Available online at www.sciencedirect.com
ScienceDirect
Ann. I. H. Poincaré – AN 26 (2009) 477–496
ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE
www.elsevier.com/locate/anihpc

Transverse nonlinear instability for two-dimensional dispersive models

Instabilité transverse nonlinéaire dans des modèles dispersifs bi-dimensionnels

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Received 12 June 2007; received in revised form 25 September 2007; accepted 27 September 2007

Available online 24 January 2008

Abstract

We present a method to prove nonlinear instability of solitary waves in dispersive models. Two examples are analyzed: we prove the nonlinear long time instability of the KdV solitary wave (with respect to periodic transverse perturbations) under a KP-I flow and the transverse nonlinear instability of solitary waves for the cubic nonlinear Schrödinger equation.

Résumé

On présente une méthode pour prouver l'instabilité non-linéaire d'ondes solitaires dans des modèles dispersifs. Deux exemples sont analysés : on prouve l'instabilité de l'onde solitaire de KdV (par rapport à des perturbations transverses périodiques) dans l'équation de KP-I et l'instabilité transverse nonlinéaire des ondes solitaires de l'équation de Schrödinger non-linéaire cubique.

Keywords: Nonlinear instability; Solitary waves; Dispersive equations

1. Introduction

There are many results (both theoretical and numerical) dealing with detecting unstable modes of dispersive equations linearized around soliton like structures. However, in most of these cases it is not clear whether one has indeed a nonlinear instability for a flow of the full nonlinear problem. The goal of this paper is to present a method showing how only a partial information about the spectrum of the linearized operator together with a suitable nonlinear analysis may indeed give the proof of the nonlinear instability in the presence of an unstable mode. Our first example is the

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0294-1449/$ – see front matter
doi:10.1016/j.anihpc.2007.09.006

nonlinear long time instability of the KdV solitary wave (with respect to periodic transverse perturbations) under a KP-I flow. We also prove a nonlinear instability result for the cubic nonlinear Schrödinger equation. We believe that the method presented here could be useful in the contexts of other dispersive equations.

Consider the Korteweg–de Vries (KdV) equation

$$u_t + uu_x + u_{xxx} = 0, \tag{1}$$

$u : \mathbb{R}^2 \to \mathbb{R}$, which is an asymptotic model, derived from the free surface Euler equation, for the propagation of long one-directional small amplitude surface waves. A famous solution of (1) is the solitary wave solution, given by

$$u(t,x) = Q(x-t), \qquad Q(x) = 3\,\mathrm{sech}^2\left(\frac{x}{2}\right).$$

Observe that $u(t,x)$ corresponds to the displacement of the profile Q from left to the right with speed one. One also has the solution

$$u_c(t,x) = cQ\big(\sqrt{c}(x-ct)\big), \quad c>0 \tag{2}$$

which corresponds to a solitary wave with a positive speed c.

A very natural question concerning the relevance of the solution $Q(x-t)$ is its stability with respect to small perturbations. It is evident that the usual Lyapounov stability cannot hold because of the translation invariance of the problem. More precisely for c close to one $cQ(\sqrt{c}x)$ is close to $Q(x)$ while for $t \gg 1$ ($t \sim |c-1|^{-1}$) the corresponding solutions of the KdV equation $u(t,x)$ and $u_c(t,x)$ separate from each other at distance independent of the smallness of $c-1$. However, the solution $u_c(t,x)$ remains close to the spatial translates of Q and thus orbital stability of Q under the flow of KdV is not excluded. It is known since the seminal paper of Benjamin [2] that Q is orbitally stable in the energy space $H^1(\mathbb{R})$ (we call $H^1(\mathbb{R})$ the energy space since this is the natural space induced by the Hamiltonian structure of (1)). Here is the precise statement.

Theorem 1. *(Benjamin [2].) For every $\varepsilon > 0$ there exists $\delta > 0$ such that if the initial data*

$$u|_{t=0} = u_0 \in H^1(\mathbb{R})$$

of the KdV equation (1) *satisfies*

$$\|u_0 - Q\|_{H^1(\mathbb{R})} < \delta$$

then the corresponding solution u (*which is well defined thanks to* [17]) *satisfies*

$$\sup_{t\in\mathbb{R}} \inf_{a\in\mathbb{R}} \big\|u(t,\cdot) - Q(\cdot - a)\big\|_{H^1(\mathbb{R})} < \varepsilon.$$

Let us notice that the phase space $H^1(\mathbb{R})$ may be replaced by $L^2(\mathbb{R})$ (see [20]).

In [15], Kadomtsev–Petviashvili studied weak transverse perturbation of the KdV flow and derived the following two-dimensional models

$$u_t + uu_x + u_{xxx} \pm \partial_x^{-1} u_{yy} = 0. \tag{3}$$

Eq. (3) with sign $+$ is called the KP-II equation while (3) with sign $-$ is the KP-I equation. Let us observe that in the derivation of the model, the signs vary in front of the u_{xxx} term and correspond to different surface tensions. However from mathematical view point the study of the models with signs varying in front of u_{xxx} is equivalent to the study of the models with signs varying in front of $\partial_x^{-1} u_{yy}$ by the variable change $u(t,x,y) \mapsto -u(-t,x,y)$. The anti-derivative ∂_x^{-1} is defined on functions which have, in a suitable sense, a zero x mean value.

Let us observe that $Q(x-t)$ is a solution of both Eqs. (3). It is conjectured in [15] that $Q(x-t)$ is stable under the KP-II flow and unstable under the KP-I flow. Of course this conjecture is very vague since one should precise the stability notion and the spatial domain for x, y. In [1], all possible unstable modes of the linearized equation are described and in particular it is shown that the linearization about Q of the KP-I flow is unstable and the linearization of the KP-II flow is spectrally stable. In this paper, we show that the spectral instability result of [1] indeed implies the nonlinear instability in the case of the KP-I equation for solutions periodic in the y variable. This result is actually not new since the equation being completely integrable (having a Lax pair representation), the instability can be shown by exhibiting explicit solutions (see Zakharov [26]). Nevertheless, we believe that our method inspired from the work of Grenier [10] in fluid mechanics to prove that spectral instability implies nonlinear instability which does not use the complete integrability is interesting and can be applied to many other dispersive equations. As an illustration, we shall also study below a transverse instability of the two-dimensional cubic nonlinear Schrödinger equation which is not completely integrable. Let us observe that the explicit solution of the KP-I equation constructed by Zakharov tends to a soliton for $t \to -\infty$ but is different from a soliton. Our method of proof does not give such type of instability.

The global well-posedness of the KP-I equation in the setting $\mathbb{R} \times \mathbb{T}$ was recently obtained by Ionescu and Kenig [13] in a space which moreover contains the solitary wave Q and hence, we state our result in the context of Ionescu–Kenig's theorem. In general it is difficult to get nonlinear instability results in natural energy norms like L^2 or H^1 for conservative equations due to the presence of strong nonlinearities. Here we shall use the general setting developed by Grenier in [10] in the context of the Euler equation which relies on the possibility of constructing a high order approximate solution more accurate than the only linear approximation. For other methods, we refer to [8,11,9]. One of the difficulty in the problem is to get a precise estimate on the growth of the semi-group of the linearized equation about the solitary wave. The main interest of the strategy of [10] is that it allows to study the semi-group on smaller spaces which are only made of functions with a finite number of Fourier modes in the transverse direction so that we basically deal with one-dimensional problems. In this part of the analysis, our argument uses in a crucial way the properties of the equations we consider and in particular the Hamiltonian structure and the properties of the 1-D solitary waves.

We consider thus the KP-I equation

$$u_t + uu_x + u_{xxx} - \partial_x^{-1} u_{yy} = 0 \tag{4}$$

for $(x, y) \in \mathbb{R} \times \mathbb{T}_L$ where $\mathbb{T}_L$ is the flat torus $\mathbb{R}/2\pi L\mathbb{Z}$. As mentioned above, a special solution of this equation is given by the KdV soliton $Q(x - t)$. Since we are interested in the stability of the soliton for (4), it is more convenient to go into a moving frame i.e. to change x into $x - t$ and to study the equation

$$u_t - u_x + uu_x + u_{xxx} - \partial_x^{-1} u_{yy} = 0, \quad (x, y) \in \mathbb{R} \times \mathbb{T}_L \tag{5}$$

so that $Q(x)$ is now a stationary solution of (5). Note that we can always change space and time scales to reduce the study of the stability of u_c, given by (2) to the study of the stability of Q for (5). Nevertheless, since we are in a bounded domain in y, the scaling changes the size of the domain, this is why we keep the parameter L in our study.

As established in [13], the Cauchy problem for (4) or equivalently (5) is globally well-posed for data in the space $Z^2(\mathbb{R} \times \mathbb{T}_L)$ defined by

$$Z^2(\mathbb{R} \times \mathbb{T}_L) = \left\{u, \left\|\hat{u}(\xi, k)\left(1 + |\xi|^2 + |k/\xi|^2\right)\right\|_{L^2(\mathbb{R}\times\mathbb{Z})} < +\infty\right\},$$

where $\hat{u}(\xi, k)$ is the Fourier transform of u:

$$\hat{u}(\xi, k) = \frac{1}{2\pi L} \int_{-\infty}^{\infty} \int_0^{2\pi L} e^{-i(x\xi + \frac{yk}{L})} u(x, y)\, dy\, dx.$$

If $u \in Z^2$, this means that u, u_x, u_{xx} and $\partial_x^{-1} u_y, \partial_x^{-2} u_{yy}$ are in L^2, where ∂_x^{-1} is defined in the natural way via the Fourier transform for functions $u \in L^2$ such that $\xi^{-1}\hat{u}(\xi, k) \in L^2$. Moreover, the propagation of H^s regularity holds: if $u_0 \in H^s \cap Z^2$ for $s > 7$, then $u(t) \in H^s \cap Z^2$ for every $t > 0$. Note that since Q does not depend of y, we have $Q \in Z^2$. The first goal of this paper is to prove the following orbital instability result.

