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Boundary controllability for the nonlinear Korteweg–de Vries equation on any critical domain ☆

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Abstract

It is known that the linear Korteweg–de Vries (KdV) equation with homogeneous Dirichlet boundary conditions and Neumann boundary control is not controllable for some critical spatial domains. In this paper, we prove in these critical cases, that the nonlinear KdV equation is locally controllable around the origin provided that the time of control is large enough. It is done by performing a power series expansion of the solution and studying the cascade system resulting of this expansion.

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1. Introduction and main result

Let $L > 0$ be fixed. Let us consider the following Neumann boundary control system for the Korteweg–de Vries (KdV) equation with homogeneous Dirichlet boundary conditions

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t,0) = y(t,L) = 0, \\ y_x(t,L) = \kappa(t), \end{cases} \tag{1}$$

where the state is $y(t,\cdot):[0,L] \to \mathbb{R}$ and the control is $\kappa(t) \in \mathbb{R}$. This equation has been introduced by Korteweg and de Vries in [16] to describe the propagation of small amplitude long waves in a uniform channel. The KdV equation also appears in the study of various physical phenomena like long internal waves in a density-stratified ocean, ionic-acoustic waves in a plasma, etc.

In this paper, we are concerned with the controllability of (1). More precisely, for a time $T > 0$, we want to prove the following local exact controllability property.

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$\mathcal{P}(T)$ *There exists $r > 0$ such that, for every $(y_0, y_T) \in L^2(0,L)^2$ with $\|y_0\|_{L^2(0,L)} < r$ and $\|y_T\|_{L^2(0,L)} < r$, there exist $\kappa \in L^2(0,T)$ and*

$$y \in C\big([0,T], L^2(0,L)\big) \cap L^2\big(0,T,H^1(0,L)\big)$$

satisfying (1), $y(0,\cdot) = y_0$ *and* $y(T,\cdot) = y_T$.

In order to deal with the nonlinear term in (1), one can perform a power series expansion of (y,κ) around 0. To find the different terms of this development, one can write, formally, for a parameter ϵ small,

$$y = \epsilon\alpha + \epsilon^2\beta + \epsilon^3\gamma + \cdots,$$
$$\kappa = \epsilon u + \epsilon^2 v + \epsilon^3 w + \cdots$$

thus, the nonlinear term can be written as

$$yy_x = \epsilon^2\alpha\alpha_x + \epsilon^3(\alpha\beta)_x + (\text{higher terms})$$

and therefore the three main orders are given by

$$\begin{cases} \alpha_t + \alpha_x + \alpha_{xxx} = 0, \\ \alpha(t,0) = \alpha(t,L) = 0, \\ \alpha_x(t,L) = u(t), \end{cases} \tag{2}$$

$$\begin{cases} \beta_t + \beta_x + \beta_{xxx} = -\alpha\alpha_x, \\ \beta(t,0) = \beta(t,L) = 0, \\ \beta_x(t,L) = v(t), \end{cases} \tag{3}$$

and

$$\begin{cases} \gamma_t + \gamma_x + \gamma_{xxx} = -(\alpha\beta)_x, \\ \gamma(t,0) = \gamma(t,L) = 0, \\ \gamma_x(t,L) = w(t). \end{cases} \tag{4}$$

In [18] Rosier studies the control system (1) by using a first order expansion, i.e. he considers the linear control system (2) where the state is $\alpha(t,\cdot) : [0,L] \to \mathbb{R}$ and the control is $u(t) \in \mathbb{R}$. First, by using multiplier technique and the HUM method (see [17]), he proves that (2) is exactly controllable if and only if

$$L \notin N := \left\{ 2\pi\sqrt{\frac{k^2 + kl + l^2}{3}};\ k,l \in \mathbb{N}^* \right\}, \tag{5}$$

and then, by means of a fixed point theorem, he gets the following result.

Theorem 1.1. *(See [18, Theorem 1.3].) If $L \notin N$, then property $\mathcal{P}(T)$ holds for every $T > 0$.*

Remark 1.2. If one is allowed to use more than one boundary control input, there is no critical spatial domain and the exact controllability holds for any $L > 0$. More precisely, let us consider the nonlinear control system

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t,0) = u_1(t), \quad y(t,L) = u_2(t), \quad y_x(t,L) = u_3(t), \end{cases} \tag{6}$$

where the controls are $u_1(t)$, $u_2(t)$ and $u_3(t)$. As it has been pointed out by Rosier in [18], for every $L > 0$ the system (6) with $u_1 \equiv 0$ is locally exactly controllable in $L^2(0,L)$ around the origin. Moreover, using all the three control inputs, Zhang proves in [22] that for every $L > 0$, the system (6) is exactly controllable in the space $H^s(0,L)$ for any $s \geqslant 0$, in a neighborhood of a given smooth solution of the KdV equation.

If $L \in N$, one says that L is a critical length since the linear control system (2) is no more controllable. Indeed, Rosier proves in [18] that there exists a finite-dimensional subspace of $L^2(0,L)$, denoted by $M = M(L)$, which is unreachable from 0 for the linear system. More precisely, for every nonzero state $\psi \in M$, for every $u \in L^2(0,T)$ and for every $\alpha \in C([0,T], L^2(0,L)) \cap L^2(0,T,H^1(0,L))$ satisfying (2) and $\alpha(0,\cdot) = 0$, one has $\alpha(T,\cdot) \neq \psi$.

Let us recall the description of M given in [4]. Let $L \in N$. There exist n distinct pairs $(k_j, l_j) \in \mathbb{N}^* \times \mathbb{N}^*$ with $k_j \geqslant l_j$ such that

$$\forall j \in \{1, \dots, n\}, \quad L = 2\pi \sqrt{\frac{k_j^2 + k_j l_j + l_j^2}{3}}. \tag{7}$$

$$\left(L = 2\pi \sqrt{\frac{k^2 + kl + l^2}{3}},\ k \geqslant l,\ k, l \in \mathbb{N}^* \right) \quad \Longrightarrow \quad (\exists j \in \{1, \dots, n\} \text{ s.t. } k = k_j \text{ and } l = l_j). \tag{8}$$

Let us introduce the notation

$$J^> := \{ j \in \{1, \dots, n\};\ k_j > l_j \}, \qquad J^= := \{ j \in \{1, \dots, n\};\ k_j = l_j \}, \quad n^> := |J^>|. \tag{9}$$

For every $j \in \{1, \dots, n\}$, we define the real number

$$p_j := (2k_j + l_j)(k_j - l_j)(2l_j + k_j)\left(\frac{2\pi}{3L}\right)^3. \tag{10}$$

We have then (see [18]),

$$\xi^3 - \xi + p_j = (\xi - \gamma_1^j)(\xi - \gamma_2^j)(\xi - \gamma_3^j)$$

with

$$\begin{cases} \gamma_1^j = -\dfrac{1}{3}(2k_j + l_j)\dfrac{2\pi}{L}, \\ \gamma_2^j = \gamma_1^j + k_j \dfrac{2\pi}{L}, \\ \gamma_3^j = \gamma_2^j + l_j \dfrac{2\pi}{L}. \end{cases} \tag{11}$$

Lemma 1.3. *With the previous notations, we get*

1. *if* $j \in J^>$, $p_j \neq 0$,
2. *if* $j \in J^=$, $p_j = 0$,
3. *if* $i \neq j$, $p_i \neq p_j$.

Proof. Items 1. and 2. are obvious with (10). Let $i, j \in J$ such that $p_i = p_j$. Then, $\gamma_k^i = \gamma_k^j$ for $k = 1, 2, 3$. With the definitions of γ_k^j, (11) we obtain $k_i = k_j$, $l_i = l_j$ and hence $i = j$. □

Remark 1.4. We can easily notice that $|J^=| \leqslant 1$.

Thus we can reorganize the indexes such that

$$p_1 > p_2 > \cdots > p_n \geqslant 0.$$

With this notation, we define,

- for $j \in J^>$, the subspace of $L^2(0, L)$

  $$M_j := \{ \lambda_1 \varphi_1^j + \lambda_2 \varphi_2^j;\ \lambda_1, \lambda_2 \in \mathbb{R} \} = \langle \varphi_1^j, \varphi_2^j \rangle,$$

  where the real-valued functions φ_1^j, φ_2^j are given by

  $$\begin{aligned} \varphi_1^j(x) &:= C_j \left(\cos(\gamma_1^j x) - \frac{\gamma_1^j - \gamma_3^j}{\gamma_2^j - \gamma_3^j} \cos(\gamma_2^j x) + \frac{\gamma_1^j - \gamma_2^j}{\gamma_2^j - \gamma_3^j} \cos(\gamma_3^j x) \right), \\ \varphi_2^j(x) &:= C_j \left(\sin(\gamma_1^j x) - \frac{\gamma_1^j - \gamma_3^j}{\gamma_2^j - \gamma_3^j} \sin(\gamma_2^j x) + \frac{\gamma_1^j - \gamma_2^j}{\gamma_2^j - \gamma_3^j} \sin(\gamma_3^j x) \right), \end{aligned} \tag{12}$$

  where C_j is a constant chosen so that $\|\varphi_1^j\|_{L^2(0,L)} = \|\varphi_2^j\|_{L^2(0,L)} = 1$;

- for $j \in J^{=}$, the subspace of $L^2(0, L)$

$$M_j := \{\lambda(1 - \cos x);\ \lambda \in \mathbb{R}\} = \langle 1 - \cos(x) \rangle.$$

Then, one can define the following subspaces of $L^2(0, L)$

$$M := \bigoplus_{j=1}^{n} M_j \quad \text{and} \quad H := M^{\perp}.$$

Note that

$$\bigcup_{j=1}^{n^>} \{\varphi_1^j, \varphi_2^j\} \quad (\text{if } L \neq 2\pi k \text{ for any } k) \quad \text{or} \quad \{1 - \cos(x)\} \bigcup_{j=1}^{n^>} \{\varphi_1^j, \varphi_2^j\} \quad (\text{if } L = 2\pi k \text{ for some } k)$$

is an orthogonal basis from M.

