

# Wellposedness and stability results for the Navier–Stokes equations in $R^3$

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## Abstract

In [J.-Y. Chemin, I. Gallagher, On the global wellposedness of the 3-D Navier–Stokes equations with large initial data, *Annales Scientifiques de l'École Normale Supérieure de Paris*, in press] a class of initial data to the three dimensional, periodic, incompressible Navier–Stokes equations was presented, generating a global smooth solution although the norm of the initial data may be chosen arbitrarily large. The aim of this article is twofold. First, we adapt the construction of [J.-Y. Chemin, I. Gallagher, On the global wellposedness of the 3-D Navier–Stokes equations with large initial data, *Annales Scientifiques de l'École Normale Supérieure de Paris*, in press] to the case of the whole space: we prove that if a certain nonlinear function of the initial data is small enough, in a Koch–Tataru [H. Koch, D. Tataru, Well-posedness for the Navier–Stokes equations, *Advances in Mathematics* 157 (2001) 22–35] type space, then there is a global solution to the Navier–Stokes equations. We provide an example of initial data satisfying that nonlinear smallness condition, but whose norm is arbitrarily large in  $C^{-1}$ . Then we prove a stability result on the nonlinear smallness assumption. More precisely we show that the new smallness assumption also holds for linear superpositions of translated and dilated iterates of the initial data, in the spirit of a construction in [H. Bahouri, J.-Y. Chemin, I. Gallagher, Refined Hardy inequalities, *Annali di Scuola Normale di Pisa, Classe di Scienze, Serie V* 5 (2006) 375–391], thus generating a large number of different examples.

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## 1. Introduction

### 1.1. On the global wellposedness of the Navier–Stokes system

We consider the three dimensional, incompressible Navier–Stokes system in  $R^3$ ,

$$(NS) \quad \begin{cases} \partial_t u - \Delta u + u \cdot \nabla u = -\nabla p, \\ \operatorname{div} u = 0, \\ u|_{t=0} = u_0. \end{cases}$$

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Here  $u$  is a three-component vector field  $u = (u_1, u_2, u_3)$  representing the velocity of the fluid,  $p$  is a scalar denoting the pressure, and both are unknown functions of the space variable  $x \in \mathbf{R}^3$  and of the time variable  $t \in \mathbf{R}^+$ . We have chosen the kinematic viscosity of the fluid equal to one for simplicity – a comment on the dependence of our results on viscosity is given further down in this introduction.

It is well known that (NS) has a global, smooth solution if the initial data is small enough in the scale invariant space  $\dot{H}^{\frac{1}{2}}$ , where we recall that  $\dot{H}^s$  is the set of tempered distributions  $f$  with Fourier transform  $\hat{f}$  in  $L^1_{loc}(\mathbf{R}^3)$  and such that

$$\|f\|_{\dot{H}^s} \stackrel{\text{def}}{=} \left( \int_{\mathbf{R}^3} |\xi|^{2s} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}$$

is finite. We recall that the scaling of (NS) is the following: for any positive  $\lambda$ , the vector field  $u$  is a solution associated with the data  $u_0$  if  $u_\lambda$  is a solution associated with  $u_{0,\lambda}$ , where

$$u_\lambda(t, x) = \lambda u(\lambda^2 t, \lambda x) \quad \text{and} \quad u_{0,\lambda}(x) = \lambda u_0(\lambda x).$$

The result in  $\dot{H}^{\frac{1}{2}}$  is due to H. Fujita and T. Kato in [10] (see also [22] for a similar result, where the smallness of  $u_0$  is measured by  $\|u_0\|_{L^2} \|\nabla u_0\|_{L^2}$ ). Since then, a number of works have been devoted to proving similar wellposedness results for larger classes of initial data; one should mention the result of T. Kato [17] where the smallness is measured in  $L^3$  (see also [15]) and the result of M. Cannone, Y. Meyer and F. Planchon (see [4]) where the smallness is measured in the Besov space  $\dot{B}^{-1+\frac{3}{p}}_{p,\infty}$ . Let us recall that, for positive  $\sigma$ ,

$$\|u\|_{\dot{B}^{-\sigma}_{p,r}} \stackrel{\text{def}}{=} \|t^{\frac{\sigma}{2}} \|S(t)u\|_{L^p(\mathbf{R}^+, \frac{dt}{t})}$$

where  $S(t) = e^{t\Delta}$  denotes the heat flow. The importance of this result can be illustrated by the following example: if  $\phi$  is a function in the Schwartz space  $\mathcal{S}(\mathbf{R}^3)$ , let us introduce the family of divergence free vector fields

$$\phi_\varepsilon(x) = \cos\left(\frac{x_3}{\varepsilon}\right) (\partial_2 \phi, -\partial_1 \phi, 0).$$

Then, for small  $\varepsilon$ , the size of  $\|\phi_\varepsilon\|_{\dot{B}^{-\sigma}_{p,r}}$  is  $\varepsilon^\sigma$ .

Let us also mention the result by H. Koch and D. Tataru in [18] where the smallness is measured in the space  $BMO^{-1}$ , defined by

$$\|u\|_{BMO^{-1}} \stackrel{\text{def}}{=} \|u\|_{\dot{B}^{-1}_{\infty,\infty}} + \sup_{\substack{x \in \mathbf{R}^3 \\ R > 0}} R^{-\frac{3}{2}} \left( \int_{P(x,R)} |S(t)u(y)|^2 dy dt \right)^{\frac{1}{2}},$$

where  $P(x, R) = [0, R^2] \times B(x, R)$  and  $B(x, R)$  denotes the ball of radius  $R$  centered at zero.

As observed by H. Koch and D. Tataru, this norm seems to be the ultimate norm for the initial data for which the classical Picard’s iterative scheme can work. Indeed the first iterate,  $S(t)u_0$  must be in  $L^2$  locally in  $\mathbf{R}^+ \times \mathbf{R}^3$ . In particular,  $S(t)u_0$  must be in  $L^2([0, 1] \times B(0, 1))$ . Then considering the norm of the space must be invariant by translation as well as by the scaling of the equation, we get the norm  $\|\cdot\|_{BMO^{-1}}$ . Moreover, let us notice that we have

$$\sup_{\substack{x \in \mathbf{R}^3 \\ R > 0}} R^{-\frac{3}{2}} \left( \int_{P(x,R)} |S(t)u(y)|^2 dy \right)^{\frac{1}{2}} \leq \|S(t)u\|_{L^2(L^\infty)}$$

and thus  $\|u\|_{\dot{B}^{-1}_{\infty,\infty}} \leq \|u\|_{BMO^{-1}} \leq \|u\|_{\dot{B}^{-1}_{\infty,2}}$ .

Moreover the space  $\dot{C}^{-1} = \dot{B}^{-1}_{\infty,\infty}$  seems to be optimal independently of the method of resolution, due to the following argument (see [1] for instance). Let  $B$  be a Banach space continuously included in the space  $S'$  of tempered distributions on  $\mathbf{R}^3$ . Let us assume that, for any  $(\lambda, a) \in \mathbf{R}^+_* \times \mathbf{R}^3$ ,

$$\|f(\lambda(\cdot - a))\|_B = \lambda^{-1} \|f\|_B.$$

Then we have that  $|\langle f, e^{-|\cdot|^2} \rangle| \leq C \|f\|_B$ . By dilation and translation, we deduce that

$$\|f\|_{\dot{C}^{-1}} = \sup_{t>0} t^{\frac{1}{2}} \|S(t)f\|_{L^\infty} \leq C \|f\|_B.$$

We have proved that any Banach space included in  $\mathcal{S}'$ , translation invariant and which has the right scaling is included in  $\dot{C}^{-1}$ .

Let us point out that none of the results mentioned so far are specific to the Navier–Stokes equations, as they do not use the special structure of the nonlinear term in (NS).

Our aim in this paper is to go beyond the smallness condition on the initial data and to exhibit arbitrarily large initial data in  $\dot{C}^{-1}$  which generate a unique, global solution. We refer among others to [9,11,13,16,19,20,23] or [24] where different types of examples are provided. This was also performed in [8] in the periodic case, where we presented a new, nonlinear smallness assumption on the initial data, which may hold despite the fact that the data is large. That result uses the structure of the nonlinear term, as it is based on the fact that the two dimensional Navier–Stokes equation is globally well posed.

The first theorem of this paper consists in a result of global existence under a nonlinear smallness hypothesis (Theorem 1 below). The proof consists mainly in introducing an idea of [6] in the proof of the Koch and Tataru Theorem. The nonlinear smallness hypothesis is, roughly speaking, that the first iterate  $S(t)u_0 \cdot \nabla S(t)u_0$  is exponentially small with respect to  $u_0$ .

Then we exhibit an example of a family of initial data with very large  $\dot{C}^{-1}$  norm which satisfies the nonlinear smallness hypothesis. This example fits the structure of the nonlinear term  $u \cdot \nabla u$ .

Then, we study the stability of this nonlinear smallness condition, but not in the usual sense of a perturbation by a small vector field. This problem of a small perturbation was solved by I. Gallagher, D. Iftimie and F. Planchon in [13] and by P. Auscher, S. Dubois and P. Tchamitchian in [1].

Our purpose is different. Once constructed an initial data generating a global solution, we want to generate a large family of global solutions that may not be close to the one we start with, in the  $\dot{C}^{-1}$  norm. This is done with a fractal type transform (see the forthcoming Definition 1.3). Roughly speaking, this is the linear superposition of an arbitrarily large number of dilated and translated iterates of the initial data, and we will see that the initial data so-transformed still satisfies the nonlinear smallness assumption. That of course enables one to construct a very large class of initial data satisfying that smallness assumption; the transformation is based on a construction of [2].

### 1.2. Definitions

Before presenting more precisely the results of this paper, let us give some definitions and notation. We shall be using Besov spaces, which are defined equivalently using the Littlewood–Paley decomposition or the heat operator. As both definitions will be useful in the following, we present them both in the next definition.

**Definition 1.1.** Let  $\varphi \in \mathcal{S}(\mathbf{R}^3)$  be such that  $\hat{\varphi}(\xi) = 1$  for  $|\xi| \leq 1$  and  $\hat{\varphi}(\xi) = 0$  for  $|\xi| > 2$ . Define, for  $j \in \mathbf{Z}$ , the function  $\varphi_j(x) \stackrel{\text{def}}{=} 2^{3j} \varphi(2^j x)$ , and the Littlewood–Paley operators  $S_j \stackrel{\text{def}}{=} \varphi_j * \cdot$  and  $\Delta_j \stackrel{\text{def}}{=} S_{j+1} - S_j$ . Let  $f$  be in  $\mathcal{S}'(\mathbf{R}^3)$ . Then  $f$  belongs to the homogeneous Besov space  $\dot{B}_{p,q}^s(\mathbf{R}^3)$  if and only if

- The partial sum  $\sum_{-m}^m \Delta_j f$  converges towards  $f$  as a tempered distribution;
- The sequence  $\varepsilon_j \stackrel{\text{def}}{=} 2^{js} \|\Delta_j f\|_{L^p}$  belongs to  $\ell^q(\mathbf{Z})$ .

