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Singular asymptotic expansions for Dirichlet eigenvalues and eigenfunctions of the Laplacian on thin planar domains ☆

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Abstract

We consider the Laplace operator with Dirichlet boundary conditions on a planar domain and study the effect that performing a scaling in one direction has on the spectrum. We derive the asymptotic expansion for the eigenvalues and corresponding eigenfunctions as a function of the scaling parameter around zero. This method allows us, for instance, to obtain an approximation for the first Dirichlet eigenvalue for a large class of planar domains, under very mild assumptions.

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1. Introduction

The study of the spectrum of the Laplace operator on thin domains has received much attention in the mathematical literature over the last few years. Apart from the connection to certain physical systems such as quantum waveguides, this limit situation is also of interest as it may provide insight into certain questions related to the spectrum of the Laplacian – see the recent paper by Friedlander and Solomyak [5] for some references on both counts.

The purpose of the present paper is to study this spectrum in the singular limit around a line segment in the plane. More precisely, given a planar domain we consider its scaling in one direction, so that in the limit we have a line segment orthogonal to this direction. In particular, and under mild smooth assumptions on the domain, we derive the

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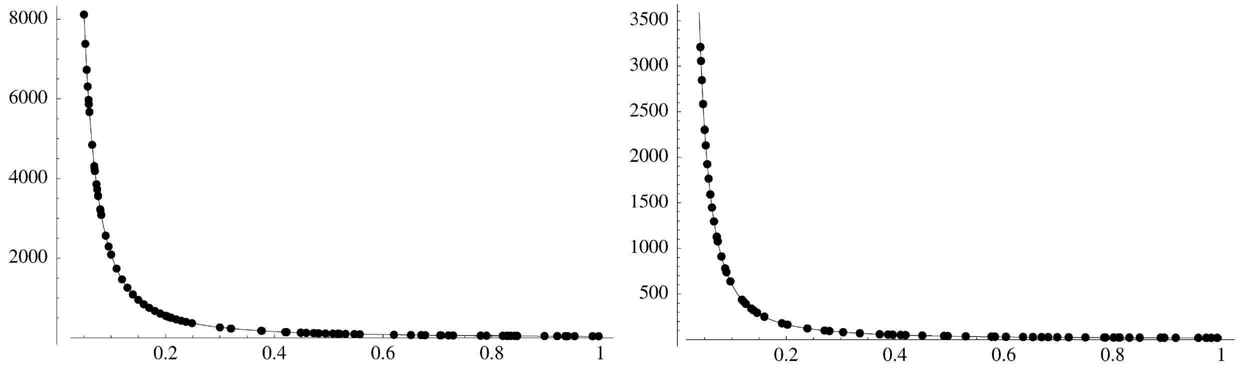

Fig. 1. Graphs of the four-term asymptotic expansions obtained for the first eigenvalue of the lemniscate and the bean curve in Examples 2 and 3 presented in Section 5. The points are numerical approximations to the eigenvalue with an error not greater than 10^{-5}.

asymptotic expansion for the first Dirichlet eigenvalue thus showing that the coefficients in this series have a simple explicit expression in terms of the functions defining the domain – see Theorem 2 below.

Due to the notorious lack of explicit expressions for Dirichlet eigenvalues of planar domains, this seems to be a possible path towards obtaining information about such eigenvalues, and indeed it was one of the motivations behind our work. As an example, consider the case of the family of ellipses centred at $(1/2, 0)$ and with axes $1/2$ and $\varepsilon/2$. In this case Theorem 2 yields that

$$\lambda_1(\varepsilon) = \frac{\pi^2}{\varepsilon^2} + \frac{2\pi}{\varepsilon} + 3 + \left(\frac{11}{2\pi} + \frac{\pi}{3}\right)\varepsilon + \mathcal{O}(\varepsilon^2), \quad \text{as } \varepsilon \to 0, \tag{1.1}$$

thus complementing the asymptotic expansion for the first eigenvalue of ellipses around the disk derived by Joseph in [6]. Further terms in the expansion may be obtained by means of Lemma 3.2. The graphs of the four-term asymptotic expansions for two other examples, namely the lemniscate and the *bean* curve are shown in Fig. 1. For a discussion of these and other examples, see the last section.

The advantages of our method are twofold. On the one hand, it does not require any a priori knowledge neither of the eigenvalues nor of the eigenfunctions of a particular domain, as the perturbation is always made around the singular case of the line segment. In the above example, for instance, Joseph's method requires knowledge of the first eigenvalue of the disk. This limits applications of the same type of argument to general examples. Furthermore, it is not evident that even in cases where the eigenvalue of the unperturbed domain is known one can determine explicitly the value of the coefficients of the perturbation around this domain in terms of known constants. This is illustrated by the example in Section 5 in [6], where one of the coefficients in the expansion gives rise to a double series. On the other hand, and by its own nature, our method is particularly suited to dealing with long and thin domains where numerical methods will tend to have problems.

The fact that we consider a singular perturbation does pose, however, some difficulties. In order to understand these, let us begin by observing that the behaviour of the asymptotic series obtained for the eigenvalues depends on what happens at the point of the line segment where the vertical width is maximal, say $\bar{x}$. More precisely, the coefficients appearing in the expansion will depend on the value that the functions defining the domain and their derivatives take at $\bar{x}$, but not on what happens away from this point. This is not surprising, since it is in this region that the first eigenfunction will tend to concentrate as the parameter goes to zero. In order to derive the asymptotics, we assumed that there existed only one such point and that the domain was $\mathcal{C}^\infty$ smooth in a small neighbourhood of $\bar{x}$. While the former condition may be somewhat relaxed, and a finite number of points of maximal height could, in principle, be considered, if the latter condition is not satisfied then the powers appearing in the asymptotic series will actually be different.

A first point is thus that the type of expansion obtained will depend on the local behaviour at $\bar{x}$. To illustrate what may happen, consider the following example studied by the second author using a different approach [3]. Let T_β be an obtuse isosceles triangle where the largest side is assumed to have fixed unit length, and where the equal angle β approaches zero. In this case we have

$$\lambda_1(T_\beta) = \frac{4\pi^2}{\beta^2} - \frac{4 \cdot 2^{2/3} a_1' \pi^{4/3}}{\beta^{4/3}} + \mathcal{O}(\beta^{-2/3}) \quad \text{as } \beta \to 0,$$

where $a_1' \approx -1.01879$ is the first negative zero of the first derivative of the Airy function of the first kind. We thus see that having the maximal width at a corner point introduces fractional powers in the expansion.

On the other hand, it can be seen from our results that what happens away from $\bar{x}$ does not influence the coefficients appearing in the expansion in Theorems 1 and 2. Thus in the example given above for ellipses, changing the domain outside a band containing the mid-region will yield precisely the same coefficients although the expression for the first eigenvalue as a function of ε will certainly be different. This means that the series described in these theorems cannot, in general, converge to the desired eigenvalue, and that there should exist a *tail* term which goes to zero faster than any power of ε, which our analysis does not allow us to recover. An interesting question is thus to determine the nature of such a term and when can we ensure that the series expansion derived here is actually convergent to the corresponding eigenvalue.

Note that although we address the same problem as in [5], both our approach and results are quite different from those in that paper. In particular, we obtain the full asymptotic expansion for the eigenvalues and eigenfunctions. Furthermore, the coefficients in these expansions are obtained as a simple explicit function of the value of the functions involved and their derivatives at the point where the function H takes its maximum. Another difference is that we consider a two-parameter set of eigenvalues, while the method used in [5] leads to a one-parameter set. On the other hand, our approach required the function H to be smooth in a neighbourhood of this point, while the results in [5] are more general in this respect, allowing for the existence of corners, for instance. We hope to be able to consider this situation in a forthcoming paper. Furthermore, in [5] the authors actually prove convergence in the norm of the resolvents, which is a stronger property than convergence of the eigenvalues and eigenfunctions.

The plan of the paper is as follows. In the next section we state the results in the paper. In Section 3 we prove the general form of the asymptotic expansions for eigenvalues and eigenfunctions. The proof is split into two parts. First we construct the asymptotics expansions formally. The main idea here is to use the ansatz of boundary layer type which localizes in a vicinity of the point $\bar{x}$ mentioned above. After formal constructing, the formal expansions are justified, i.e., the estimates for the error term are obtained. Here the main tool is Vishik–Lyusternik's lemma. Section 4 contains the study of the expansion for the first eigenvalue. This consists in identifying the eigenvalue which corresponds to a positive eigenfunction, and then obtaining the explicit form of the terms in the expansion of the eigenvalue. In the last section we consider some applications of our results to specific domains.