The subspace H is the space of reachable states for the linear control system. More precisely, from the work of Rosier one has the exact controllability in H for the control system (2).

Theorem 1.5. *Let $L > 0$ and $T > 0$. For every $(y_0, y_T) \in H \times H$, there exist $u \in L^2(0, T)$ and $\alpha \in C([0, T], L^2(0, L)) \cap L^2(0, T, H^1(0, L))$ satisfying* (2), *$\alpha(0, \cdot) = y_0$ and $\alpha(T, \cdot) = y_T$.*

In [11], Coron and Crépeau study the first critical case:

$$n = 1 \quad \text{and} \quad k_1 = l_1 = k.$$

In this case the subspace M is one-dimensional. First, they prove that one can reach all the missed directions lying in M, i.e. $(1 - \cos(x))$ and $(\cos(x) - 1)$, with a third order power series expansion.

Proposition 1.6. *(See [11, Proposition 8].) Let $L \in N$ be such that $\dim M(L) = 1$. Let $T > 0$. There exist $(u^{\pm}, v^{\pm}, w^{\pm}) \in L^2(0, T)^3$ such that if $\alpha^{\pm}, \beta^{\pm}, \gamma^{\pm}$ are the solutions of* (2), (3) *and* (4) *with initial conditions*

$$\alpha^{\pm}(0, \cdot) = 0, \qquad \beta^{\pm}(0, \cdot) = 0, \qquad \gamma^{\pm}(0, \cdot) = 0,$$

then

$$\alpha^{\pm}(T, \cdot) = 0, \qquad \beta^{\pm}(T, \cdot) = 0, \qquad \gamma^{\pm}(T, \cdot) = \pm(1 - \cos(x)).$$

Then, using Theorem 1.5 and a fixed point theorem, they prove that property $\mathcal{P}(T)$ holds for every $T > 0$ [11, Theorem 2]. They also prove that for this first critical case, a second order expansion is not sufficient to enter into the subspace M [11, Corollary 19].

Remark 1.7. The proof of $\mathcal{P}(T)$ given in [11] requires that the subspace M is one-dimensional, but this is not implied by the fact that $L = 2k\pi$ for some $k \in \mathbb{N}^*$. It is necessary to add a condition as the following one

$$\left(m^2 + mn + n^2 = 3k^2, m \in \mathbb{N}^*,\ n \in \mathbb{N}^*\right) \quad \Rightarrow \quad (m = n = k). \tag{13}$$

This condition, not explicitly given in [11], appears in [10]. In this book it is also proved that there are infinitely many positive integers k satisfying (13) and therefore there are infinitely many lengths L such that M is one-dimensional.

In [4], the same approach is used to treat the second critical case:

$$n = 1 \quad \text{and} \quad k_1 > l_1.$$

In this case, the space M is two-dimensional and a second order expansion allows to enter into the subspace M.

Proposition 1.8. *(See [4, Proposition 3.1].) Let $L \in N$ be such that* $\dim M(L) = 2$. *Let $T > 0$. There exist $u, v \in L^2(0,T)$ such that if α, β are the solutions of* (2) *and* (3) *with initial conditions*

$$\alpha(0,\cdot) = 0, \qquad \beta(0,\cdot) = 0,$$

then

$$\alpha(T,\cdot) = 0, \qquad \beta(T,\cdot) \in M \setminus \{0\}.$$

It is also proved that if the time of control is large enough, one can reach all the missed directions. Using this and a fixed point argument, one obtains property $\mathcal{P}(T)$ provided that the time of control T is large enough [4, Theorem 1.4].

The aim of this paper is to prove $\mathcal{P}(T)$ in the critical cases for which $n > 1$, i.e. when the dimension of the subspace M is higher than 2. We use an expansion to the second order if $L \neq 2\pi k$ for any $k \in \mathbb{N}^*$ and an expansion to the third order if $L = 2\pi k$ for some $k \in \mathbb{N}^*$. Our main result is the following.

Theorem 1.9. *Let $L \in N$. Then, there exists $T_L \geqslant 0$ such that $\mathcal{P}(T)$ holds provided that $T > T_L$.*

The paper is organized as follows. First, in Section 2, we recall the well-posedness results for both linear and nonlinear KdV control systems. Next, in Section 3, we prove by using a second order power series expansion, that one can reach all the missed states in the subspaces M_j for $j \in J^>$. Then, in Section 4, we prove that if $L = 2\pi k$, one can reach the missed states $\pm(1 - \cos(x))$ with a third order expansion and finally, in Section 5 we get Theorem 1.9 by using a fixed point argument.

Remark 1.10. From our proof of Theorem 1.9, it follows that there exists a constant $C > 0$ such that for every $y_0, y_T \in L^2(0,L)$ small enough, the solution y and the control κ given by property $\mathcal{P}(T)$ satisfy

$$\|y\|_{C([0,T],L^2(0,L))} + \|y\|_{L^2(0,T,H^1(0,L))} + \|\kappa\|_{L^2(0,T)} \leqslant C\big(\|y_0\|_{L^2(0,L)} + \|y_T\|_{L^2(0,L)}\big)^{1/3}$$

if $L = 2k\pi$ for some $k \in \mathbb{N}^*$ and

$$\|y\|_{C([0,T],L^2(0,L))} + \|y\|_{L^2(0,T,H^1(0,L))} + \|\kappa\|_{L^2(0,T)} \leqslant C\big(\|y_0\|_{L^2(0,L)} + \|y_T\|_{L^2(0,L)}\big)^{1/2}$$

if $L \neq 2k\pi$ for any $k \in \mathbb{N}^*$. The power $1/3$ and $1/2$ come from the order of the series expansion needed in each case.

Remark 1.11. One can find other results on the controllability of KdV control systems in [14,19–22] and the references therein.

Remark 1.12. The power series expansion method is a classical tool to study finite-dimensional control systems. It has been used for the first time in infinite dimension in [11]; see also [4] as well as [2] for a Schrödinger equation. This method and others such as quasi-static deformations (see [1,12,13] and [10, Chapter 7]) and the return method (see [1,6–8] and [10, Chapter 6]) are very useful to deal with nonlinear systems and to get properties which are not a consequence of the linearized system behavior.

2. Well-posedness results

The aim of this section is to precise what we mean by "a solution" of the KdV equations appearing in this paper and to recall the existence and uniqueness results we will use.

Let us introduce the space $\mathcal{B} := C([0,T], L^2(0,L)) \cap L^2(0,T,H^1(0,L))$ endowed with the norm

$$\|y\|_{\mathcal{B}} := \max_{t\in[0,T]} \big\|y(t)\big\|_{L^2(0,L)} + \left(\int_0^T \big\|y(t)\big\|^2_{H^1(0,L)}\,dt\right)^{1/2}.$$

Let us begin with the linear case.

Definition 2.1. Let $T>0$, $f\in L^1(0,T,L^2(0,L))$, $y_0\in L^2(0,L)$ and $\kappa\in L^2(0,T)$ be given. A solution of the Cauchy problem

$$\begin{cases} y_t+y_x+y_{xxx}=f,\\ y(t,0)=y(t,L)=0,\\ y_x(t,L)=\kappa(t),\\ y(0,\cdot)=y_0, \end{cases} \tag{14}$$

is a function $y\in\mathcal{B}$ such that, for every $\tau\in[0,T]$ and for every $\phi\in C^3([0,\tau]\times[0,L])$ satisfying

$$\phi(t,0)=\phi(t,L)=\phi_x(t,0)=0,\quad \forall t\in[0,\tau],$$

one has

$$\begin{aligned} &-\int_0^\tau\!\!\int_0^L(\phi_t+\phi_x+\phi_{xxx})y\,dx\,dt-\int_0^\tau\kappa(t)\phi_x(t,L)\,dt+\int_0^L y(\tau,x)\phi(\tau,x)\,dx-\int_0^L y_0(x)\phi(0,x)\,dx\\ &\quad=\int_0^\tau\!\!\int_0^L f\phi\,dx\,dt. \end{aligned}$$

With this definition and from the work of Rosier in [18], we have the following result.