In that case

$$\|f\|_{\dot{B}_{p,q}^s} \stackrel{\text{def}}{=} \left( \sum_{j \in \mathbf{Z}} 2^{jsq} \|\Delta_j f\|_{L^p}^q \right)^{\frac{1}{q}}$$

and if  $s < 0$ , the one has the equivalent norm

$$\|f\|_{\dot{B}_{p,q}^s} \sim \|t^{-\frac{s}{2}} \|S(t)f\|_{L^p} \|_{L^q(\mathbf{R}^+; \frac{dt}{t})}. \tag{1.1}$$

Let us notice that the above equivalence comes from the inequality, proved for instance in [6],

$$\|S(t)\Delta_j a\|_{L^p} \leq C e^{-C^{-1}2^{2j}t} \|\Delta_j a\|_{L^p}. \tag{1.2}$$

Note that the following Sobolev-type continuous embeddings hold:

$$\dot{B}_{p_1, q_1}^{s_1} \subset \dot{B}_{p_2, q_2}^{s_2}, \quad \text{as soon as } s_1 - \frac{d}{p_1} = s_2 - \frac{d}{p_2} \quad \text{with } p_1 \leq p_2 \text{ and } q_1 \leq q_2.$$

We shall denote by  $\mathbf{P}$  the Leray projector onto divergence free vector fields

$$\mathbf{P} = \text{Id} - \nabla \Delta^{-1} \text{div}.$$

Before stating the first result of this paper, let us introduce the following space.

**Definition 1.2.** We shall denote by  $E$  the space of functions  $f$  in  $L^1(\mathbf{R}^+; \dot{B}_{\infty, 1}^{-1})$  such that

$$\sum_{j \in \mathbf{Z}} 2^{-j} \|\|\Delta_j f(t)\|_{L^\infty}\|_{L^2(\mathbf{R}^+; t dt)} < \infty$$

equipped with the norm

$$\|f\|_E \stackrel{\text{def}}{=} \|f\|_{L^1(\mathbf{R}^+; \dot{B}_{\infty, 1}^{-1})} + \sum_{j \in \mathbf{Z}} 2^{-j} \|\|\Delta_j f(t)\|_{L^\infty}\|_{L^2(\mathbf{R}^+; t dt)}.$$

Let us remark that, for any homogeneous function  $\sigma$  of order 0 smooth outside 0, we have

$$\forall p \in [1, \infty], \quad \|\sigma(D)\Delta_j f\|_{L^p} \leq C \|\Delta_j f\|_{L^p}.$$

Thus the Leray projection  $\mathbf{P}$  onto divergence free vectors fields maps continuously  $E$  into  $E$ .

### 1.3. Statement of the results

#### 1.3.1. Global existence results

The first result we shall prove is a new global wellposedness result, under a nonlinear smallness assumption on the initial data.

**Theorem 1.** *There is a constant  $C_0$  such that the following result holds. Let  $u_0 \in \dot{H}^{\frac{1}{2}}(\mathbf{R}^3)$  be a divergence free vector field. Suppose that*

$$\|\mathbf{P}(S(t)u_0 \cdot \nabla S(t)u_0)\|_E \leq C_0^{-1} \exp(-C_0 \|u_0\|_{\dot{B}_{\infty, 2}^{-1}}^4). \tag{1.3}$$

*Then there is a unique, global solution to (NS) associated with  $u_0$ , satisfying*

$$u \in C_b(\mathbf{R}^+; \dot{H}^{\frac{1}{2}}) \cap L^2(\mathbf{R}^+; \dot{H}^{\frac{3}{2}}).$$

**Remark.** For the sake of simplicity, we state the theorem for initial data in  $\dot{H}^{\frac{1}{2}}$ , but it obviously works for initial data in  $\dot{B}_{\infty, 2}^{-1}$ .

The proof of Theorem 1 is given in Section 2 below; it consists in writing the solution  $u$  (which exists for a short time at least), as  $u = S(t)u_0 + R$  and in proving a global wellposedness result for the perturbed Navier–Stokes equation satisfied by  $R$ , under assumption (1.3). While the proof follows the lines of that of Koch and Tataru (see [18]), a small modification of classical Picard’s argument is needed to control the linear term, which is not small.

In fact, the main point of the paper is to exhibit examples of applications of this theorem which go beyond the assumption of smallness of  $\|u_0\|_{\dot{B}_{\infty, 2}^{-1}}$ . The problem we have to solve is the construction of large initial data  $u_0$  such that a quadratic functional, namely  $\mathbf{P}(S(t)u_0 \cdot \nabla S(t)u_0)$  is small. This demands a careful use of the structure of this quadratic functional.

The situation here is different from our previous work (see [8]), devoted to the periodic case, where the structure of the equation was used through the fact that the bidimensional Navier–Stokes equation is globally wellposed.

Now let us state the theorem that ensures that Theorem 1 is relevant.

**Theorem 2.** *Let  $\phi \in \mathcal{S}(\mathbf{R}^3)$  be a given function, and consider two real numbers  $\varepsilon$  and  $\alpha$  in  $]0, 1[$ . Define*

$$\varphi_\varepsilon(x) = \frac{(-\log \varepsilon)^{\frac{1}{5}}}{\varepsilon^{1-\alpha}} \cos\left(\frac{x_3}{\varepsilon}\right) \phi\left(x_1, \frac{x_2}{\varepsilon^\alpha}, x_3\right).$$

*There is a constant  $C > 0$  such that for  $\varepsilon$  small enough, the smooth, divergence free vector field*

$$u_{0,\varepsilon}(x) = (\partial_2 \varphi_\varepsilon(x), -\partial_1 \varphi_\varepsilon(x), 0)$$

*satisfies*

$$C^{-1}(-\log \varepsilon)^{\frac{1}{5}} \leq \|u_{0,\varepsilon}\|_{\dot{B}_{\infty,\infty}^{-1}} \leq C(-\log \varepsilon)^{\frac{1}{5}},$$

*and*

$$\|S(t)u_{0,\varepsilon} \cdot \nabla S(t)u_{0,\varepsilon}\|_E \leq C\varepsilon^{\frac{\alpha}{3}}(-\log \varepsilon)^{\frac{2}{5}}. \tag{1.4}$$

*Thus for  $\varepsilon$  small enough, the vector field  $u_{0,\varepsilon}$  generates a unique, global solution to (NS).*

The proof of Theorem 2 is the purpose of Section 3.

**Remark.** One can also write this example in terms of the Reynolds number of the fluid: let  $\text{Re} > 0$  be the Reynolds number, and define the rescaled velocity field  $v(t, x) = \nu u(\nu t, x)$  where  $\nu = 1/\text{Re}$ . Then  $v$  satisfies the Navier–Stokes equation

$$\partial_t v + \mathbf{P}(v \cdot \nabla v) - \nu \Delta v = 0$$

and Theorem 2 states the following: the vector field

$$v_{0,\nu}(x) = (-\log \nu)^{\frac{1}{5}} \cos\left(\frac{x_3}{\nu}\right) \left( (\partial_2 \phi)\left(x_1, \frac{x_2}{\nu^\alpha}, x_3\right), \nu^\alpha (-\partial_1 \phi)\left(x_1, \frac{x_2}{\nu^\alpha}, x_3\right) \right)$$

satisfies

$$\|v_{0,\nu}\|_{\dot{B}_{\infty,\infty}^{-1}} \sim C\nu(-\log \nu)^{\frac{1}{5}}$$

and generates a global solution to the Navier–Stokes equations if  $\nu$  is small enough. Compared with the usual theory of global existence for the Navier–Stokes equations, we have gained a (power of a) logarithm in the smallness assumption in terms of the viscosity, since classically one expects the initial data to be small with respect to  $\nu$ .

### 1.3.2. Stability results

The second aim of this paper is to give some stability properties of global solutions. It is known since [13] that any initial data in  $\dot{B}_{p,\infty}^{-1+\frac{3}{p}}$  giving rise to a unique global solution is stable: a small perturbation of that data also generates a global solution (see [1] for the case of  $BMO^{-1}$ ). Here we present a stability result where the perturbation is as large as the initial data but has a special form: it consists in the superposition of dilated and translated duplicates of the initial data, in the spirit of profile decompositions of P. Gérard (see [14]). This transform is a version of the fractal transform used in [2] in the study of refined Sobolev and Hardy inequalities. Let us be more precise and define the transformation. We shall only be considering compactly supported initial data for this study, and up to a rescaling we shall suppose to simplify that the support of the initial data is restricted to the unit cube  $Q$  of  $\mathbf{R}^3$  centered at 0.

**Definition 1.3.** Let  $X = (x_1, \dots, x_K)$  be a set of  $K$  distinct points in  $\mathbf{R}^3$ . For  $\Lambda \in 2^{\mathbf{N}}$ , let us define

$$T_{\Lambda,X} \begin{cases} \mathcal{S}' \rightarrow \mathcal{S}' \\ f \mapsto T_{\Lambda,X} f \stackrel{\text{def}}{=} \sum_{J \in \{1, \dots, K\}} T_{\Lambda}^J f \end{cases} \quad \text{with} \quad T_{\Lambda}^J f(x) \stackrel{\text{def}}{=} \Lambda f(\Lambda(x - x_J)).$$

It can be noted that this is a generalization of the fractal transformation  $T^k$  studied in [2].

The next statement is quite easy to prove: it shows that this transformation on the initial data preserves global wellposedness, as soon as the scaling parameter  $\Lambda$  is large enough (the threshold  $\Lambda$  being unknown as a function of the initial data). The theorem following that statement gives a quantitative approach to that stability: if the initial data  $u_0$  satisfies the smallness assumption (1.3) of Theorem 1, then so does  $T_{\Lambda, X}u_0$  as soon as  $\Lambda$  is large enough (the threshold being an explicit function of norms of  $u_0$ ).

More precisely we have the following results.

**Proposition 1.1.** *Let  $u_0$  be a divergence free vector field in  $\dot{H}^{\frac{1}{2}}(\mathbf{R}^3)$  generating a unique, global solution to the Navier–Stokes equations and  $X$  be a finite sequence of distinct points. There is  $\Lambda_0 > 0$  such that, for any  $\Lambda \geq \Lambda_0$ , the vector field  $T_{\Lambda, X}u_0$  also generates a unique, global solution.*

**Remarks.**

- Using the global stability of global solutions proved in [13], a global solution associated to an initial data in  $\dot{H}^{\frac{1}{2}}$  is always in  $L^\infty(\mathbf{R}^+; \dot{H}^{\frac{1}{2}}) \cap L^2(\mathbf{R}^+; \dot{H}^{\frac{3}{2}})$ .
- As the proof of that result in Section 4.1 will show (see Proposition 4.1), Proposition 1.1 can be generalized to the case where the vector field  $u_0$  is replaced by any finite sequence of vector fields in  $\dot{H}^{\frac{1}{2}}$  generating a global solution.
- As we shall see in the proof of Theorem 3 stated below, the functions  $T_{\Lambda, X}u_0$  and  $u_0$  have essentially the same norm in  $\dot{C}^{-1}$ .

Now let us state the quantitative stability theorem, in particular in the case of an initial data satisfying the assumptions of Theorem 1. In order to avoid excessive heaviness, we shall assume from now on that the initial data is compactly supported, and after scaling, supported in the unit cube  $Q = ]-\frac{1}{2}, \frac{1}{2}[^d$ . We shall consider sequences  $X$  such that

$$\inf_{\substack{(J, J') \in \{1, \dots, K\}^2 \\ J \neq J'}} \{d(x_J, x_{J'}); d(x_J, {}^cQ)\} \geq \delta > 0. \tag{1.5}$$

We shall prove the following theorem.

