to\mathbb{R}$ is continuous too.

Now let $H:[0,1]\times(I_a^b\setminus\Sigma_\lambda)\to W_0^{1,p}(\Omega)$ be defined by $H(t,u)=(t\delta(u)+1-t)u$. The proof is completed, as we see immediately that:

- H is continuous;
- $H(0,u)=u\ \forall u$;
- $I_\lambda(H(1,u))=a\ \forall u$;
- $H(t,u)\in I_a^b\setminus\Sigma_\lambda\ \forall t,\ \forall u$;
- $H(t,u)=u\ \forall u\in I_\lambda^{-1}\{a\}\ \forall t$. □

We now give a technical lemma (see [7, Lemma 5.3] for the proof).

Lemma 3.8. *Let $\mathcal{M}$ be a manifold and $\mathcal{N} \subset \mathcal{M}$ be a closed oriented submanifold of codimension d. If W is a subset of $\mathcal{N}$ closed in $\mathcal{N}$, then*

$$\mathcal{P}_t(\mathcal{M}, \mathcal{M} \setminus W) = t^d \mathcal{P}_t(\mathcal{N}, \mathcal{N} \setminus W).$$

Proposition 3.9. *If $a \in (0, c_\lambda)$ and $b \geqslant a$ is a noncritical level for I_λ, then*

$$\mathcal{P}_t(I_\lambda^b, I_\lambda^a) = t\mathcal{P}_t(I_a^b \cap \Sigma_\lambda).$$

Proof. If we set $\mathcal{M} = \operatorname{int}(I_a^b)$, $\mathcal{N} = \mathcal{M} \cap \Sigma_\lambda$ and $W = \mathcal{N}$, from Lemma 3.8 we get

$$\mathcal{P}_t\big(\operatorname{int}(I_a^b), \operatorname{int}(I_a^b) \setminus \Sigma_\lambda\big) = t\mathcal{P}_t\big(\operatorname{int}(I_a^b) \cap \Sigma_\lambda\big). \tag{3.5}$$

Furthermore, a and b being not critical values for I_λ, we have

$$\mathcal{P}_t(I_a^b, I_a^b \setminus \Sigma_\lambda) = \mathcal{P}_t\big(\operatorname{int}(I_a^b), \operatorname{int}(I_a^b) \setminus \Sigma_\lambda\big) \quad \text{and} \quad \mathcal{P}_t(I_a^b \cap \Sigma_\lambda) = \mathcal{P}_t\big(\operatorname{int}(I_a^b) \cap \Sigma_\lambda\big). \tag{3.6}$$

So, since

$$\mathcal{P}_t(I_\lambda^b, I_\lambda^a) = \mathcal{P}_t\big(I_a^b, I_\lambda^{-1}(a)\big),$$

the assert comes by (3.5), (3.6) and Lemma 3.7. □

Corollary 3.10. *There exists $\lambda^* > 0$ such that if $\lambda \in (0, \lambda^*)$ and $a \in (0, c_\lambda)$, then*

$$\mathcal{P}_t(I_\lambda^{m_r(\lambda)}, I_\lambda^a) = t\big(\mathcal{P}_t(\Omega) + \mathcal{Z}_\lambda(t)\big),$$

where $\mathcal{Z}_\lambda(t)$ is a polynomial with nonnegative integer coefficients.

Proof. Let λ^* be as in Lemma 3.6 and let us fix $\lambda \in (0, \lambda^*)$ and $a \in (0, c_\lambda)$.

By Remark 3.4, we can assume that $m_r(\lambda)$ is a noncritical value for I_λ, so the assert derives from Lemma 3.6 and Proposition 3.9. □

Proof of Theorem 1.4. Let λ^* be chosen in accordance with Corollary 3.10 and $\lambda \in (0, \lambda^*)$. Let u_j $(1 \leqslant j \leqslant m)$ be the critical points of I in the strip $(I)_a^{m_r(\lambda)}$, where $a \in (0, c_\lambda)$. Since I satisfies $(P.S.)_c$ condition for all $c \in (0, S^{N/p}/N)$ (see Lemma 3.1), the global Morse relation (2.1) gives

$$\sum_{k=0}^{+\infty} a_k t^k = \sum_{k=0}^{+\infty} \dim H^k(I^{m_r(\lambda)}, I^a) t^k + (1+t)Q_\lambda(t), \tag{3.7}$$

where $a_k = \sum_{j=1}^m \dim C_k(f_\lambda, u_j)$ and $Q_\lambda(t)$ is a formal series with coefficients in $\mathbb{N} \cup \{+\infty\}$. Corollary 3.10 implies

$$\sum_{k=0}^{+\infty} a_k t^k = t\big(\mathcal{P}_t(\Omega) + \mathcal{Z}_\lambda(t)\big) + (1+t)Q_\lambda(t)$$

whence, for $t = 1$, we get

$$\sum_{j=1}^{m} \mathcal{P}_1(I, u_j) = \mathcal{P}_1(\Omega) + \mathcal{Z}_\lambda(1) + 2Q_\lambda(1). \tag{3.8}$$