Theorem 2.2. *Let $T>0$, $f\in L^1(0,T,L^2(0,L))$, $y_0\in L^2(0,L)$ and $\kappa\in L^2(0,T)$. Then, there exists one and only one solution of the Cauchy problem* (14)*.*

Let us now give the definition of a solution for the nonlinear equation.

Definition 2.3. Let $T>0$, $g\in L^1(0,T,L^2(0,L))$, $y_0\in L^2(0,L)$ and $\kappa\in L^2(0,T)$ be given. A solution of the Cauchy problem

$$\begin{cases} y_t+y_x+y_{xxx}+yy_x=g,\\ y(t,0)=y(t,L)=0,\\ y_x(t,L)=\kappa(t),\\ y(0,\cdot)=y_0, \end{cases} \tag{15}$$

is a function $y\in\mathcal{B}$ satisfying (14) with $f=g-yy_x$.

Remark 2.4. Note that if $y\in\mathcal{B}$, then $yy_x\in L^1(0,T,L^2(0,L))$ and therefore $(g-yy_x)$ as well.

Theorem 2.5. *(See [11, Appendix].) Let $T>0$. Then there exists $\epsilon>0$ such that, for every $g\in L^1(0,T,L^2(0,L))$, $y_0\in L^2(0,L)$ and $\kappa\in L^2(0,T)$ satisfying*

$$\|g\|_{L^1(0,T,L^2(0,L))}+\|y_0\|_{L^2(0,L)}+\|\kappa\|_{L^2(0,T)}\leqslant\epsilon, \tag{16}$$

the Cauchy problem (15) *has one and only one solution. Furthermore, there exists a constant $C>0$ such that this solution satisfies*

$$\|y\|_{\mathcal{B}}\leqslant C\big(\|g\|_{L^1(0,T,L^2(0,L))}+\|y_0\|_{L^2(0,L)}+\|\kappa\|_{L^2(0,T)}\big). \tag{17}$$

Remark 2.6. In [3] and [15], one can find some well-posedness results in the case where there are nonhomogeneous Dirichlet boundary conditions.

Remark 2.7. Recently, in [5] the author proved Theorem 2.5 with $\epsilon=\infty$, that is, without a smallness condition on the data.

3. Motion in the missed subspaces M_j, for $j \in J^>$

Here and in the sequel, we denote by L a critical length such that $\dim M(L) > 2$ and by P_A the orthogonal projection on a subspace A in $L^2(0,L)$. We also adopt the notations introduced in Section 1.

The first point is that for any $j \in J^>$, we can *enter* into the two-dimensional subspace M_j. The strategy is the same as in [11] and [4]. We consider a power series expansion of (y,κ) with the same scaling on the state y and on the control κ. One has the following result that can be proved in the same way as in [4, Proposition 3.1].

Proposition 3.1. *Let $T > 0$. For every $i = 1,\dots,n^>$, there exists $(u_i,v_i) \in L^2(0,T)^2$ such that if $\alpha_i = \alpha_i(t,x)$ and $\beta_i = \beta_i(t,x)$ are the solutions of*

$$\begin{cases} \alpha_{it} + \alpha_{ix} + \alpha_{ixxx} = 0, \\ \alpha_i(t,0) = \alpha_i(t,L) = 0, \\ \alpha_{ix}(t,L) = u_i(t), \\ \alpha_i(0,\cdot) = 0, \end{cases} \tag{18}$$

and

$$\begin{cases} \beta_{it} + \beta_{ix} + \beta_{ixxx} = -\alpha_i\alpha_{ix}, \\ \beta_i(t,0) = \beta_i(t,L) = 0, \\ \beta_{ix}(t,L) = v_i(t), \\ \beta_i(0,\cdot) = 0, \end{cases} \tag{19}$$

then

$$\alpha_i(T,\cdot) = 0, \quad P_H\big(\beta_i(T,\cdot)\big) = 0 \quad \text{and} \quad P_{M_i}\big(\beta_i(T,\cdot)\big) \neq 0.$$

Let us denote, for $j = 1,\dots,n^>$,

$$\phi_i^j := P_{M_j}\big(\beta_i(T,\cdot)\big).$$

From Proposition 3.1, $\phi_i^i \neq 0$. Consequently, using scaling on the controls, we can assume that $\|\phi_i^i\|_{L^2(0,L)} = 1$. Notice that the previous proposition says nothing about ϕ_i^j for $j \neq i$.

Now, we shall prove that we can reach all the states lying in the subspace

$$M^> := \bigoplus_{i \in J^>} M_i,$$

in any time $T > T^>$, where

$$T^> := \pi \sum_{i=1}^{n^>} \big(n^> + 1 - i\big)\frac{1}{p_i}.$$

In order to do that, we will strongly use the fact (proved in [4]) that if there is no control (i.e. $\kappa = 0$) and if the initial condition lies in M_j for $j \in J^>$ (i.e. $y_0 \in M_j$), then the solution y of the linear KdV equation only turns in the two-dimensional subspace M_j with an angular velocity equal to p_j (defined in (10)) and conserves its L^2-norm. More precisely, we have the following result.

Lemma 3.2. *Let $j \in J^>$. Let $y_0 \in M_j$. Let $\lambda \geqslant 0$ and $\delta \in [0,2\pi)$ be such that*

$$y_0 = \lambda\cos(\delta)\varphi_1^j + \lambda\sin(\delta)\varphi_2^j. \tag{20}$$

Then the solution of

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t,0) = y(t,L) = y_x(t,L) = 0, \\ y(0,\cdot) = y_0 \end{cases} \tag{21}$$

is given by

$$y(t,x) = \lambda\cos(p_jt + \delta)\varphi_1^j + \lambda\sin(p_jt + \delta)\varphi_2^j. \tag{22}$$

For the sake of brevity we introduce, for $j \in J^{>}$, $\theta \in \mathbb{R}$ and $y_0 \in M_j$ reading as (20), the notation

$$R^j(y_0, \theta) := \lambda \cos(\theta + \delta)\varphi_1^j + \lambda \sin(\theta + \delta)\varphi_2^j, \tag{23}$$

i.e. $R^j(\cdot, \theta)$ represents a rotation of an angle θ in the subspace M_j. Thus, the solution of (21) can be written as

$$y(t, x) = R^j(y_0, p_j t).$$

Proposition 3.3. *Let $T > T^{>}$. Let $\psi \in M^{>}$. There exists $(u_\psi, v_\psi) \in L^2(0, T)^2$ such that if $\alpha_\psi = \alpha_\psi(t, x)$ and $\beta_\psi = \beta_\psi(t, x)$ are the solutions of* (18) *and* (19)*, then*

$$\alpha_\psi(T, \cdot) = 0, \qquad \beta_\psi(T, \cdot) = \psi.$$

Proof. First at all, let us notice that if $L = 2k\pi$ for some $k \in \mathbb{N}^*$, then $M_n = \langle 1 - \cos x \rangle$ and *a priori* $P_{M_n}(\beta_\psi(T, \cdot))$ may be non-null. However, we know from [11, Corollary 19] that a second order expansion is not sufficient to enter into the subspace M_n and therefore $P_{M_n}\beta_\psi(T.\cdot) = 0$. That is the reason why we do not care about the projection on M_n of second-order trajectories.

The case $n^{>} = 1$ has already been studied in [4]. Let us consider the case $n^{>} = 2$, i.e. where we have 2 subspaces, M_1 and M_2 associated to (k_1, l_1) and (k_2, l_2) with $p_1 > p_2 > 0$ (for instance, $L = 2\pi\sqrt{91}$ leads to the couples $(k_1, l_1) = (16, 1)$ and $(k_2, l_2) = (11, 8)$).

Let $T > \frac{2\pi}{p_1} + \frac{\pi}{p_2}$. Let T_1 be such that

$$T_1 > \frac{\pi}{p_1} \quad \text{and} \quad T - T_1 > \frac{\pi}{p_1} + \frac{\pi}{p_2}.$$

Let $T_\theta > 0$ and $T_c > 0$ be such that

$$T_c < T_\theta, \qquad T_c < \frac{\pi}{p_1},$$
$$T_c + T_\theta < \min\left(T - T_1 - \frac{\pi}{p_1} - \frac{\pi}{p_2}, \frac{\pi}{p_2} - \frac{\pi}{p_1}, T_1 - \frac{\pi}{p_1}\right).$$

Thanks to Proposition 3.1, there exist two pairs of controls, (u_1, v_1) and (u_2, v_2) in $L^2(0, T_c)$ such that the respective solutions of (18) and (19), (α_1, β_1) and (α_2, β_2), satisfy $P_{M_1}(\beta_1(T_c, \cdot)) \neq 0$ and $P_{M_2}(\beta_2(T_c, \cdot)) \neq 0$. With the notations introduced before,

$$\begin{cases} (\phi_1^1, \phi_1^2) = (P_{M_1}(\beta_1(T, \cdot)), P_{M_2}(\beta_1(T, \cdot))), \\ (\phi_2^1, \phi_2^2) = (P_{M_1}(\beta_2(T, \cdot)), P_{M_2}(\beta_2(T, \cdot))). \end{cases}$$

We now use the rotation phenomena explained before and Proposition 3.1 to reach a basis for the missed directions lying in $M^{>}$. For the seek of clarity in our control strategy, we define for a time t_1, the following control in $L^2(0, T)$

$$(U_{t_1}, V_{t_1})(t) := \begin{cases} (0, 0) & \text{if } t \in (0, t_1), \\ (u_1(t - t_1), v_1(t - t_1)) & \text{if } t \in (t_1, t_1 + T_c), \\ (0, 0) & \text{if } t \in (t_1 + T_c, T). \end{cases}$$

This control becomes active at time $t = t_1$, between $t = t_1$ and $t = t_2$, it drives the system to enter into the space M_1 and after $t = t_2$, it becomes inactive, producing a rotation in M_1.