**Theorem 3.** *Let  $u_0$  be a smooth  $\dot{H}^{\frac{1}{2}}$  divergence free vector field, compactly supported in the cube  $Q$ . Suppose that  $u_0$  satisfies (1.3) in the following slightly looser sense: there is  $\eta \in ]0, 1[$  such that*

$$\|\mathbf{P}(S(t)u_0 \cdot \nabla S(t)u_0)\|_E \leq C_0^{-1} \exp(-C_0(\|u_0\|_{\dot{B}_{\infty, 2}^{-1}} + \eta)^4) - \eta. \tag{1.6}$$

*Then there is a positive  $\Lambda_0$  (depending only on  $\eta, K, \delta, \|u_0\|_{\dot{H}^{-1}}$  and  $\|u_0\|_{\dot{B}_{\infty, \infty}^{-3}}$ ) such that for any  $\Lambda \geq \Lambda_0$ , the vector field  $T_{\Lambda, X}u_0$  satisfies (1.3) and in particular generates a global solution to (NS). Moreover, for all  $r$  in  $[1, \infty]$ ,*

$$\|u_0\|_{\dot{B}_{\infty, r}^{-1}} - \eta \leq \|T_{\Lambda, X}u_0\|_{\dot{B}_{\infty, r}^{-1}} \leq \|u_0\|_{\dot{B}_{\infty, r}^{-1}} + \eta.$$

**Remarks.**

- The factor  $\eta$  appearing in (1.6) means that  $u_0$  must not saturate the nonlinear smallness assumption (1.3) of Theorem 1.
- The proof of this theorem is based on the fact that the Besov norm of index  $-1$  as well as  $\|\mathbf{P}(S(t)u_0 \cdot \nabla S(t)u_0)\|_E$  are invariant under the action of  $T_{\Lambda, X}$ , up to some small error terms.

As a conclusion of this introduction, let us state the following result, which describes the action of  $T_{\Lambda, X}$  on the family  $u_{0, \varepsilon}$  introduced in Theorem 2.

**Theorem 4.** Let  $u_{0,\varepsilon}$  be the family introduced in Theorem 2. For any  $K$  and  $\delta$ , a constant  $\Lambda_0$  exists, which is independent of  $\varepsilon$ , such that the following result holds. For any family  $X$  and any  $\Lambda \geq \Lambda_0$ , there is a global smooth solution of (NS) with initial data  $T_{\Lambda,X}u_{0,\varepsilon}$ .

**Remark.** Let us point out that as opposed to Proposition 1.1, Theorem 3 (or rather Lemmas 4.1 and 4.2 which are the key to its proof) provides precise bounds on  $\Lambda_0$  so that the constant  $\Lambda_0$  appearing in Theorem 4 may be chosen independently of  $\varepsilon$ .

**2. Proof of Theorem 1**

2.1. Main steps of the proof

Let us start by remarking that in the case when  $u_0$  is small then there is nothing to be proved, so in the following we shall suppose that  $\|u_0\|_{\dot{B}_{\infty,2}^{-1}}$  is not small, say  $\|u_0\|_{\dot{B}_{\infty,2}^{-1}} \geq 1$ .

We follow the method introduced by H. Koch and D. Tataru in [18] in order to look for the solution  $u$  under the form  $u_F + R$ , where  $u_F(t) \stackrel{\text{def}}{=} S(t)u_0$ . Let us denote by  $\mathcal{Q}$  the bilinear operator defined by

$$\mathcal{Q}(a, b)(t) \stackrel{\text{def}}{=} -\frac{1}{2} \int_0^t S(t-t') \mathbf{P}(a(t') \cdot \nabla b(t') + b(t') \cdot \nabla a(t')) dt'.$$

Then  $R$  is the solution of

$$(MNS) \quad R = \mathcal{Q}(u_F, u_F) + 2\mathcal{Q}(u_F, R) + \mathcal{Q}(R, R).$$

To prove the global existence of  $u$ , we are reduced to proving the global wellposedness of (MNS); that relies on the following easy lemma, the proof of which is omitted.

**Lemma 2.1.** Let  $X$  be a Banach space, let  $L$  be a continuous linear map from  $X$  to  $X$ , and let  $B$  be a bilinear map from  $X \times X$  to  $X$ . Let us define

$$\|L\|_{\mathcal{L}(X)} \stackrel{\text{def}}{=} \sup_{\|x\|=1} \|Lx\| \quad \text{and} \quad \|B\|_{\mathcal{B}(X)} \stackrel{\text{def}}{=} \sup_{\|x\|=\|y\|=1} \|B(x, y)\|.$$

If  $\|L\|_{\mathcal{L}(X)} < 1$ , then for any  $x_0$  in  $X$  such that

$$\|x_0\|_X < \frac{(1 - \|L\|_{\mathcal{L}(X)})^2}{4\|B\|_{\mathcal{B}(X)}},$$

the equation

$$x = x_0 + Lx + B(x, x)$$

has a unique solution in the ball of center 0 and radius  $\frac{1 - \|L\|_{\mathcal{L}(X)}}{2\|B\|_{\mathcal{B}(X)}}$ .

Let us introduce the functional space for which we shall apply the above lemma. We define the quantity

$$U(t) \stackrel{\text{def}}{=} \|u_F(t)\|_{L^\infty}^2 + t \|u_F(t)\|_{L^\infty}^4,$$

which satisfies

$$\begin{aligned} \int_0^\infty U(t) dt &\leq C \|u_0\|_{\dot{B}_{\infty,2}^{-1}}^2 + C \|u_0\|_{\dot{B}_{\infty,4}^{-1}}^4 \\ &\leq C \|u_0\|_{\dot{B}_{\infty,2}^{-1}}^4 \end{aligned} \tag{2.1}$$

recalling that we have supposed that  $\|u_0\|_{\dot{B}_{\infty,2}^{-1}} \geq 1$  to simplify the proof.

For all  $\lambda \geq 0$ , let us denote by  $X_\lambda$  the set of functions on  $\mathbf{R}^+ \times \mathbf{R}^3$  such that

$$\|v\|_\lambda \stackrel{\text{def}}{=} \sup_{t>0} \left( t^{\frac{1}{2}} \|v_\lambda(t)\|_{L^\infty} + \sup_{\substack{x \in \mathbf{R}^3 \\ R>0}} R^{-\frac{3}{2}} \left( \int_{P(x,R)} |v_\lambda(t,y)|^2 dy \right)^{\frac{1}{2}} \right) < \infty, \quad (2.2)$$

where

$$v_\lambda(t,x) \stackrel{\text{def}}{=} v(t,x) \exp \left( -\lambda \int_0^t U(t') dt' \right)$$

while  $P(x,R) = [0, R^2] \times B(x,R)$  and  $B(x,R)$  denotes the ball of  $\mathbf{R}^3$  of center  $x$  and radius  $R$ . Let us point out that, in the case when  $\lambda = 0$ , this is exactly the space introduced by H. Koch and D. Tataru in [18], and that for any  $\lambda \geq 0$  we have due to (2.1),

$$\|v\|_\lambda \leq \|v\|_0 \leq C \|v\|_\lambda \exp(C\lambda \|u_0\|_{\dot{B}_{\infty,2}^{-1}}^4). \quad (2.3)$$

From Lemmas 3.1 and 3.2 of [18] together with the above equivalence of norms, we infer that

$$\|\mathcal{Q}(v,w)\|_\lambda \leq C \|v\|_\lambda \|w\|_\lambda \exp(C\lambda \|u_0\|_{\dot{B}_{\infty,2}^{-1}}^4). \quad (2.4)$$

Theorem 1 follows from the following two lemmas.

**Lemma 2.2.** *There is a constant  $C > 0$  such that the following holds. For any nonnegative  $\lambda$ , for any  $t \geq 0$  and any  $f \in E$ , we have*

$$\left\| \int_0^t \mathcal{S}(t-t') f(t') dt' \right\|_\lambda \leq C \|f\|_E.$$

**Lemma 2.3.** *Let  $u_0 \in \dot{B}_{\infty,2}^{-1}$  be given, and define  $u_F(t) = \mathcal{S}(t)u_0$ . There is a constant  $C > 0$  such that the following holds. For any  $\lambda \geq 1$ , for any  $t \geq 0$  and any  $v \in X_\lambda$ , we have*

$$\|\mathcal{Q}(u_F, v)(t)\|_\lambda \leq \frac{C}{\lambda^{\frac{1}{4}}} \|v\|_\lambda.$$

**End of the proof of Theorem 1.** Let us apply Lemma 2.1 to Eq. (MNS) satisfied by  $R$ , in a space  $X_\lambda$ . We choose  $\lambda$  so that according to Lemma 2.3,

$$\|\mathcal{Q}(u_F, \cdot)(t)\|_{\mathcal{L}(X_\lambda)} \leq \frac{1}{4}.$$

Then according to Lemma 2.1, there is a unique solution  $R$  to (MNS) in  $X_\lambda$  as soon as  $\mathcal{Q}(u_F, u_F)$  satisfies

$$\|\mathcal{Q}(u_F, u_F)\|_{X_\lambda} \leq \frac{1}{16 \|\mathcal{Q}\|_{\mathcal{B}(X_\lambda)}}.$$

But (2.4) guarantees that

$$\|\mathcal{Q}\|_{\mathcal{B}(X_\lambda)} \leq C \exp(C\lambda \|u_0\|_{\dot{B}_{\infty,2}^{-1}}^4),$$

so it is enough to check that for some constant  $C$ ,

$$\|\mathcal{Q}(u_F, u_F)\|_{X_\lambda} \leq C^{-1} \exp(-C\lambda \|u_0\|_{\dot{B}_{\infty,2}^{-1}}^4).$$

By Lemma 2.2, this is precisely condition (1.3) of Theorem 1, so under assumption (1.3), there is a unique, global solution  $R$  to (MNS), in the space  $X_\lambda$ . This implies immediately that there is a unique, global solution  $u$  to the Navier–Stokes system in  $X_\lambda$ . The fact that  $u$  belongs to  $C_b(\mathbf{R}^+; \dot{H}^{\frac{1}{2}}) \cap L^2(\mathbf{R}^+; \dot{H}^{\frac{3}{2}})$  is then simply an argument of propagation of regularity (see for instance [21]).  $\square$



2.2. Proof of Lemma 2.2

Thanks to (2.3), it is enough to prove Lemma 2.2 for  $\lambda = 0$ .

Let us start by proving that  $\int_0^t S(t-t')f(t') dt'$  belongs to  $L^2(\mathbf{R}^+; L^\infty)$ ; that will give in particular the boundedness of the second norm entering in the definition of  $X_\lambda$ .