Since both $\mathcal{Z}_\lambda(1)$ and $Q_\lambda(1)$ have nonnegative coefficients, problem (P_λ) has at least $\mathcal{P}_1(\Omega)$ positive solutions, each counted with its own multiplicity. □

Remark 3.11. If we consider $\Omega = A \setminus \bigcup_{i=1}^k \overline{C}_i$, where A and C_i $(i = 1, 2, \ldots, k)$ are contractible, open, smooth and bounded nonempty sets in $\mathbb{R}^n$, $\overline{C}_i \subset A$ for any $i = 1, 2, \ldots, k$ and $\overline{C}_i \cap \overline{C}_j = \emptyset$ for any $i \neq j$, Theorem 1.4 guarantees that (P_λ) has at least $k + 1$ solutions, each counted with its own multiplicity.

4. Nondegeneracy and local Morse theory

Theorem 1.4 assures that problem (P_λ) has at least $\mathcal{P}_1(\Omega)$ solutions, which can be distinct or, if not, counted with their multiplicities. However the evaluation of the multiplicity of a critical point is not easy, in general.

In a Hilbert space, the local behavior of the functional near a critical point is quite clear if the critical point is nondegenerate and computing the critical groups of a nondegenerate critical point is possible via its Morse index. Successively, Gromoll and Meyer generalized these ideas in order to compute the critical groups of an isolated critical point u, possibly degenerate, having finite Morse index, if the second derivative of the functional in u is a Fredholm operator. The generalized Morse lemma is a basic tool for computing the critical groups and the theory of Fredholm operators provides a natural setting for this lemma. Moreover we emphasize that such critical groups estimates seem to require a Hilbert space structure.

We remark that when $p \neq 2$, several conceptual difficulties arise for developing a local Morse theory for I_λ. Firstly, all the critical points of I_λ are degenerate in the classical sense given in Hilbert spaces, as the Banach space $W_0^{1,p}(\Omega)$ is not isomorphic to its dual space. Moreover the second derivative of I_λ in each critical point is not Fredholm, so that the (generalized) Morse Lemma does not hold, and relations between differentiable notions, like Morse index, and critical groups are not available in general. Our idea is to perform an approximation of I_λ in terms of functionals on $W_0^{1,p}(\Omega)$, for which we are able to develop a local Morse theory.

In what follows, we assume that $p \geqslant 2$. In this case it is standard to check that I_λ is a C^2 functional. If $p = q = 2$ the functional I_λ is not C^2, nevertheless this case can be easily covered just by replacing I_λ with the functional

$$\tilde{I}_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{\lambda}{2}\int_\Omega u^2\,dx - \frac{1}{2^*}\int_\Omega (u^+)^{2^*}\,dx.$$

In the sequel of the work we will simply refer to I_λ, as all the arguments analogously work for $\tilde{I}_\lambda$.

For any $2 \leqslant p \leqslant q < p^*$, $\alpha \geqslant 0$ we consider the following C^2 functionals

$$T_\alpha : W_0^{1,p}(\Omega) \to \mathbb{R}, \quad T_\alpha(u) = \frac{1}{p}\int_\Omega \left(\alpha + |\nabla u|^2\right)^{p/2} - \frac{\lambda}{q}\int_\Omega (u^+)^q\,dx - \frac{1}{p^*}\int_\Omega (u^+)^{p^*} \tag{4.1}$$

which approximate I_λ in $C^1(A)$ for $\alpha \to 0$, if A is a bounded subset of $W_0^{1,p}(\Omega)$. For $p = q = 2$ we replace T_α with

$$\tilde{T}_\alpha : W_0^{1,p}(\Omega) \to \mathbb{R}, \quad \tilde{T}_\alpha(u) = \frac{1}{2}\int_\Omega \left(\alpha + |\nabla u|^2\right) - \frac{\lambda}{2}\int_\Omega u^2\,dx - \frac{1}{2^*}\int_\Omega (u^+)^{2^*}. \tag{4.2}$$

Moreover we consider a functional $J_\alpha : W_0^{1,p}(\Omega) \to \mathbb{R}$ of the type

$$J_\alpha(u) = T_\alpha(u) - \int_\Omega f u$$

with $f \in C^1(\overline{\Omega})$. For $p = q = 2$ we set $J_\alpha = \tilde{T}_\alpha - \int_\Omega f u$.

We begin to establish that, for any $\alpha \geqslant 0$, J_α satisfy a local Palais–Smale condition on each level. It can be proved reasoning as in Lemma 3.2 of [18]. For reader's convenience, we sketch the proof.