Theorem 2. *Assuming that $L > 4/\sqrt{3}$, then for every $s \geqslant 2$, there exists $\eta > 0$ such that for every $\delta > 0$, there exists $u_0^\delta \in Z^2 \cap H^s$ and a time $T^\delta \sim |\log \delta|$ such that*

$$\|u_0^\delta - Q\|_{H^s(\mathbb{R}\times\mathbb{T}_L)} < \delta$$

and the solution u^δ of (5) with initial value u_0^δ satisfies

$$\inf_{a\in\mathbb{R}} \left\|u^\delta\left(T^\delta, \cdot\right) - Q(\cdot - a)\right\|_{L^2(\mathbb{R}\times\mathbb{T}_L)} \geqslant \eta.$$

Remark 3. If $u(t,x,y)$ is a solution of the KP equation (3) then so is u_λ defined by

$$u_\lambda(t,x,y) = \lambda^2 u\big(\lambda^3 t, \lambda x, \lambda^2 y\big).$$

Thus in the context of (4) solutions of period one in y transform into solutions of period λ^{-2} and solitary waves of speed c transform into solitary waves of speed $\lambda^2 c$. Consequently Theorem 2 implies that if we fix period one perturbations in y then one needs to consider solitary waves of sufficiently large speed to get the instability. Let us also remark that the restriction $L > 4/\sqrt{3}$ in Theorem 2 is imposed by the spectral considerations of [1] and is needed for the existence of unstable eigenmodes. It would be interesting to decide what happens for $L \leqslant 4/\sqrt{3}$ (or equivalently for small speed solitary waves for period one perturbations).

Remark 4. Let us recall that a three-dimensional analogue of (4)

$$u_t + uu_x + u_{xxx} - \partial_x^{-1}(u_{yy} + u_{zz}) = 0, \quad (x,y,z) \in \mathbb{R}^3 \tag{6}$$

has solutions blowing up in finite time (see [19] and also [22]) and thus for the three-dimensional versions of the KP-I equation a stronger form of the instability appears. It is however an open problem to prove the existence of blow-up solutions for (6) with u periodic in y, z.

Let us outline the main steps of the proof Theorem 2. First, we need to use the result of [1] concerning the existence of unstable eigenmodes for the linearized about Q operator. Next, following the idea of Grenier [10], we perform the construction of an approximate solution. The approximate solution is defined iteratively. At the first step we put the unstable eigenmode. At each further step, we get linear problems with source terms involving the previous iterates (the procedure is closely related to the Picard iteration). We need to control precisely the eventual growth in time of each iterate. By applying a Laplace transform, we reduce the matters to showing estimates on a resolvent equation which are uniform on some straight line $\lambda = \gamma + i\tau$, $\tau \in \mathbb{R}$. For bounded frequencies (i.e. $|\tau|$ bounded), a classical ODE argument combined with the absence of unstable modes coming from [1] suffices to get the needed bound. The main difficulty is to get uniform resolvent estimates for large τ. They will result from conservation (or almost conservation) laws. We finally perform an energy estimate to the nonlinear problem to show that the constructed approximate solution is indeed close to the actual solution for suitable time scales. This in turn implies the nonlinear instability claimed in Theorem 2.

The second example that we consider in this paper is the two-dimensional Nonlinear Schrödinger equation (NLS)

$$iv_t + \Delta_{x,y} v + |v|^2 v = 0, \tag{7}$$

where v is a complex valued function. A famous solution of this equation is the solitary wave $Q(x)e^{it}$ with Q given by

$$Q(x) = \frac{\sqrt{2}}{\operatorname{ch}(x)}.$$

This solitary wave is orbitally stable when submitted to one-dimensional perturbations i.e. perturbations which depend on x only (see [6]). Here orbital stability means that

$$\forall \varepsilon > 0,\ \exists \delta > 0\colon\ \big\|v(0,\cdot) - Q\big\|_{H^1(\mathbb{R})} < \delta \quad \Longrightarrow \quad \inf_{a\in\mathbb{R},\gamma\in\mathbb{R}} \big\|v(t,\cdot) - e^{i\gamma}e^{it}Q(\cdot - a)\big\|_{H^1(\mathbb{R})} < \varepsilon.$$

We shall prove that, similarly to the KdV soliton as a solution of the KP-I equation, this stationary solution of (8) which is orbitally stable when submitted to one-dimensional perturbation is nonlinearly unstable when it is submitted to two-dimensional perturbation. As previously, it is more convenient to set $v = e^{it}u$ and to study the equation

$$iu_t + \Delta u - u + |u|^2 u = 0, \tag{8}$$

for $(x,y) \in \mathbb{R} \times \mathbb{T}_L$. A stationary solution of this equation is now given by the ground state $Q(x)$. Since the solitary waves modelled on $Q(x)$ for (7) are given by

$$u_\lambda(t,x) = \lambda Q(\lambda x)e^{i\lambda^2 t}$$

we can always reduce by scaling the study of the stability of u_λ to the study of the stability of Q in (8), but it is again important to keep L as a parameter. Here is our result.

Theorem 5. *There exists L_0 such that for $L \geqslant L_0$, for every $s \geqslant 2$, there exists $\eta > 0$ such that for every $\delta > 0$, there exists $u_0^\delta \in H^s$ and a time $T^\delta \sim |\log \delta|$ such that*

$$\left\| u_0^\delta - Q \right\|_{H^s(\mathbb{R}\times\mathbb{T}_L)} < \delta$$

and the solution u^δ of (8) *with initial value u_0^δ belongs to $\mathcal{C}([0, T^\delta], H^s)$ and satisfies*

$$\inf_{a\in\mathbb{R},\gamma\in\mathbb{R}} \left\| u^\delta\left(T^\delta, \cdot\right) - Q(\cdot - a)e^{i\gamma} \right\|_{L^2(\mathbb{R}\times\mathbb{T}_L)} \geqslant \eta.$$

Remark 6. Let us observe that the cubic two-dimensional NLS is not known to be integrable (in the sense of Lax pairs representation) and thus it is hard to expect that the instability result presented in Theorem 5 can be displayed by an explicit family of solutions in the spirit of [26].

Remark 7. It is likely that the method presented here may be applied to the following two-dimensional perturbation of the Boussinesq equation

$$u_{tt} + \left(u_{xx} + u^2 - u\right)_{xx} - u_{yy} = 0. \tag{9}$$

The stability, for suitable values of the propagation speed, with respect to one-dimensional perturbation of the soliton of the Boussinesq equation is obtained in [4]. The analysis for an unstable mode in 2D in the context of (9) is essentially the same as the corresponding analysis for the KdV soliton as a solution of KP (see [3]). One thus may perform the analysis of [1] (see also Appendix A of this paper) combined with the nonlinear analysis of this paper to get statements in the spirit of Theorems 2, 5 for Eq. (9).

The assumption $L \geqslant L_0$ in Theorem 5 is used to get the spectral instability of the solitary wave. A difference with Theorem 2 is that for the two-dimensional Schrödinger equation in $\mathbb{R} \times \mathbb{T}$ a global existence result of large data strong solutions is not known so that Theorem 5 contains the fact that our unstable solution u^δ remains well-defined on a sufficiently long time scale. In fact, small data global existence for (8), posed on $\mathbb{R} \times \mathbb{T}$, is obtained in [23]. For general large data we may not have the global existence for (8), posed on $\mathbb{R} \times \mathbb{T}$, since one may localize the well-known explicit blow-up solution for the cubic NLS on $\mathbb{R}^2$ (see [5] for details on this argument).

Let us complete this introduction by several remarks. The instability we describe is due to isolated point spectrum of the linearized operator. Nevertheless, in principle, as in the work of Grenier [10], our approach may be applied to the case of fully localized perturbations (in this case, there is only continuous spectrum) by constructing a wave packet near the most unstable mode. The only new difficulty in the analysis would be to control the resolvent estimates below in the regime $j \to 0$. Let us finally observe that if one can prove that the growth in $H^s(\mathbb{R} \times \mathbb{T})$ for s sufficiently large of the semi-group generated by the linearized equation about the solitary wave is bounded by $e^{\gamma t}$ where γ is the real part of the most unstable eigenvalue, then a more classical approach to instability (see e.g. [12]) may give our result for NLS which is a weakly nonlinear problem (the nonlinearity does not contain derivatives). We however do not see how such an approach may be applied to the case of quasi-linear problems as the KP-I equation.

The rest of the paper is organized as follows. In the next section, we give a detailed proof of all the steps of the proof of Theorem 2. Then we give a less detailed proof of Theorem 5 since the method is the same. Finally, Appendix A is devoted to the linear instability results.

2. Proof of Theorem 2

2.1. Existence of a most unstable eigenmode

The linearized equation about the soliton Q reads

$$u_t + Au = 0, \quad Au = -u_x + (Qu)_x + u_{xxx} - \partial_x^{-1}\partial_{yy}u, \quad (x, y) \in \mathbb{R} \times \mathbb{T}_L. \tag{10}$$

This last linear equation can be solved, for instance by a classical energy method, for initial data in H^s such that its anti-derivative exists. The main result of [1] is the characterization of all the unstable eigenmodes associated to A. An unstable eigenmode is a solution of (10) under the form

$$\varphi_{\sigma,k}(t, x, y) = e^{\sigma t} e^{\frac{iky}{L}} V(x),$$

with $\operatorname{Re}\sigma > 0$, $V \in L^2(\mathbb{R})$. The result of [1] adapted to our framework reads:

Theorem 8. *(See [1].) There exist unstable eigenmodes if and only if $L > \frac{4}{\sqrt{3}}$. Moreover, for an unstable eigenmode, σ and $k \in \mathbb{Z}$ are parametrized by*

$$2\sigma = \mu(\mu - 1)(2 - \mu), \quad k = \frac{\sqrt{3}L}{4}\mu(2 - \mu), \quad \mu \in (1, 2) \tag{11}$$

and there exists $g \in H^\infty(\mathbb{R})$ such that

$$V = g_{xx}. \tag{12}$$

For the sake of completeness, we recall the main steps of the proof of this result in Appendix A.