Using (1.2), we get

$$\left\| \int_0^t \Delta_j S(t-t')f(t') dt' \right\|_{L^\infty} \leq C \int_0^t e^{-C^{-1}2^{2j}(t-t')} \|\Delta_j f(t')\|_{L^\infty} dt'.$$

Young’s inequality then gives

$$\left\| \int_0^t \Delta_j S(t-t')f(t') dt' \right\|_{L^2(\mathbf{R}^+; L^\infty)} \leq C2^{-j} \|\Delta_j f\|_{L^1(\mathbf{R}^+; L^\infty)},$$

thus the series  $(\Delta_j \int_0^t S(t-t')f(t') dt')_{j \in \mathbf{Z}}$  converges in  $L^2(\mathbf{R}^+; L^\infty)$ , and

$$\left\| \int_0^t S(t-t')f(t') dt' \right\|_{L^2(\mathbf{R}^+; L^\infty)} \leq C \|f\|_E.$$

This implies in particular that

$$\sup_{\substack{x \in \mathbf{R}^3 \\ R > 0}} R^{-\frac{3}{2}} \left( \int_{P(x,R)} \left| \int_0^t (S(t-t')f(t'))(y) dt' \right|^2 dy \right)^{\frac{1}{2}} \leq C \|f\|_E. \tag{2.5}$$

The second part of the norm defining  $\|\cdot\|_{X_\lambda}$  in (2.2) is therefore controlled by the norm of  $f$  in  $E$ .

To estimate the first part of that norm, let us write that for any  $t \geq 0$  and any  $j \in \mathbf{Z}$ ,

$$t^{\frac{1}{2}} \Delta_j \int_0^t S(t-t')f(t') dt' = G_j^{(1)}(t) + G_j^{(2)}(t) \quad \text{with}$$

$$G_j^{(1)}(t) \stackrel{\text{def}}{=} t^{\frac{1}{2}} \int_0^{\frac{t}{2}} S(t-t') \Delta_j f(t') dt' \quad \text{and}$$

$$G_j^{(2)}(t) \stackrel{\text{def}}{=} t^{\frac{1}{2}} \int_{\frac{t}{2}}^t S(t-t') \Delta_j f(t') dt'.$$

Using again (1.2) we have, since  $t \leq 2(t-t')$ ,

$$\begin{aligned} \|G_j^{(1)}(t)\|_{L^\infty} &\leq C \int_0^{\frac{t}{2}} (t-t')^{\frac{1}{2}} 2^j e^{-C^{-1}2^{2j}(t-t')} 2^{-j} \|\Delta_j f(t')\|_{L^\infty} dt' \\ &\leq 2^{-j} \|\Delta_j f\|_{L^1(\mathbf{R}^+; L^\infty)}. \end{aligned}$$

In order to estimate  $\|G_j^{(2)}(t)\|_{L^\infty}$ , let us write, since  $t \leq 2t'$ ,

$$\|G_j^{(2)}(t)\|_{L^\infty} \leq C \int_0^t e^{-C^{-1}2^{2j}(t-t')} t'^{\frac{1}{2}} \|\Delta_j f(t')\|_{L^\infty} dt'.$$

Using the Cauchy–Schwarz inequality, we get

$$\|G_j^{(2)}(t)\|_{L^\infty} \leq C 2^{-j} \|t^{\frac{1}{2}} \Delta_j f(t)\|_{L^2(\mathbf{R}^+; L^\infty)}.$$

Then using (2.5) and summing over  $j \in \mathbf{Z}$  concludes the proof of Lemma 2.2.  $\square$

### 2.3. Proof of Lemma 2.3

We have (see for instance [18] or [7]) that

$$\begin{aligned} \mathcal{Q}(v, w)(t, x) &= \int_0^t \int_{\mathbf{R}^3} k(t-t', y) v(t', x-y) w(t', x-y) dy dt' \\ &= k \star (vw)(t, x) \quad \text{with } |k(\tau, \zeta)| \leq \frac{C}{(\sqrt{\tau} + |\zeta|)^4}. \end{aligned}$$

The proof relies now mainly on the following proposition.

**Proposition 2.1.** *Let  $u_0 \in \dot{B}_{\infty,2}^{-1}$  be given, and define  $u_F(t) = S(t)u_0$ . There is a constant  $C$  such that the following holds. Consider, for any positive  $R$  and for  $(\tau, \zeta) \in \mathbf{R}^+ \times \mathbf{R}^3$ , the following functions:*

$$K_R^{(1)}(\tau, \zeta) \stackrel{\text{def}}{=} \mathbf{1}_{|\zeta| \geq R} \frac{1}{|\zeta|^4} \quad \text{and} \quad K_R^{(2)}(\tau, \zeta) \stackrel{\text{def}}{=} \mathbf{1}_{|\zeta| \leq R} \frac{1}{(\sqrt{\tau} + |\zeta|)^4}.$$

Then for any  $\lambda \geq 1$  and any  $R > 0$ ,

$$\|e^{-\lambda \int_0^t U(t') dt'} K_R^{(1)} \star (u_F v)\|_{L^\infty([0, R^2] \times \mathbf{R}^3)} \leq \frac{C}{\lambda^{\frac{1}{2}} R} \|v\|_\lambda. \tag{2.6}$$

Moreover, for any  $\lambda \geq 1$  and any  $R > 0$ ,

$$\|e^{-\lambda \int_0^t U(t') dt'} K_R^{(2)} \star (u_F v)\|_{L^\infty([R^2, 2R^2] \times \mathbf{R}^3)} \leq \frac{C}{\lambda^{\frac{1}{4}} R} \|v\|_\lambda. \tag{2.7}$$

**Proof of Proposition 2.1.** Let us write that

$$\begin{aligned} V_\lambda^{(1)}(t, x) &\stackrel{\text{def}}{=} e^{-\lambda \int_0^t U(t') dt'} |K_R^{(1)} \star (u_F v)(t, x)| \\ &\leq \int_0^t \int_{cB(0,R)} \frac{1}{|y|^4} e^{-\lambda \int_{t'}^t U(t'') dt''} \|u_F(t', \cdot)\|_{L^\infty} |v_\lambda(t', x-y)| dt' dy. \end{aligned}$$

By the Cauchy–Schwarz inequality and by definition of  $U$ , we infer that

$$\begin{aligned} V_\lambda^{(1)}(t, x) &\leq \left( \int_0^t \int_{cB(0,R)} \frac{1}{|y|^4} e^{-2\lambda \int_{t'}^t U(t'') dt''} \|u_F(t', \cdot)\|_{L^\infty}^2 dt' dy \right)^{\frac{1}{2}} \left( \int_0^t \int_{cB(0,R)} \frac{1}{|y|^4} |v_\lambda(t', x-y)|^2 dt' dy \right)^{\frac{1}{2}} \\ &\leq \left( \frac{C}{\lambda R} \right)^{\frac{1}{2}} \left( \int_0^t \int_{cB(0,R)} \frac{1}{|y|^4} |v_\lambda(t', x-y)|^2 dt' dy \right)^{\frac{1}{2}}. \end{aligned} \tag{2.8}$$

Now let us decompose the integral on the right on rings; this gives

$$\begin{aligned} \int_0^t \int_{cB(0,R)} \frac{1}{|y|^4} |v_\lambda(t', x - y)|^2 dt' dy &= \sum_{p=0}^\infty \int_0^t \int_{B(0,2^{p+1}R) \setminus B(0,2^pR)} \frac{1}{|y|^4} |v_\lambda(t', x - y)|^2 dt' dy \\ &\leq \frac{1}{R} \sum_{p=0}^\infty 2^{-p+3} (2^{p+1}R)^{-3} \\ &\quad \times \int_0^t \int_{B(0,2^{p+1}R)} |v_\lambda(t, x - y)|^2 dt dy. \end{aligned}$$

As  $t \leq R^2$  and  $p$  is nonnegative, we have

$$\begin{aligned} \int_0^t \int_{cB(0,R)} \frac{1}{|y|^4} |v_\lambda(t', x - y)|^2 dt' dy &\leq \frac{C}{R} \sum_{p=0}^\infty 2^{-p} (2^{p+1}R)^{-3} \int_{P(x,2^{p+1}R)} |v_\lambda(t, z)|^2 dt dz \\ &\leq \frac{C}{R} \sum_{p=0}^\infty 2^{-p} \sup_{R'>0} \frac{1}{R'^3} \int_{P(x,R')} |v_\lambda(t, z)|^2 dt dz. \end{aligned}$$

By definition of  $\|\cdot\|_\lambda$ , we infer that

$$\int_0^t \int_{cB(0,R)} \frac{1}{|y|^4} |v_\lambda(t', x - y)|^2 dt' dy \leq \frac{C}{R} \|v\|_\lambda^2.$$

Then, using (2.8), we conclude the proof of (2.6).

In order to prove the second inequality, let us observe that

$$\begin{aligned} e^{-\lambda \int_0^t U(t') dt'} |(K_R^{(2)} \star (u_F v))(t, x)| &\leq \mathcal{K}_R^{(21)}(t, x) + \mathcal{K}_R^{(21)}(t, x) \quad \text{with} \\ \mathcal{K}_R^{(22)}(t, x) &\stackrel{\text{def}}{=} \int_0^{\frac{t}{2}} \int_{B(0,R)} \frac{1}{(\sqrt{t-t'} + |y|)^4} e^{-\lambda \int_{t'}^t U(t'') dt''} \|u_F(t', \cdot)\|_{L^\infty} |v_\lambda(t', x - y)| dt' dy, \\ \mathcal{K}_R^{(22)}(t, x) &\stackrel{\text{def}}{=} \int_{\frac{t}{2}}^t \int_{B(0,R)} \frac{1}{(\sqrt{t-t'} + |y|)^4} e^{-\lambda \int_{t'}^t U(t'') dt''} \|u_F(t', \cdot)\|_{L^\infty} |v_\lambda(t', x - y)| dt' dy. \end{aligned}$$

Using the Cauchy–Schwarz inequality, as  $t \in [R^2, 2R^2]$  and  $t \leq 2(t - t')$ , we infer that

$$\begin{aligned} \mathcal{K}_R^{(21)}(t, x) &\leq \left( \int_0^{\frac{t}{2}} \int_{B(0,R)} \frac{1}{(\sqrt{t-t'} + |y|)^8} e^{-2\lambda \int_{t'}^t U(t'') dt''} \|u_F(t', \cdot)\|_{L^\infty}^2 dt' dy \right)^{\frac{1}{2}} \\ &\quad \times \left( \int_0^{\frac{t}{2}} \int_{B(0,R)} |v_\lambda(t', x - y)|^2 dt' dy \right)^{\frac{1}{2}} \\ &\leq \frac{C}{\lambda^{\frac{1}{2}}} \left( \int_{B(0,R)} \frac{dy}{(R + |y|)^8} \right)^{\frac{1}{2}} \left( \int_0^{\frac{t}{2}} \int_{B(0,R)} |v_\lambda(t', x - y)|^2 dt' dy \right)^{\frac{1}{2}} \\ &\leq \frac{C}{(t\lambda)^{\frac{1}{2}}} R^{-\frac{3}{2}} \left( \int_0^R \int_{B(0,R)} |v_\lambda(t', x - y)|^2 dt' dy \right)^{\frac{1}{2}}, \end{aligned}$$

so that

$$K_R^{(21)}(t, x) \leq \frac{C}{(t\lambda)^{\frac{1}{2}}} \|v\|_\lambda. \tag{2.9}$$

In order to estimate  $\mathcal{K}_R^{(22)}$ , let us write that

$$\begin{aligned} \mathcal{K}_R^{(22)}(t, x) &\leq \int_{\frac{t}{2}}^t \int_{\mathbf{R}^3} \frac{1}{(\sqrt{t-t'}+|y|)^4} e^{-\lambda \int_{t'}^t U(t'') dt''} \|u_F(t', \cdot)\|_{L^\infty} \|v_\lambda(t', \cdot)\|_{L^\infty} dt' dy \\ &\leq C \|v\|_\lambda \int_{\frac{t}{2}}^t \frac{1}{\sqrt{t-t'}} e^{-\lambda \int_{t'}^t U(t'') dt''} \frac{\|u_F(t', \cdot)\|_{L^\infty}}{t'^{\frac{1}{2}}} dt'. \end{aligned}$$

By definition of  $U$  and using the fact that  $t \leq 2t'$ , Hölder’s inequality implies that

$$\begin{aligned} \mathcal{K}_R^{(22)}(t, x) &\leq \frac{C}{t^{\frac{1}{2}}} \|v\|_\lambda \left( \int_0^t e^{-4\lambda \int_{t'}^t U(t'') dt''} t' \|u_F(t', \cdot)\|_{L^\infty}^4 dt' \right)^{\frac{1}{4}} \\ &\leq \frac{C}{\lambda^{\frac{1}{4}} t^{\frac{1}{2}}} \|v\|_\lambda. \end{aligned}$$

Together with (2.9), this concludes the proof of the proposition.  $\square$

From this proposition, we infer immediately the following corollary. This corollary proves directly one half of Lemma 2.3, as it gives a control of  $\mathcal{Q}(u_F, v)$  in the first norm out of the two entering in the definition of  $X_\lambda$ .