Lemma 4.1. *Assume $p \geqslant 2$. There exists $R > 0$ such that, for any fixed $\alpha \geqslant 0$, $f \in C^1(\overline{\Omega})$ and any $u \in W_0^{1,p}(\Omega)$, the functional J_α satisfies $(P.S.)$ condition on $\overline{B_R(u)} = \{v \in W_0^{1,p}(\Omega)\colon \|v - u\| \leqslant R\}$.*

Proof. For convenience we fix $\alpha \geqslant 0$ and denote $J_\alpha = J$. Fixing $R \in (0, \frac{S^{N/p^2}}{2})$, if $(u_m) \subset \overline{B_R(u)}$ is a sequence such that $J'(u_m) \to 0$, then (u_m) is bounded, thus converges to some $\bar{u} \in \overline{B_R(u)}$, weakly in $W_0^{1,p}(\Omega)$ and strongly in each $L^r(\Omega)$, with $r < p^*$. Moreover, arguing as in Lemma 3.1 in [27], one can prove that $(\alpha + |\nabla u_m|^2)^{\frac{p-2}{2}} \nabla u_m$ converges to $(\alpha + |\nabla \bar{u}|^2)^{\frac{p-2}{2}} \nabla \bar{u}$ weakly in $L^{p/(p-1)}(\Omega)$ and a.e. in Ω.

Therefore, for any $z \in W_0^{1,p}(\Omega)$,

$$\langle J'(\bar{u}), z\rangle = \lim_{m\to+\infty} \langle J'(u_m), z\rangle = 0$$

so that $\bar{u}$ is a critical point and, in particular,

$$\langle J'(u_m), u_m\rangle - \langle J'(\bar{u}), \bar{u}\rangle = \mathrm{o}(1). \tag{4.3}$$

Using [8] (cf. [35]), we have that

$$\left|(u_m - \bar{u})^+\right|_{p^*}^{p^*} = \left|(u_m)^+\right|_{p^*}^{p^*} - \left|(\bar{u})^+\right|_{p^*}^{p^*} + \mathrm{o}(1). \tag{4.4}$$

Moreover arguing as in Lemma 3.2 in [18] we can infer that

$$\int_\Omega \left(\alpha + |\nabla u_m - \nabla \bar{u}|^2\right)^{\frac{p-2}{2}} |\nabla u_m - \nabla \bar{u}|^2\, dx = \int_\Omega \left(\alpha + |\nabla u_m|^2\right)^{\frac{p-2}{2}} |\nabla u_m|^2\, dx - \int_\Omega \left(\alpha + |\nabla \bar{u}|^2\right)^{\frac{p-2}{2}} |\nabla \bar{u}|^2\, dx + \mathrm{o}(1). \tag{4.5}$$

From (4.3), (4.4) and (4.5) we deduce

$$\int_\Omega |\nabla u_m - \nabla \bar{u}|^p\, dx - \int_\Omega |u_m - \bar{u}|^{p^*}\, dx \leqslant \int_\Omega \left(\alpha + |\nabla u_m - \nabla \bar{u}|^2\right)^{\frac{p-2}{2}} |\nabla u_m - \nabla \bar{u}|^2\, dx - \int_\Omega \left((u_m - \bar{u})^+\right)^{p^*} dx = \langle J'(u_m), u_m\rangle - \langle J'(\bar{u}), \bar{u}\rangle + \mathrm{o}(1) = \mathrm{o}(1). \tag{4.6}$$

Denoting $a = \limsup_{m\to+\infty} \|u_m - \bar{u}\|^p$, by (4.6) and the definition of S we have

$$a \leqslant \limsup_{m\to+\infty} \int_\Omega |u_m - \bar{u}|^{p^*} \leqslant S^{-p^*/p} a^{p^*/p}.$$

Therefore, if $a > 0$, this implies $a \geqslant S^{N/p}$, hence

$$S^{N/p} \leqslant a \leqslant \limsup_{m\to+\infty} \left(\|u_m - u\| + \|u - \bar{u}\|\right)^p \leqslant (2R)^p < S^{N/p}$$

which is absurd. Therefore it must be $a = 0$ and thus u_m strongly converges to $\bar{u}$ in $W_0^{1,p}(\Omega)$. □

Now we state two results concerning critical group computations via Morse index. For the proofs, we refer the reader to Theorems 1.3 and 1.4 of [18] (see also [16]).

Theorem 4.2. *Let $p > 2$ and $\alpha > 0$. Let $u \in W_0^{1,p}(\Omega)$ be a nondegenerate* (*in the sense of Definition* 1.5) *critical point of J_α. Then the Morse index $m(J_\alpha, u)$ is finite and*

$$C_j(J_\alpha, u) \cong \mathbb{K} \quad \text{if } j = m(J_\alpha, u), \tag{4.7}$$

$$C_j(J_\alpha, u) = \{0\} \quad \text{if } j \neq m(J_\alpha, u). \tag{4.8}$$

Theorem 4.3. *Let $p > 2$ and $\alpha > 0$. Let $u \in W_0^{1,p}(\Omega)$ be an isolated critical point of J_α. Then $m(J_\alpha, u)$ and $m^*(J_\alpha, u)$ are finite and*

$$C_j(J_\alpha, u) = \{0\} \quad \text{for any } j \leqslant m(J_\alpha, u) - 1 \text{ and } j \geqslant m^*(J_\alpha, u) + 1.$$

Moreover, $\dim C_j(J_\alpha, u) < \infty$ *for any* $j \in \mathbb{N}$.