2. Statement of results

Let $h_\pm = h_\pm(x_1) \in \mathcal{C}[0,1]$ be arbitrary functions and write $H(x_1) := h_+(x_1) + h_-(x_1)$ for x_1 in $[0,1]$. We shall consider the thin domain defined by

$$\Omega_\varepsilon = \{x\colon\ x_1 \in (0,1),\ -\varepsilon h_-(x_1) < x_2 < \varepsilon h_+(x_1)\},$$

for which we assume that the function h attains its global maximum at a single point $\bar{x} \in (0,1)$, and that $H(x_1) > 0$ for $x_1 \in (0,1)$. Note that the cases where either $H(0) = 0$ or $H(1) = 0$ are not excluded. We also assume that the functions $h_\pm$ are infinitely differentiable in a small neighbourhood of $\bar{x}$, say $(\bar{x} - \delta, \bar{x} + \delta)$. For the sake of brevity, in what follows we shall write $H_0 := H(\bar{x})$ and denote by H_i the ith derivative of H at $\bar{x}$. In the same way we denote the derivatives of h_- by h_i. We shall assume that there exists $k \geqslant 1$ such that

$$H_i = 0, \quad i = 1, \ldots, 2k-1, \qquad H_{2k} < 0. \tag{2.1}$$

Our aim is to study the asymptotic behaviour of the eigenvalues and the eigenfunctions of the Dirichlet Laplacian $-\Delta^D_{\Omega_\varepsilon}$ in Ω_ε. Let $\chi = \chi(t) \in \mathcal{C}^\infty(\mathbb{R})$ be a non-negative cut-off function which equals one when $|\xi_1 - \bar{x}| < \delta/3$ and vanishes for $|t - \bar{x}| > \delta/2$. We denote $\Omega^\delta_\varepsilon := \Omega_\varepsilon \cap \{x\colon\ |t - \bar{x}| < \delta\}$.

The main results of the paper are contained in the following two theorems.

Theorem 1. *Under the above conditions, there exist eigenvalues $\lambda_{n,m}(\varepsilon)$, $n, m \in \mathbb{N}$, of the operator $-\Delta^D_{\Omega_\varepsilon}$ whose asymptotic expansions as ε goes to zero read as*

$$\lambda_{n,m}(\varepsilon) = \varepsilon^{-2} c_0^{(n,m)} + \varepsilon^{-2} \sum_{i=2k}^{\infty} \eta^i c_i^{(n,m)}, \quad \eta := \varepsilon^\alpha, \ \alpha := \frac{1}{k+1}, \tag{2.2}$$

where

$$\begin{aligned} c_0^{(n,m)} &= \frac{\pi^2 n^2}{H_0^2}, \qquad c_{2k}^{(n,m)} = \Lambda_{n,m}, \qquad c_{2k+1}^{(n,m)} = 0, \\ c_{2k+2}^{(n,m)} &= \frac{\pi^2 n^2 h_1^2}{H_0^2} - \frac{2\pi^2 n^2 H_{2k+1}}{(2k+1)! H_0^3} \big(\xi_1^{2k+1} \Psi_1^{(n,m)}, \Phi_{n,m}\big)_{L_2(\mathbb{R})} \\ &\quad - \frac{2\pi^2 n^2 H_{2k+2}}{(2k+2)! H_0^3} \big\| \xi_1^{k+1} \Phi_{n,m} \big\|^2_{L_2(\mathbb{R})} + \boldsymbol{\delta}_{1k} \frac{3\pi^2 n^2 H_{2k}^2}{((2k)!)^2 H_0^4} \big\| \xi_1^{2k} \Phi_{n,m} \big\|^2_{L_2(\mathbb{R})}. \end{aligned} \tag{2.3}$$

Here $\Lambda_{n,m}$ and $\Phi_{n,m}$ are the eigenvalues and the associated orthonormalized in $L_2(\mathbb{R})$ eigenfunctions of the operator

$$G_n := -\frac{d^2}{d\xi_1^2} - \frac{2\pi^2 n^2 H_{2k}}{(2k)! H_0^3} \xi_1^{2k}$$

in $L_2(\mathbb{R})$, $\boldsymbol{\delta}_{1k}$ is the Kronecker delta, and $\Psi_1^{(n,m)} \in C^\infty(\mathbb{R}) \cap L_2(\mathbb{R})$ is the exponentially decaying solution to

$$(G_n - \Lambda_{n,m}) \Psi_1^{(n,m)} = \frac{2\pi^2 n^2 H_{2k+1} \xi_1^{2k+1}}{(2k+1)! H_0^3} \Phi_{n,m},$$

which is orthogonal to $\Phi_{n,m}$ in $L_2(\mathbb{R})$.

Given an eigenvalue $\lambda_{n,m}(\varepsilon)$, let $\lambda_{n,m}^{(i)}(\varepsilon)$ be the eigenvalues of $-\Delta^D_{\Omega_\varepsilon}$ having the same asymptotic expansions (2.2) *as $\lambda_{n,m}(\varepsilon)$, and $\lambda_{n,m}^{(1)}(\varepsilon) := \lambda_{n,m}(\varepsilon)$. Then there exists a linear combination $\psi_{n,m}(x, \varepsilon)$ of the eigenfunctions associated with $\lambda_{n,m}^{(i)}(\varepsilon)$, whose asymptotic expansion reads as follows*

$$\psi_{n,m}(x, \varepsilon) = \chi(x_1) \sum_{i=0}^{\infty} \eta^i \psi_i^{(n,m)}(\xi), \tag{2.4}$$

in the $W_2^1(\Omega_\varepsilon)$-norm and $W_2^2(\Omega_\varepsilon^{\delta/3})$-norm, where

$$\begin{aligned} \xi_1 &= \frac{x_1 - \bar{x}}{\varepsilon^\alpha}, \qquad \xi_2 = \frac{x_2 + \varepsilon h_-(x_1)}{\varepsilon H(x_1)}, \\ \psi_0^{(n,m)}(\xi) &= \Phi_{n,m}(\xi_1) \sin \pi n \xi_2, \qquad \psi_1^{(n,m)}(\xi) = \Psi_1^{(n,m)}(\xi_1) \sin \pi n \xi_2. \end{aligned}$$

The remaining coefficients of the series (2.2), (2.4) *are determined by Lemma* 3.2.

The second result gives explicit expressions for the first four non-vanishing terms in the asymptotic expansion of the first eigenvalue in terms of the functions $h_\pm$ and their derivatives at $\bar{x}$ in the case where H_2 is negative.

Theorem 2. *For any $N \geqslant 1$ there exists $\varepsilon_0 = \varepsilon_0(N) > 0$ such that for $\varepsilon < \varepsilon_0$ the first N eigenvalues are $\lambda_{1,m}(\varepsilon)$, $m = 1, \ldots, N$. These eigenvalues are simple, and the asymptotic expansions of the associated eigenfunctions are given by* (2.4)*, where $\psi_{1,m}$ stands for the corresponding eigenfunction. In particular, if $k = 1$, the lowest eigenvalue will have the expansion*

$$\lambda_{1,1}(\varepsilon) = \frac{c_0^{(1,1)}}{\varepsilon^2} + \frac{c_2^{(1,1)}}{\varepsilon} + c_4^{(1,1)} + c_6^{(1,1)} \varepsilon + \mathcal{O}\big(\varepsilon^2\big), \quad \text{as } \varepsilon \to 0,$$

where

$$
\begin{aligned}
c_0^{(1,1)} &= \frac{\pi^2}{H_0^2}, \qquad c_2^{(1,1)} = \frac{\pi(-H_2)^{1/2}}{H_0^{3/2}},\\
c_4^{(1,1)} &= \frac{\pi^2 h_1^2}{H_0^2} - \frac{9H_2}{16H_0} - \frac{11H_3^2}{144H_2^2} + \frac{H_4}{16H_2},\\
c_6^{(1,1)} &= \frac{H_0^{3/2}}{\pi(-H_2)^{1/2}}\biggl(\frac{\pi^2 h_2^2}{2H_0^2} - \frac{\pi^2(h_1H_3 + H_2^2)h_2}{2H_0^2 H_2} + \frac{83H_2^2}{256H_0^2} + \frac{19H_3^2 H_4}{384H_2^3} - \frac{155H_3^4}{6912H_2^4} + \frac{29H_3^2}{384H_0H_2}\\
&\quad - \frac{9H_4}{128H_0} - \frac{13H_3H_5}{576H_2^2} - \frac{7H_4^2}{768H_2^2} + \frac{H_6}{192H_2} + \frac{\pi^2 H_2^2}{6H_0^2} - \frac{\pi^2 H_2 h_1^2}{2H_0^3} + \frac{\pi^2 h_3 h_1}{2H_0^2}\biggr).
\end{aligned} \tag{2.5}
$$

In the case of higher eigenvalues, which of the eigenvalues $\lambda_{n,m}(\varepsilon)$ corresponds to a given eigenvalue λ_ℓ will depend on the value of ε. Furthermore, generally speaking, the eigenvalues given in Theorem 1 do not exhaust the whole spectrum. Indeed, assume that the function H has a local maximum at a point $\tilde{x} \neq \bar{x}$. Then one can reproduce the proof of Theorem 1 and show that there exists an additional infinite two-parametric set of the eigenvalues associated with $\tilde{x}$; the corresponding eigenfunctions are localized in a vicinity of $\tilde{x}$.