**Corollary 2.1.** *Under the assumptions of Proposition 2.1, we have*

$$t^{\frac{1}{2}} e^{-\lambda \int_0^t U(t') dt'} \|\mathcal{Q}(u_F, v)(t, \cdot)\|_{L^\infty} \leq \frac{C}{\lambda^{\frac{1}{4}}} \|v\|_\lambda.$$

**Proof of Corollary 2.1.** Let us write that

$$k \star (u_F v)(t, x) = k \star (u_F \mathbf{1}_{c_{B(x, 2\sqrt{t})}} v)(t, x) + k \star (u_F \mathbf{1}_{B(x, 2\sqrt{t})} v)(t, x).$$

From Proposition 2.1, we infer that

$$\begin{aligned} e^{-\lambda \int_0^t U(t') dt'} |k \star (u_F \mathbf{1}_{c_{B(x, 2\sqrt{t})}} v)(t, x)| &\leq e^{-\lambda \int_0^t U(t') dt'} K_{2\sqrt{t}}^{(1)} \star (|u_F \mathbf{1}_{B(x, 2\sqrt{t})} v|)(t, x) \\ &\leq \frac{C}{(t\lambda)^{\frac{1}{2}}} \|v\|_\lambda. \end{aligned}$$

Moreover, thanks to Proposition 2.1, we have also

$$\begin{aligned} e^{-\lambda \int_0^t U(t') dt'} |k \star (u_F \mathbf{1}_{B(x, 2\sqrt{t})} v)(t, x)| &\leq e^{-\lambda \int_0^t U(t') dt'} K_{2\sqrt{t}}^{(2)} \star (|u_F \mathbf{1}_{B(x, 2\sqrt{t})} v|)(t, x) \\ &\leq \frac{C}{\lambda^{\frac{1}{4}} t^{\frac{1}{2}}} \|v\|_\lambda. \end{aligned}$$

This proves the corollary.  $\square$

In order to conclude the proof of Lemma 2.3, let us estimate  $\|k \star (u_F v)\|_{L^2(P(x, R))}$ , for an arbitrary  $x \in \mathbf{R}^3$ . Let us write that

$$k \star (u_F v) = k \star (u_F \mathbf{1}_{c_{B(x, 2R)}} v) + k \star (u_F \mathbf{1}_{B(x, 2R)} v).$$

Observing that, for any  $y \in B(x, R)$ , we have

$$|k \star (u_F \mathbf{1}_{cB(x,2R)} v)(t, y)| \leq C K_R^{(1)} \star (|u_F| \mathbf{1}_{cB(x,2R)} |v|)(t, y),$$

and using inequality (2.6) of Proposition 2.1, we get

$$e^{-\lambda \int_0^t U(t') dt'} \|k \star (u_F \mathbf{1}_{cB(x,2R)} v)\|_{L^\infty(P(x,R))} \leq \frac{C}{\lambda^{\frac{1}{2}} R} \|v\|_\lambda.$$

As the volume of  $P(x, R)$  is proportional to  $R^5$ , we infer that

$$\|k \star (u_F v_\lambda)\|_{L^2(P(x,R))} \leq \frac{C}{\lambda^{\frac{1}{2}}} R^{\frac{3}{2}} \|v\|_\lambda.$$

The following inequality is easy and classical, so its proof is omitted.

$$\|e^{-\lambda \int_0^t U(t') dt'} \mathcal{Q}(u_F, v)(t)\|_{L^2([0,T] \times \mathbf{R}^3)} \leq \frac{C}{\lambda^{\frac{1}{2}}} \|v_\lambda\|_{L^2([0,T] \times \mathbf{R}^3)}.$$

We deduce that

$$\begin{aligned} \|e^{-\lambda \int_0^t U(t') dt'} k \star (u_F \mathbf{1}_{B(x,2R)} v)\|_{L^2(P(x,R))} &\leq \|e^{-\lambda \int_0^t U(t') dt'} k \star (u_F \mathbf{1}_{B(x,2R)} v)\|_{L^2([0,R^2] \times \mathbf{R}^3)} \\ &\leq \frac{C}{\lambda^{\frac{1}{2}}} \|\mathbf{1}_{B(x,2R)} v_\lambda\|_{L^2([0,R^2] \times \mathbf{R}^3)} \\ &\leq \frac{C}{\lambda^{\frac{1}{2}}} \|v_\lambda\|_{L^2(P(x,2R))}. \end{aligned}$$

This concludes the proof of Lemma 2.3.  $\square$

### 3. Proof of Theorem 2

In this paragraph we shall check that the vector field  $u_{0,\varepsilon}$  introduced in Theorem 2 satisfies the nonlinear smallness assumption of Theorem 1, and we shall also show that its  $\dot{B}_{\infty,\infty}^{-1}$  norm is equivalent to  $(-\log \varepsilon)^{\frac{1}{5}}$ . Let us start by proving the following lemma.

**Lemma 3.1.** *Let  $f \in \mathcal{S}(\mathbf{R}^3)$  be given and  $\sigma \in ]0, 3(1 - \frac{1}{p})[$ . There is a constant  $C > 0$  such that for any  $\varepsilon \in ]0, 1[$ , the function*

$$f_\varepsilon(x) \stackrel{\text{def}}{=} e^{i \frac{x_3}{\varepsilon}} f\left(x_1, \frac{x_2}{\varepsilon^\alpha}, x_3\right)$$

satisfies, for all  $p \geq 1$ ,

$$\|f_\varepsilon\|_{\dot{B}_{p,1}^{-\sigma}} \leq C \varepsilon^{\sigma + \frac{\alpha}{p}} \quad \text{and} \quad \|f_\varepsilon\|_{\dot{B}_{\infty,\infty}^{-\sigma}} \geq C^{-1} \varepsilon^\sigma.$$

**Proof.** Let us recall that

$$\|f_\varepsilon\|_{\dot{B}_{p,1}^{-\sigma}} = \sum_{j \in \mathbf{Z}} 2^{-j\sigma} \|\Delta_j f_\varepsilon\|_{L^p}.$$

We shall start by estimating the high frequencies, defining a threshold  $j_0 \geq 0$  to be determined later on. We have

$$\begin{aligned} \sum_{j \geq j_0} 2^{-j\sigma} \|\Delta_j f_\varepsilon\|_{L^p} &\leq C 2^{-j_0\sigma} \|f_\varepsilon\|_{L^p} \\ &\leq C 2^{-j_0\sigma} \varepsilon^{\frac{\alpha}{p}} \|f\|_{L^p}. \end{aligned} \tag{3.1}$$

On the other hand, we have

$$\begin{aligned} \Delta_j f_\varepsilon(x) &= 2^{3j} \int_{\mathbf{R}^3} h(2^j(x-y)) f_\varepsilon(y) dy \\ &= 2^{3j} \int_{\mathbf{R}^3} h(2^j(x-y)) e^{i\frac{y_3}{\varepsilon}} f\left(y_1, \frac{y_2}{\varepsilon^\alpha}, y_3\right) dy, \end{aligned}$$

so noticing that  $e^{i\frac{y_3}{\varepsilon}} = (-i\varepsilon\partial_3)^N (e^{i\frac{y_3}{\varepsilon}})$ , we get for any  $N \in \mathbf{N}$ ,

$$\Delta_j f_\varepsilon(x) = (i\varepsilon)^N 2^{3j} \sum_{\ell=0}^N C_N^\ell \int e^{i\frac{y_3}{\varepsilon}} \partial_3^\ell (h(2^j(x-y))) \partial_3^{N-\ell} f\left(y_1, \frac{y_2}{\varepsilon^\alpha}, y_3\right) dy.$$

Young’s inequality enables us to infer that

$$2^{-j\sigma} \|\Delta_j f_\varepsilon\|_{L^p} \leq C\varepsilon^N 2^{j(3-\sigma)} \min\left(\sum_{\ell=0}^N 2^{j(\ell-3)} \varepsilon^{\frac{\alpha}{p}}, \sum_{\ell=0}^N 2^{j(\ell-\frac{3}{p})} \varepsilon^\alpha\right).$$

So, choosing  $N$  large enough and since  $\sigma < 3(1 - \frac{1}{p})$ , we get

$$\begin{aligned} \sum_{j \leq j_0} 2^{-j\sigma} \|\Delta_j f_\varepsilon\|_{L^p} &\leq \sum_{j \leq 0} 2^{-j\sigma} \|\Delta_j f_\varepsilon\|_{L^p} + \sum_{j \geq 0} 2^{-j\sigma} \|\Delta_j f_\varepsilon\|_{L^p} \\ &\leq C \sum_{j \leq 0} 2^{-j(\sigma+3(1-\frac{1}{p}))} \varepsilon^{N+\alpha} + C \sum_{0 \leq j \leq j_0} 2^{j(N-\sigma)} \varepsilon^{N+\frac{\alpha}{p}} \\ &\leq C\varepsilon^{N+\alpha} + C2^{j_0(N-\sigma)} \varepsilon^{N+\frac{\alpha}{p}}. \end{aligned} \tag{3.2}$$

Finally choosing  $2^{-j_0} = \varepsilon$  in (3.1) and (3.2) ends the proof of the bound on  $\|f_\varepsilon\|_{\dot{B}_{p,1}^{-\sigma}}$ .