Remark 4.4. For $p = 2$ Theorems 4.2 and 4.3 hold for the functional J_α, as consequence of classical results in Morse theory, based on Morse Lemma. We refer to Theorem 4.1 and Corollary 5.1 in [11].

5. Interpretation of the multiplicity

This section is devoted to prove the following result, which furnishes an interpretation of the *multiplicity* for each solution of (P_λ).

Theorem 5.1. *Assume that $\partial\Omega$ satisfies the interior sphere condition and that $N \geqslant p^2$, $2 \leqslant p \leqslant q < p^*$, $p^* = pN/(N-p)$. Let us fix $\lambda > 0$. If u_0 is an isolated positive solution to (P_λ) having multiplicity $m = P_1(I_\lambda, u_0) > 1$ and $\mathcal{N}$ is an open set such that u_0 is the only solution to (P_λ) in $\mathcal{N}$, then for any sequence (α_n), with $\alpha_n > 0$ and $\alpha_n \to 0$, there exists a sequence $(f_n) \subset C^1(\overline{\Omega})$ such that $\|f_n\|_{C^1} \to 0$ and problem*

$$(P_n)\quad \begin{cases} -\mathrm{div}\big(\big(|\nabla u|^2 + \alpha_n\big)^{(p-2)/2}\nabla u\big) = \lambda u^{q-1} + u^{p^*-1} + f_n & \text{in } \Omega,\\ u > 0 & \text{in } \Omega,\\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least m distinct solutions in $\mathcal{N}$, for n large enough.

Remark 5.2. Taking into account Remark 4.4, it follows that if $p = 2$, Theorem 5.1 still holds, if $\alpha_n \geqslant 0$ for any $n \in \mathbb{N}$.

Let $R > 0$ be defined by Lemma 4.1. Let us consider $A = \mathcal{N} \cap B_R(u_0)$ and U an open neighborhood of u_0 such that $\overline{U} \subset A$. Let $\overline{\mu} > 0$ be defined by Theorem 2.4 in correspondence of I_λ, u_0, A and U. Let (α_n) be a sequence such that $\alpha_n > 0$ and $\alpha_n \to 0$, so that $\|I_\lambda - T_{\alpha_n}\|_{C^1(A)} < \overline{\mu}$ if n is sufficiently large. We can suppose that T_{α_n} has a finite number of critical points in U, otherwise we simply choose $f_n = 0$.

Let us denote by $u_1, \ldots, u_j$ the critical points of T_{α_n} in U. For simplicity, we omit the dependence of $u_1, \ldots, u_j$ (and their related objects) from n. Let m_i be the multiplicity of each u_i. By Theorem 2.4 $j \geqslant 1$ and

$$\sum_{i=1}^{j} m_i \geqslant m. \tag{5.1}$$

If $j \geqslant m$, then we can choose again $f_n = 0$, while, if $j < m$, there is at least one $i \in \{1, \ldots, j\}$ such that $m_i \geqslant 2$ and we really need to introduce f_n.

In order to do that we give the following result.

Theorem 5.3. *There are V and W subspaces of $W_0^{1,p}(\Omega)$, $r > 0$, $\varrho \in (0, r)$ such that*

(1) $W_0^{1,p}(\Omega) = V \oplus W$;
(2) $V \subset C^1(\overline{\Omega})$ *is finite dimensional*;
(3) *V and W are orthogonal in $L^2(\Omega)$*;
(4) *for any $M > 0$ there exist $r_0 > 0$ and $C > 0$ such that if $z \in W_0^{1,p}(\Omega) \cap L^\infty(\Omega)$, $\|z\|_\infty \leqslant M$ and $\|z - u_i\| < r_0$ for some $i \in \{1, \ldots, j\}$, then*

$$\big\langle T''_{\alpha_n}(z)w, w\big\rangle \geqslant C\|w\|_{1,2}^2 \quad \forall w \in W;$$

(5) *for any $i \in \{1, \ldots, j\}$ and $v \in V \cap B_\varrho(0)$ there exists one and only one $\bar{w}_i = \bar{w}_i(v) \in W \cap B_r(0)$ such that*

$$\big\langle T'_{\alpha_n}(u_i + v + \bar{w}_i), w\big\rangle = 0 \quad \forall w \in W. \tag{5.2}$$

Moreover, denoting by $U_i = u_i + (V \cap B_\varrho(0)) + (W \cap B_r(0))$, for any $i \in \{1, \ldots, j\}$ $U_i \subset U$ and $U_{i_1} \cap U_{i_2} = \emptyset$ if $i_1 \neq i_2$.