Note that for $\mu \in (0, 2)$, $\mu(2 - \mu) \in (0, 1)$ hence one can find an integer such that $k = \frac{\sqrt{3}L}{4}\mu(2 - \mu)$ only if $\frac{\sqrt{3}L}{4} > 1$. Moreover, for L fixed, there is only a finite number of k which verify this property, this allows us to choose σ_0 and k_0 such that φ_{σ_0,k_0} is the most unstable eigenmode i.e.

$$\sigma_0 = \sup\{\sigma, (\sigma, k) \text{ verifying (11)}\}$$

and k_0 is the corresponding integer such that (11) holds with $(\sigma, k) = (\sigma_0, k_0)$. Let us define

$$u^0(t, x, y) \equiv \varphi_{\sigma_0,k_0}(t, x, y) + \overline{\varphi_{\sigma_0,k_0}}(t, x, y).$$

To prove Theorem 2, we shall use $Q + \delta u^0(0, x, y)$ as an initial data for (5). As remarked before, we have $Q \in Z^2 \cap H^s$ for every s, but thanks to (12) in Theorem 8, we also have that $u^0(0, x, y) \in Z^2 \cap H^s$ consequently, thanks to the result of [13] there is a unique global solution u^δ of (5) in $Z^2 \cap H^s$ with initial value $Q + \delta u^0(0, x, y)$. So the only problem that remains is to estimate from below

$$\inf_{a\in\mathbb{R}} \left\| u^\delta\left(T^\delta, \cdot\right) - Q(\cdot - a)\right\|_{L^2(\mathbb{R}\times\mathbb{T}_L)}.$$

Towards this, we shall use the method of [10] which relies on the construction of an high order unstable solution. This is the aim of the next section.

2.2. Construction of an high order unstable approximate solution

Let us set $v = u^\delta - Q$, then v solves

$$v_t + Av = -vv_x. \tag{13}$$

We define V^s_K as the space:

$$V^s_K = \left\{u, u = \sum_{j\in\frac{k_0}{L}\mathbb{Z},\ |jL/k_0|\leqslant K} u_j(x)e^{ijy},\ u_j \in H^s(\mathbb{R})\right\}$$

and we define a norm on V^s_K by

$$|u|_{V^s_K} = \sup_j |u_j|_s$$

where $|\cdot|_s$ is the standard $H^s(\mathbb{R})$ norm. Let us notice that u^0 is such that $u^0 \in V^s_1$ for all $s \in \mathbb{N}$. Following the strategy of [10], for $s \gg 1$, we look for an high order solution under the form:

$$u^{ap} = \delta\left(u^0 + \sum_{k=1}^{M} \delta^k u^k\right), \quad u^k \in V^{s-k}_{k+1} \tag{14}$$

such that $u^k_{/t=0} = 0$ and $M \geqslant 1$ is to be fixed later. Once the value of M is fixed, then we fix the integer s so that $s > M$. By plugging the expansion in (13), cancelling the terms involving δ^k, $1 \leqslant k \leqslant M + 1$, we choose u^k so that u^k solves the problem

$$\partial_t u^k + Au^k = -\frac{1}{2}\left(\sum_{j+l=k-1} u^j u^l\right)_x, \quad u^k_{/t=0} = 0. \tag{15}$$

The main point in the analysis of u^{ap} is the following estimate.

Proposition 9. *Let u^k the solution of* (15), *if $s-k \geqslant 1$, we have the estimate*:

$$\left|u^k(t)\right|_{V_{k+1}^{s-k}} \leqslant C_{k,s} e^{(k+1)\sigma_0 t}, \quad \forall t \geqslant 0. \tag{16}$$

The proof of the proposition will follow easily by induction from the following theorem.

Theorem 10. *Consider the solution u of the linear problem*

$$\partial_t u + Au = F_x, \quad u_{/t=0} = 0 \tag{17}$$

with a source term $F \in V_K^{s+1}$ with

$$\left|F(t)\right|_{V_K^{s+1}} \leqslant C_{K,s}^F e^{\gamma t}, \quad \gamma \geqslant 2\sigma_0 \tag{18}$$

then u belongs to V_K^s and satisfies the estimate

$$\left|u(t)\right|_{V_K^s} \leqslant C_{K,s} e^{\gamma t}, \quad \forall t \geqslant 0. \tag{19}$$

We first observe that under our hypothesis on F the solution of (10) is well-defined and the only point is to prove the quantitative bound (19). The estimate (19) relies on the fact that on V_K^s, the real part of the spectrum of the operator $-A$ is bounded by σ_0. Nevertheless for such a dispersive operator, there is no general theory to convert an information on the position of the spectrum into an estimate on the semi-group like it is the case for example for sectorial operators. To get the result, we need to estimate the resolvent of $-A$ on V_K^s. At first, we can perform some reductions on the problem. Indeed, since F has a finite number of Fourier modes, we can expand u in Fourier modes and hence we only need to study the problem

$$\partial_t v + A_j v = \partial_x F_j(t,x), \quad v_{/t=0} = 0 \tag{20}$$

where

$$A_j v = -v_x + (Qv)_x + v_{xxx} + j^2 \partial_x^{-1} v, \tag{21}$$

$j \in \frac{k_0}{L}\mathbb{Z}$, $|jL/k_0| \leqslant K$ and $v(t,x) = u_j(t,x)$, and to establish that v satisfies

$$\left|v(t)\right|_s \leqslant C_{j,s} e^{\gamma t}$$

under the assumption

$$\left|F_j(t)\right|_{s+1} \leqslant C_{j,s} e^{\gamma t}. \tag{22}$$

In what follows, we fix γ_0 such that $\sigma_0 < \gamma_0 < \gamma$ and we shall use the Laplace transform. For $T > 0$, we first introduce G such that

$$G = 0, \quad t < 0, \qquad G = 0, \quad t > T, \qquad G = F_j, \quad t \in [0,T]$$

and we notice that the solution of

$$\partial_t \tilde{v} + A_j \tilde{v} = G_x, \quad \tilde{v}_{/t=0} = 0$$

coincides with v on $[0,T]$ so that it is sufficient to study $\tilde{v}$. Next, we set

$$w(\tau,x) = \mathcal{L}\tilde{v}(\gamma_0 + i\tau), \quad H(\tau,x) = \mathcal{L}G(\gamma_0 + i\tau), \quad (\tau,x) \in \mathbb{R}^2$$

where $\mathcal{L}$ stands for the Laplace transform in time:

$$\mathcal{L}f(\gamma_0 + i\tau) = \int_{\mathbb{R}} e^{-\gamma_0 t - i\tau t} f(t) \mathbf{1}_{t \geqslant 0}\, dt.$$

We get that w solves the resolvent equation

$$(\gamma_0 + i\tau)w + A_j w = H_x. \tag{23}$$

In the sequel, for complex valued functions depending on x, we define

$$(f,g)\equiv\int_{\mathbb{R}} f(x)\bar{g}(x)\,dx,\qquad |f|^2\equiv\|f\|^2_{L^2(\mathbb{R})}=(f,f),\qquad |f|_s^2\equiv\|f\|^2_{H^s(\mathbb{R})}=\sum_{0\leqslant m\leqslant s}\left|\partial_x^m f\right|^2.$$

Towards the proof of Theorem 10, we first need to study (23). Our main estimate on the resolvent will be

Theorem 11 *(Resolvent estimates). Let $s\geqslant 1$ be an integer. Let $w(\tau)$ be the solution of* (23) *for j, $|j|\leqslant k_0K/L$, then there exists $C(s,\gamma_0,K)>0$ such that for every τ, we have the estimate*

$$\left|w(\tau)\right|_s^2\leqslant C(s,\gamma_0,K)\left|H(\tau)\right|_{s+1}^2. \tag{24}$$

2.2.1. Proof of Theorem 11

We shall split the proof in various lemmas. To estimate w, we shall deal differently with large and bounded frequencies.

Lemma 12. *There exists $M>0$ (which depends on K) and $C(s,\gamma_0,K)$ such that for $|\tau|\geqslant M$, we have the estimate*

$$\left|w(\tau)\right|_s^2\leqslant C(s,\gamma_0,K)\left|H(\tau)\right|_{s+1}^2. \tag{25}$$

2.2.2. Proof of Lemma 12

We first prove (25) for $s=1$. Note that Eq. (23) can be rewritten as

$$(\gamma_0+i\tau)w-(\mathcal{L}w)_x+j^2\partial_x^{-1}w=H_x \tag{26}$$

where $\mathcal{L}$ is defined by

$$\mathcal{L}w=-w_{xx}-Qw+w.$$

Note that $\mathcal{L}$ is a self-adjoint operator in L^2 which is very useful in the proof of the stability of the soliton for the KdV equation. Since it is self-adjoint, the spectrum is real. Moreover, since Q goes to zero exponentially fast, the essential spectrum of $\mathcal{L}$ is in $[1,+\infty)$. For $\lambda<1$ there are only eigenvalues of finite multiplicity. Finally by Sturm–Liouville theory, since Q_x is in the kernel of $\mathcal{L}$ and has only one zero, we get that $\mathcal{L}$ has only one negative eigenvalue. Moreover, 0 is a simple eigenvalue. Consequently we can define an orthogonal decomposition:

$$w=\alpha(\tau)\varphi_{-1}+\beta(\tau)\varphi_0+w_\perp \tag{27}$$

where

$$\mathcal{L}\varphi_{-1}=\mu\varphi_{-1},\ \mu<0,\quad \mathcal{L}\varphi_0=0,\quad (\mathcal{L}w_\perp,w_\perp)\geqslant c_0|w_\perp|^2,\quad c_0>0. \tag{28}$$

Note that the eigenvectors φ_{-1} and φ_0 are smooth. The important role of $\mathcal{L}$ is due to the following conservation law

$$\gamma_0\big((w,\mathcal{L}w)+j^2\left|\partial_x^{-1}w\right|^2\big)=\operatorname{Re}\big((H_x,\mathcal{L}w)+j^2\big(H,\partial_x^{-1}w\big)\big) \tag{29}$$

which can be checked by a straightforward computation. Consequently, we can use (27), (28) and integrate by parts the right-hand side to get

$$\gamma_0\big(\mu\alpha(\tau)|\varphi_{-1}|^2+c_0|w_\perp|^2+j^2\left|\partial_x^{-1}w\right|^2\big)\leqslant C|H|_2|w|_1+j^2|H|\left|\partial_x^{-1}w\right|.$$