In order to bound from below  $\|f_\varepsilon\|_{\dot{B}_{\infty,\infty}^{-\sigma}}$ , let us first observe that, as the space of smooth compactly supported functions is dense in  $\mathcal{S}$  and the Fourier transform is continuous on  $\mathcal{S}$ . Thus, for any positive  $\eta$ , a function  $g$  exists, the Fourier transform of which is smooth and compactly supported such that, denoting as before  $g_\varepsilon(x) = e^{i\frac{x_3}{\varepsilon}} g(x_1, \frac{x_2}{\varepsilon^\alpha}, x_3)$ ,

$$\|f_\varepsilon - g_\varepsilon\|_{\dot{B}_{\infty,\infty}^{-\sigma}} \leq \eta\varepsilon^\sigma \quad \text{and} \quad \|f - g\|_{L^\infty} \leq \eta. \tag{3.3}$$

As the support of the Fourier transform of  $g$  is included in the ball  $B(0, R)$  for some positive  $R$ , that of  $g(x_1, \varepsilon^{-\alpha}x_2, x_3)$  is included in the ball  $B(0, R\varepsilon^{-\alpha})$ . Then the support of  $\mathcal{F}g_\varepsilon$  is included in the ball  $B(\varepsilon^{-1}(0, 0, 1), \varepsilon^{-\alpha}R)$ . This ball is included in  $\varepsilon^{-1}\mathcal{C}$  for some ring  $\mathcal{C}$ . Thanks to (1.1) we shall use the heat flow. Let us write that

$$\begin{aligned} \|g_\varepsilon\|_{\dot{B}_{\infty,\infty}^{-\sigma}} &\sim \sup_{t>0} t^{\frac{\sigma}{2}} \|S(t)g_\varepsilon\|_{L^\infty} \\ &\geq C\varepsilon^\sigma \|S(\varepsilon^2)g_\varepsilon\|_{L^\infty}. \end{aligned}$$

For any function  $h$  such that the support of  $\hat{h}$  is included in  $\varepsilon^{-1}\mathcal{C}$ , we have

$$\|\mathcal{F}^{-1}(e^{\varepsilon^2|\xi|^2}h)\|_{L^\infty} \leq C\|h\|_{L^\infty}.$$

Applied with  $h = S(\varepsilon^2)g_\varepsilon$ , this inequality gives

$$\|g_\varepsilon\|_{L^\infty} \leq C\|S(\varepsilon^2)g_\varepsilon\|_{L^\infty} \quad \text{and thus} \quad \|g_\varepsilon\|_{\dot{B}_{\infty,\infty}^{-\sigma}} \geq C^{-1}\varepsilon^\sigma \|g_\varepsilon\|_{L^\infty} = C^{-1}\varepsilon^\sigma \|g\|_{L^\infty}.$$

Now let us write that

$$\begin{aligned} \|f_\varepsilon\|_{\dot{B}_{\infty,\infty}^{-\sigma}} &\geq \|g_\varepsilon\|_{\dot{B}_{\infty,\infty}^{-\sigma}} - \eta\varepsilon^\sigma \\ &\geq C^{-1}\varepsilon^\sigma (\|f\|_{L^\infty} - 2\eta). \end{aligned}$$

This ends the proof of the lemma.  $\square$

This enables us to infer immediately the following corollary.

**Corollary 3.1.** *A constant  $C$  exists such that, for any  $p \geq 1$ , we have*

$$\|u_{0,\varepsilon}\|_{\dot{B}_{p,1}^{-1}} \leq C\varepsilon^{\frac{\alpha}{p}}(-\log \varepsilon)^{\frac{1}{5}} \quad \text{and} \quad \|u_{0,\varepsilon}\|_{\dot{B}_{\infty,\infty}^{-1}} \geq C^{-1}(-\log \varepsilon)^{\frac{1}{5}}.$$

The last verification to be made is the nonlinear assumption (1.4). It is based on the following lemma.

**Lemma 3.2.** *There is a constant  $C$  such that the following result holds. Let  $f$  and  $g$  be in  $\dot{B}_{\infty,2}^{-1} \cap \dot{H}^{-1}$ . Then we have*

$$\|\mathbf{P}(S(t)fS(t)g)\|_E \leq C(\|f\|_{\dot{B}_{\infty,2}^{-1}}\|g\|_{\dot{B}_{\infty,2}^{-1}})^{\frac{2}{3}}(\|f\|_{\dot{H}^{-1}}\|g\|_{\dot{H}^{-1}})^{\frac{1}{3}}.$$

**Proof.** As the Leray projection  $\mathbf{P}$  is continuous on  $E$ , it is enough to prove the lemma without  $\mathbf{P}$ . Using Bernstein’s estimate, we get that

$$\|\Delta_j(S(t)fS(t)g)\|_{L^\infty} \leq C2^{3j}\|S(t)fS(t)g\|_{L^1}.$$

Then, using the Cauchy–Schwarz inequality, we infer that

$$\begin{aligned} E_j &\stackrel{\text{def}}{=} \|\Delta_j(S(t)fS(t)g)\|_{L^1(\mathbf{R}^+;L^\infty)} + \|t^{\frac{1}{2}}\Delta_j(S(t)fS(t)g)\|_{L^2(\mathbf{R}^+;L^\infty)} \\ &\leq C2^{3j}(\|S(t)f\|_{L^2(\mathbf{R}^+;L^2)} + \|t^{\frac{1}{2}}S(t)f\|_{L^\infty(\mathbf{R}^+;L^2)})\|S(t)g\|_{L^2(\mathbf{R}^+;L^2)}. \end{aligned}$$

So using (1.1), we deduce that

$$E_j \leq C2^{3j}\|f\|_{\dot{H}^{-1}}\|g\|_{\dot{H}^{-1}}. \tag{3.4}$$

Let us observe that we also have

$$\begin{aligned} E_j &\leq C(\|S(t)f\|_{L^2(\mathbf{R}^+;L^\infty)} + \|t^{\frac{1}{2}}S(t)f\|_{L^\infty(\mathbf{R}^+;L^\infty)})\|S(t)g\|_{L^2(\mathbf{R}^+;L^\infty)} \\ &\leq C\|f\|_{\dot{B}_{\infty,2}^{-1}}\|g\|_{\dot{B}_{\infty,2}^{-1}}. \end{aligned}$$

Using this estimate for high frequencies and (3.4) for low frequencies, we get, for any  $j_0$  in  $\mathbf{Z}$ ,

$$\begin{aligned} \|S(t)fS(t)g\|_E &= \sum_j 2^{-j}E_j \\ &\leq C\left(\|f\|_{\dot{H}^{-1}}\|g\|_{\dot{H}^{-1}} \sum_{j \leq j_0} 2^{2j} + \|f\|_{\dot{B}_{\infty,2}^{-1}}\|g\|_{\dot{B}_{\infty,2}^{-1}} \sum_{j \geq j_0} 2^{-j}\right) \\ &\leq C(\|f\|_{\dot{H}^{-1}}\|g\|_{\dot{H}^{-1}}2^{2j_0} + \|f\|_{\dot{B}_{\infty,2}^{-1}}\|g\|_{\dot{B}_{\infty,2}^{-1}}2^{-j_0}). \end{aligned}$$

Choosing  $j_0$  such that

$$2^{3j_0} \sim \frac{\|f\|_{\dot{B}_{\infty,2}^{-1}}\|g\|_{\dot{B}_{\infty,2}^{-1}}}{\|f\|_{\dot{H}^{-1}}\|g\|_{\dot{H}^{-1}}}$$

gives the result.  $\square$

Finally we are ready to prove estimate (1.4). Note that the proof relies heavily on the special structure of the nonlinear term in the system. We indeed start by remarking that there is no derivative in the third direction since  $u_{0,\varepsilon}$  does not have a third component. Then denoting  $u_F(t) = S(t)u_{0,\varepsilon}$ , we have by an easy computation and with the notation as in Lemma 3.1,

$$u_F^1 \partial_1 u_F^1 + u_F^2 \partial_2 u_F^1 = \frac{1}{\varepsilon^2}(-\log \varepsilon)^{\frac{2}{5}}S(t)f_\varepsilon S(t)g_\varepsilon \quad \text{and}$$

$$u_F^1 \partial_1 u_F^2 + u_F^2 \partial_2 u_F^2 = \frac{1}{\varepsilon^{2-\alpha}}(-\log \varepsilon)^{\frac{2}{5}}S(t)\tilde{f}_\varepsilon S(t)\tilde{g}_\varepsilon,$$

where  $f, \tilde{f}, g$  and  $\tilde{g}$  are smooth functions. The result follows immediately using Lemma 3.2 and Corollary 3.1 together with the fact that the Leray projection onto divergence free vector fields maps continuously  $E$  into  $E$ .

#### 4. Stability results

In this section we shall prove Proposition 1.1, as well as Theorems 3 and 4 stated in the introduction. The proof of Proposition 1.1 is rather easy and is given for the sake of completeness in the next section. The proof of Theorem 3 is the object of Section 4.2 below. Finally Theorem 4 is an easy consequence of the methods developed in the proof of Theorem 3 and is postponed to the end of Section 4.2.

##### 4.1. Proof of Proposition 1.1

Proposition 1.1 is an immediate consequence of the following more general result.

**Proposition 4.1.** *Let  $X = (x_1, \dots, x_K)$  be a family of  $K$  distinct points, and  $(u_{0,1}, \dots, u_{0,K})$  a family of divergence free vector fields in  $\dot{H}^{\frac{1}{2}}$ , each generating a unique, global solution to the Navier–Stokes equations. Then there is  $\Lambda_0 > 0$  such that for any  $\Lambda \geq \Lambda_0$ , the vector field*

$$u_{0,\Lambda} \stackrel{\text{def}}{=} \sum_{J \in \{1, \dots, K\}} T_\Lambda^J(u_{0,J})$$

also generates a unique, global solution to the Navier–Stokes equations.

**Proof.** The proof of that result is similar to methods of [3] concerning profile decompositions (see [12] for the case of the Navier–Stokes equations). Let us denote by  $u_J$  the solution of (NS) associated with  $u_{0,J}$ , and define

$$u_{\Lambda,J}(t, x) = \Lambda u_J(\Lambda^2 t, \Lambda(x - x_J)),$$

which solves (NS) with data  $u_{0,\Lambda,J} = T_\Lambda^J(u_{0,J})$ . Then we define the solution  $u_\Lambda$  of (NS) with data  $u_{0,\Lambda}$ , which a priori exists only for a short time. We can decompose

$$u_\Lambda = \sum_{J \in \{1, \dots, K\}} u_{\Lambda,J} + R_\Lambda = u_\Lambda^{(1)} + R_\Lambda,$$

and  $R_\Lambda$  solves the following perturbed Navier–Stokes equation

$$\partial_t R_\Lambda - \Delta R_\Lambda + \mathbf{P}(R_\Lambda \cdot \nabla R_\Lambda) + \mathbf{P}(u_\Lambda^{(1)} \cdot \nabla R_\Lambda) + \mathbf{P}(R_\Lambda \cdot \nabla u_\Lambda^{(1)}) = F_\Lambda$$

with initial data zero, and where

$$F_\Lambda = -\mathbf{P} \sum_{J \neq J'} u_{\Lambda,J} \cdot \nabla u_{\Lambda,J'}.$$

It is not difficult to prove (see for instance [12], Proposition A.2) that  $R_\Lambda$  is globally defined and unique in  $L^\infty(\mathbf{R}^+; \dot{H}^{\frac{1}{2}}) \cap L^2(\mathbf{R}^+; \dot{H}^{\frac{3}{2}})$  under the condition that

$$\|F_\Lambda\|_{L^2(\mathbf{R}^+; \dot{H}^{-\frac{1}{2}})} \leq C_0^{-1} \exp(-C_0 \|u_\Lambda^{(1)}\|_{L^4(\mathbf{R}^+; \dot{H}^1)}^4), \quad (4.1)$$

so let us compute  $\|F_\Lambda\|_{L^2(\mathbf{R}^+; \dot{H}^{-\frac{1}{2}})}$  and  $\|u_\Lambda^{(1)}\|_{L^4(\mathbf{R}^+; \dot{H}^1)}$ .