Proof. In the case $p = 2$, the proof follows by standard arguments. We focus on the case $p > 2$. Arguing as in [18], for any $i = 1, \ldots, j$ we introduce a Hilbert space, depending on the critical point u_i, in which $W_0^{1,p}(\Omega)$ is embedded, so that a suitable splitting can be obtained.

Precisely, set $b_i(x) = \nabla u_i(x)$, let H_i be the closure of $C_0^\infty(\Omega)$ under the scalar product

$$(v,w)_i = \int_\Omega \left[\left(\alpha_n + |b_i(x)|^2\right)^{(p-2)/2}(\nabla v|\nabla w) + (p-2)\left(\alpha_n + |b_i(x)|^2\right)^{(p-4)/2}\left(b_i(x) \mid \nabla v\right)\left(b_i(x) \mid \nabla w\right)\right]dx.$$

Since $u_i \in C^{1,\alpha}(\overline{\Omega})$, with $\alpha \in (0,1)$, it follows that $b_i(x) \in C^{0,\alpha}(\overline{\Omega})$ and the norm $\|\cdot\|_i$ induced by $(\cdot,\cdot)_i$ is equivalent to the usual norm of $W_0^{1,2}(\Omega)$. Hence H_i is isomorphic to $W_0^{1,2}(\Omega)$ and the embedding $W_0^{1,p}(\Omega) \hookrightarrow H_i$ is continuous.

Denoting by $\langle\cdot,\cdot\rangle : H_i^* \times H_i \to \mathbb{R}$ the duality pairing in H_i, $T''_\alpha(u_i)$ can be extended to the operator $L_i : H_i \to H_i^*$ defined by setting

$$\langle L_i v, w\rangle = (v,w)_i + \langle K_i v, w\rangle$$

where $\langle K_i v, w\rangle = -\lambda \int_\Omega (q-1)(u_i)^{q-2} vw - (p^*-1)\int_\Omega (u_i)^{p^*-2} vw$, for any $v, w \in H_i$. L_i is a compact perturbation of the Riesz isomorphism from H_i to H_i^*. Since L_i is a Fredholm operator in H_i, we can consider the natural splitting

$$H_i = H_i^- \oplus H_i^0 \oplus H_i^+$$

where H_i^-, H_i^0, H_i^+ are, respectively, the negative, null and positive spaces, according to the spectral decomposition of L_i in $L^2(\Omega)$.

Since $u_i \in C^1(\overline{\Omega})$, we can deduce from standard regularity theory (see Theorems 8.15, 8.24 and 8.29 in [22]) that $H_i^- \oplus H_i^0 \subset C^{1,\alpha}(\overline{\Omega})$. Consequently, we get the splitting $W_0^{1,p}(\Omega) = V_i \oplus W_i$ where $W_i = H_i^+ \cap W_0^{1,p}(\Omega)$ and $V_i = H_i^- \oplus H_i^0 \subset C^1(\overline{\Omega})$ is finite dimensional. Moreover V_i and W_i are orthogonal in $L^2(\Omega)$.

Now we set

$$V = V_1 + V_2 + \cdots + V_j \quad \text{and} \quad W = \bigcap_{i=1}^{j} W_i.$$

In [18] Lemma 4.3 gives (4), while Lemma 4.6 assures that for any $i \in \{1,\dots,j\}$ there are $r_i > 0$, $\varrho_i \in (0, r_i)$ such that for any $v \in V_i \cap B_{\varrho_i}(0)$ there exists one and only one $\bar{w}_i = \bar{w}_i(v) \in W_i \cap B_{r_i}(0)$ which verifies

$$\left\langle T'_{\alpha_n}(u_i + v + \bar{w}_i), w\right\rangle = 0 \quad \forall w \in W_i.$$

It is easy to see that this result still holds replacing V_i with V and W_i with W, so that, choosing r and ϱ suitably small, also (5) is completely proved. □

Moreover, reasoning as in [17, Lemma 2.2] we infer the following result.

Proposition 5.4. *For any $i = 1,\dots,j$, let us introduce the maps*

$$\psi_i : V \cap B_\varrho(0) \to W \cap B_r(0), \qquad \phi_i = V \cap B_\varrho(0) \to \mathbb{R}$$

where $\psi_i(v)$ is the only element $\bar{w}_i \in W \cap B_r(0)$ satisfying (5.2) *and $\phi_i(v) = T_{\alpha_n}(u_i + v + \psi_i(v))$. The map ϕ_i is C^2 and, for any $v \in V \cap B_\varrho(0)$, $z_1, z_2 \in V$*

$$\left\langle \phi_i'(v), z_1\right\rangle = \left\langle T'_{\alpha_n}\left(u_i + v + \psi_i(v)\right), z_1\right\rangle \tag{5.3}$$

$$\left\langle \phi_i''(v) z_1, z_2\right\rangle = \left\langle T''_{\alpha_n}\left(u_i + v + \psi_i(v)\right)\left(z_1 + \psi_i{}'(v) z_1\right), z_2\right\rangle. \tag{5.4}$$

Furthermore $\phi_i{}''(v)$ is an isomorphism if and only if $T''_{\alpha_n}(u_i + v + \psi_i(v))$ is injective.