Now, we define the controls

$$(u_1^1, v_1^1) := (U_{t_1}, V_{t_1}) \quad \text{with } t_1 = T - T_c,$$
$$(u_1^2, v_1^2) := (U_{t_1}, V_{t_1}) \quad \text{with } t_1 = T - T_c - \frac{\pi}{2p_1},$$
$$(u_1^3, v_1^3) := (U_{t_1}, V_{t_1}) \quad \text{with } t_1 = T - T_c - \frac{\pi}{p_1},$$
$$(u_1^4, v_1^4) := (U_{t_1}, V_{t_1}) \quad \text{with } t_1 = T - T_c - \frac{\pi}{p_1} - T_\theta.$$

Let $\alpha_1^j, \beta_1^j \in B$ be the solutions of (18) and (19) with controls u_1^j and v_1^j for $j = 1, \ldots, 4$ and let us denote

$$\psi_1^j := P_{M_1}\beta_1^j(T,\cdot) \quad \text{and} \quad \tilde{\psi}_2^j := P_{M_2}\beta_1^j(T,\cdot).$$

It is easy to see that

$$\begin{aligned}
&\psi_1^1 = \phi_1^1, && \tilde{\psi}_2^1 = \phi_1^2,\\
&\psi_1^2 = R^1\left(\phi_1^1, \frac{\pi}{2}\right), && \tilde{\psi}_2^2 = R^2\left(\phi_1^2, \frac{p_2\pi}{2p_1}\right),\\
&\psi_1^3 = R^1\left(\phi_1^1, \pi\right) = -\phi_1^1, && \tilde{\psi}_2^3 = R^2\left(\phi_1^2, \frac{p_2\pi}{p_1}\right),\\
&\psi_1^4 = R^1\left(-\phi_1^1, p_1 T_\theta\right), && \tilde{\psi}_2^4 = R^2\left(\phi_1^2, p_2\left(T_\theta + \frac{\pi}{p_1}\right)\right).
\end{aligned}$$

Thus, we have constructed some controls allowing to reach the missed states

$$\psi_1^1 + \tilde{\psi}_2^1, \quad \psi_1^2 + \tilde{\psi}_2^2, \quad \psi_1^3 + \tilde{\psi}_2^3, \quad \text{and} \quad \psi_1^4 + \tilde{\psi}_2^4.$$

Now, we define for a time t_2, the following control in $L^2(0,T)$

$$\left(U^{t_2}, V^{t_2}\right)(t) := \begin{cases}
(0,0) & \text{if } t \in (0, t_2),\\
(u_1(t-t_2), v_1(t-t_2)) & \text{if } t \in (t_2, t_2 + T_c),\\
(0,0) & \text{if } t \in (t_2 + T_c, t_2 + \frac{\pi}{p_1}),\\
(u_1(t - t_2 - \frac{\pi}{p_1}), v_1(t - t_2 - \frac{\pi}{p_1})) & \text{if } t \in (t_2 + \frac{\pi}{p_1}, t_2 + \frac{\pi}{p_1} + T_c),\\
(0,0) & \text{if } t \in (t_2 + \frac{\pi}{p_1} + T_c, T),
\end{cases}$$

which is the superposition of two controls of type (U_{t_1}, V_{t_1})

$$\left(U^{t_2}, V^{t_2}\right)(t) = (U_{t_2+\frac{\pi}{p_1}}, V_{t_2+\frac{\pi}{p_1}}) + (U_{t_2}, V_{t_2}).$$

This fact means that the solution corresponding to the controls (U^{t_2}, V^{t_2}) is the addition of two trajectories which enter into M and then turn during different times.

We define the following controls in $L^2(0,T)$,

$$\begin{aligned}
&\left(u_2^1, v_2^1\right) = \left(U^{t_2}, V^{t_2}\right) \quad \text{with } t_2 = T - T_1 - \frac{\pi}{p_1} - T_c,\\
&\left(u_2^2, v_2^2\right) = \left(U^{t_2}, V^{t_2}\right) \quad \text{with } t_2 = T - T_1 - \frac{\pi}{p_1} - T_c - T_\theta,\\
&\left(u_2^3, v_2^3\right) = \left(U^{t_2}, V^{t_2}\right) \quad \text{with } t_2 = T - T_1 - \frac{\pi}{p_1} - \frac{\pi}{p_2} - T_c,\\
&\left(u_2^4, v_2^4\right) = \left(U^{t_2}, V^{t_2}\right) \quad \text{with } t_2 = T - T_1 - \frac{\pi}{p_1} - \frac{\pi}{p_2} - T_c - T_\theta.
\end{aligned}$$

Let $\alpha_2^j, \beta_2^j \in B$ be the solutions of (18) and (19) with controls u_2^j and v_2^j for $j = 1, \ldots, 4$ and let us denote

$$\psi_2^j := P_{M_2}\beta_2^j(T,\cdot).$$

Here, it is very important to note that, by construction and since $p_1 > p_2$, one has

$$P_{M_1}\beta_2^1(T,\cdot) = 0 \quad \text{and} \quad \psi_2^1 = R^2\left(\phi_1^2, p_2 T_1\right) + R^2\left(\phi_1^2, p_2(T_1 + \pi/p_1)\right) \neq 0.$$

Thus, we have constructed some controls allowing to reach the following missed states

$$\psi_2^1, \quad \psi_2^2, \quad \psi_2^3, \quad \text{and} \quad \psi_2^4,$$

where

$$\begin{aligned}
&\psi_2^2 = R^2\left(\psi_2^1, p_2 T_\theta\right),\\
&\psi_2^3 = R^2\left(\psi_2^1, \pi\right) = -\psi_2^1,\\
&\psi_2^4 = R^2\left(-\psi_2^2, p_2 T_\theta\right).
\end{aligned}$$

Furthermore, we have for $k = 1, 2$

$$M_k = \bigcup_{j=1}^{4} M_k^j \tag{24}$$

where

$$\begin{aligned}
M_k^1 &:= \{d_k^1 \psi_k^1 + d_k^2 \psi_k^2;\ d_k^1 > 0, d_k^2 \geqslant 0\},\\
M_k^2 &:= \{d_k^1 \psi_k^2 + d_k^2 \psi_k^3;\ d_k^1 > 0, d_k^2 \geqslant 0\},\\
M_k^3 &:= \{d_k^1 \psi_k^3 + d_k^2 \psi_k^4;\ d_k^1 > 0, d_k^2 \geqslant 0\},\\
M_k^4 &:= \{d_k^1 \psi_k^4 + d_k^2 \psi_k^1;\ d_k^1 > 0, d_k^2 \geqslant 0\}.
\end{aligned}$$

Let $\psi \in M^>$. From (24), we know that $P_{M_1}(\psi) \in M_1^i$ for some $i \in \{1, \dots, 4\}$. Hence, there exist $d_1^1 > 0, d_1^2 \geqslant 0$, such that

$$\psi = d_1^1 \psi_1^i + d_1^2 \psi_1^{i+1} + P_{M_2}(\psi).$$

Let us write ψ as follows

$$\psi = d_1^1 \psi_1^i + d_1^2 \psi_1^{i+1} + d_1^1 \tilde{\psi}_2^i + d_1^2 \tilde{\psi}_2^{i+1} + \left(P_{M_2}(\psi) - d_1^1 \tilde{\psi}_2^i - d_1^2 \tilde{\psi}_2^{i+1}\right).$$

Since the states $\tilde{\psi}_2^i, \tilde{\psi}_2^{i+1}$ lie in M_2, there exists $j \in \{1, \dots, 4\}$ such that

$$P_{M_2}(\psi) - d_1^1 \tilde{\psi}_2^i - d_1^2 \tilde{\psi}_2^{i+1} \in M_2^j$$

and therefore there exist $d_2^1 > 0, d_2^2 \geqslant 0$ such that

$$\psi = d_1^1\left(\psi_1^i + \tilde{\psi}_2^i\right) + d_1^2\left(\psi_1^{i+1} + \tilde{\psi}_2^{i+1}\right) + d_2^1 \psi_2^j + d_2^2 \psi_2^{j+1}.$$