Therefore, using the inequality

$$ab\leqslant\varepsilon a^2+\frac{1}{4\varepsilon}b^2,\quad \forall\varepsilon>0,\ \forall(a,b)\in\mathbb{R}^2, \tag{30}$$

with ε small enough, we can incorporate $|\partial_x^{-1}w|$ in the left-hand side and arrive at

$$|w_\perp|^2+j^2\left|\partial_x^{-1}w\right|^2\leqslant C\big(|\alpha|^2+|H|^2+|H|_2|w|_1\big). \tag{31}$$

In what follows C is a large number which may change from lines to lines and depend on γ and K but not on τ. The next step is to estimate α and β. We use the decomposition (27) and take the scalar product of (26) with φ_{-1} and with φ_0 respectively to get

$$(\gamma_0 + i\tau)\alpha = -\big(w, \mathcal{L}\partial_x(\varphi_{-1})\big) - j^2\big(\partial_x^{-1} w, \varphi_{-1}\big) + (H_x, \varphi_{-1}),$$
$$(\gamma_0 + i\tau)\beta = -\big(w, \mathcal{L}\partial_x(\varphi_0)\big) - j^2\big(\partial_x^{-1} w, \varphi_0\big) + (H_x, \varphi_0)$$

and hence, we can take the modulus and add the two identities to get

$$\big(\gamma_0 + |\tau|\big)\big(|\alpha| + |\beta|\big) \leqslant C\big(|\alpha| + |\beta| + |w_\perp| + j^2\big|\partial_x^{-1} w\big| + |H|_1\big).$$

Next, we multiply by $|\alpha| + |\beta|$ and use (30) to get

$$\big(\gamma_0 + |\tau| - C\big)\big(|\alpha|^2 + |\beta|^2\big) \leqslant C\big(|w_\perp|^2 + j^4\big|\partial_x^{-1} w\big|^2 + |H|_1^2\big). \tag{32}$$

Note that this last estimate is a good estimate when τ is large. Next, we can consider B(31) + (32) with B a large number to be chosen to get

$$(B - C)\big(|w_\perp|^2 + \big(Bj^2 - Cj^4\big)\big|\partial_x^{-1} w\big|^2\big) + \big(\gamma_0 + |\tau| - C - BC\big)\big(|\alpha|^2 + |\beta|^2\big) \leqslant CB\big(|H|_2|w|_1 + |H|_1^2\big).$$

Consequently, we can first choose B sufficiently large (such that $B > C$, and $B > Cj^2$) and then consider τ sufficiently large (for example $|\tau| \geqslant 2(C + BC)$) to get the estimate

$$|w|^2 + j^2\big|\partial_x^{-1} w\big|^2 \leqslant C\big(|H|_2|w|_1 + |H|_1^2\big), \quad |\tau| \geqslant M. \tag{33}$$

To conclude we just need to estimate $|\partial_x w|$. It suffices to look again at (29). Indeed, we can use that $(w, \mathcal{L}w) = |w_x|^2 - \mathcal{O}(1)|w|^2$ in (29) to get

$$|w_x|^2 + j^2\big|\partial_x^{-1} w\big|^2 \leqslant C\big(|w|^2 + |H|^2 + |H|_2|w|_1\big). \tag{34}$$

Consequently, the combination of a sufficiently large constant times (33) and (34) gives

$$|w|_1^2 + j^2\big|\partial_x^{-1} w\big|^2 \leqslant C\big(|H|_2|w|_1 + |H|_1^2\big), \quad |\tau| \geqslant M$$

and hence by using the inequality (30), we get

$$|w|_1^2 + j^2\big|\partial_x^{-1} w\big|^2 \leqslant C|H|_2^2. \tag{35}$$

This proves (25) for $s = 1$. Note that moreover (35) gives a control of $j^2|\partial_x^{-1} w|^2$ which is interesting when $j \neq 0$.

To estimate higher order derivatives, we shall use higher order approximate conservation laws for the linearized KdV equation. Namely, we define a self-adjoint operator

$$\mathcal{L}_{s+1} w = \partial_x^{2s+2} w + \partial_x^s\big(r_{s+1}(x)\partial_x^s w\big),$$

where r_{s+1} is real valued and will be chosen in order that the following cancellation property occurs:

$$\operatorname{Re}\big((\mathcal{L}w)_x, \mathcal{L}_{s+1} w\big) = \mathcal{O}(1)|w|_s^2. \tag{36}$$

By making repeated integration by parts, we easily establish that

$$\operatorname{Re}\big(\partial_x^{2s+2} w, w_{xxx}\big) = (-1)^s \operatorname{Re}\big(\partial_x^{s+2} w, \partial_x \partial_x^{s+2} w\big) = 0,$$

$$\begin{aligned}\operatorname{Re}\big(\partial_x^{2s+2} w, Qw_x\big) &= (-1)^{s+1} \operatorname{Re}\big(\big(\partial_x^{s+1} w, Q\partial_x^{s+2} w\big) + (s+1)\big(\partial_x^{s+1} w, Q_x \partial_x^{s+1} w\big) + \mathcal{O}(1)|w|_s^2\big)\\ &= (-1)^{s+1} \operatorname{Re}\left(\left(s + \frac{1}{2}\right)\big(\partial_x^{s+1} w, Q_x \partial_x^{s+1} w\big) + \mathcal{O}(1)|w|_s^2\right),\end{aligned}$$
$$\operatorname{Re}\big(\partial_x^{2s+2} w, Q_x w\big) = (-1)^{s+1} \operatorname{Re}\big(\big(\partial_x^{s+1} w, Q_x \partial_x^{s+1} w\big) + \mathcal{O}(1)|w|_s^2\big),$$

$$\begin{aligned}\operatorname{Re}\big(\partial_x^s\big(r_{s+1}\partial_x^s w\big), w_{xxx}\big) &= (-1)^{s-1} \operatorname{Re}\big(\partial_x\big(r_{s+1}\partial_x^s w\big), \partial_x^{s+2} w\big)\\ &= (-1)^{s-1} \operatorname{Re}\left(-\frac{3}{2}\big(\partial_x^{s+1} w, \partial_x r_{s+1} \partial_x^{s+1} w\big) + \mathcal{O}(1)|w|_s^2\right)\end{aligned}$$

and that all the other terms which appear in the product $\operatorname{Re}((\mathcal{L}w)_x, \mathcal{L}_{s+1} w)$ are $\mathcal{O}(1)|w|_s^2$. Consequently, we get

$$\begin{aligned}\operatorname{Re}\bigl(-(\mathcal{L}w)_x, \mathcal{L}_{s+1}w\bigr) &= (-1)^{s+1}\left(\left(s+\frac{3}{2}\right)\bigl(\partial_x^{s+1}w, Q_x\partial_x^{s+1}w\bigr)\right.\\ &\quad \left.+\left(s-\frac{3}{2}\right)\bigl(\partial_x^{s+1}w, \partial_x r_{s+1}\partial_x^{s+1}w\bigr)\right)+\mathcal{O}(1)|w|_s^2\\ &= \mathcal{O}(1)|w|_s^2\end{aligned}$$

with the choice

$$r_{s+1} = \frac{s+3/2}{3/2}Q = \left(\frac{2s}{3}+1\right)Q.$$

Note that s is an integer so that r_{s+1} is always well-defined.

Finally, we can take the scalar product of (26) by $(-1)^{s+1}\mathcal{L}_{s+1}w$ and then take the real part to get thanks to the above cancellation property

$$\gamma_0\bigl|\partial_x^{s+1}w\bigr|^2 \leqslant C\bigl(|w|_s^2 + j^4\bigl|\partial_x^{-1}w\bigr|^2 + |H|_{s+2}\bigl|\partial_x^{s+1}w\bigr| + |H|_{s+1}|w|_s\bigr)$$

since $\operatorname{Re}(\partial_x^{-1}w, \partial_x^{2s+2}w) = 0$ (the constant C is independent of τ). We finally obtain

$$\bigl|\partial_x^{s+1}w\bigr|^2 \leqslant C\bigl(|w|_s^2 + j^4\bigl|\partial_x^{-1}w\bigr|^2 + |H|_{s+2}^2\bigr)$$

thanks to the inequality (30) and hence we get (25) by induction and the control of $j^2|\partial_x^{-1}w|^2$ given by (35).

Next, we need to estimate w for $|\tau| \leqslant M$. This is the aim of the following lemma.

Lemma 13. *For $|\tau| \leqslant M$, we have the estimate*

$$\bigl|w(\tau)\bigr|_s^2 \leqslant C(s, \gamma_0, K, M)\bigl|H(\tau)\bigr|_{s+1}^2. \tag{37}$$

2.2.3. Proof of Lemma 13

Note that here we actually give a proof of the fact that if λ is not an eigenvalue then λ is not in the spectrum. To prove (37), we need to treat differently the cases $j = 0$ and $j \neq 0$.