Proof. For $p = 2$ the result is well-known. We assume $p > 2$. Firstly we show that for any fixed $v \in V \cap \overline{B}_\varrho(0)$, we have $\psi_i(v) \in C^1(\overline{\Omega})$ and the map $\psi_i : V \cap \overline{B}_\varrho(0) \to W$ is C^1 with respect to the norm $\|\cdot\|_{1,2}$.

Indeed, from Lemma 4.4 in [18] we know that $z_v = u_i + v + \psi_i(v) \in L^\infty(\Omega)$ and $\|z_v\|_\infty$ is bounded from above, uniformly with respect to v. By [36,37], we can infer that $z_v \in C^1(\overline{\Omega})$, and thus $\psi_i(v) \in C^1(\overline{\Omega})$, as $V \subset C^1(\overline{\Omega})$.

Moreover, in consequence of the regularity results in [23], we have that $\|\psi_i(v)\|_{C^1(\overline{\Omega})}$ is bounded from above by a suitable constant. Therefore there exists a constant $R_1 > 0$ such that $\|u_i + v + \psi_i(v)\|_{C^1} \leqslant R_1$ for any $v \in V \cap \overline{B}_\varrho(0)$. Now, fixed $R_2 > R_1$, let us consider a nonincreasing C^∞ function $\omega : [0, +\infty) \to \mathbb{R}$ such that $\omega(t) = 1$ if $t \in [0, R_1]$,

$\omega(t) = 0$ if $t \geqslant R_2$. By a suitable choose of R_2, we can build a function $\theta : \mathbb{R}^N \to \mathbb{R}$ such that $\theta(\xi) = \frac{1}{2}|\xi|^2 + \omega(|\xi|)(\frac{1}{p}(\alpha_n + |\xi|^2)^{p/2} - \frac{1}{2}|\xi|^2)$ is strictly convex in $\mathbb{R}^N$. Moreover we consider the function $G : [0, +\infty) \to \mathbb{R}$ defined by $G(t) = \omega(t)(-\frac{\lambda}{q} t^q - \frac{1}{p^*} t^{p^*})$ and we can introduce the following C^2 functional

$$F(u) = \int_\Omega \theta(\nabla u)\,dx + \int_\Omega G(u^+)\,dx \quad \forall u \in W_0^{1,2}(\Omega).$$

We underline that for any $u \in C^1(\overline{\Omega})$ with $\|u\|_{C^1} \leqslant R_1$, we have

$$F(u) = T_{\alpha_n}(u), \quad F'(u)|_{W_0^{1,p}(\Omega)} = T'_{\alpha_n}(u), \quad F''(u)|_{(W_0^{1,p}(\Omega))^2} = T''_{\alpha_n}(u). \tag{5.5}$$

Now, denoting by $H^+ = \bigcap_{i=1}^j H_i^+$, we have that $W_0^{1,2}(\Omega) = V \oplus H^+$. So applying the Implicit Function Theorem to the map $B : (V \cap \overline{B}_\varrho(0)) \times H^+ \to (H^+)^*$ given by $B(v, w) = F'(u_i + v + w)|_{H^+}$, we can infer that $\psi_i : V \cap \overline{B}_\varrho(0) \mapsto W$ is C^1 with respect to $\|\cdot\|_{1,2}$. Now by (5.5), it follows that ϕ_i is a C^1 functional. By the chain rule, we infer (5.3), as $F'(u_i + v + \psi_i(v))|_{H^+} = 0$ and $\psi_i'(v)(z) \in W$. This shows that ϕ_i is also a C^2 functional and, again by the chain rule, (5.4) derives.