Let $\gamma := \lambda_{1,2} - \lambda_{1,1}$ denote the length of the first spectral gap. The above result allows us to obtain the first term in the asymptotic expansion for γ.

Corollary 2.1. *Under the above hypothesis, the quantity $\varepsilon\gamma(\varepsilon)$ remains bounded as ε goes to zero. If $k = 1$, then*

$$
\gamma(\varepsilon) = \frac{2\pi(-H_2)^{1/2}}{H_0^{3/2}\varepsilon} + \mathcal{O}(1), \quad \text{as } \varepsilon \to 0.
$$

We remark that, in general, the spectral gap is unbounded when either the diameter or the area are kept constant, the simplest example being that of a circular sector where the opening angle approaches zero. From the results for obtuse isosceles triangles in [3] we know this to be the case also for one-parameter families of domains of the type considered here. What this result shows is that under the conditions above the gap will remain bounded as ε goes to zero if we keep the area fixed. To see this, it is sufficient to note that $|\Omega_\varepsilon| = \varepsilon|\Omega_1|$ and thus $|\Omega_\varepsilon|\gamma(\varepsilon)$ is also bounded. In particular, this shows that regularity at the point of maximum height plays an important role in bounding the gap. Note that this boundedness is not uniform on the domain. Also, if we fix the diameter instead of the area then the gap will still go to infinity as ε goes to zero. For a sharp upper bound for the gap and a numerical study of the same quantity, see [4,1], respectively.

3. Proof of Theorem 1

Let λ_ε and ψ_ε be an eigenvalue and an associated eigenfunction of $-\Delta_{\Omega_\varepsilon}^D$, respectively. We construct the asymptotics for λ_ε and ψ_ε as the series

$$
\lambda_\varepsilon = \varepsilon^{-2}\mu_\varepsilon, \qquad \mu_\varepsilon = c_0 + \sum_{i=2k}^{\infty} \eta^i c_i, \qquad \psi_\varepsilon(\xi) = \sum_{i=0}^{\infty} \eta^i \psi_i(\xi), \tag{3.1}
$$

where c_i and ψ_i are the coefficients and functions to be determined.

In what follows we will show that the function ψ_ε is exponentially small (with respect to ε) outside $\Omega_\varepsilon^\delta$. This is why we are interested only in determining ψ_i on $\Omega_\varepsilon^\delta$. After passing to the variables ξ, the domain $\Omega_\varepsilon^\delta$ becomes $\{\xi\colon |\xi_1| < \delta\eta^{-1},\ 0 < \xi_2 < 1\}$. As $\eta \to 0$, the latter domain "tends" to the strip $\Pi := \{\xi\colon 0 < \xi_2 < 1\}$. This is why we will consider the functions ψ_i as defined on Π. The mentioned exponential decaying of the eigenfunction is implied by the fact that all the coefficients ψ_i decay exponentially as $|\xi_1| \to \pm\infty$, in other words, we postulate the latter for ψ_i.

Having the made assumptions in mind, we rewrite the eigenvalue equation for ψ_ε considered in $\Omega_\varepsilon^\delta$ in the variables ξ,

$$
\left(-K_{11}\frac{\partial^2}{\partial\xi_1^2} - 2K_{12}\frac{\partial^2}{\partial\xi_1\partial\xi_2} - K_{22}\frac{\partial^2}{\partial\xi_2^2} - K_2\frac{\partial}{\partial\xi_2}\right)\psi_\varepsilon = \mu_\varepsilon\psi_\varepsilon \quad \text{in } \Pi, \qquad \psi_\varepsilon = 0 \quad \text{on } \partial\Pi, \tag{3.2}
$$

where $K_{ij} = K_{ij}(\xi_1\eta, \xi_2, \eta)$, $K_i = K_i(\xi_1\eta, \xi_2, \eta)$, and

$$
\begin{aligned}
&K_{11}(z,\eta) = \eta^{2k}, \qquad K_{12}(z,\eta) = \eta^{2k+1}\frac{h'_-(\bar{x}+z_1) - z_2 H'(\bar{x}+z_1)}{H(\bar{x}+z_1)},\\
&K_{22}(z,\eta) = \frac{1+\eta^{2k+2}(h'_-(\bar{x}+z_1) - z_2 H'(\bar{x}+z_1))^2}{h^2(\bar{x}+z_1)},\\
&K_2(z,\eta) = \frac{\eta^{2k+2}}{h^2(\bar{x}+z_1)}\big(h''_-(\bar{x}+z_1)H(\bar{x}+z_1) - 2h'_-(\bar{x}+z_1)H'(\bar{x}+z_1)\\
&\qquad\qquad + 2z_2\big(H'(\bar{x}+z_1)\big)^2 - z_2 H''(\bar{x}+z_1)H(\bar{x}+z_1)\big).
\end{aligned}
\tag{3.3}
$$

Now we expand the coefficients K_{12}, K_{22}, and K_2 into the Taylor series with respect to η, taking into account (2.1),

$$
\begin{aligned}
&K_{12}(\xi_1\eta,\xi_2,\eta) = \sum_{i=2k+1}^{\infty}\eta^i\big(P_{12}^{(i)}(\xi_1) + \xi_2 Q_{12}^{(i)}(\xi_1)\big),\\
&K_{22}(\xi_1\eta,\xi_2,\eta) = H_0^{-2} + \sum_{i=2k}^{\infty}\eta^i\big(P_{22}^{(i)}(\xi_1) + \xi_2 Q_{22}^{(i)}(\xi_1) + \xi_2^2 R_{22}^{(i)}(\xi_1)\big),\\
&K_2(\xi_1\eta,\xi_2,\eta) = \sum_{i=2k+2}^{\infty}\eta^i\big(P_2^{(i)}(\xi_1) + \xi_2 Q_2^{(i)}(\xi_1)\big),
\end{aligned}
\tag{3.4}
$$

where P_{12}^i, P_{22}^i, P_2^i, Q_{12}^i, Q_{22}^i, Q_2^i are certain polynomials with respect to ξ_1, and, in particular,

$$
\begin{aligned}
&P_{12}^{(2k1)} = \frac{h_1}{H_0}, \qquad P_{12}^{(2k+2)} = \frac{h_2}{H_0}\xi_1,\\
&Q_{12}^{(2k+1)} = 0, \qquad Q_{12}^{(2k+2)} = -\frac{H_2}{H_0}\xi_1,\\
&P_{22}^{(2k)} = -\frac{2H_{2k}\xi_1^{2k}}{(2k)!H_0^3}, \qquad P_{22}^{(2k+1)} = -\frac{2H_{2k+1}\xi_1^{2k+1}}{(2k+1)!H_0^3},\\
&P_{22}^{(2k+2)} = -\frac{2H_{2k+2}\xi_1^{2k+2}}{(2k+2)!H_0^3} + \delta_{1k}\frac{3H_{2k}^2\xi_1^{4k}}{((2k)!)^2H_0^4} + \frac{h_1^2}{H_0^2},\\
&Q_{22}^{(2k)} = Q_{22}^{(2k+1)} = Q_{22}^{(2k+2)} = 0, \qquad R_{22}^{(2k)} = R_{22}^{(2k+1)} = R_{22}^{(2k+2)} = 0,\\
&P_2^{(2k+2)} = \frac{h_2}{H_0}, \qquad Q_2^{(2k+2)} = -\frac{H_2}{H_0}.
\end{aligned}
\tag{3.5}
$$