As mentioned in the introduction, any global solution belongs to  $L^4(\mathbf{R}^+; \dot{H}^1)$ . Thus, by definition of  $u_\Lambda^{(1)}$ , we have

$$\|u_\Lambda^{(1)}\|_{L^4(\mathbf{R}^+; \dot{H}^1)} \leq \sum_{J \in \{1, \dots, K\}} \|u_{\Lambda,J}\|_{L^4(\mathbf{R}^+; \dot{H}^1)}.$$

Using a scaling argument, we infer

$$\begin{aligned} \|u_\Lambda^{(1)}\|_{L^4(\mathbf{R}^+; \dot{H}^1)} &\leq \sum_{J \in \{1, \dots, K\}} \|u_J\|_{L^4(\mathbf{R}^+; \dot{H}^1)} \\ &\leq K \sup_{J \in \{1, \dots, K\}} C_J \quad \text{with} \end{aligned} \quad (4.2)$$

$$C_J \stackrel{\text{def}}{=} \|u_J\|_{L^\infty(\mathbf{R}^+; \dot{H}^{\frac{1}{2}})} + \|u_J\|_{L^2(\mathbf{R}^+; \dot{H}^{\frac{3}{2}})}.$$



In order to estimate  $\|F_\Lambda\|_{L^2(\mathbf{R}^+; \dot{H}^{-\frac{1}{2}})}$ , let us start by noticing that  $F_\Lambda$  is bounded uniformly in  $\Lambda$  in the space  $L^{\frac{4}{3}}(\mathbf{R}^+; L^2)$ , by a constant depending on  $K$  and on the initial data. Indeed Hölder’s inequality and Sobolev embeddings give

$$\begin{aligned} \|u_{\Lambda, J} \cdot \nabla u_{\Lambda, J'}\|_{L^{\frac{4}{3}}(\mathbf{R}^+; L^2)} &\leq \|u_{\Lambda, J}\|_{L^4(\mathbf{R}^+; L^6)} \|\nabla u_{\Lambda, J'}\|_{L^2(\mathbf{R}^+; L^3)} \\ &\leq C \|u_{\Lambda, J}\|_{L^4(\mathbf{R}^+; \dot{H}^1)} \|\nabla u_{\Lambda, J'}\|_{L^2(\mathbf{R}^+; \dot{H}^{\frac{1}{2}})}, \end{aligned}$$

so that by scale invariance

$$\begin{aligned} \|F_\Lambda\|_{L^{\frac{4}{3}}(\mathbf{R}^+; L^2)} &\leq C \|u_J\|_{L^4(\mathbf{R}^+; \dot{H}^1)} \|\nabla u_{J'}\|_{L^2(\mathbf{R}^+; \dot{H}^{\frac{1}{2}})} \\ &\leq CK^2 \sup_{J, J'} (C_J C_{J'}). \end{aligned}$$

So by interpolation it is enough to prove that

$$\lim_{\Lambda \rightarrow \infty} \|F_\Lambda\|_{L^4(\mathbf{R}^+; \dot{H}^{-1})} = 0. \tag{4.3}$$

Let  $J \neq J'$  be two integers in  $\{1, \dots, K\}$ , and let  $\varepsilon > 0$  be given. There exists a positive  $R$  and two vector fields  $\psi_\varepsilon$  and  $\varphi_\varepsilon$  in  $\mathcal{D}(\mathbf{R} \times B(0, R))$  such that

$$\|\psi_\varepsilon - u_J\|_{L^4(\mathbf{R}^+; \dot{H}^1)} + \|\varphi_\varepsilon - u_{J'}\|_{L^4(\mathbf{R}^+; \dot{H}^1)} \leq \varepsilon.$$

The support of  $T_{\Lambda, J}\psi_\varepsilon$  (resp.  $T_{\Lambda, J'}\varphi_\varepsilon$ ) is included in the ball  $B(x_J, R\Lambda^{-1})$  (resp.  $B(x_{J'}, R\Lambda^{-1})$ ). Thus we have

$$\Lambda \geq 4\delta^{-1}R \implies T_{\Lambda, J}\psi_\varepsilon T_{\Lambda, J'}\varphi_\varepsilon = 0. \tag{4.4}$$

Then Sobolev embeddings as above give the estimate

$$\begin{aligned} \|u_J \otimes (\varphi_\varepsilon - u_{J'})\|_{L^4(\mathbf{R}^+; L^2)} + \|(\psi_\varepsilon - u_J) \otimes \varphi_\varepsilon\|_{L^4(\mathbf{R}^+; L^2)} \\ \leq C (\|u_J\|_{L^\infty(\mathbf{R}^+; \dot{H}^{\frac{1}{2}})} \|\varphi_\varepsilon - u_{J'}\|_{L^4(\mathbf{R}^+; \dot{H}^1)} + \|\varphi_\varepsilon\|_{L^\infty(\mathbf{R}^+; \dot{H}^{\frac{1}{2}})} \|\psi_\varepsilon - u_J\|_{L^4(\mathbf{R}^+; \dot{H}^1)}), \end{aligned}$$

so that, using the scaling,

$$\|u_{\Lambda, J} \otimes (T_{\Lambda, J'}\varphi_\varepsilon - u_{J'})\|_{L^4(\mathbf{R}^+; L^2)} + \|(T_{\Lambda, J}\psi_\varepsilon - u_J) \otimes T_{\Lambda, J'}\varphi_\varepsilon\|_{L^4(\mathbf{R}^+; L^2)} \leq C(C_J + C_{J'})\varepsilon. \tag{4.5}$$

Using (4.4), it follows that for  $\Lambda$  large enough,

$$\|F_\Lambda\|_{L^4(\mathbf{R}^+; \dot{H}^{-1})} \leq CK^2\varepsilon \sup_{J \in \{1, \dots, K\}} C_J,$$

and (4.3) is proved. Plugging together that estimate with (4.2) gives (4.1) for  $\Lambda$  large enough, and Proposition 4.1 is proved.  $\square$

#### 4.2. Proof of Theorems 3 and 4

Before starting the proofs, let us make a few comments on the transformation  $T_{\Lambda, X}$  and state its main properties. In all that follows, we shall consider only the action of  $T_{\Lambda, X}$  on functions compactly supported in  $Q$ . First, one can notice that if the family  $X$  of points satisfies (1.5), then if  $\Lambda \geq 4\delta^{-1}$ ,

$$\text{supp } T_\Lambda^J f \subset Q_\Lambda^J \stackrel{\text{def}}{=} \{x \mid d(x, x_J) \leq \Lambda^{-1}\} \subset Q_\delta^J \stackrel{\text{def}}{=} \left\{x \mid d(x, x_J) \leq \frac{1}{4}\delta\right\}.$$

This implies immediately that

$$\|T_{\Lambda, X} f\|_{L^p} = \Lambda^{1-\frac{3}{p}} K^{\frac{1}{p}} \|f\|_{L^p}. \tag{4.6}$$

Then let us state the following two lemmas, which are crucial for the proof of Theorem 3 and will be proved in Section 4.2.2.

**Lemma 4.1.** *Let  $K \geq 1$  be an integer and  $\delta > 0$  a real number. There is a constant  $C_{K,\delta}$  such that the following results hold. Let  $r$  be in  $[1, \infty]$  and consider a family  $X$  as in Definition 1.3. Then for any real number  $\Lambda$  in  $2^{\mathbf{N}}$  greater than  $4\delta^{-1}$  and for any  $f \in \mathcal{D}(Q)$ , we have*

$$\|f\|_{\dot{B}_{\infty,r}^{-1}} - C_{K,\delta}\Lambda^{-2}\|f\|_{\dot{B}_{\infty,\infty}^{-3}} \leq \|T_{\Lambda,X}f\|_{\dot{B}_{\infty,r}^{-1}} \leq \|f\|_{\dot{B}_{\infty,r}^{-1}} + C_{K,\delta}\Lambda^{-2}\|f\|_{\dot{B}_{\infty,\infty}^{-3}}.$$

Moreover the following estimate holds, where the constant  $C$  is universal:

$$\|T_{\Lambda,X}f\|_{\dot{H}^{-1}} \leq C\sqrt{K}\Lambda^{-\frac{3}{2}}\|f\|_{\dot{H}^{-1}}. \tag{4.7}$$

**Remark.** Let us point out that  $L^1$  is continuously included in  $\dot{B}_{\infty,\infty}^{-3}$ .

**Lemma 4.2.** *Let  $K \geq 1$  be an integer and  $\delta > 0$  a real number. There is a constant  $C_{K,\delta}$  such that the following results hold. Consider a family  $X$  as in Definition 1.3. Then for any real number  $\Lambda$  in  $2^{\mathbf{N}}$  greater than  $4\delta^{-1}$  and for all divergence free vector fields  $f$  and  $g$  in  $\mathcal{D}(Q)$ , we have*

$$\|\mathbf{P}(S(t)T_{\Lambda,X}f \cdot \nabla S(t)T_{\Lambda,X}g)\|_E \leq \|\mathbf{P}(S(t)f \cdot \nabla S(t)g)\|_E + C_{K,\delta}\Lambda^{-3}\|f\|_{\dot{H}^{-1}}\|g\|_{\dot{H}^{-1}}.$$

4.2.1. End of the proof of Theorem 3

Let us consider a vector field  $u_0 \in \mathcal{D}(Q)$  satisfying (1.6) for some  $\eta \in ]0, 1[$ . We know from Lemma 4.1 that for any  $r \in [1, \infty]$  and any  $\eta \in ]0, 1[$ , for any  $\Lambda$  greater than some  $\Lambda_0$  (depending on  $K$  and  $\delta$  only), we have

$$\|u_0\|_{\dot{B}_{\infty,r}^{-1}} - \eta \leq \|T_{\Lambda,X}u_0\|_{\dot{B}_{\infty,r}^{-1}} \leq \|u_0\|_{\dot{B}_{\infty,r}^{-1}} + \eta. \tag{4.8}$$

Next let us consider the smallness condition (1.3). By Lemma 4.2 we know that as soon as  $\Lambda_0$  is large enough, then for any  $\Lambda \geq \Lambda_0$ ,

$$\|\mathbf{P}(S(t)T_{\Lambda,X}u_0 \cdot \nabla S(t)T_{\Lambda,X}u_0)\|_E \leq \|\mathbf{P}(S(t)u_0 \cdot \nabla S(t)u_0)\|_E + \eta.$$

So we infer that

$$\begin{aligned} \|\mathbf{P}(S(t)T_{\Lambda,X}u_0 \cdot \nabla S(t)T_{\Lambda,X}u_0)\|_E &\leq C_0^{-1} \exp(-C_0(\|u_0\|_{\dot{B}_{\infty,2}^{-1}} + \eta)^4) \\ &\leq C_0^{-1} \exp(-C_0\|T_{\Lambda,X}u_0\|_{\dot{B}_{\infty,2}^{-1}}^4) \end{aligned}$$

due to (4.8). So Theorem 3 is proved, up to the proof of Lemmas 4.1 and 4.2 which is the object of the coming section.  $\square$

4.2.2. The properties of  $T_{\Lambda,X}$

In this section, we are going to prove the properties of the transformation  $T_{\Lambda,X}$  required in the proof of Theorem 3, namely Lemmas 4.1 and 4.2. Before starting the proofs, let us give some more notation and prove preliminary results which will be used many times in the rest of this section.