Thus, we have decomposed ψ in terms of reachable directions for the second-order expansion. Now, we take the controls u_ψ, v_ψ defined by

$$(u_\psi, v_\psi) = \left(\sqrt{d_1^1} u_1^i + \sqrt{d_1^2} u_1^{i+1} + \sqrt{d_2^1} u_2^j + \sqrt{d_2^2} u_2^{j+1}, d_1^1 v_1^i + d_1^2 v_1^{i+1} + d_2^1 v_2^j + d_2^2 v_2^{j+1}\right),$$

and $\alpha_\psi, \beta_\psi \in \mathcal{B}$ the corresponding solutions of (18) and (19) respectively. Here, it is important to note that, with the choices of T, T_1, T_c and T_θ, the supports of the trajectories α_k^j for $k = 1, 2$ and $j = 1, \dots, 4$ are disjoint and that all these trajectories go from 0 at $t = 0$ to 0 at $t = T$, i.e.

$$\alpha_k^j(0, \cdot) = \alpha_k^j(T, \cdot) = 0.$$

Thus, it is not difficult to verify that

$$\alpha_\psi(T, \cdot) = 0 \quad \text{and} \quad \beta_\psi(T, \cdot) = \psi$$

which ends the proof in the case $n^> = 2$. The previous method can be easily adapted to the case where $n^> > 2$. In order to construct the controls needed in the general case, our method requires a time of control T greater than $T^>$. □

4. Motion in the missed directions $\pm(1 - \cos x)$ if $L = 2k\pi$

We assume in this section that $L = 2k\pi$ for some $k \in \mathbb{N}^*$. Let us recall that in this case we have

$$M_n = \langle 1 - \cos x \rangle \quad \text{and} \quad n^> = n - 1. \tag{25}$$

Thanks to [11], we have the following result that one can prove in a similar way to [11, Proposition 8].

Proposition 4.1. *Let $T_c > 0$. There exists (u, v, w) in $L^2(0, T_c)^3$ such that, if α, β, γ are the mild solutions of*

$$\begin{cases} \alpha_t + \alpha_x + \alpha_{xxx} = 0, \\ \alpha(t, 0) = \alpha(t, L) = 0, \\ \alpha_x(t, L) = u(t), \\ \alpha(0, \cdot) = 0, \end{cases} \tag{26}$$

$$\begin{cases} \beta_t + \beta_x + \beta_{xxx} = -\alpha\alpha_x, \\ \beta(t, 0) = \beta(t, L) = 0, \\ \beta_x(t, L) = v(t), \\ \beta(0, \cdot) = 0, \end{cases} \tag{27}$$

$$\begin{cases} \gamma_t + \gamma_x + \gamma_{xxx} = -(\alpha\beta)_x, \\ \gamma(t, 0) = \gamma(t, L) = 0, \\ \gamma_x(t, L) = w(t), \\ \gamma(0, \cdot) = 0, \end{cases} \tag{28}$$

then

$$\alpha(T_c, \cdot) = 0, \quad \beta(T_c, \cdot) = 0 \quad \textit{and} \quad \gamma(T_c, \cdot) = (1 - \cos x) + \sum_{i=1}^{n^>} P_{M_i}\big(\gamma(T_c, \cdot)\big).$$

The idea to vanish the projections of $\gamma(T_c, \cdot)$ on M_i, and thus to reach the direction $(1 - \cos(x))$, is the same as before, that is, to use the rotation phenomena given in Lemma 3.2. In addition, we use the fact that the function $(1 - \cos x)$ satisfies

$$\begin{cases} y_x + y_{xxx} = 0, \\ y(0) = y(2k\pi) = y_x(2k\pi) = 0. \end{cases}$$

The case $n = 1$ has already been considered in [11]. We deal with the case $n = 2$ (for example, $L = 14\pi$ leads to the couples $(k_1, l_1) = (11, 2)$ and $(k_2, l_2) = (7, 7)$).

Let us define the following control lying in $L^2(0, T)^3$, where $T > \pi/p_1$. (Here, we omit the time translation needed for the controls u, v and w which are defined in $(0, T_c)$.)

$$(u_+, v_+, w_+)(t) = \begin{cases} (0, 0, 0) & \text{if } t \in (0, T - T_c - \frac{\pi}{p_1}), \\ (u, v, w) & \text{if } t \in (T - T_c - \frac{\pi}{p_1}, T - \frac{\pi}{p_1}), \\ (0, 0, 0) & \text{if } t \in (T - \frac{\pi}{p_1}, T - T_c), \\ (u, v, w) & \text{if } t \in (T - T_c, T). \end{cases}$$

By defining $\alpha_+, \beta_+, \gamma_+ \in \mathcal{B}$ as the solutions of (26) with control u_+, (27) with control v_+ and (28) with control w_+ respectively, it is not difficult to see that

$$\alpha_+(T, \cdot) = 0, \qquad \beta_+(T, \cdot) = 0, \qquad \gamma_+(T, \cdot) = 2(1 - \cos x). \tag{29}$$

Now, if we consider the control $(u_-, v_-, w_-) := (-u_+, v_+, -w_+)$ we get

$$\alpha_-(T, \cdot) = 0, \qquad \beta_-(T, \cdot) = 0, \qquad \gamma_-(T, \cdot) = -2(1 - \cos x), \tag{30}$$

where obviously $\alpha_-, \beta_-, \gamma_- \in \mathcal{B}$ are the solutions of (26), (27) and (28) with controls u_-, v_- and w_- respectively. Thus we can reach all directions in M_2 in a time $T > \frac{\pi}{p_1}$.

We can easily deduce the same result in the case $n > 2$. We just have to construct a control that vanishes the components in the other missed subspaces M_j, $j \in J^>$. In order to do that, a time of control T, with

$$T > T^n := \pi \sum_{i=1}^{n-1} \frac{1}{p_i}, \tag{31}$$

is sufficient.

5. Fixed point argument

If $L \neq 2k\pi$, then we can use the same proof as in [4] and get property $\mathcal{P}(T)$ for every $T > T^>$. Thus the only interesting case we detail here is when $L = 2k\pi$ and $\dim M(L) > 2$.

5.1. Preliminaries

Recall that for $L \in N$, we have n pairs (k_j, l_j) such that (7) and (8) hold. We have introduced some important notations

$$J^> := \{j; k_j > l_j\}, \qquad n^> := |J^>|, \qquad M^> := \bigoplus_{j=1}^{n^>} M_j.$$

In this section, we consider the case where $n^> = (n-1)$ and consequently where $M_n = \langle 1 - \cos x\rangle$. Thus we can write any $z \in L^2(0, L)$ as

$$z = P_H(z) + \rho_z \psi_z + d_z(1 - \cos x), \tag{32}$$

where

$$\rho_z := \|P_{M^>}(z)\|_{L^2(0,L)}, \quad \rho_z\psi_z := P_{M^>}(z), \quad \text{and} \quad d_z(1-\cos x) = P_{M_n}(z).$$

Let us also denote, for $D > 0$ and $R > 0$,

$$B_R^D := \{\xi \in L^2(0, D);\ \|\xi\|_{L^2(0,D)} \leqslant R\}.$$

From the work of Rosier in [18], we know that for every $y_0 \in L^2(0, L)$ there exists a continuous linear affine map

$$\Gamma_0 : h \in H \subset L^2(0, L) \longmapsto \Gamma_0(h) \in L^2(0, T),$$

such that the solution of

$$\begin{cases} y_t + y_x + y_{xxx} = 0, \\ y(t, 0) = y(t, L) = 0, \\ y_x(t, L) = \Gamma_0(h), \\ y(0, \cdot) = P_H(y_0), \end{cases}$$

satisfies $y(T, \cdot) = h$. Moreover, there exist two constants $D_1, D_2 > 0$ such that

$$\forall y_0 \in L^2(0, L),\ \forall h \in H, \quad \|\Gamma_0(h)\|_{L^2(0,T)} \leqslant D_1\big(\|h\|_{L^2(0,L)} + \|y_0\|_{L^2(0,L)}\big), \tag{33}$$

$$\forall y_0 \in L^2(0, L),\ \forall h, g \in H, \quad \|\Gamma_0(h) - \Gamma_0(g)\|_{L^2(0,T)} \leqslant D_2\|h - g\|_{L^2(0,L)}. \tag{34}$$

From Sections 3 and 4, we have the existence of the controls $u_\pm, v_\pm, w_\pm \in L^2(0, T^n)$ and for every $\psi \in M^>$, the controls $u_\psi, v_\psi \in L^2(0, T^>)$. As we shall see later, we need that the corresponding trajectories of first order $\alpha_\pm$ and α_ψ are disjoint and therefore for every $z \in L^2(0, L)$ written as (32), and for every T satisfying

$$T > T_L := T^n + T^>,$$

we define the following controls lying in $L^2(0, T)$

$$(\tilde{u}, \tilde{v}, \tilde{w})(t) := \begin{cases} (0, 0, 0) & \text{if } t \in (0, T - T_L), \\ (u_{\mathrm{sign}(d_z)}, v_{\mathrm{sign}(d_z)}, w_{\mathrm{sign}(d_z)})|_{(t-T+T_L)} & \text{if } t \in (T - T_L, T - T^>), \\ (0, 0, 0) & \text{if } t \in (T - T^>, T) \end{cases}$$

and

$$(\hat{u}, \hat{v})(t) := \begin{cases} (0, 0) & \text{if } t \in (0, T - T^>), \\ (u_{\psi_z}, v_{\psi_z})|_{(t-T+T^>)} & \text{if } t \in (T - T^>, T), \end{cases}$$

where we use the notation

$$\mathrm{sign}(d_z) = \begin{cases} + & \text{if } d_z \geqslant 0, \\ - & \text{if } d_z < 0. \end{cases} \tag{35}$$