We substitute (3.1) and (3.4) into (3.2) and evaluate the coefficients of the same powers of η taking into account (3.5). It leads us to the system of the boundary value problems,

$$
\frac{1}{H_0^2}\frac{\partial^2\psi_i}{\partial\xi_2^2} + c_0\psi_i = 0 \quad \text{in } \Pi, \qquad \psi_i = 0 \quad \text{on } \partial\Pi, \quad i = 0,\dots,2k-1,
\tag{3.6}
$$

$$
-\frac{1}{H_0^2}\frac{\partial^2\psi_{2k}}{\partial\xi_2^2} - \frac{\pi^2n^2}{H_0^2}\psi_{2k} = \frac{\partial^2\psi_0}{\partial\xi_1^2} - \frac{2H_{2k}\xi_1^{2k}}{(2k)!H_0^3}\frac{\partial^2\psi_0}{\partial\xi_2^2} + c_{2k}\psi_0 \quad \text{in } \Pi, \qquad \psi_{2k} = 0 \quad \text{on } \partial\Pi,
\tag{3.7}
$$

$$
\begin{aligned}
&-\frac{1}{H_0^2}\frac{\partial^2\psi_i}{\partial\xi_2^2} - c_0\psi_i = c_i\psi_0 + \frac{\partial^2\psi_{i-2k}}{\partial\xi_1^2} + c_{2k}\psi_{i-2k} - \frac{2H_{2k}\xi_1^{2k}}{(2k)!H_0^3}\frac{\partial^2\psi_{i-2k}}{\partial\xi_2^2} + F_i \quad \text{in } \Pi,\\
&\psi_i = 0 \quad \text{on } \partial\Pi, \quad i \geqslant 2k+1,
\end{aligned}
\tag{3.8}
$$

where the functions ψ_0, ψ_i are assumed to decay exponentially as $\xi_1 \to \pm\infty$, and

$$F_i = 2\sum_{j=2k+1}^{i}\big(P_{12}^{(j)}(\xi_1)+\xi_2 Q_{12}^{(j)}(\xi_1)\big)\frac{\partial^2\psi_{i-j}}{\partial\xi_1\partial\xi_2} + \sum_{j=2k+1}^{i}\big(P_{22}^{(j)}(\xi_1)+\xi_2 Q_{22}^{(j)}(\xi_1)+\xi_2^2 R_{22}^{(j)}(\xi_1)\big)\frac{\partial^2\psi_{i-j}}{\partial\xi_2^2}$$
$$+\sum_{j=2k+2}^{i}\big(P_{2}^{(j)}(\xi_1)+\xi_2 Q_{2}^{(j)}(\xi_1)\big)\frac{\partial\psi_{i-j}}{\partial\xi_2} + \sum_{j=2k+1}^{i-1} c_j\psi_{i-j}.$$

Problems (3.6) can be solved by separation of variables. It gives the formulas for ψ_i, $i = 0, \dots, 2k-1$, and c_0,

$$\psi_i(\xi) = \Psi_i(\xi_1)\sin n\pi\xi_2, \quad c_0 = \frac{\pi^2 n^2}{H_0^2}, \quad n \in \mathbb{N}, \tag{3.9}$$

where $i = 0, \dots, 2k-1$, and the functions Ψ_i are to be determined. We can consider the problem (3.7) as posed on the interval (0, 1) and depending on the parameter ξ_1. Hence, this problem is solvable, if the right-hand side is orthogonal to $\sin n\pi\xi_2$ in $L_2(0,1)$ for each ξ_1, where the scalar product is taken with respect to ξ_2. Evaluating this scalar product and taking into account (3.9), we arrive at the equation

$$\left(-\frac{d^2}{d\xi_1^2} - \frac{2\pi^2 n^2 H_{2k}\xi_1^{2k}}{(2k)!H_0^3}\right)\Psi_0 = c_{2k}\Psi_0 \quad \text{in } \mathbb{R}. \tag{3.10}$$

The exponential decaying of ψ_0 as $|\xi_1| \to \infty$ is possible, if the same is true for Ψ_0. Hence, Ψ_0 is an eigenfunction of G_n, and therefore, c_{2k} is the corresponding eigenvalue. Thus, $\Psi_0 = \Phi_{n,m}$, and the formula (2.3) for c_{2k} is valid.

Lemma 3.1. *The spectrum of G_n consists of infinitely many simple positive isolated eigenvalues $\Lambda_{n,m}$. The associated eigenfunctions are infinitely differentiable, and decay exponentially at infinity, namely,*

$$\Phi_{n,m}(\xi_1) = C_\pm|\xi_1|^{\frac{\lambda}{2\sqrt{A}}-\frac{1}{2}}\mathrm{e}^{-\frac{\sqrt{A}}{2}|\xi_1|^2}\big(1+\mathrm{o}(1)\big), \quad \text{if } k=1,$$
$$\Phi_{n,m}(\xi_1) = C_\pm|\xi_1|^{-\frac{k}{2}}\mathrm{e}^{-\frac{\sqrt{A}}{k+1}|\xi_1|^{k+1}}\big(1+\mathrm{o}(1)\big), \quad \text{if } k>1.$$

The last formulas can be differentiated with respect to ξ_1.

The equation $(G_n - \Lambda_{n,m})u = f$, $f \in L_2(\mathbb{R})$, is solvable if and only if

$$(f, \Phi_{n,m})_{L_2(\mathbb{R})} = 0. \tag{3.11}$$

In this case, there is a unique solution u orthogonal to $\Phi_{n,m}$ in $L_2(\mathbb{R})$. If $f \in C^\infty(\mathbb{R})$ is an exponentially decaying function satisfying

$$f = \mathcal{O}\big(|\xi_1|^\beta \exp^{-\frac{\sqrt{A}}{k+1}|\xi_1|^{k+1}}\big), \quad |\xi_1| \to \infty, \tag{3.12}$$

and this identity can be differentiated with respect to ξ_1, then the solution u is infinitely differentiable and decays exponentially,

$$u = \mathcal{O}\big(|\xi_1|^{\tilde\beta} \exp^{-\frac{\sqrt{A}}{k+1}|\xi_1|^{k+1}}\big), \quad |\xi_1| \to \infty, \tag{3.13}$$

where $\tilde\beta$ is some number. This identity can be differentiated with respect to ξ_1.

Proof. The statement on the eigenfunctions follows from [2, Chapter II, Section 2.3, Theorem 3.1, Section 2.4, Theorem 4.6]. The solvability condition (3.11) is due to self-adjointness of G_n. Theorem 4.6 in [2, Chapter II, Section 2.4] gives also the asymptotic behaviour of the fundamental system of Eq. (3.10). Using these formulas and representing the solution u via Green function, one can easily prove (3.13). □

Taking into account (3.9), (3.10), we can rewrite (3.7) as

$$-\frac{1}{H_0^2}\frac{\partial^2\psi_{2k}}{\partial\xi_2^2} - \frac{\pi^2 n^2}{H_0^2}\psi_{2k} = 0 \quad \text{in } \Pi, \qquad \psi_{2k} = 0 \quad \text{on } \partial\Pi.$$

Hence, the formula (3.9) is valid for ψ_{2k} as well. In what follows, and for the sake of brevity we will write simply Λ and Φ instead of $\Lambda_{n,m}$ and $\Psi_{n,m}$.

Lemma 3.2. *The problems* (3.6)–(3.8) *are solvable, and their solutions read as follows*:

$$\psi_i(\xi) = \tilde{\psi}_i(\xi) + \Psi_i(\xi_1)\sin\pi n\xi_2, \qquad \tilde{\psi}_i(\xi) = \sum_j \phi_{i,j}^{(1)}(\xi_1)\phi_{i,j}^{(2)}(\xi_2), \tag{3.14}$$

where the sum with respect to j is finite, $\phi_{i,j}^{(1)} \in C^\infty(\mathbb{R})$ are exponentially decaying functions satisfying (3.13)*, $\phi_{i,j}^{(2)} \in C^\infty[0,\pi]$ are orthogonal to $\sin\pi n\xi_2$ in $L_2(0,\pi)$ for each $\xi_1 \in \mathbb{R}$ and vanish on $\partial\Pi$. As $i = 0,\dots,2k$, the functions $\tilde{\psi}_i$ are identically zero, while for $i > 2k$ they solve the boundary value problems*

$$-\frac{1}{H_0^2}\frac{\partial^2\tilde{\psi}_i}{\partial\xi_2^2} - \frac{\pi^2 n^2}{H_0^2}\tilde{\psi}_i = \frac{\partial^2\tilde{\psi}_{i-2k}}{\partial\xi_1^2} - \frac{2H_{2k}\xi_1^{2k}}{(2k)!H_0^3}\frac{\partial^2\tilde{\psi}_{i-2k}}{\partial\xi_2^2} + \Lambda\tilde{\psi}_{i-2k} + \sum_{j=2k+1}^{i-1} c_j\tilde{\psi}_{i-j} + \widetilde{F}_i - f_i\sin\pi n\xi_2 \quad \text{in } \Pi,$$

$$\tilde{\psi}_i = 0 \quad \text{on } \partial\Pi, \tag{3.15}$$

$$\widetilde{F}_i = 2\sum_{j=2k+1}^{i}\big(P_{12}^{(j)} + \xi_2 Q_{12}^{(j)}\big)\frac{\partial^2\psi_{i-j}}{\partial\xi_1\partial\xi_2} + \sum_{j=2k+1}^{i}\big(P_{22}^{(j)} + \xi_2 Q_{22}^{(j)} + \xi_2^2 R_{22}^{(j)}\big)\frac{\partial^2\psi_{i-j}}{\partial\xi_2^2}$$
$$+ \sum_{j=2k+2}^{i}\big(P_2^{(j)} + \xi_2 Q_2^{(j)}\big)\frac{\partial\psi_{i-j}}{\partial\xi_2}, \tag{3.16}$$

$$f_i = 2(\widetilde{F}_i, \sin\pi n\xi_2)_{L_2(0,\pi)}. \tag{3.17}$$

The functions $\Psi_i \in C^\infty(\mathbb{R})$ satisfy (3.13)*, are orthogonal to Φ in $L_2(\mathbb{R})$ and solve the equations*

$$(G_n - \Lambda)\Psi_{i-2k} = f_i + \sum_{j=2k+1}^{i-1} c_j\Psi_{i-j} + c_i\Phi. \tag{3.18}$$