Let us start with the case $j \neq 0$. In this case, we take the derivative of (26) to get

$$(\gamma_0 + i\tau)w_x - (\mathcal{L}w)_{xx} + j^2 w = H_{xx} \tag{38}$$

and we introduce $V = (w, w_x, w_{xx}, w_{xxx})^t \in \mathbb{C}^4$ and $\mathbb{H} = (0, 0, 0, H_{xx})^t$ to rewrite the problem as

$$V_x = \mathbb{A}(q, x)V + \mathbb{H} \tag{39}$$

where $\mathbb{A}$ is a 4×4 matrix that one may easily find from Eq. (38) and the parameter $q = (\gamma_0 + i\tau, j^2)$ is in the compact set $\mathcal{K}$ defined by

$$\mathcal{K} = \bigl\{(\gamma_0 + i\tau, b),\ |\tau| \leqslant M, k_0^2/L^2 \leqslant |b| \leqslant K^2k_0^2/L^2\bigr\}.$$

Let us denote by $T(q, x, x')$ the fundamental solution of $V_x = \mathbb{A}V$ i.e. the solution such that $T(q, x', x') = I_4$. Next, since $Q(x)$ tends to zero exponentially fast when $x \to \pm\infty$, there exists a matrix $\mathbb{A}_\infty(q)$ such that

$$\mathbb{A}(q, x) - \mathbb{A}_\infty(q) = \mathcal{O}\bigl(e^{-|x|}\bigr), \quad x \to \pm\infty.$$

Moreover the eigenvalues of $\mathbb{A}_\infty$ are the roots of the polynomial P defined in (63) below and hence are not purely imaginary. By classical arguments of ODE (namely the roughness of exponential dichotomy, see [7] for example), the equation $V_x = \mathbb{A}V$ has an exponential dichotomy on $\mathbb{R}_+$ and $\mathbb{R}_-$, i.e., there exist projections $P^+(q, x)$, $P^-(q, x)$ which are smooth in the parameter with the invariance property

$$T(q, x, x')P^\pm(q, x') = P^\pm(q, x)T(q, x, x') \tag{40}$$

and such that there exist C and $\alpha > 0$ such that for every $U \in \mathbb{C}^4$, and $q \in \mathcal{K}$, we have

$$\begin{aligned}
&\left|T(q,x,x')P^+(q,x')U\right| \leqslant Ce^{-\alpha(x-x')}\left|P^+(q,x')U\right|, \quad x \geqslant x' \geqslant 0,\\
&\left|T(q,x,x')\big(I-P^+(q,x')\big)U\right| \leqslant Ce^{\alpha(x-x')}\left|\big(I-P^+(q,x')\big)U\right|, \quad 0 \leqslant x \leqslant x',\\
&\left|T(q,x,x')P^-(q,x')U\right| \leqslant Ce^{\alpha(x-x')}\left|P^-(q,x')U\right|, \quad x \leqslant x' \leqslant 0,\\
&\left|T(q,x,x')\big(I-P^-(q,x')\big)U\right| \leqslant Ce^{-\alpha(x-x')}\left|\big(I-P^-(q,x')\big)U\right|, \quad 0 \geqslant x \geqslant x'.
\end{aligned}$$

In particular, note that a solution $T(q,x,0)V^0$ is decaying when x tend to $\pm\infty$ if and only if V^0 belongs to $\mathcal{R}(P^\pm(q,0))$. Since by the analysis of [1] recalled in Appendix A there is no eigenvalue of A_j (see (21) for the definition of A_j) for $q \in \mathcal{K}$, we have no nontrivial solution decaying in both sides and hence we have

$$\mathcal{R}\big(P^+(q,0)\big) \cap \mathcal{R}\big(P^-(q,0)\big) = \{0\}. \tag{41}$$

Let us choose bases $(r_1^\pm, r_2^\pm)$ of $\mathcal{R}(P^\pm(q,0))$ which depend on the parameters in a smooth way (see [16] for example) then we can define

$$M(q) = \big(r_1^+, r_2^+, r_1^-, r_2^-\big)$$

and we note that $M(q)$ is invertible for $q \in \mathcal{K}$ because of (41). This allows us to define a new projection $P(q)$ by

$$P(q) = M(q)\begin{pmatrix} I_2 & 0 \\ 0 & 0 \end{pmatrix} M(q)^{-1}$$

and next

$$P(q,x) = T(q,x,0)P(q).$$

The main interest of these definitions is that we have $\mathcal{R}(P(q)) = \mathcal{R}(P^+(q,0))$ and $\mathcal{R}(I-P(q)) = \mathcal{R}(P^-(q,0))$. Therefore thanks to (40), we have for every x that $\mathcal{R}(P(q,x)) = \mathcal{R}(P^+(q,x))$ and similarly that

$$\mathcal{R}\big(I - P(q,x)\big) = \mathcal{R}\big(P^-(q,x)\big).$$

Consequently, we have the estimates

$$\left|T(q,x,x')P(q,x')\right| \leqslant Ce^{-\alpha(x-x')}, \quad x, x' \in \mathbb{R},\ x \geqslant x',\ \forall q \in \mathcal{K}, \tag{42}$$

$$\left|T(q,x,x')\big(I - P(q,x')\big)\right| \leqslant Ce^{\alpha(x-x')}, \quad x, x' \in \mathbb{R},\ x \leqslant x',\ \forall q \in \mathcal{K}. \tag{43}$$

By using this property, the unique bounded solution of (39) reads by Duhamel formula

$$V(x) = \int_{-\infty}^{x} T(q,x,x')P(q,x')\mathbb{H}(x')\,dx' - \int_{x}^{+\infty} T(q,x,x')\big(I - P(q,x')\big)\mathbb{H}(x')\,dx'$$

and hence, we get thanks to (42), (43) that

$$\left|V(x)\right| \leqslant C \int_{\mathbb{R}} e^{-\alpha|x-x'|}\left|\mathbb{H}(x')\right| dx'$$

which yields by standard convolution estimates

$$|V| \leqslant C|H|.$$

The estimates of high order derivatives are very easy, it suffices to write

$$\partial_x^{s+1} V = \mathbb{A}\partial_x^s V + \big[\partial_x^s, \mathbb{A}\big]V + \partial_x^s \mathbb{H},$$

and to write Duhamel formula considering $[\partial_x^s, \mathbb{A}]V$ as part of the source term.

It remains the case $j = 0$. In this case, we do not take the derivative of (26), we directly define $W = (w, w_x, w_{xx})$ and we rewrite (26) under the form

$$W_x = \mathbb{B}(\lambda, x)W + \tilde{\mathbb{H}}.$$

Then the proof of the estimate follows the same line, we find that $\mathbb{B}_\infty$ has no eigenvalue on the imaginary axis. This yields that there is an exponential dichotomy on $\mathbb{R}_+$ and $\mathbb{R}_-$ for this system. Next since, the spectrum of the linearized KdV equation about the soliton is on the imaginary axis, we get that the system has an exponential dichotomy on the real line. We do not detail more since the proof is similar to the previous case.

2.2.4. End of the proof of Theorem 11

To get (24), it suffices to combine Lemmas 12 and 13.

2.2.5. End of the proof of Theorem 10

By using Theorem 11 and Bessel–Parseval identity, we get that for every $T > 0$,

$$\int_0^T e^{-2\gamma_0 t} \left|v(t)\right|_s^2 dt \leqslant \int_0^{+\infty} e^{-2\gamma_0 t} \left|\tilde{v}(t)\right|_s^2 dt = \int_{\mathbb{R}} \left|w(\tau)\right|_s^2 d\tau \leqslant C \int_{\mathbb{R}} \left|H(\tau)\right|_{s+1}^2 d\tau = \int_0^T e^{-2\gamma_0 t} \left|F_j(t)\right|_{s+1}^2 dt$$

and finally thanks to (18), we get

$$\int_0^T e^{-2\gamma_0 t} \left|v(t)\right|_s^2 dt \leqslant C \int_0^T e^{2(\gamma-\gamma_0)t} dt \leqslant C e^{2(\gamma-\gamma_0)T} \tag{44}$$

since γ_0 was fixed such that $\gamma > \gamma_0$. To finish the proof, we notice that the energy estimate for Eq. (20) gives

$$\frac{d}{dt}\left|v(t)\right|_s^2 \leqslant C\left(\left|v(t)\right|_s^2 + \left|F_j(t)\right|_{s+1}^2\right).$$

Consequently, we can multiply the last estimate by $e^{-2\gamma_0 t}$ and use (22) to get

$$\frac{d}{dt}\left(e^{-2\gamma_0 t}\left|v(t)\right|_s^2\right) \leqslant C\left(e^{-2\gamma_0 t}\left|v(t)\right|_s^2 + e^{2(\gamma-\gamma_0)t}\right).$$

Next, we integrate in time and use (44) and again the fact that $\gamma > \gamma_0$, this yields

$$e^{-2\gamma_0 t}\left|v(t)\right|_s^2 \leqslant C e^{2(\gamma-\gamma_0)t}.$$

This ends the proof of Theorem 10 .

2.2.6. Proof of Proposition 9

By induction, it suffices to use Theorem 10 and the fact that $H^s(\mathbb{R})$ is an algebra for $s \geqslant 1$.

2.3. Nonlinear instability: end of the proof of Theorem 2

Of course, we only need to prove the statement for δ small enough. Let us define w by setting $v = u^{ap} + w$, where u^{ap} is defined by (14). Therefore we have that the solution u^δ may be decomposed as follows

$$u^\delta = Q + u^{ap} + w.$$

If we set

$$F \equiv (\partial_t + A)u^{ap} + u^{ap}u_x^{ap},$$

where A is defined in (10), then thanks to Proposition 9,

$$\left\|F(t,\cdot)\right\|_{L^2(\mathbb{R}\times\mathbb{T}_L)} \leqslant C_{M,s}\delta^{M+2}e^{(M+2)\sigma_0 t}, \quad t \in \left[0, \left|\log(\delta)\right|/\sigma_0\right].$$

We have that w solves the problem

$$(\partial_t + A)w + \partial_x\left(u^{ap}w\right) + ww_x + F = 0, \quad w_{|t=0} = 0. \tag{45}$$

We now estimate the solution of (45). Using that

$$\left|\int_{\mathbb{R}\times\mathbb{T}_L} Fw\right| \leqslant \left\|F(t,\cdot)\right\|_{L^2(\mathbb{R}\times\mathbb{T}_L)}^2 + \left\|w(t,\cdot)\right\|_{L^2(\mathbb{R}\times\mathbb{T}_L)}^2,$$

multiplying (45) by w and integrating $\mathbb{R}\times\mathbb{T}_L$, we get after several integrations by parts

$$\frac{d}{dt}\left\|w(t,\cdot)\right\|_{L^2}^2 \leqslant \left(\|Q'\|_{L^\infty} + \left\|\partial_x u_{ap}(t,\cdot)\right\|_{L^\infty} + 1\right)\left\|w(t,\cdot)\right\|_{L^2}^2 + \left\|F(t,\cdot)\right\|_{L^2}^2. \tag{46}$$