In order to complete the proof, fix $v \in V \cap \overline{B}_\varrho(0)$ and suppose that $\phi_i''(v)$ is an isomorphism. By contradiction, if $T''_{\alpha_n}(u_i + v + \psi_i(v))$ is not injective, there exists $\bar{z} \in W_0^{1,p}(\Omega)$, $\bar{z} \neq 0$ such that

$$\langle T''_{\alpha_n}(u_i + v + \psi_i(v))(z + \psi_i'(v)(z)), \bar{z}\rangle = 0 \quad \forall z \in V. \tag{5.6}$$

Let us write $\bar{z}$ as $\bar{v} + \bar{w}$ with $\bar{v} \in V$ and $\bar{w} \in W$. For any fixed $w \in W$, the function $v \in V \cap \overline{B}_\varrho(0) \mapsto \langle T'_{\alpha_n}(u_i + v + \psi_i(v), w\rangle \in \mathbb{R}$ is constantly equal to zero. Hence by (5.4) and (5.6), we deduce $\langle \phi_i''(v)\bar{v}, z\rangle = 0$ for any $z \in V$, so that $\bar{v} = 0$ and $\bar{z} \in W$. By (4) of Theorem 5.3 $\bar{z} = 0$ which is a contradiction. In a similar way we can deduce that if $T''_{\alpha_n}(u_i + v + \psi_i(v))$ is injective, then $\phi_i''(v)$ is an isomorphism. □

From the previous result we get a crucial implication of the nondegeneracy notion.

Corollary 5.5. *If u is a nondegenerate critical point of T_α, then u is isolated.*

Proof of Theorem 5.1. Let $\{e_1, \ldots, e_l\}$ be an L^2-orthonormal basis of V, where $l = \dim V$. For any $v' \in V'$ we introduce $L_{v'} : W_0^{1,p}(\Omega) \to \mathbb{R}$ the functional defined by

$$L_{v'}(u) = \int_\Omega f_{v'} u\,dx, \quad \text{where } f_{v'} = \sum_{k=1}^l \langle v', e_k\rangle e_k.$$

Let μ_i be defined by Theorem 2.4 relatively to T_{α_n}, u_i, A and U_i, for any $i = 1, \ldots, j$, and μ be equal to $\min\{\mu_1, \ldots, \mu_j\}$. Let $\varepsilon > 0$ be such that $\|L_{v'}\|_{C^1(A)} < \mu/j$ if $v' \in V'$ and $\|v'\|_{V'} \leqslant \varepsilon$.

Denoting by $\varepsilon_1 = \min\{\varepsilon, 1/n\}$, by Sard's Lemma there exists $v_1' \in V'$ such that $\|v_1'\|_{V'} < \varepsilon_1$ and if $\phi_1'(v) = v_1'$ then $\phi_1''(v)$ is an isomorphism. Moreover there is $\beta_1 > 0$ such that if $v' \in V'$, $\|v'\|_{V'} \leqslant \beta_1$ and $\phi_1'(v) = v_1' + v'$ then $\phi_1''(v)$ is an isomorphism.

Analogously, for $i = 2, \ldots, j$, there exist $\beta_i > 0$, $\varepsilon_i = \min\{\varepsilon_1, \beta_1/(j-1), \ldots, \beta_{i-1}/(j-i+1)\}$ and $v_i' \in V'$ such that $\|v_i'\|_{V'} < \varepsilon_i$ and if $v' \in V'$, $\|v'\|_{V'} \leqslant \beta_i$ and $\phi_i'(v) = v_1' + \cdots + v_i' + v'$ then $\phi_i''(v)$ is an isomorphism.

So it is sufficient to choose

$$f_n = \sum_{i=1}^j \sum_{k=1}^l \langle v_i', e_k\rangle e_k = \sum_{i=1}^j f_{v_i'}.$$

In fact, solutions to

$$(P_n) \quad \begin{cases} -\mathrm{div}\big((|\nabla u|^2 + \alpha_n)^{(p-2)/2}\nabla u\big) = \lambda u^{q-1} + u^{p^*-1} + f_n & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

are critical points of the functional

$$T_n = u \in W_0^{1,p}(\Omega) \mapsto T_{\alpha_n}(u) - \int_\Omega f_n u\,dx \in \mathbb{R}.$$

Denoting by $K_i = \{u \in U_i\colon\ T_n'(u) = 0\}$, we will see that any $u \in K_i$ is nondegenerate, hence, by Theorem 4.2 (see also Remark 4.4) $P_1(u, T_n) = 1$. Moreover as $\|T_n - T_{\alpha_n}\|_{C^1(A)} < \mu$, Theorem 2.4 and (5.1) assure that

$$\sum_{i=1}^{j} \sum_{u \in K_i} P_1(u, T_n) \geqslant \sum_{i=1}^{j} m_i \geqslant m$$

so that (P_n) has at least m distinct solutions.

Let us prove that the critical points of T_n in $U_1 \cup \cdots \cup U_j$ are nondegenerate.

Firstly observe that

$$\int_\Omega f_n w\,dx = 0 \quad \forall w \in W \quad \text{and} \quad \int_\Omega f_n v\,dx = \sum_{i=1}^{j} \langle v_i', v\rangle \quad \forall v \in V. \tag{5.7}$$

If $\bar u \in K_i$ there exists $(\bar v, \bar w) \in V \times W$ such that $\bar u = u_i + \bar v + \bar w$.

By (5.7), for any $w \in W$

$$\big\langle T_{\alpha_n}'(u_i + \bar v + \bar w), w\big\rangle = \big\langle T_n'(\bar u), w\big\rangle = 0$$

so that $\bar w = \psi_i(\bar v)$.