The numbers c_i, $i \geqslant 2k+1$, are given by the formulas

$$c_i = -(f_i, \Phi)_{L_2(\mathbb{R})}. \tag{3.19}$$

Proof. We prove the lemma by induction. The statement of the lemma for $i = 0$ follows from (3.9). This identity also implies the formulas (3.14) for $i = 1,\dots,2k$, where $\tilde{\psi}_i \equiv 0$, and Ψ_i are functions to be determined.

Assume that the statement of the lemma holds true for $i < p$, $p \geqslant 2k+1$. Then, in view of (3.14), the right-hand side in Eq. (3.8) can be rewritten as

$$\left(c_p\Phi - (G_n - \Lambda)\Psi_{p-2k} + \sum_{j=2k+1}^{p-1} c_j\Psi_{p-j}\right)\sin\pi n\xi_2 + \frac{\partial^2\tilde{\psi}_{p-2k}}{\partial\xi_1^2} + \Lambda\tilde{\psi}_{p-2k}$$
$$- \frac{2H_{2k}\xi_1^{2k}}{(2k)!H_0^3}\frac{\partial^2\tilde{\psi}_{p-2k}}{\partial\xi_2^2} + \sum_{j=2k+1}^{p-1} c_j\tilde{\psi}_{p-j} + \widetilde{F}_p. \tag{3.20}$$

The solvability condition of (3.8) is the orthogonality of this right-hand side to $\sin\pi n\xi_2$ in $L_2(0,\pi)$ for each $\xi_1 \in \mathbb{R}$. We write this condition, taking into account the orthogonality of $\tilde{\psi}_j$, $j < p$, to $\sin\pi n\xi_2$ in $L_2(0,\pi)$, and the relation

$$\left(\frac{\partial^2\tilde{\psi}_{p-2k}}{\partial\xi_2^2}, \sin\pi n\xi_2\right)_{L_2(0,\pi)} = -\pi^2 n^2(\tilde{\psi}_{p-2k}, \sin\pi n\xi_2)_{L_2(0,\pi)} = 0.$$

This procedure leads us to (3.18). By Lemma 3.1, the solvability condition of (3.18) is exactly the formula (3.19), since the functions Ψ_{i-2k}, $i < p$, are orthogonal to Φ in $L_2(\mathbb{R})$. We choose the solution of this equation to be orthogonal to Φ in $L_2(\mathbb{R})$ and note that by the formula (3.14) for $\widetilde{\Psi}_i$, $i < p$, the function f_i satisfies (3.12). Hence, by Lemma 3.1, the function Ψ_{p-2k} satisfies (3.13). The formulas (3.14) and (3.20) yield that the right-hand side of the equation in (3.8) with $i = p$ can be represented as a finite sum $\sum_j f_{p,j}^{(1)}(\xi_1)f_{p,j}^{(2)}(\xi_2)$, where $f_{p,j}^{(2)} \in C^\infty[0,1]$ are orthogonal to $\sin\pi n\xi_2$ in $L_2(0,\pi)$, while the functions $f_{p,j}^{(1)} \in C^\infty(\mathbb{R})$ satisfy (3.12). This fact implies the formula (3.14) for $i = p$. $\square$

Let us prove the remaining formulas in (2.3). It follows from (3.16), (3.5) that

$$\widetilde{F}_{2k+1} = \frac{2\pi n h_1}{H_0}\frac{\mathrm{d}\Phi}{\mathrm{d}\xi_1}\cos\pi n\xi_2 + \frac{2\pi^2 n^2 H_{2k+1}\xi_1^{2k+1}}{(2k+1)!H_0^3}\Phi\sin\pi n\xi_2. \tag{3.21}$$

Therefore,

$$f_{2k+1} = \frac{2\pi^2 n^2 H_{2k+1}\xi_1^{2k+1}}{(2k+1)!H_0^3}\Phi. \tag{3.22}$$

The eigenfunction Φ is either odd or even with respect to ξ_1, due to the evenness of the potential in the operator G_n. Thus, the function Φ^2 is even, and this is why $(f_{2k+1},\Phi)_{L_2(\mathbb{R})} = 0$, proving the formula for c_{2k+1}.

Employing (3.5), by direct calculation we check that

$$f_{2k+2} = -Q_{12}^{(2k+2)}\frac{\mathrm{d}\Phi}{\mathrm{d}\xi_1} - \pi^2 n^2 P_{22}^{(2k+1)}\Psi_1 - \pi^2 n^2 P_{22}^{(2k+2)}\Phi - \frac{1}{2}Q_2^{(2k+2)}\Phi. \tag{3.23}$$

By integrating by parts we obtain

$$\left(\xi_1\frac{\mathrm{d}\Phi}{\mathrm{d}\xi_1},\Phi\right)_{L_2(\mathbb{R})} = \frac{1}{2}\int_{\mathbb{R}}\xi_1\,\mathrm{d}\Phi^2 = -\frac{1}{2}.$$

Now the formula (2.3) for c_{2k+2} follows from the last two identities and (3.5).

We proceed to the justification of the expansions (3.1). For any $N > 7(k+1)$ we denote

$$\psi_{\varepsilon,N}(x) := \chi(x_1)\sum_{i=0}^{N}\eta^i\psi_i(\xi), \quad \mu_{\varepsilon,N} := c_0 + \sum_{i=2k}^{N}\eta^i c_i, \quad \lambda_{\varepsilon,N} := \varepsilon^{-2}\mu_{\varepsilon,N}.$$

Lemma 3.3. *The pair $\psi_{\varepsilon,N}$, $\lambda_{\varepsilon,N}$ satisfies the boundary value problem*

$$-\Delta\psi_{\varepsilon,N} = \lambda_{\varepsilon,N}\psi_{\varepsilon,N} + g_{\varepsilon,N} \quad \text{in } \Omega_\varepsilon, \qquad \psi_{\varepsilon,N} = 0 \quad \text{on } \partial\Omega_\varepsilon, \tag{3.24}$$

and

$$\|g_{\varepsilon,N}\|_{L_2(\Omega_\varepsilon)} = \mathcal{O}\big(\varepsilon^{\frac{N}{k+1}-3}\big).$$

This lemma follows from the definition of the problems (3.6), (3.8), Lemma 3.2, and the exponential decaying of ψ_i as $|\xi_1| \to \infty$.

We can rewrite the problem (3.24) as

$$\psi_{\varepsilon,N} = (\lambda_{\varepsilon,N}+1)A_\varepsilon\psi_{\varepsilon,N} + A_\varepsilon g_{\varepsilon,N}, \tag{3.25}$$

where A_ε indicates the inverse of $-\Delta_{\Omega_\varepsilon}^D + 1$. It is clear that the operator A_ε is self-adjoint, has a compact resolvent, and satisfies the estimate $\|A_\varepsilon\| \leqslant 1$, uniformly in ε. We rewrite (3.25) as

$$\frac{1}{\lambda_{\varepsilon,N}+1}\psi_{\varepsilon,N} = A_\varepsilon\psi_{\varepsilon,N} + \frac{1}{\lambda_{\varepsilon,N}+1}A_\varepsilon g_{\varepsilon,N}.$$

It follows from Lemma 3.3 and the estimate $\|A_\varepsilon\| \leqslant 1$ that

$$\left\|\frac{1}{\lambda_{\varepsilon,N}+1}A_\varepsilon g_{\varepsilon,N}\right\|_{L_2(\Omega_\varepsilon)} \leqslant C_N\varepsilon^{\frac{N}{k+1}-7},$$

where C_N are some constants. We apply Lemma 1.1 in [8, Chapter III, Section 1.1], where we let $\alpha = C_N\varepsilon^{\frac{N}{k+1}-7}$ (see inequality (1.1) in [8, Chapter II, Section 1.1]). This lemma yields that given N for each ε there exists an eigenvalue $\tau_N(\varepsilon)$ of A_ε such that

$$\big|\tau_N(\varepsilon) - (1+\lambda_{\varepsilon,N})^{-1}\big| = \mathcal{O}\big(\varepsilon^{\frac{N}{k+1}-3}\big).$$

Hence, there exists an eigenvalue $\tilde{\lambda}_N(\varepsilon)$ of $-\Delta^D_{\Omega_\varepsilon}$ such that

$$\left|\tilde{\lambda}_N(\varepsilon)-\lambda_{\varepsilon,N}\right|\leqslant \widetilde{C}_N\varepsilon^{\frac{N}{k+1}-7}, \tag{3.26}$$

where $\widetilde{C}_N$ are some constants.