Observe that

$$\left\|\partial_x u_{ap}(t,\cdot)\right\|_{L^\infty(\mathbb{R}\times\mathbb{T}_L)} \leqslant \sum_{k=0}^{M} C_{k,s}\delta^{k+1}e^{(k+1)\sigma_0 t}.$$

Next, we set

$$T^\delta \equiv \frac{\log(\kappa/\delta)}{\sigma_0},$$

where $\kappa \in]0,1]$ is small enough to be chosen after the several restrictions we will impose in the next lines. The number T^δ represents the time when the instability occurs. Coming back to (46), we observe that there exist a constant $\Lambda_{M,s}$ depending on s and M but independent of κ and t and an absolute constant C (C is essentially $\|Q'\|_{L^\infty}$) such that for $0\leqslant t\leqslant T^\delta$,

$$\frac{d}{dt}\left\|w(t,\cdot)\right\|_{L^2}^2 \leqslant (C+\kappa\Lambda_{M,s})\left\|w(t,\cdot)\right\|_{L^2}^2 + C_{M,s}\delta^{2(M+2)}e^{2(M+2)\sigma_0 t}.$$

Therefore

$$\frac{d}{dt}\left(e^{-(\kappa\Lambda_{M,s}+C)t}\left\|w(t,\cdot)\right\|_{L^2}^2\right) \leqslant C_{M,s}\delta^{2(M+2)}e^{2(M+2)\sigma_0 t-\kappa\Lambda_{M,s}t-Ct}, \quad t\in\left[0,T^\delta\right]. \tag{47}$$

Now we choose M large enough and κ small enough so that

$$2(M+2)\sigma_0 - \kappa\Lambda_{M,s} - C > 1.$$

At this place we fix the value of M (and of s, for example $s=M+1$) while we will make two more restrictions on κ. Since w vanishes for $t=0$ an integration of (47) yields

$$\left\|w(t,\cdot)\right\|_{L^2(\mathbb{R}\times\mathbb{T}_L)} \leqslant C_{M,s}\delta^{M+2}e^{(M+2)\sigma_0 t}, \quad t\in\left[0,T^\delta\right].$$

Therefore

$$\left\|w(T^\delta,\cdot)\right\|_{L^2(\mathbb{R}\times\mathbb{T}_L)} \leqslant C_{M,s}\kappa^{M+2}. \tag{48}$$

Let us denote by Π the projection on the nonzero modes in y i.e.

$$(\Pi v)(x,y) \equiv v(x,y) - \frac{1}{2\pi L}\int_0^{2\pi L} v(x,y)\,dy.$$

Then for every $a\in\mathbb{R}$ one has $\Pi(Q(x-a))=0$. On the other hand the first term of u^{ap} satisfies $\Pi(u^0)=u^0$ and therefore

$$\left\|\Pi\left(u^{ap}(t,\cdot)\right)\right\|_{L^2} \geqslant c_s\delta e^{\sigma_0 t} - \sum_{k=1}^{M}\delta^{k+1}\left\|\Pi\left(u^k\right)\right\|_{L^2} \geqslant c_s\delta e^{\sigma_0 t} - \sum_{k=1}^{M} C_{k,s}\delta^{k+1}e^{(k+1)\sigma_0 t},$$

where c_s is the $H^s(\mathbb{R}\times\mathbb{T}_L)$ norm of u^0. Therefore for κ small enough one has

$$\left\|\Pi\left(u^{ap}\left(T^\delta,\cdot\right)\right)\right\|_{L^2(\mathbb{R}\times\mathbb{T}_L)} \geqslant \frac{c_s\kappa}{2}. \tag{49}$$

Using (48) and (49), we may write that for every $a\in\mathbb{R}$,

$$\begin{aligned}
\left\|u^\delta\left(T^\delta,\cdot\right)-Q(\cdot-a)\right\|_{L^2} &\geqslant \left\|\Pi\left(u^\delta\left(T^\delta,\cdot\right)-Q(\cdot-a)\right)\right\|_{L^2}\\
&= \left\|\Pi\left(u^\delta\left(T^\delta,\cdot\right)-Q(\cdot)\right)\right\|_{L^2} = \left\|\Pi\left(u^{ap}\left(T^\delta,\cdot\right)+w\left(T^\delta,\cdot\right)\right)\right\|_{L^2}\\
&\geqslant \frac{c_s\kappa}{2} - \left\|\Pi\left(w\left(T^\delta,\cdot\right)\right)\right\|_{L^2} \geqslant \frac{c_s\kappa}{2} - \left\|w\left(T^\delta,\cdot\right)\right\|_{L^2} \geqslant \frac{c_s\kappa}{2} - C_{M,s}\kappa^{M+2}.
\end{aligned}$$

A final restriction on κ may insure that the right-hand side of the last inequality is bounded from below by a fixed positive constant η depending only on s (in particular η is independent of δ). This completes the proof of Theorem 2.

Remark 14. Let us observe that the analysis in the proof of Theorem 2 is quite different from the high frequency instabilities studied in [18]. In [18], the approximated solution is a high frequency linear wave with modified speed, perturbed by a low frequency wave. In Theorem 2, the approximated solution is a low frequency object modelled on the profile u^0.

3. Proof of Theorem 5

The proof follows exactly the same lines as the proof of Theorem 2 and thus we shall only sketch it. We again look for u^δ under the form $u^\delta = Q + u^{ap} + w$. At first, we need to find a most unstable eigenmode for the linearized equation to begin the construction of u^{ap}. The linearized equation about Q reads

$$iu_t + Au = 0, \qquad Au = \Delta u - u + 2uQ^2 + \bar{u}Q^2.$$

It is more convenient to introduce $U = (\operatorname{Re} u, \Im u)^t$ and to rewrite the equation as the system:

$$U_t + \begin{pmatrix} 0 & -\mathcal{L}^- \\ \mathcal{L}^+ & 0 \end{pmatrix} U = 0, \tag{50}$$
$$\mathcal{L}^- u = -\Delta u + u - Q^2 u, \qquad \mathcal{L}^+ u = -\Delta u + u - 3Q^2 u.$$

We seek unstable eigenmodes under the form

$$\Phi_{\sigma,k}(t,x,y) = e^{\sigma t} e^{\frac{iky}{L}} V(x) + e^{\bar{\sigma} t} e^{\frac{-iky}{L}} \overline{V}(x), \quad \operatorname{Re}\sigma > 0, \tag{51}$$

where $V(x) \in \mathbb{C}^2$ so that we have to solve

$$\sigma V + \begin{pmatrix} 0 & -L^- - \frac{k^2}{L^2} \\ L^+ + \frac{k^2}{L^2} & 0 \end{pmatrix} V = 0 \tag{52}$$

where

$$L^- u = -u_{xx} + u - Q^2 u, \qquad L^+ u = -u_{xx} + u - 3Q^2 u.$$

We set $\varepsilon = \frac{k}{L}$ and we look for nontrivial solutions of (52) with $\operatorname{Re}\sigma > 0$ for $\varepsilon > 0$. The first result we shall use is that

Lemma 15. *For $\varepsilon > 0$, there is at most one unstable eigenmode and there exists ε_0 such that for $0 < \varepsilon \leqslant \varepsilon_0$, there is exactly one unstable eigenmode.*

In Ref. [14], it is claimed that the result of this lemma is due to Zakharov and Rubenchik. Unfortunately, we were not able to find a copy of the paper by Zakharov and Rubenchik as this paper is quoted in [14]. We give a proof of this lemma in Appendix A.

Now, thanks to Lemma 15, for $k = 1$ and L sufficiently large there exists an unstable eigenmode. We now consider L as fixed. For every k, we have by Lemma 15 that there exists at most one $\sigma(k)$ such that $\operatorname{Re}\sigma(k) > 0$ and (52) has a solution in $L^2(\mathbb{R}; \mathbb{C}^2)$ with $\sigma = \sigma(k)$. Moreover we can easily get that the solutions of (52) satisfy the conservation law

$$\operatorname{Re}\sigma\left(\left(L^+ V_1, V_1\right) + \left(L^- V_2, V_2\right) + \frac{k^2}{L^2}|V|^2\right) = 0.$$

Therefore for large k (depending only on Q) there is no nontrivial solution of (52) with $\operatorname{Re}\sigma > 0$. Consequently, we can choose an eigenmode $\Phi_{\sigma,k}$ under the form (51) such that

$$\operatorname{Re}\sigma = \sup\{\operatorname{Re}\sigma(k)\} := \sigma_0$$

and we set $u^0 = (\Phi_{\sigma,k})_1 + i(\Phi_{\sigma,k})_2$. Observe that thanks to (52) we have $(i\partial_t + A)u^0 = 0$. The next step towards the proof of Theorem 5 is the construction on an high order unstable solution. We use the same method as previously, we use the same spaces V_K^s and we build an approximate solution under the form (14). For $1 \leqslant k \leqslant M+1$, we need to solve

$$i\partial_t u^k + Au^k = -\sum_{j+l=k-1} \left(2Qu^j \bar{u}^l + Qu^j u^l\right) - \sum_{j+l+m=k-2} u^j \bar{u}^l u^m, \quad \left(u^k\right)_{|t=0} = 0 \tag{53}$$

where the last sum is zero for $k=1$. We have the estimates:

Proposition 16. *Let u^k the solution of* (53)*, we have the estimate*

$$\left|u^k(t)\right|_{V^s_{k+1}} \leqslant C_{k,s} e^{(k+1)\sigma_0 t}, \quad \forall t \geqslant 0.$$