By construction $\phi_i'(\bar v) = v_1' + \cdots + v_i' + v'$, where $v' = v_{i+1}' + \cdots + v_j'$ and $\|v'\|_{V'} < \beta_i$, so that $\phi_i''(\bar v)$ is an isomorphism and, by Proposition 5.4, $\bar u$ is nondegenerate.

Therefore for n large enough, there exist at least $u_n^1, \ldots, u_n^m$ solutions of the equation in (P_n). We remain to prove that u_n^i are positive $i = 1, \ldots, m$. Firstly we notice that for any $i = 1, \ldots, k$, u_n^i tends to u_0 in $W_0^{1,p}(\Omega)$ as $n \to +\infty$.

Moreover since $V \subset C^{1,\alpha}(\overline{\Omega})$, the regularity results in [23,25] assure that the solutions u_n^i are uniformly bounded in $C^{1,\alpha}(\overline{\Omega})$, and then, up to subsequence, u_n^i converges in $C^1(\overline{\Omega})$ to $u_0 > 0$ as $n \to +\infty$. By the Strong Maximum Principle (see Lemma 3.4 in [22] for $p = 2$ and Theorems 1 and 5 in [39] for $p > 2$), we know that $\frac{\partial u_0}{\partial \nu}(x_0) > 0$ being $x_0 \in \partial\Omega$, ν is the interior normal of x_0. This implies $u_n^i > 0$ on Ω, for n sufficiently large. □

Proof of Theorem 1.6. Let $\lambda^* > 0$ be defined by Theorem 1.4 and $\lambda \in (0, \lambda^*)$. By Theorem 1.4, problem (P_λ) has at least $\mathcal{P}_1(\Omega)$ solutions, possibly counted with their multiplicities. If (P_λ) has less than $\mathcal{P}_1(\Omega)$ distinct positive solutions, this means, in particular, that I_λ has a finite number of critical points $u_1, \ldots, u_k$, having multiplicities $\tilde m_i \geqslant 1$, where $1 \leqslant k < \mathcal{P}_1(\Omega)$ and

$$\sum_{i=1}^{k} \tilde m_i \geqslant \mathcal{P}_1(\Omega). \tag{5.8}$$

If $k = 1$, then $\tilde m_1 \geqslant \mathcal{P}_1(\Omega)$ and the thesis follows from Theorem 5.1.

Let us consider the case $1 < k$. Let $R > 0$ be defined by Lemma 4.1. For any $i = 1, \ldots, k$, u_i is isolated, so let $\gamma_i \in (0, R)$ be such that I_λ has not critical points other than u_i in $\overline{B_{\gamma_i}(u_i)}$. Denoting by $U_i = B_{\gamma_i}(u_i)$, let $\bar\mu_i$ be defined by Theorem 2.4 relatively to U_i and $A_i = B_R(u_i)$. Moreover we call $A = \bigcup_{i=1}^k A_i$ and $\bar\mu = \min\{\bar\mu_1, \ldots, \bar\mu_k\}$.

Let (α_n) be a sequence such that $\alpha_n > 0$, $\alpha_n \to 0$, so that $\|I_\lambda - T_{\alpha_n}\|_{C^1(A)} < \bar\mu$, if n is sufficiently large.

We can suppose that T_{α_n} has a finite number of critical points in $U = \bigcup_{i=1}^k U_i$, otherwise we simply choose $f_n = 0$.

Let us denote by $u_1, \ldots, u_j$ the critical points of T_{α_n} in U, and by $m_1, \ldots, m_j$ their multiplicities. By Theorem 2.4 and by (5.8) we have

$$\sum_{i=1}^{j} m_i \geqslant \sum_{i=1}^{k} \tilde m_i \geqslant \mathcal{P}_1(\Omega). \tag{5.9}$$

We immediately see that, if $j \geqslant \mathcal{P}_1(\Omega)$, then we can choose again $f_n = 0$, while, if $j < \mathcal{P}_1(\Omega)$, there is at least one $i \in \{1, \dots, j\}$ such that $m_i \geqslant 2$ and we really need to introduce f_n.

At this point, reasoning as in the proof of Theorem 5.1, we build a function $f_n \in C^1(\overline{\Omega})$ such that $\|f_n\|_{C^1} \to 0$ and

$$(P_n) \qquad \begin{cases} -\mathrm{div}\big(\big(|\nabla u|^2 + \alpha_n\big)^{(p-2)/2}\nabla u\big) = \lambda u^{q-1} + u^{p^*-1} + f_n & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

has at least $\mathcal{P}_1(\Omega)$ distinct positive solutions. □

Proof of Theorem 1.7. Taking into account Remarks 4.4 and 5.2, one can derive Theorem 1.7, arguing as in the proof of Theorem 1.6. □

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