Let ε_N be a monotone sequence such that $C_N\varepsilon^{\frac{1}{k+1}}\leqslant C_{N-1}$ as $\varepsilon\leqslant\varepsilon_N$. Given n and m corresponding to the series (3.1) (see (3.9), (3.10)), we chose the eigenvalue $\lambda_{n,m}(\varepsilon)$, letting $\lambda_{n,m}(\varepsilon):=\tilde{\lambda}_N(\varepsilon)$ as $\varepsilon\in[\varepsilon_N,\varepsilon_{N+1})$. Employing (3.26), one can check easily that the eigenvalue $\lambda_{n,m}$ satisfies (2.2).

For $\varepsilon\in[\varepsilon_N,\varepsilon_{N+1})$, we employ Lemma 1.1 in [8, Chapter III, Section 1.1] once again, and we let in this lemma $\alpha=C_N\varepsilon^{\frac{N}{k+1}-7}$, $d=2C_N\varepsilon^{\frac{N}{k+1}-4}$. It implies the existence of a linear combination $\psi_{n,m}$ described in the statement of the theorem such that

$$\|\psi_{n,m}-\psi_{\varepsilon,N}\|_{L_2(\Omega_\varepsilon)}=\mathcal{O}\big(\varepsilon^{\frac{N}{k+1}-3}\big) \tag{3.27}$$

for each N.

We denote $\zeta_1:=x_1$, $\zeta_2:=x_2\varepsilon^{-1}$, $\Omega:=\{\zeta\colon \zeta_1\in(0,1), -h_-(\zeta_1)<\zeta_2<h_+(\zeta_1)\}$, $\Omega^\delta:=\Omega\cap\{\zeta\colon |\zeta_1-\bar{x}|<\delta\}$,

$$\hat{\psi}_N(\zeta):=\psi_{n,m}(x,\varepsilon)-\psi_{\varepsilon,N}(x). \tag{3.28}$$

Lemma 3.4. *The relations*

$$\|\nabla\hat{\psi}_N\|_{L_2(\Omega_\varepsilon)}=\mathcal{O}\big(\varepsilon^{\frac{N}{2(k+1)}-\frac{7}{2}}\big),\qquad \|\hat{\psi}_N\|_{W_2^2(\Omega^{\delta/3})}=\mathcal{O}\big(\varepsilon^{\frac{N}{2(k+1)}-\frac{7}{2}}\big)$$

hold true.

Proof. Employing Lemma 3.3 and (3.27), (3.26), by integrating by parts one can easily check that

$$\|\nabla\hat{\psi}_N\|_{L_2(\Omega_\varepsilon)}=\mathcal{O}\big(\varepsilon^{\frac{N}{2(k+1)}-\frac{7}{2}}\big). \tag{3.29}$$

It follows from Lemma 3.3, the equation for $\psi_{\varepsilon,N}$ and (3.27), (3.26), (3.29) that the function

$$\phi(\zeta):=\chi(x_1)\psi_{n,m}(x,\varepsilon)-\psi_{\varepsilon,N}(x)$$

satisfies the boundary value problem

$$-\left(\varepsilon^2\frac{\partial^2}{\partial\zeta_1^2}+\frac{\partial^2}{\partial\zeta_2^2}\right)\phi=\hat{g}\quad\text{in }\Omega^\delta,\qquad \hat{\psi}_N=0\quad\text{on }\partial\Omega^\delta, \tag{3.30}$$

where

$$\|\hat{g}\|_{L_2(\Omega^\delta)}=\mathcal{O}\big(\varepsilon^{\frac{N}{k+1}-\frac{9}{2}}\big),\quad \|\nabla\phi\|_{L_2(\Omega^\delta)}=\mathcal{O}\big(\varepsilon^{\frac{N}{2(k+1)}-3}\big). \tag{3.31}$$

Integrating by parts in the same way as in the proof of Lemma 7.1 in [7, Chapter 3, Section 7], one can check that

$$\int\limits_{\Omega^\delta}\frac{\partial^2\phi}{\partial\zeta_1^2}\frac{\partial^2\phi}{\partial\zeta_2^2}\,\mathrm{d}\zeta=\int\limits_{\Gamma^\delta}\frac{\partial\phi}{\partial\zeta_1}\left(\nu_1\frac{\partial^2\phi}{\partial\zeta_2^2}-\nu_2\frac{\partial^2\phi}{\partial\zeta_1\partial\zeta_2}\right)\mathrm{d}s+\int\limits_{\Omega^\delta}\left(\frac{\partial^2\phi}{\partial\zeta_1\partial\zeta_2}\right)^2\mathrm{d}\zeta. \tag{3.32}$$

Here $\Gamma^\delta:=\partial\Omega^\delta\setminus\{\zeta\colon \zeta_1=-\delta\}\setminus\{\zeta\colon \zeta_1=\delta\}$, $\nu=\nu(s)$, $\nu=(\nu_1,\nu_2)$ is the outward normal to Γ^δ and s is the arc length of Γ^δ. Employing the identity

$$\frac{\partial\phi}{\partial s}=0\quad\text{on }\Gamma^\delta,$$

we continue the calculations

$$\begin{aligned}\int\limits_{\Gamma^\delta}\frac{\partial\phi}{\partial\zeta_1}\left(\nu_1\frac{\partial^2\phi}{\partial\zeta_2^2}-\nu_2\frac{\partial^2\phi}{\partial\zeta_1\partial\zeta_2}\right)\mathrm{d}s&=\int\limits_{\Gamma^\delta}\nu_1\frac{\partial\phi}{\partial\nu}\frac{\partial}{\partial s}\frac{\partial\phi}{\partial\zeta_2}\,\mathrm{d}s=\int\limits_{\Gamma^\delta}\nu_1\frac{\partial\phi}{\partial\nu}\frac{\partial}{\partial s}\nu_2\frac{\partial\phi}{\partial\nu}\,\mathrm{d}s\\&=\int\limits_{\Gamma^\delta}\nu_1\nu_2'\left(\frac{\partial\phi}{\partial\nu}\right)^2\mathrm{d}s+\frac{1}{2}\int\limits_{\Gamma^\delta}\nu_1\nu_2\frac{\partial}{\partial s}\left(\frac{\partial\phi}{\partial\nu}\right)^2\mathrm{d}s\\&=\frac{1}{2}\int\limits_{\Gamma^\delta}(\nu_1\nu_2'-\nu_1'\nu_2)\left(\frac{\partial\phi}{\partial\nu}\right)^2\mathrm{d}s\end{aligned}$$

where $'$ denotes the derivative with respect to s. The obtained formula implies that

$$\left|\int\limits_{\Gamma^\delta} \frac{\partial\phi}{\partial\zeta_1}\left(\nu_1\frac{\partial^2\phi}{\partial\zeta_2^2}-\nu_2\frac{\partial^2\phi}{\partial\zeta_1\partial\zeta_2}\right)\mathrm{d}s\right| \leqslant \varepsilon^3\|\phi\|^2_{W_2^2(\Omega^\delta)}+C\varepsilon^{-3}\|\phi\|^2_{W_2^1(\Omega^\delta)},$$

where the constant C is independent of ε. Hence, by (3.32),

$$\left(\frac{\partial^2\phi}{\partial\zeta_1^2},\frac{\partial^2\phi}{\partial\zeta_2^2}\right)_{L_2(\Omega^\delta)} \geqslant \left\|\frac{\partial^2\phi}{\partial\zeta_1\partial\zeta_2}\right\|^2-\varepsilon^3\|\phi_{\zeta\zeta}\|^2_{L_2(\Omega^\delta)}-C\varepsilon^{-3}\|\phi\|^2_{W_2^1(\Omega^\delta)},$$

$$\|\phi_{\zeta\zeta}\|^2_{L_2(\Omega^\delta)}:=\left\|\frac{\partial^2\phi}{\partial\zeta_1^2}\right\|^2_{L_2(\Omega^\delta)}+\left\|\frac{\partial^2\phi}{\partial\zeta_2^2}\right\|^2_{L_2(\Omega^\delta)}+\left\|\frac{\partial^2\phi}{\partial\zeta_1\partial\zeta_2}\right\|^2_{L_2(\Omega^\delta)}.$$