Note that here we do not loose regularity at each step because the nonlinear term does not involve derivatives. To prove Proposition 16, we need to prove the equivalent of Theorem 10. By using Laplace transform, we can still reduce the problem to the proof of a resolvent estimate as in Theorem 11. The proof of the low frequencies estimates rely on the same ODE argument and we shall not detail it. We shall just explain how to get the high frequencies estimates. As in Lemma 15, it is more convenient to work on the system form of the problem, and thus we consider the equation

$$(\gamma_0 + i\tau)W + \begin{pmatrix} 0 & -L^- - \frac{k^2}{L^2} \\ L^+ + \frac{k^2}{L^2} & 0 \end{pmatrix} W = H \tag{54}$$

and we want to prove that $W(\tau)$ satisfies the estimate

$$\left|W(\tau)\right|_s^2 \leqslant C(s,\gamma_0,K)\left|H(\tau)\right|_s^2 \tag{55}$$

for $\gamma_0 > \sigma_0$, $|\tau| \geqslant M \gg 1$ and $s \geqslant 1$. We first give the proof for $s=1$. The conservation law reads for $W=(w_1,w_2)$

$$\gamma_0\left(\left(L^+ w_1, w_1\right) + \left(L^- w_2, w_2\right) + \frac{k^2}{L^2}|W|^2\right) = \mathrm{Re}\left(\left(H_1, L^+ w_1\right) + \left(H_2, L^- w_2\right)\right). \tag{56}$$

At this stage, we shall use the description of the spectrum of $L^\pm$ recalled in Appendix A of this paper. We can write

$$w_2 = \alpha Q + w_2^\perp, \qquad \left(L^- w_2^\perp, w_2^\perp\right) \geqslant c_0\left|w_2^\perp\right|^2.$$

Similarly, we can write

$$w_1 = \beta\varphi_{-1} + \gamma Q_x + w_1^\perp, \qquad \left(L^+ w_1^\perp, w_1^\perp\right) \geqslant c_0\left|w_1^\perp\right|^2$$

($\varphi_{-1} \equiv Q^2$). Setting $W^\perp = (w_1^\perp, w_2^\perp)^t$ and $W_F = (\alpha,\beta,\gamma)^t \in \mathbb{C}^3$, we get from (56)

$$\gamma_0\left|W^\perp(\tau)\right|^2 \leqslant C\left(|H|_1|W|_1 + |W_F|^2\right). \tag{57}$$

Next, we can take the projection of the equation on the finite dimensional subspace generated by $(0,Q)$, $(Q_x,0)$, $(\varphi_{-1},0)$ to get

$$\left(\gamma_0 + |\tau| - C\right)|W_F|^2 \leqslant C(K)\left(\left|W^\perp\right|^2 + |H|^2\right). \tag{58}$$

As for the KP-I equation, a suitable combination of (56)–(58) with the use of (30) gives (55) for $s=1$ for $|\tau|$ large enough. To get higher order derivatives, we use approximate higher order conservation laws. Namely, if we set $L_{s+1}w = \partial_x^{2(s+1)} w$ then

$$-\mathrm{Re}\left(L^- w_2 + \frac{k^2}{L^2} w_2, L_{s+1} w_1\right) + \mathrm{Re}\left(L^+ w_1 + \frac{k^2}{L^2} w_1, L_{s+1} w_2\right) = \mathcal{O}(1)\left(|W|_s^2 + |W|_s\left|\partial_x^{s+1} W\right|\right)$$

which implies

$$\gamma_0\left|\partial_x^{s+1} W\right|^2 \leqslant C\left(|W|_s^2 + |W|_s|W|_{s+1} + |H|_{s+1}\left|\partial_x^{s+1} W\right| + |H|_s|W|_s\right)$$

and we conclude thanks to (30) via an induction argument.

To end the proof of Theorem 5, we seek for a solution of (8) under the form $u^\delta = Q + u^{ap} + w$, with $w_{/t=0} = 0$ so that w solves the equation

$$i w_t + A w + 2\left|u^{ap}\right|^2 w + \left(u^{ap}\right)^2 \overline{w} + \mathcal{N}\left(u^{ap}, w\right) + |w|^2 w = F \tag{59}$$

with

$$\|F\|_{H^s(\mathbb{R}\times\mathbb{T}_L)} \leqslant C_{M,s}\delta^{M+2} e^{(M+2)\sigma_0 t},$$

and the bilinear term satisfies

$$\left\|\mathcal{N}\left(u^{ap}, w\right)\right\|_{H^s(\mathbb{R}\times\mathbb{T}_L)} \leqslant C\left|u^{ap}\right|_{W^{s,\infty}}\|w\|_s^2.$$

Since here we do not have a global existence result available, we shall first prove that this last equation has a smooth solution w which remains defined on a time scale sufficiently long to see the instability.

A classical existence result for this equation based on Duhamel formula and Sobolev embedding gives that there exists a local solution $w \in \mathcal{C}([0,T], H^s)$ for $s > 1$. Moreover, we can define a maximum time T^* such that

$$T^* = \sup\left\{T,\ \forall t \in [0,T],\ \left\|w(t)\right\|_{H^s} \leqslant 1\right\}.$$

The H^s energy estimate for (59) gives for $t \in [0, T^*)$ that

$$\frac{d}{dt}\left\|w(t)\right\|_{H^s}^2 \leqslant C\left(1 + \left|u^{ap}\right|_{W^{s,\infty}}\right)\|w\|_{H^s} + C_{M,s}\delta^{2(M+2)}e^{2(M+2)\sigma_0 t}$$

where C is an absolute constant (which depends on Q). Consequently for

$$t \leqslant \operatorname{Min}\left(T^\delta := \frac{\log(\kappa/\delta)}{\sigma_0}, T^*\right),$$

we get

$$\frac{d}{dt}\left\|w(t)\right\|_{H^s}^2 \leqslant (C + \kappa\Lambda_{M,s})\|w\|_{H^s} + C_{M,s}\delta^{2(M+2)}e^{2(M+2)\sigma_0 t}$$

and hence by the choice

$$2(M+2)\sigma_0 - \kappa\Lambda_{M,s} - C > 0,$$

we get that

$$\left\|w(t)\right\|_{H^s(\mathbb{R}\times\mathbb{T}_L)} \leqslant C_{M,s}\kappa^{M+2}, \quad t \leqslant \operatorname{Min}\left(T^\delta, T^*\right). \tag{60}$$

In particular for κ sufficiently small, we get that

$$\left\|w(t)\right\|_{H^s(\mathbb{R}\times\mathbb{T}_L)} \leqslant \frac{1}{2}, \quad t \leqslant \operatorname{Min}\left(T^\delta, T^*\right).$$

By definition of T^*, this proves that $T^* \geqslant T^\delta$ so that the time of existence of a smooth solution is in any case large enough to see an instability. The end of the proof follows the same lines as previously, using again the projection Π on nonzero modes in y, we write for every $a \in \mathbb{R}$, $\gamma \in \mathbb{R}$,

$$\begin{aligned}\left\|u^\delta\left(T^\delta,\cdot\right) - e^{i\gamma}Q(\cdot - a)\right\|_{L^2} &\geqslant \left\|\Pi\left(u^\delta\left(T^\delta,\cdot\right) - e^{i\gamma}Q(\cdot - a)\right)\right\|_{L^2} = \left\|\Pi\left(u^{ap}\left(T^\delta,\cdot\right) + w\left(T^\delta,\cdot\right)\right)\right\|_{L^2} \\ &\geqslant \frac{c_s\kappa}{2} - \left\|\Pi\left(w\left(T^\delta,\cdot\right)\right)\right\|_{H^s} \geqslant \frac{c_s\kappa}{2} - \left\|w\left(T^\delta,\cdot\right)\right\|_{H^s} \geqslant \frac{c_s\kappa}{2} - C_{M,s}\kappa^{M+2},\end{aligned}$$

where we have used (60) in the last inequality. A final restriction of κ gives the instability result.

Acknowledgement

We are grateful to Robert Pego for pointing out to us the reference [26]. A previous version of this text, before we were aware of the Zakharov work [26], was posted to the arxiv of preprints on December 2006. We also thank the referees for their valuable remarks.

Appendix A

A.1. *Proof of Theorem 8*

In order to have the same equations as in [1], we look for solutions of (10) under the form

$$u(t,x,y) = e^{\frac{\lambda t}{2}}e^{\frac{iky}{L}}U\left(\frac{x}{2}\right)$$

with $U \in L^2$, $\operatorname{Re}\lambda > 0$ and $k \neq 0$. Note that this last condition is natural since for $k = 0$, we cannot find instability since the KdV soliton is stable in the KdV equation. We get for U the equation

$$4\lambda U_z + 4(\Phi U)_{zz} + U_{zzzz} - 4U_{zz} + 3\eta^2 U = 0 \tag{61}$$

where we have set

$$3\eta^2 = \frac{16k^2}{L^2} \tag{62}$$

and $\Phi = 3\operatorname{sech}^2 z$. Since Φ and its derivatives tend to zero exponentially fast when $z \to \pm\infty$, the solutions of (61) have the same behaviour as the solutions of

$$4\lambda U_z + U_{zzzz} - 4U_{zz} + 3\eta^2 U = 0$$

when $z \to \pm\infty$. The characteristic values μ of this linear equation are the roots of the polynomial P defined by

$$P(\mu) = \mu^4 - 4\mu^2 + 4\lambda\mu + 3\eta^2. \tag{63}$$

Consequently for $\eta \neq 0$ and $\gamma = \operatorname{Re}\lambda > 0$, $\mu \notin i\mathbb{R}$. Indeed, if $\mu = i\xi \in i\mathbb{R}$, then ξ should solve

$$\xi^4 + 4\xi^2 + 4\lambda\xi i + 3\eta^2 = 0$$

which cannot have a real root ξ for $\eta \neq 0$ and $\operatorname{Re}\lambda \neq 0$. A consequence of this is that the number of roots μ of positive real part of P is independent of the parameters. Since the limit $\eta \to +\infty$ gives

$$\mu = 3^{\frac{1}{4}}\omega\sqrt{\eta} + \mathcal{O}(1), \quad \omega^4 = -1$$

we finally get that P has two roots of positive real parts and two roots of negative real parts. This proves that the solutions of (61) either tend to zero or blows-up exponentially fast when $z \to \pm\infty$. Moreover, the stable manifold and the unstable manifold have the same dimension 2. Finally, there will be a nontrivial bounded solution of (61) if and only if U belongs simultaneously to the stable and the unstable manifold.