Let $y_0 \in L^2(0,L)$ be such that $\|y_0\|_{L^2(0,L)} < r$, where $r > 0$ has to be chosen later. Using (32), we define the functions G and F by

$$
\begin{aligned}
&G : L^2(0,L) \longrightarrow L^2(0,T),\\
&\qquad\qquad z \longmapsto G(z) := \Gamma_0\big(P_H(z)\big) + \rho_z^{1/2}\hat{u} + \rho_z\hat{v} + |d_z|^{1/3}\tilde{u} + |d_z|^{2/3}\tilde{v} + |d_z|\tilde{w},
\end{aligned}
$$

$$
\begin{aligned}
&F : B^T_{\epsilon_1} \cap L^2(0,T) \longrightarrow L^2(0,L),\\
&\qquad\qquad \kappa \longmapsto F(\kappa) := y(T,\cdot),
\end{aligned}
$$

where $y = y(t,x)$ is the solution of

$$
\begin{cases}
y_t + y_x + y_{xxx} + yy_x = 0,\\
y(t,0) = y(t,L) = 0,\\
y_x(t,L) = \kappa(t),\\
y(0,\cdot) = y_0,
\end{cases}
\tag{36}
$$

and ϵ_1 is small enough so that the function F is well defined.

Let $y_T \in L^2(0,L)$ be such that $\|y_T\| < r$. Let Λ_{y_0,y_T} denotes the map

$$
\begin{aligned}
&\Lambda_{y_0,y_T} : B^L_{\epsilon_2} \cap L^2(0,L) \longrightarrow L^2(0,L),\\
&\qquad\qquad z \longmapsto \Lambda_{y_0,y_T}(z) := z + y_T - F \circ G(z),
\end{aligned}
$$

where ϵ_2 is small enough so that Λ_{y_0,y_T} is well defined.

Let us remark that if we find a fixed point $\tilde{z} \in L^2(0,L)$ of the map Λ_{y_0,y_T}, then we will have

$$F \circ G(\tilde{z}) = y_T$$

which means that the control

$$\kappa := G(\tilde{z}) \in L^2(0,T)$$

drives the solution of (36) from y_0 at $t = 0$ to y_T at $t = T$. In the following sections, we prove that such a fixed point does exist.

5.2. A technical lemma

Let us assert the following technical result which will be needed to study the map Λ_{y_0,y_T}.

Lemma 5.1. *There exist $\epsilon_3 > 0$ and $C_1 > 0$ such that, for every $z, y_0 \in B^L_{\epsilon_3}$, the following estimate holds*

$$\big\|z - F \circ G(z)\big\|_{L^2(0,L)} \leqslant C_1\big(\|y_0\|_{L^2(0,L)} + \|z\|^{4/3}_{L^2(0,L)}\big).$$

Proof. Let $z, y_0 \in L^2(0,L)$. Let $y = y(t,x)$ be the solution of

$$
\begin{cases}
y_t + y_x + y_{xxx} + yy_x = 0,\\
y(t,0) = y(t,L) = 0,\\
y_x(t,L) = G(z),\\
y(0,\cdot) = y_0.
\end{cases}
\tag{37}
$$

From (33) and the fact that $\rho_z \leqslant \|z\|_{L^2(0,L)}$, one deduces that if $\|z\|_{L^2(0,L)}$ is smaller than 1 (and therefore $\|z\|_{L^2(0,L)} \leqslant \|z\|^{1/2}_{L^2(0,L)}$), then there exists a constant C_2 such that

$$\big\|G(z)\big\|_{L^2(0,T)} \leqslant C_2\big(\|y_0\|_{L^2(0,L)} + \|z\|^{1/3}_{L^2(0,L)}\big). \tag{38}$$

Thus, one can find $\epsilon_4, C_3 > 0$ such that for every $z, y_0 \in B^L_{\epsilon_4}$, the unique solution of (37) satisfies

$$\|y\|_{\mathcal{B}} \leqslant C_3\big(\|y_0\|_{L^2(0,L)} + \|z\|^{1/3}_{L^2(0,L)}\big). \tag{39}$$

Let $\tilde{y}$, $\hat{\alpha}$, $\hat{\beta}$, $\tilde{\alpha}$, $\tilde{\beta}$, $\tilde{\gamma}$ and $\hat{y}$ be the solutions of

$$\begin{cases} \tilde{y}_t + \tilde{y}_x + \tilde{y}_{xxx} = 0, \\ \tilde{y}(t,0) = \tilde{y}(t,L) = 0, \\ \tilde{y}_x(t,L) = \Gamma_0\big(P_H(z)\big), \\ \tilde{y}(0,\cdot) = P_H(y_0), \end{cases} \tag{40}$$

$$\begin{cases} \hat{\alpha}_t + \hat{\alpha}_x + \hat{\alpha}_{xxx} = 0, \\ \hat{\alpha}(t,0) = \hat{\alpha}(t,L) = 0, \\ \hat{\alpha}_x(t,L) = \hat{u}(t), \\ \hat{\alpha}(0,\cdot) = 0, \end{cases} \tag{41}$$

$$\begin{cases} \hat{\beta}_t + \hat{\beta}_x + \hat{\beta}_{xxx} = -\hat{\alpha}\hat{\alpha}_x, \\ \hat{\beta}(t,0) = \hat{\beta}(t,L) = 0, \\ \hat{\beta}_x(t,L) = \hat{v}(t), \\ \hat{\beta}(0,\cdot) = 0, \end{cases} \tag{42}$$

$$\begin{cases} \tilde{\alpha}_t + \tilde{\alpha}_x + \tilde{\alpha}_{xxx} = 0, \\ \tilde{\alpha}(t,0) = \tilde{\alpha}(t,L) = 0, \\ \tilde{\alpha}_x(t,L) = \tilde{u}(t), \\ \tilde{\alpha}(0,\cdot) = 0, \end{cases} \tag{43}$$

$$\begin{cases} \tilde{\beta}_t + \tilde{\beta}_x + \tilde{\beta}_{xxx} = -\tilde{\alpha}\tilde{\alpha}_x, \\ \tilde{\beta}(t,0) = \tilde{\beta}(t,L) = 0, \\ \tilde{\beta}_x(t,L) = \tilde{v}(t), \\ \tilde{\beta}(0,\cdot) = 0, \end{cases} \tag{44}$$

$$\begin{cases} \tilde{\gamma}_t + \tilde{\gamma}_x + \tilde{\gamma}_{xxx} = -(\tilde{\alpha}\tilde{\beta})_x, \\ \tilde{\gamma}(t,0) = \tilde{\gamma}(t,L) = 0, \\ \tilde{\gamma}_x(t,L) = \tilde{w}(t), \\ \tilde{\gamma}(0,\cdot) = 0, \end{cases} \tag{45}$$

$$\begin{cases} \hat{y}_t + \hat{y}_x + \hat{y}_{xxx} = 0, \\ \hat{y}(t,0) = \hat{y}(t,L) = 0, \\ \hat{y}_x(t,L) = 0, \\ \hat{y}(0,\cdot) = P_M(y_0). \end{cases} \tag{46}$$

Let us define

$$\phi := y - \tilde{y} - \rho_z^{1/2}\hat{\alpha} - \rho_z\hat{\beta} - |d_z|^{1/3}\tilde{\alpha} - |d_z|^{2/3}\tilde{\beta} - |d_z|\tilde{\gamma} - \hat{y}.$$

Then $\phi = \phi(t,x)$ satisfies

$$\begin{cases} \phi_t + \phi_x + \phi_{xxx} + \phi\phi_x = -(\phi a)_x - b, \\ \phi(t,0) = \phi(t,L) = 0, \\ \phi_x(t,L) = 0, \\ \phi(0,\cdot) = 0, \end{cases} \tag{47}$$

where $a := y - \phi$,

$$\begin{aligned} b :=\ & \tilde{y}\tilde{y}_x + \hat{y}\hat{y}_x + \rho_z^2\hat{\beta}\hat{\beta}_x + \rho_z^{3/2}(\hat{\alpha}\hat{\beta})_x + |d_z|^{4/3}\tilde{\beta}\tilde{\beta}_x + |d_z|^{5/3}(\tilde{\beta}\tilde{\gamma})_x + |d_z|^{4/3}(\tilde{\alpha}\tilde{\gamma})_x \\ & + |d_z|^2\tilde{\gamma}\tilde{\gamma}_x + \big(\tilde{y}\big(\rho_z^{1/2}\hat{\alpha} + \rho_z\hat{\beta} + |d_z|^{1/3}\tilde{\alpha} + |d_z|^{2/3}\tilde{\beta} + |d_z|\tilde{\gamma} + \hat{y}\big)\big)_x \\ & + \big(\big(\rho_z^{1/2}\hat{\alpha} + \rho_z\hat{\beta}\big)\big(|d_z|^{1/3}\tilde{\alpha} + |d_z|^{2/3}\tilde{\beta} + |d_z|\tilde{\gamma} + \hat{y}\big)\big)_x + \big(\hat{y}\big(|d_z|^{1/3}\tilde{\alpha} + |d_z|^{2/3}\tilde{\beta} + |d_z|\tilde{\gamma}\big)\big)_x. \end{aligned}$$

Here, in order to use Eq. (47) we need some estimates on its right-hand side.