Employing this estimate, we obtain

$$\begin{aligned}\|\hat g\|^2_{L_2(\Omega^\delta)}&=\varepsilon^4\left\|\frac{\partial^2\phi}{\partial\zeta_1^2}\right\|^2_{L_2(\Omega^\delta)}+2\varepsilon^2\left(\frac{\partial^2\phi}{\partial\zeta_1^2},\frac{\partial^2\phi}{\partial\zeta_2^2}\right)_{L_2(\Omega^\delta)}+\left\|\frac{\partial^2\phi}{\partial\zeta_2^2}\right\|^2_{L_2(\Omega^\delta)}\\&\geqslant\varepsilon^4(1-2\varepsilon)\|\phi_{\zeta\zeta}\|^2_{L_2(\Omega^\delta)}-C\varepsilon^{-1}\|\phi\|^2_{L_2(\Omega^\delta)}.\end{aligned}$$

Combining this estimate with (3.31), we complete the proof. □

The proven lemma implies the asymptotics (2.4).

4. Proof of Theorem 2

We begin by showing that for sufficiently small ε the first N eigenvalues are $\lambda_{1,m}(\varepsilon)$, $m=1,\dots,N$. If $h_-\equiv 0$, this statement follows from Theorem 1 and the arguments of Section 6.1 in [5]. If h_- is not identically zero, we cannot apply directly the above mentioned results from [5], but it is possible to extend their proof to the case $h_-\not\equiv 0$ with minor changes. Below we list the required changes and refer to [5] for the detailed proof.

The first change is that in our case by the function h in [5] one should mean H. Suppose for a while that H is strictly positive. The space $\mathfrak{L}_\varepsilon$ is defined as consisting of the functions

$$\psi(x)=\chi(x_1)\sqrt{\frac{2}{\varepsilon H(x_1)}}\sin\frac{\pi(x_2+\varepsilon h_-(x_1))}{\varepsilon H(x_1)}.$$

The function $v(x)$ in the potential W_ε defined in [5, Eq. (1.5)] should be introduced as

$$v(x_1)=\frac{\pi^2(H'(x_1))^2}{3H^2(x_1)}+\frac{(2\pi h_-'(x_1)-H'(x_1))^2}{4H^2(x_1)}.$$

The number M, as in [5], should be H_0. The definition of the operators $\mathbf{Q}_\varepsilon$ and $\widehat{\mathbf{Q}}_\varepsilon$ remain the same as in [5], and the operator $\mathbf{H}$ is our operator G_n. Under such changes Theorems 1.2, 1.3 in [5] remain true. Throughout the proofs of these theorems the function $\sin\frac{\pi y}{\varepsilon h(x)}$ should be replaced by $\sin\frac{\pi(x_2+\varepsilon h_-(x_1))}{\varepsilon H(x_1)}$ and in all the integrals w.r.t. x_2 the limits of the integrations are $-\varepsilon h_-(x_1)$, $\varepsilon h_+(x_1)$. The other arguments in the proofs of Theorems 1.2, 1.3 remain unchanged. Thus, these theorems are valid for the case $h_-\not\equiv 0$ under the additional assumption $H>0$.

Employing the proven Theorems 1.2, 1.3 for the case $h_-\not\equiv 0$ and proceeding as in [5, Section 6.1] one can check that in the case $H\geqslant 0$ the eigenvalues $\lambda_{1,m}(\varepsilon)$, $m=1,\dots,N$, are the first eigenvalues of $-\Delta^D_{\Omega_\varepsilon}$ for ε small enough. We also conclude that these eigenvalues are simple. Thus, for each $m=1,\dots,N$ the eigenvalue $\lambda_{1,m}(\varepsilon)$ is the unique eigenvalue of $-\Delta^D_{\Omega_\varepsilon}$ having the asymptotics (2.2) for $n=1$ and given m. Hence, the linear combinations $\psi_{1,m}$ introduced in Theorem 1 are the eigenfunctions associated with the eigenvalues $\lambda_{1,m}$.

It remains to prove the formulas (2.5). In the case considered $n=m=k=1$. The operator G_1 is the harmonic oscillator, and its eigenvalues and eigenfunctions are known explicitly. Namely,

$$\Lambda_{1,1}=\theta,\quad \Phi_{1,1}(\xi_1)=\frac{\theta^{1/4}\mathrm{e}^{-\theta\xi_1^2/2}}{\pi^{1/4}},\quad \theta:=\frac{\pi(-H_2)^{1/2}}{H_0^{3/2}}. \tag{4.1}$$

The first identity proves the formula for $c_2^{(1,1)}$. The formula for $c_3^{(1,1)}$ follows from (2.3).

It is easy to check by direct calculation that

$$\Psi_1^{(1,1)} = \frac{1}{18}\frac{\pi^2 H_3}{\theta^2 H_0^3}\xi_1\big(\theta\xi_1^2+3\big)\Phi_{1,1},$$

$$\big\|\xi_1^2\Phi_{1,1}\big\|^2_{L_2(\mathbb{R})} = \frac{3}{4\theta^2}, \qquad \big(\xi_1^3\Psi_1^{(1,1)},\Phi_{1,1}\big)_{L_2(\mathbb{R})} = \frac{11}{48}\frac{\pi^2 H_3}{\theta^4 H_0^3}. \tag{4.2}$$

Now we substitute these identities into formula (2.3) for c_{2k+2} and arrive at formula (2.5) for $c_4^{(1,1)}$.

To proceed further, we need the formulas for the coefficients of the series (3.4) up to the order η^6. They read as follows,

$$P_{12}^{(5)} = \frac{h_3H_0-h_1H_2}{2H_0^2}\xi_1^2, \qquad P_{12}^{(6)} = \frac{h_4H_0-h_1H_3-3h_2H_2}{6H_0^2}\xi_1^3,$$

$$Q_{12}^{(5)} := -\frac{H_3}{2H_0}\xi_1^2, \qquad Q_{12}^{(6)} := -\frac{H_4H_0-3H_2^2}{6H_0^2}\xi_1^3,$$

$$P_{22}^{(5)} := \frac{2h_1h_2}{H_0^2}\xi_1 + \frac{H_2H_3}{2H_0^4}\xi_1^5 - \frac{H_5}{60H_0^3}\xi_1^5,$$

$$P_{22}^{(6)} := \frac{45H_4H_2H_0-180H_2^3-H_6H_0^2+30H_3^2H_0}{360H_0^5}\xi_1^6 + \frac{h_3h_1H_0-h_1^2H_2+h_2^2H_0}{H_0^3}\xi_1^2,$$

$$Q_{22}^{(5)} = -\frac{2h_1H_2\xi_1}{H_0^2}, \qquad Q_{22}^{(6)} = -\frac{h_1H_3+2h_2H_2}{H_0^2}\xi_1^2, \qquad R_{22}^{(5)} = 0, \qquad R_{22}^{(6)} = \frac{H_2^2\xi_1^2}{H_0^2},$$

$$P_2^{(5)} = \frac{h_3H_0-2h_1H_2}{H_0^2}\xi_1, \qquad P_2^{(6)} = \frac{h_4H_0-5h_2H_2-2h_1H_3}{2H_0^2}\xi_1^2,$$

$$Q_2^{(5)} = -\frac{H_3}{H_0}\xi_1, \qquad Q_2^{(6)} = \frac{5H_2^2-H_4H_0}{2H_0^2}\xi_1^2. \tag{4.3}$$

It follows from (3.15), (3.21), (3.22) that

$$\tilde\psi_3^{(1,1)}(\xi) = \frac{1}{2}H_0h_1(1-2\xi_2)\sin\pi\xi_2. \tag{4.4}$$