In our case, this condition can be computed explicitly. Indeed, we notice that for $\gamma > 0$, $\eta \neq 0$ there is a bounded solution of (61) if and only if $U = g_{zz}$ with g bounded which solves

$$g_{zzzz} + 4\Phi g_{zz} + 4\lambda g_z - 4g_{zz} + 3\eta^2 g = 0. \tag{64}$$

Note that the asymptotic behaviour of the solutions of this equation is also determined by the characteristic values given by the roots of P so that this equation also has stable and unstable manifolds of dimension 2. Moreover, if μ is a root of P, then

$$g_\mu(z) = e^{\mu z}\big(\mu^3 + 2\mu + \lambda - 3\mu^2 \tanh z\big) \tag{65}$$

is a solution of (64). In particular, if $\operatorname{Re}\mu > 0$, then g_μ is in the unstable manifold. Moreover, when P has two simple roots μ_1, μ_2 of positive real parts, then one can prove (see [1] for details) that g_{μ_1}, g_{μ_2} are linearly independent so that they constitute a basis of the unstable manifold. Consequently, any bounded solution of (64) must be a linear combination of g_{μ_1}, and g_{μ_2}.

Now, let us define

$$C_+(\mu) = \lim_{z \to +\infty} e^{-\mu z} g_\mu = \mu^3 + 2\mu + \lambda - 3\mu^2.$$

Then, if $C_+(\mu_i) \neq 0$, $i = 1, 2$, we cannot have nontrivial solutions which tend to zero when $z \to +\infty$. Consequently, this proves that when the positive real part roots of P are simple, then a necessary condition to have bounded solutions of (64) is that $C_+(\mu) = 0$ for some root μ of P of positive real part. In the case where μ is a double root, then one can check that the same condition holds. Indeed it suffices to take g_μ and $\partial_\mu g$ as a basis of the unstable manifold (again, we refer to [1] for details).

It remains to study the equation $C_+(\mu) = 0$ with μ a root of P of positive real part. This yields the system of algebraic equation

$$P(\mu) = 0, \qquad \mu^3 + 2\mu + \lambda - 3\mu^2 = 0, \tag{66}$$

with the constraint $\operatorname{Re}\mu > 0$. The elimination of λ between the two algebraic equations gives

$$\lambda = -\mu(\mu-1)(\mu-2), \qquad \eta^2 = \mu^2(\mu-2)^2. \tag{67}$$

The analysis of this system gives that there is a solution with $\operatorname{Re}\lambda > 0$, $\operatorname{Re}\mu > 0$, if and only if given $\mu \in (0,2)$, η and λ are given by

$$\eta = \mu(2-\mu), \qquad \lambda = -\mu(\mu-1)(\mu-2).$$

Finally, we notice that when $C^+(\mu) = 0$, we have

$$g_\mu(z) = 3\mu^2 e^{\mu z}(1-\tanh z) = \mathcal{O}\big(e^{-(2-\mu)z}\big)$$

and hence $\lim_{z\to+\infty} g_\mu = 0$ since $2-\mu > 0$. This proves that $C_+(\mu) = 0$ with μ a root of P of positive real part is also a sufficient condition to have a bounded solution on $\mathbb{R}$. This ends the proof.

A.2. Proof of Lemma 15

Set $V(x) = (u(x), v(x))^t$ with u, v real valued functions. Then (52) implies that

$$L^+u + \varepsilon^2 u = -\sigma v, \qquad L^-v + \varepsilon^2 v = \sigma u. \tag{68}$$

Observe that if (u,v) is a solution of (68) corresponding to a complex number σ then $(u,-v)$ is a solution of (68) corresponding to $-\sigma$. The operators L^+ and L^- have classical self adjoint realizations on $L^2(\mathbb{R})$ and their spectra are well-known (see e.g. [24,25]). The operator L^+ has exactly two simple eigenvalues -3 and 0 with corresponding eigenfunctions Q^2 and Q'. The continuous spectrum of L^+ is $[1,\infty[$. The operator L^- has only the simple eigenvalue 0 with corresponding eigenfunction Q and the continuous spectrum of L^- is $[1,\infty[$. Observe that (68) may be written as

$$\mathcal{L}\begin{pmatrix} u \\ v \end{pmatrix} := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} L^+ + \varepsilon^2 & 0 \\ 0 & L^- + \varepsilon^2 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = -\sigma\begin{pmatrix} u \\ v \end{pmatrix}. \tag{69}$$

Thanks to the above discussion on the spectrum of L^+ and L^-, we obtain that

$$\begin{pmatrix} L^+ + \varepsilon^2 & 0 \\ 0 & L^- + \varepsilon^2 \end{pmatrix}$$

has at most one negative eigenvalue which should be simple. Therefore, thanks to [21, Theorem 3.1], there cannot be more than one unstable mode.

For $\varepsilon \ll 1$, the bifurcation of the eigenvalue zero in the case $\varepsilon = 0$ can be explicitly computed. Note that zero is an isolated eigenvalue so that we can use perturbation methods as in finite dimension (see [16] Theorem 1.8, Chapter 7). In the case $\varepsilon = 0$, we have that zero is an eigenvalue of multiplicity 4 for the linear map introduced in the left-hand side of (69) (see [25]). The generalized eigenspace splits into two two-dimensional invariant sub-spaces corresponding to the eigenvectors $(u,v) = (Q',0)$ and $(u,v) = (0,Q)$ respectively. As generalized eigenvectors, we can take $\frac{1}{2}(Q + xQ_x, 0)$ and $(0, \frac{1}{2}xQ)$ which verify

$$\mathcal{L}\begin{pmatrix} \frac{1}{2}(Q + xQ_x) \\ 0 \end{pmatrix} = -\begin{pmatrix} 0 \\ Q \end{pmatrix}, \qquad \mathcal{L}\begin{pmatrix} 0 \\ \frac{1}{2}xQ \end{pmatrix} = \begin{pmatrix} Q_x \\ 0 \end{pmatrix}.$$

Thanks to the analytic dependence in ε (see [16]), we look for a σ in (68) of the form $\sigma = \omega_1\varepsilon + \omega_2\varepsilon^2 + \cdots$ with $\operatorname{Re}(\omega_1) > 0$ which corresponds to an unstable mode. We will see below that the invariant subspace corresponding to $(u,v) = (Q',0)$ splits to two one-dimensional invariant spaces corresponding to eigenvalues with ω_1 purely imaginary and, what is of importance for our purposes, the invariant subspace corresponding to $(u,v) = (0,Q)$ splits to two one-dimensional invariant spaces corresponding to eigenvalues with positive and negative ω_1. The eigenvector corresponding to a positive ω_1 provides the unstable eigenmode. Assume that u and v are expanded as

$$u = u_0 + u_1\varepsilon + u_2\varepsilon^2 + \cdots, \qquad v = v_0 + v_1\varepsilon + v_2\varepsilon^2 + \cdots.$$

Then (u_0, v_0) satisfy $L^+(u_0) = L^-(v_0) = 0$. Thus there exist two numbers α_0 and β_0 such that $u_0 = \alpha_0 Q'$ and $v_0 = \beta_0 Q$. Then (u_1, v_1) are solutions of $L^+(u_1) = -\omega_1\beta_0 Q$, $L^-(v_1) = \omega_1\alpha_0 Q'$. Therefore there exist two numbers α_1 and β_1 such that

$$u_1(x) = \frac{\omega_1\beta_0}{2}\big(xQ'(x) + Q(x)\big) + \alpha_1 Q'(x), \qquad v_1(x) = -\frac{\omega_1\alpha_0}{2}\big(xQ(x)\big) + \beta_1 Q(x).$$

Next, (u_2, v_2) are solutions of

$$L^+(u_2) = -\alpha_0 Q' - \omega_1\left(-\frac{\omega_1\alpha_0}{2}(xQ) + \beta_1 Q\right) - \omega_2\beta_0 Q, \tag{70}$$

$$L^-(v_2) = -\beta_0 Q + \omega_1\left(\frac{\omega_1\beta_0}{2}(xQ' + Q) + \alpha_1 Q'\right) + \omega_2\alpha_0 Q'. \tag{71}$$

The first equation of (70) can be solved if the right-hand side is orthogonal to Q' (the kernel of L^+). This imposes that either $\alpha_0 = 0$ or

$$\int_{-\infty}^{\infty}\left(-\alpha_0 Q'(x) - \omega_1\left(-\frac{\omega_1\alpha_0}{2}\big(xQ(x)\big) + \beta_1 Q(x)\right) - \omega_2\beta_0 Q(x)\right)Q'(x)\,dx = 0,$$

which implies that $\omega_1^2 = -4\theta^2$, where $\theta \equiv \|Q'\|_{L^2(\mathbb{R})}/\|Q\|_{L^2(\mathbb{R})}$, i.e. $\omega_1 = \pm i\theta$. Hence if $\alpha_0 \neq 0$ we have an eigenmode with purely imaginary ω_1.

The second equation of (70) can be solved only if the right-hand side is orthogonal to the kernel of L^-, i.e. to Q. This imposes that either $\beta_0 = 0$ or

$$\int_{-\infty}^{\infty}\left(-\beta_0 Q(x) + \omega_1\left(\frac{\omega_1\beta_0}{2}\big(xQ'(x) + Q(x)\big) + \alpha_1 Q'(x)\right) + \omega_2\alpha_0 Q'(x)\right)Q(x)\,dx = 0$$

which implies that $\omega_1^2 = 4$, i.e. $\omega_1 = \pm 2$. From the above discussion, we have that either $\alpha_0 = 0$ or $\beta_0 = 0$. If $\alpha_0 \neq 0$ (and thus $\beta_0 = 0$) we obtain purely imaginary ω_1 and have the bifurcation of $(Q', 0)$. These modes are not of interest for us. If $\beta_0 \neq 0$ (and thus $\alpha_0 = 0$) we indeed have an eigenvalue with positive ω_1. This mode corresponds to the eigenvector which is the bifurcation of $(u, v) = (0, Q)$ to the unstable mode of the form (51) for the linearized about Q cubic NLS equation.

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