Lemma 5.2. *There exists $C_4 > 0$ such that for every $z, y_0 \in B^L_{\epsilon_4}$,*

$$\|\phi\|_{\mathcal{B}} \leqslant C_4\big(\|y_0\|_{L^2(0,L)} + \|z\|^{1/3}_{L^2(0,L)}\big), \tag{48}$$

$$\|a\|_{\mathcal{B}} \leqslant C_4\big(\|y_0\|_{L^2(0,L)} + \|z\|^{1/3}_{L^2(0,L)}\big), \tag{49}$$

$$\|b\|_{L^1(0,T,L^2(0,L))} \leqslant C_4\big(\|y_0\|_{L^2(0,L)} + \|z\|^{4/3}_{L^2(0,L)}\big). \tag{50}$$

Proof of Lemma 5.2. Let us prove (48). One has

$$\begin{aligned}
\|\phi\|_{\mathcal{B}} &\leqslant \|y\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} + \rho_z^{1/2}\|\hat{\alpha}\|_{\mathcal{B}} + \rho_z\|\hat{\beta}\|_{\mathcal{B}} + |d_z|^{1/3}\|\tilde{\alpha}\|_{\mathcal{B}} + |d_z|^{2/3}\|\tilde{\beta}\|_{\mathcal{B}} + |d_z|\|\tilde{\gamma}\|_{\mathcal{B}} + \|\hat{y}\|_{B} \\
&\leqslant C\big(\big\|G(z)\big\|_{L^2(0,T)} + \|y_0\|_{L^2(0,L)}\big) + C\big(\big\|\Gamma_0\big(P_H(z)\big)\big\|_{L^2(0,T)} + \|y_0\|_{L^2(0,L)}\big) \\
&\quad + C\rho_z^{1/2}\|\hat{u}\|_{L^2(0,T)} + C\rho_z\big(\|\hat{v}\|_{L^2(0,T)} + \|\hat{\alpha}\hat{\alpha}_x\|_{L^1(0,T,L^2(0,L))}\big) + C|d_z|^{1/3}\|\tilde{u}\|_{L^2(0,T)} \\
&\quad + C|d_z|^{2/3}\big(\|\tilde{v}\|_{L^2(0,T)} + \|\tilde{\alpha}\tilde{\alpha}_x\|_{L^1(0,T,L^2(0,L))}\big) \\
&\quad + C|d_z|\big(\|\tilde{w}\|_{L^2(0,T)} + \big\|(\tilde{\alpha}\tilde{\beta})_x\big\|_{L^1(0,T,L^2(0,L))}\big) + C\big\|P_M(y_0)\big\|_{L^2(0,L)}.
\end{aligned}$$

One needs at this point the following trivial estimate

$$\exists C_5 > 0,\ \forall f, g \in \mathcal{B}, \quad \big\|(fg)_x\big\|_{L^1(0,T,L^2(0,L))} \leqslant C_5\|f\|_{\mathcal{B}}\|g\|_{\mathcal{B}}. \tag{51}$$

By noticing that if $z = P_H(z) + \rho_z\psi_z + d_z(1-\cos(x))$, then

$$\|z\|^2_{L^2(0,L)} = \big\|P_H(z)\big\|^2_{L^2(0,L)} + \rho_z^2 + d_z^2\big\|1-\cos(x)\big\|^2_{L^2(0,L)},$$

and using (38) and (51), one gets (48). Estimate (49) follows from (48) and the definition of the function a. To prove (50), one uses (51) being very careful with the powers which appear. For instance, looking at the function b, one finds the term $(\rho_z^{1/2}\hat{\alpha}|d_z|^{1/3}\tilde{\alpha})$ which apparently is not bounded by $C_4\|z\|^{4/3}_{L^2(0,L)}$ for $z \in B^L_1$. This is the reason why one takes the trajectories $\tilde{\alpha}$ and $\hat{\alpha}$ disjoint. □

Thus, from (47) one obtains the existence of $C_6 > 0$ such that

$$\|\phi\|^2_{\mathcal{B}} \leqslant C_6\big(\|\phi\|^2_{\mathcal{B}}\|a\|^2_{\mathcal{B}} + \|y_0\|^2_{L^2(0,L)} + \|z\|^{8/3}_{L^2(0,L)}\big),$$

i.e. one has

$$\|\phi\|^2_{\mathcal{B}}\big(1 - C_6\|a\|^2_{\mathcal{B}}\big) \leqslant C_6\big(\|y_0\|^2_{L^2(0,L)} + \|z\|^{8/3}_{L^2(0,L)}\big),$$

which, together with (49), implies the existence of ϵ_5 and C_7 such that for every $z, y_0 \in B^L_{\epsilon_5}$

$$\|\phi\|_{\mathcal{B}} \leqslant C_7\big(\|y_0\|_{L^2(0,L)} + \|z\|^{4/3}_{L^2(0,L)}\big). \tag{52}$$

Finally, from (52) one obtains

$$\begin{aligned}
\big\|z - F\circ G(z)\big\|_{L^2(0,L)} &\leqslant \big\|z - F\circ G(z) + \hat{y}(T,\cdot)\big\|_{L^2(0,L)} + \big\|-\hat{y}(T,\cdot)\big\|_{L^2(0,L)} \\
&= \big\|\phi(T,\cdot)\big\|_{L^2(0,L)} + \big\|\hat{y}(0,\cdot)\big\|_{L^2(0,L)} \\
&\leqslant \|\phi\|_{\mathcal{B}} + \|y_0\|_{L^2(0,L)} \\
&\leqslant C_7\big(\|y_0\|_{L^2(0,L)} + \|z\|^{4/3}_{L^2(0,L)}\big) + \|y_0\|_{L^2(0,L)} \\
&\leqslant (C_7+1)\big(\|y_0\|_{L^2(0,L)} + \|z\|^{4/3}_{L^2(0,L)}\big),
\end{aligned}$$

which ends the proof of Lemma 5.1 with $C_1 := C_7 + 1$ and $\epsilon_3 := \epsilon_5$. □

5.3. Fixed point in H

For $w = (w_1^1, w_1^2, \dots, w_{n-1}^1, w_{n-1}^2, w_n) \in \mathbb{R}^{2n-1}$ fixed, let us denote

$$\Psi_w := w_n(1-\cos x) + \sum_{j=1}^{n-1}\big(w_j^1\varphi_j^1 + w_j^2\varphi_j^2\big), \tag{53}$$

where the functions φ_j^i for $i=1,2,\ j=1,\dots,n-1$ are given in (12). Let us study the map

$$\Pi := P_H \circ \Lambda_{y_0,y_T}(\cdot + \Psi_w)$$

on the subspace H

$$\begin{aligned} \Pi : H &\longrightarrow H,\\ h &\longmapsto \Pi(h) = h + P_H(y_T) - P_H\big(F\circ G(h+\Psi_w)\big). \end{aligned}$$

In order to prove the existence of a fixed point of the map Π, we will apply the Banach fixed point theorem to the restriction of Π to the closed ball $B_R^L \cap H$ with $R>0$ small enough. Using Lemma 5.1 we see that

$$\begin{aligned} \big\|\Pi(h)\big\|_{L^2(0,L)} &\leqslant \|y_T\|_{L^2(0,L)} + \big\|h+\Psi_w - F\circ G(h+\Psi_w)\big\|_{L^2(0,L)}\\ &\leqslant \|y_T\|_{L^2(0,L)} + C_1\big(\|y_0\|_{L^2(0,L)} + \|h+\Psi_w\|_{L^2(0,L)}^{4/3}\big)\\ &\leqslant (C_1+1)\big(\|y_0\|_{L^2(0,L)} + \|y_T\|_{L^2(0,L)} + |w|^{4/3}\big) + C_1\|h\|_{L^2(0,L)}^{4/3}\\ &\leqslant (C_1+1)\big(2r+|w|^{4/3}\big) + C_1\|h\|_{L^2(0,L)}^{4/3}. \end{aligned}$$

Hence, if we choose R, r and w such that

$$R^{4/3} \leqslant \frac{R}{2C_1} \quad \text{and} \quad \big(2r+|w|^{4/3}\big) \leqslant \frac{R}{2(C_1+1)},$$

then it follows that

$$\big\|\Pi(h)\big\|_{L^2(0,L)} \leqslant R \quad \text{and so} \quad \Pi\big(B_R^L\cap H\big) \subset \big(B_R^L \cap H\big).$$

It remains to prove that the map Π is a contraction. Let $g,h\in B_R^L\cap H$. Let $y=y(t,x)$, $q=q(t,x)$, $\tilde y = \tilde y(t,x)$ and $\tilde q = \tilde q(t,x)$ be the solutions of the following problems