In view of (3.23), (3.5) we can also find the function $\Psi_2^{(1,1)}$ explicitly,

$$\Psi_2^{(1,1)} = \bigg(-\frac{\pi^2H_3^2}{648H_0^3H_2}\xi_1^6 + \frac{\pi^2\xi_1^4}{864H_0^4H_2\theta}\big(9H_4H_2H_0-11H_3^2H_0-81H_2^3\big)$$
$$+ \frac{9H_2^3-9H_4H_2H_0+11H_3^2H_0}{288H_0H_2^2}\xi_1^2 + \frac{3H_4}{128H_2\theta} - \frac{109H_3^2}{3456H_2^2\theta} - \frac{11H_2}{128H_0\theta}\bigg)\Phi_{1,1}. \tag{4.5}$$

Now we use the formulas (2.5) for $c_i^{(1,1)}$, $i \leqslant 4$, and (3.5), (3.9), (3.16), (3.17), (4.1)–(4.5) and obtain

$$f_5 = \bigg(-\frac{\pi^4H_3^3\xi_1^9}{1944H_2H_0^6} - \frac{\pi^4H_3\xi_1^7}{2592H_2H_0^7\theta}\big(-21H_4H_0H_2+189H_2^3+11H_3^2H_0\big)$$
$$- \frac{\pi^2\xi_1^5}{4320H_2^2H_0^4}\big(105H_5H_3H_2H_0+1815H_2^3H_3-72H_2^2H_5H_0-55H_3^3H_0\big)$$
$$+ \frac{\xi_1^3\pi^2H_3}{10368H_0^5H_2^2\theta}\big(5175H_2^3H_0-576\pi^2h_1^2h2^2+81H_4H_2H_0^2-109H_3^2H_0^2\big)$$
$$+ \frac{\xi_1}{12H_2H_0^2}\big(2\pi^2H_3h_1^2-24\pi^2H_2h_1h_2+3H_3H_2H_0+12\pi^2H_2^2h_1\big)\bigg)\Phi_{1,1}. \tag{4.6}$$

The function f_5 is odd with respect to ξ_1, and therefore, in view of (3.19) and evenness of $\Phi_{1,1}$, the formula (2.5) is valid for $c_5^{(1,1)}$. Employing the obtained identity for f_5, we can solve explicitly Eq. (3.18) for $\Psi_3^{(1,1)}$:

$$\begin{aligned}\Psi_3^{(1,1)} = \bigg(&-\frac{\pi^4 H_3^3}{34992 H_0^6 \theta}\xi_1^9 + \frac{\pi^2 \xi_1^7 H_3}{15552 H_0^4 H_2^2}\big(7H_3^2 H_0 + 81 H_2^3 - 9 H_4 H_2 H_0\big) \\ &+ \frac{\pi^2 \xi_1^5}{64800 H_0^4 H_2^2 \theta}\big(205 H_3^3 H_0 - 315 H_4 H_3 H_2 H_0 - 1305 H_3 H_2^3 + 108 H_5 H_2^2 H_0\big) \\ &+ \frac{\xi_1^3}{311040 H_0 H_2^3}\big(4455 H_4 H_3 H_2 H_0 - 2515 H_3^3 H_0 - 3375 H_3 H_2^3 - 1728 H_2^2 H_5 H_0\big) \\ &+ \frac{\xi}{103680 H_0^2 H_2^3 \theta}\big(-1855 H_3^3 H_0^2 + 3915 H_4 H_3 H_2 H_0^2 + 14445 H_3 H_2^3 H_0 \\ &- 1728 H_5 H_2^2 H_0^2 - 103680 \pi^2 H_2^3 h_1 h_2 + 51840 \pi^2 H_2^4 h_1\big)\bigg)\Phi_{1,1}.\end{aligned}$$

We substitute the relation obtained, the formulas (2.5) for $c_i^{(1,1)}$, $i \leqslant 5$, (3.5), (3.9), (3.16), (3.17), (4.1)–(4.6) into (3.19) and arrive at the formula (2.5) for $c_6^{(1,1)}$. The proof is complete.

5. Examples

We shall now apply our results to obtain the expansion of the first eigenvalue for different domains, and compare the values obtained with a numerical approximation. We are indebted to Pedro Antunes for carrying out the necessary numerical computations.

We have chosen five examples illustrating several possibilities for the functions H, h_+ and h_-. The first three correspond to algebraic curves, namely the circle, the lemniscate and the bean curve. For these $h_+ = h_-$, and the fit between the four-term asymptotic expansion and the numerical approximation is very good: for the lemniscate and the bean curve the error is always below 2% and for the disk the maximum error is around 5%.

The last two examples have $h_+ \neq h_-$, the second being non-convex. Here we see that the error may become much larger – in the last example it can go up to 50%.

Example 1 *(Disk).* Consider the disk centred at $(1/2, 0)$ and radius $1/2$ for which we have $H(x_1) = 2h_+(x_1) = 2h_-(x_1) = 2(x_1 - x_1^2)^{1/2}$. The maximum of H occurs at $x_1 = 1/2$ and we obtain the expansion given by (1.1). Comparing these results with those of Daymond referred to in [6] we see that the error up to ε equals one is maximal at one and is around five per cent.

Example 2 *(Lemniscate).* Consider the lemniscate defined by

$$\big(x_1^2 + x_2^2\big)^2 = x_1^2 - x_2^2.$$

In this case we have

$$H(x_1) = 2h_+(x_1) = 2h_-(x_1) = 2\bigg[-\frac{1}{2} - x_1^2 + \frac{1}{2}\big(1 + 8x_1^2\big)^{1/2}\bigg]^{1/2},$$

and the maximum of H is now situated at $\sqrt{3}/(2\sqrt{2})$. This yields

$$\lambda_1(\varepsilon) = \frac{2\pi^2}{\varepsilon^2} + \frac{2\sqrt{3}\pi}{\varepsilon} + \frac{97}{24} + \bigg(\frac{593}{64\sqrt{3}\pi} + \frac{\sqrt{3}\pi}{4}\bigg)\varepsilon + \mathcal{O}\big(\varepsilon^2\big), \quad \text{as } \varepsilon \to 0.$$

Example 3 *(Bean curve).* As a third example we consider the quartic curve defined by $x_1(x_1 - 1)(x_1^2 + x_2^2) + x_2^4 = 0$. We now have

$$H(x_1) = 2h_+(x_1) = 2h_-(x_1) = (2x_1)^{1/2}\big[1 - x_1 + (1 - x_1)^{1/2}(1 + 3x_1)^{1/2}\big]^{1/2}.$$

The maximum of H is situated at $x_1 = 2/3$ and we obtain

$$\lambda_1(\varepsilon) = \frac{9\pi^2}{16\varepsilon^2} + \frac{3\sqrt{15}\pi}{8\varepsilon} + \frac{127}{40} + \left(\frac{24229}{600\sqrt{15}\pi} + \frac{5\sqrt{5}\pi}{16\sqrt{3}}\right)\varepsilon + \mathcal{O}(\varepsilon^2), \quad \text{as } \varepsilon \to 0.$$

Example 4 *(Convex, with $h_+ \neq h_-$).* Let $h_+(x_1) = \sin(\pi x_1)$ and $h_-(x_1) = \pi(1-x_1)/2$, yielding $H(x_1) = \sin(\pi x_1) + \pi(1-x_1)/2$, which has a maximum at $x_1 = 1/3$. The expression for the eigenvalue asymptotics now becomes

$$\lambda_1(\varepsilon) = \frac{36\pi^2}{(3\sqrt{3}+2\pi)^2\varepsilon^2} + \frac{63^{3/4}\pi^2}{(3\sqrt{3}+2\pi)^{3/2}\varepsilon} + \pi^2\left[\frac{9(27+6\sqrt{3}\pi+16\pi^2)}{16(3\sqrt{3}+2\pi)^2} - \frac{19}{216}\right] + \left(\frac{13273}{4608} + \frac{1807\pi}{3^{1/2}2304} + \frac{5465\pi^2}{1296} + \frac{17257\pi^3}{3^{1/2}11664}\right)\frac{3^{-1/4}\pi^2\varepsilon}{(3\sqrt{3}+2\pi)^{3/2}} + \mathcal{O}(\varepsilon^2), \quad \text{as } \varepsilon \to 0.$$

Example 5 *(Non-convex).* Let $h_+(x_1) = 1 + \sin(7\pi x_1/2)$ and $h_-(x_1) = 7\pi(1-x_1)/4$, yielding $H(x_1) = 1 + \sin(7\pi x_1/2) + 7\pi(1-x_1)/4$, which has its global maximum on $(0, 1)$ at $x_1 = 2/21$. We have

$$\lambda_1(\varepsilon) = \frac{0.210941}{\varepsilon^2} + \frac{1.79692}{\varepsilon} + 4.35119 + 60.5706\varepsilon + \mathcal{O}(\varepsilon^2), \quad \text{as } \varepsilon \to 0,$$

where, for simplicity, we only presented the numerical values of the coefficients.

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