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0,\\ y(t,0) = y(t,L) = 0,\\ y_x(t,L) = G(g+\Psi_w),\\ y(0,\cdot) = y_0, \end{cases}$$

$$\begin{cases} q_t + q_x + q_{xxx} + qq_x = 0,\\ q(t,0) = q(t,L) = 0,\\ q_x(t,L) = G(h+\Psi_w),\\ q(0,\cdot) = y_0, \end{cases}$$

$$\begin{cases} \tilde y_t + \tilde y_x + \tilde y_{xxx} = 0,\\ \tilde y(t,0) = \tilde y(t,L) = 0,\\ \tilde y_x(t,L) = \Gamma_0(g),\\ \tilde y(0,\cdot) = P_H(y_0), \end{cases}$$

$$\begin{cases} \tilde q_t + \tilde q_x + \tilde q_{xxx} = 0,\\ \tilde q(t,0) = \tilde q(t,L) = 0,\\ \tilde q_x(t,L) = \Gamma_0(h),\\ \tilde q(0,\cdot) = P_H(y_0). \end{cases}$$

Let us define $\phi := y - \tilde{y}$, $\psi := q - \tilde{q}$ and $\gamma := \phi - \psi$. One sees that γ satisfies

$$\begin{cases} \gamma_t + \gamma_x + \gamma_{xxx} + \gamma\gamma_x = -(\gamma a)_x - b, \\ \gamma(t,0) = \gamma(t,L) = 0, \\ \gamma_x(t,L) = 0, \\ \gamma(0,\cdot) = 0, \end{cases} \tag{54}$$

where

$$a := \tilde{y} + \psi \quad \text{and} \quad b := \big(q(\tilde{y} - \tilde{q})\big)_x + (\tilde{y} - \tilde{q})(\tilde{y} - \tilde{q})_x.$$

It is easy to see that there exists a constant C_8 such that

$$\|b\|_{L^1(0,T,L^2(0,L))} \leqslant C_8\big(\|q\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}}\big)\|\tilde{y} - \tilde{q}\|_{\mathcal{B}}, \tag{55}$$

$$\big\|(a\gamma)_x\big\|_{L^1(0,T,L^2(0,L))} \leqslant C_8\big(\|q\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}}\big)\|\gamma\|_{\mathcal{B}}. \tag{56}$$

Thus, we get the existence of $C_9 > 0$ such that

$$\|\gamma\|_{\mathcal{B}}^2 \leqslant C_9\big(\|q\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}}\big)^2\big(\|\tilde{y} - \tilde{q}\|_{\mathcal{B}}^2 + \|\gamma\|_{\mathcal{B}}^2\big). \tag{57}$$

In addition, since $w := \tilde{y} - \tilde{q}$ satisfies the following linear equation

$$\begin{cases} w_t + w_x + w_{xxx} = 0, \\ w(t,0) = w(t,L) = 0, \\ w_x(t,L) = \Gamma_0(g) - \Gamma_0(h), \\ w(0,\cdot) = 0, \end{cases}$$

there exists $C_{10} > 0$ such that

$$\|\tilde{y} - \tilde{q}\|_{\mathcal{B}} \leqslant C_{10}\big\|\Gamma_0(g) - \Gamma_0(h)\big\|_{L^2(0,T)}$$

and so, from (34), one gets

$$\|\tilde{y} - \tilde{q}\|_{\mathcal{B}} \leqslant C_{10} D_2 \|g - h\|_{L^2(0,L)}. \tag{58}$$

Moreover, it is easy to see that there exists a constant $C_{11} > 0$ such that

$$\|q\|_{\mathcal{B}} + \|\tilde{q}\|_{\mathcal{B}} + \|\tilde{y}\|_{\mathcal{B}} \leqslant C_{11}\big(\|y_0\|_{L^2(0,L)} + \|h\|_{L^2(0,L)} + \|g\|_{L^2(0,L)} + |w|^{1/3}\big). \tag{59}$$

Thus, using (57), (58) and (59) we see that if $R, |w|, r$ are small enough, it follows that

$$\|\gamma\|_{\mathcal{B}} \leqslant \frac{1}{2}\|g - h\|_{L^2(0,L)}.$$

Therefore, we have

$$\begin{aligned} \big\|\Pi(g) - \Pi(h)\big\|_{L^2(0,L)} &\leqslant \big\|g - F \circ G(g + \Psi_w) - h + F \circ G(h + \Psi_w)\big\|_{L^2(0,L)} \\ &= \big\|\gamma(T)\big\|_{L^2(0,L)} \leqslant \|\gamma\|_{\mathcal{B}} \\ &\leqslant \frac{1}{2}\|g - h\|_{L^2(0,L)}, \end{aligned}$$

which implies the existence of a unique fixed point $h(y_0, y_T, w) \in B_R^L \cap H$ of the map $\Pi|_{B_R^L \cap H}$.

5.4. Fixed point in M

We now apply the Brouwer fixed point theorem to the restriction of the map

$$\begin{aligned} &\tau : M \longrightarrow M, \\ &\quad \Psi_w \longmapsto P_M\big(\Psi_w + y_T - F \circ G\big(\Psi_w + h(y_0, y_T, w)\big)\big), \end{aligned}$$

to the closed ball $B_{\hat{R}}^L \cap M$ with $\hat{R}$ small enough.

In Section 5.1, the controls $\hat{u}$, $\hat{v}$, $\tilde{u}$, $\tilde{v}$ and $\tilde{w}$ were chosen in such a way so that the function G is continuous. Thus, it is easy to see that the map $(y_0, y_T, w) \mapsto h(y_0, y_T, w)$ is also continuous in a neighborhood of $0 \in L^2(0, L)^2 \times \mathbb{R}^{2n-1}$. Using this continuity, Lemma 5.1, and choosing r small enough, we get the existence of a radius $\hat{R} > 0$ such that $\tau(B^L_{\hat{R}} \cap M) \subset B^L_{\hat{R}} \cap M$. This inclusion and the continuity of the map τ allow us to apply the Brouwer fixed point theorem. Therefore, there exists $\tilde{w} \in \mathbb{R}^{2n-1}$ with $|\tilde{w}| \leqslant \hat{R}$ such that $\tilde{h} := h(y_0, y_T, \tilde{w})$ satisfies

$$P_M\big(y_T - F \circ G(\tilde{h} + \Psi_{\tilde{w}})\big) = 0. \tag{60}$$

Using the fact that

$$\Pi(\tilde{h}) = P_H\big(\tilde{h} + y_T - F \circ G(\tilde{h} + \Psi_{\tilde{w}})\big) = \tilde{h},$$

we obtain

$$P_H\big(y_T - F \circ G(\tilde{h} + \Psi_{\tilde{w}})\big) = 0,$$

which together with (60), implies that

$$y_T = F \circ G(\tilde{h} + \Psi_{\tilde{w}}),$$

which ends the proof of Theorem 1.9.

6. Conclusion

In this article, we have proved that in the last remaining critical cases, i.e. when $\dim M > 2$, the nonlinear KdV equation is controllable in a time large enough. First, we have performed a power series expansion of the solution and of the control. Next, we have constructed special controls allowing to reach a basis of missed directions and thus all the missed states. Then if $\dim M$ is even, the fixed point theorems used in [4] are directly applicable. If $\dim M$ is odd, we prove the controllability using fixed point mixing proofs of [11] and [4].

The following open problem arises naturally from the results of this work.

Open Problem 1. Let $L \in N$ such that the dimension of the subspace M is higher than 1. Does $\mathcal{P}(T)$ holds for every $T > 0$?

This is an interesting question since even if the speed of propagation of the KdV equation is infinite, it may exist a minimal time of control. For example, in [2] Beauchard and Coron proved, for a time large enough, the local exact controllability along the ground state trajectory of a Schrödinger equation and Coron proved in [9] and [10, Theorem 9.8] that this local controllability does not hold in small time, even if the Schrödinger equation has an infinite speed of propagation. Our guess, based in second order computations in some particular critical cases where the space M is two-dimensional, is that there exists a minimal time of control, this means there exists a time T_0 such that for any time $T < T_0$, $\mathcal{P}(T)$ does not hold. Thus, the answer to Open Problem 1 should be negative.

We have seen that the nonlinearity gives us the controllability in the critical cases even if the linear system is not controllable. We may wonder if the nonlinearity gives us the stability.

Open Problem 2. Let $L \in N$. Let $y_0 \in L^2(0, L)$ and y the solution of

$$\begin{cases} y_t + y_x + y_{xxx} + yy_x = 0, \\ y(t, 0) = y(t, L) = 0, \\ y_x(t, L) = 0, \\ y(0, \cdot) = y_0. \end{cases} \tag{61}$$

Does the solution y decay to zero as t goes to infinity?

In order to answer this question, a really nonlinear method is needed because with a first-order approximation one obtains the linear system which has some solution conserving its L^2-norm. On the other hand, it is not clear that our method applies. It strongly needs the controls to be able to use higher-order approximations.

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