We define

$$\tilde{Q}_\delta = \bigcup_{J \in \{1, \dots, K\}} \tilde{Q}_\delta^J, \quad \text{where } \tilde{Q}_\delta^J \stackrel{\text{def}}{=} \left\{ x \mid d(x, Q_\delta^J) \leq \frac{1}{32}\delta \right\}, \tag{4.9}$$

and we notice that this is a disjoint reunion.

The proof of Lemmas 4.1 and 4.2 relies on the fact that the Littlewood–Paley theory is almost local. More precisely, let us recall Lemma 9.2.2 of [5].

**Lemma 4.3.** *For any positive integer  $N$  and any real number  $r$ , a constant  $C_N$  exists such that the following result holds. Let  $F$  be a closed subset of  $\mathbf{R}^3$  and  $u$  a distribution in  $\dot{B}_{\infty,\infty}^r$  supported in  $F$ ; then for any couple  $(j, h)$  in  $\mathbf{Z} \times \mathbf{R}^+$  such that  $2^j h$  is greater than 1, we have*

$$\|\Delta_j u\|_{L^\infty(F_h)} \leq C_N 2^{-jr} (2^j h)^{-N} \|u\|_{\dot{B}_{\infty,\infty}^r},$$

where  $F_h = \{x \in \mathbf{R}^3 \mid d(x, F) \leq h\}$ .

From this lemma, we deduce the following corollary.

**Corollary 4.1.** *Let  $K, \delta$  and  $X$  be as in Definition 1.3 and let  $M \in \mathbf{N}$  be given. There is a constant  $C_M$  (depending only on  $M$ ) such that the following holds. For any  $\Lambda \geq 4\delta^{-1}$ , for any distribution  $f$  in  $\dot{B}_{\infty,\infty}^{-3}$ , compactly supported in  $Q$  and for any  $J \in \{1, \dots, K\}$ , one has the following estimates:*

$$\forall j \in \mathbf{Z}, \quad \|\Delta_j T_\Lambda^J f\|_{L^\infty(c\tilde{Q}_\delta^j)} \leq C_M \delta^{-(M+3)} \Lambda^{-2} 2^{-jM} \|f\|_{\dot{B}_{\infty,\infty}^{-3}}. \tag{4.10}$$

Moreover there is a universal constant  $C$  such that for any positive  $R$ ,

$$\|f\|_{\dot{B}_{\infty,r}^{-1}} \leq \left\| (2^{-j} \|\Delta_j f\|_{L^\infty(Q_R)})_j \right\|_{\ell^r} + CR^{-2} \|f\|_{\dot{B}_{\infty,\infty}^{-3}}, \tag{4.11}$$

where  $Q_R = \{x \in \mathbf{R}^3 \mid d(x, Q) \leq R\}$ .

**Proof.** The first inequality is obvious when  $j$  is negative or when  $2^j \delta \leq 1$ . Indeed we have the scaling property

$$\Delta_j (f(\Lambda(\cdot - x_J))) (x) = (\Delta_{j-\log_2 \Lambda} f)(\Lambda(x - x_J)), \tag{4.12}$$

so that for any  $s \in \mathbf{R}$ ,

$$\|f(\Lambda(\cdot - x_J))\|_{\dot{B}_{\infty,\infty}^s} = \Lambda^s \|f\|_{\dot{B}_{\infty,\infty}^s}.$$

Thus let us assume that  $2^j$  and  $2^j \delta$  are greater than 1. Using Lemma 4.3, we get

$$\begin{aligned} \|\Delta_j T_\Lambda^J f\|_{L^\infty(c\tilde{Q}_\delta^j)} &\leq C_M 2^{3j} (2^j \delta)^{-M} \|T_\Lambda^J f\|_{\dot{B}_{\infty,\infty}^{-3}} \\ &\leq C_M 2^{-j(M-3)} \delta^{-M} \Lambda^{-2} \|f\|_{\dot{B}_{\infty,\infty}^{-3}}. \end{aligned}$$

In order to prove the second inequality, let us note that, thanks to the triangle inequality and to the fact that  $\|\cdot\|_{\ell^r} \leq \|\cdot\|_{\ell^1}$ , we have for any integer  $j_0$ ,

$$\|f\|_{\dot{B}_{\infty,r}^{-1}} \leq \left\| (2^{-j} \|\Delta_j f\|_{L^\infty(Q_R)})_j \right\|_{\ell^r} + \sum_{j < j_0} 2^{-j} \|\Delta_j f\|_{L^\infty(\mathbf{R}^3)} + \sum_{j \geq j_0} 2^{-j} \|\Delta_j f\|_{L^\infty(cQ_R)}.$$

Lemma 4.3 claims in particular that, if  $2^j R \geq 1$ ,

$$\|\Delta_j f\|_{L^\infty(cQ_R)} \leq CR^{-3} \|f\|_{\dot{B}_{\infty,\infty}^{-3}}.$$

Thus, if  $j_0$  is such that  $2^{j_0} R \geq 1$ , we have, by definition of the norm of  $\dot{B}_{\infty,\infty}^{-1}$ ,

$$\begin{aligned} \|f\|_{\dot{B}_{\infty,r}^{-1}} &\leq \left\| (2^{-j} \|\Delta_j f\|_{L^\infty(Q_R)})_j \right\|_{\ell^r} + \left( \sum_{j < j_0} 2^{2j} + R^{-3} \sum_{j \geq j_0} 2^{-j} \right) \|f\|_{\dot{B}_{\infty,\infty}^{-3}} \\ &\leq \left\| (2^{-j} \|\Delta_j f\|_{L^\infty(Q_R)})_j \right\|_{\ell^r} + (2^{2j_0} + 2^{-j_0} R^{-3}) \|f\|_{\dot{B}_{\infty,\infty}^{-3}}. \end{aligned}$$

Choosing  $2^{j_0} \sim R^{-1}$  gives the result.  $\square$

#### 4.2.3. Proof of Lemma 4.1

We shall start by proving the second inequality, namely that

$$\|T_{\Lambda,X} f\|_{\dot{B}_{\infty,r}^{-1}} \leq \|f\|_{\dot{B}_{\infty,r}^{-1}} + C_{K,\delta} \Lambda^{-2} \|f\|_{\dot{B}_{\infty,\infty}^{-3}}. \tag{4.13}$$

Let us start with low frequencies. We can write that

$$\begin{aligned} \sum_{j < 0} 2^{-j} \|\Delta_j T_{\Lambda,X} f\|_{L^\infty} &\leq \sum_{j < 0} 2^{-j} \sum_J \|\Delta_j T_\Lambda^J f\|_{L^\infty} \\ &\leq \sum_J \left( \sum_{j < 0} 2^{2j} \right) \|T_\Lambda^J f\|_{\dot{B}_{\infty,\infty}^{-3}}. \end{aligned}$$

Using the scaling equality (4.12) we get that

$$\sum_{j < 0} 2^{-j} \|\Delta_j T_{\Lambda, X} f\|_{L^\infty} \leq K \Lambda^{-2} \|f\|_{\dot{B}_{\infty, \infty}^{-3}}. \tag{4.14}$$

Now let us concentrate on the high frequencies. Recalling the definition given in (4.9), let us start by considering the case when  $x \notin \tilde{Q}_\delta$ . Using Inequality (4.10) of Corollary 4.1, we can write (choosing  $M = 0$ )

$$\begin{aligned} \|\Delta_j T_{\Lambda, X} f\|_{L^\infty(c\tilde{Q}_\delta)} &\leq \sum_J \|\Delta_j T_\Lambda^J f\|_{L^\infty(c\tilde{Q}_\delta)} \\ &\leq CK \delta^{-3} \Lambda^{-2} \|f\|_{\dot{B}_{\infty, \infty}^{-3}}. \end{aligned}$$

Then we infer that

$$\sum_{j \geq 0} 2^{-j} \|\Delta_j T_{\Lambda, X} f\|_{L^\infty(c\tilde{Q}_\delta)} \leq CK \delta^{-3} \Lambda^{-2} \|f\|_{\dot{B}_{\infty, \infty}^{-3}}. \tag{4.15}$$

Now let us consider the case when  $x \in \tilde{Q}_\delta$ . We can write

$$\|\Delta_j T_{\Lambda, X} f\|_{L^\infty(\tilde{Q}_\delta)} \leq \sup_J \|\Delta_j T_{\Lambda, X} f\|_{L^\infty(\tilde{Q}_\delta^J)},$$

and let us fix some  $J \in \{1, \dots, K\}$ . We recall that

$$T_{\Lambda, X} f = T_\Lambda^J f + \sum_{J' \neq J} T_\Lambda^{J'} f,$$

and let us start with the estimate of  $T_{\Lambda, X} f - T_\Lambda^J f$ . We have

$$\|\Delta_j (T_{\Lambda, X} f - T_\Lambda^J f)\|_{L^\infty(\tilde{Q}_\delta^J)} \leq \sum_{J' \neq J} \|\Delta_j T_\Lambda^{J'} f\|_{L^\infty(c\tilde{Q}_\delta^{J'})}.$$

Using Inequality (4.10) of Corollary 4.1, we get that

$$\|\Delta_j (T_{\Lambda, X} f - T_\Lambda^J f)\|_{L^\infty(c\tilde{Q}_\delta^J)} \leq CK \delta^{-3} \Lambda^{-2} \|f\|_{\dot{B}_{\infty, \infty}^{-3}}.$$

Thus we infer

$$\sum_{j \geq 0} 2^{-j} \|\Delta_j (T_{\Lambda, X} f - T_\Lambda^J f)\|_{L^\infty(\tilde{Q}_\delta^J)} \leq CK \delta^{-3} \Lambda^{-2} \|f\|_{\dot{B}_{\infty, \infty}^{-3}}. \tag{4.16}$$

Now let us examine the term  $\|T_\Lambda^J f\|_{L^\infty(\tilde{Q}_\delta^J)}$ . From (4.12) we get

$$\begin{aligned} \|(\mathbf{1}_{j \geq 0} 2^{-j} \|\Delta_j T_\Lambda^J f\|_{L^\infty(\tilde{Q}_\delta^J)})_j\|_{\ell^r} &\leq \Lambda \|(\mathbf{1}_{j \geq 0} 2^{-j} \|\Delta_{j - \log_2 \Lambda} f\|_{L^\infty(\mathbf{R}^3)})_j\|_{\ell^r} \\ &\leq \|f\|_{\dot{B}_{\infty, r}^{-1}}. \end{aligned}$$

Once noticed that  $\|\cdot\|_{\ell^r} \leq \|\cdot\|_{\ell^1}$ , we plug together that estimate with (4.14), (4.15) and (4.16) to conclude the proof of (4.13).

Let us bound from below  $\|T_{\Lambda, X} f\|_{\dot{B}_{\infty, r}^{-1}}$ . As  $\|g\|_{L^\infty(\tilde{Q}_\delta)} = \sup_J \|g\|_{L^\infty(\tilde{Q}_\delta^J)}$ , we have

$$\begin{aligned} \|T_{\Lambda, X} f\|_{\dot{B}_{\infty, r}^{-1}} &\geq \|(2^{-j} \|\Delta_j T_{\Lambda, X} f\|_{L^\infty(\tilde{Q}_\delta)})_j\|_{\ell^r} \\ &\geq \|(2^{-j} \sup_J \|\Delta_j T_{\Lambda, X} f\|_{L^\infty(\tilde{Q}_\delta^J)})_j\|_{\ell^r} \\ &\geq \|(2^{-j} \|\Delta_j T_{\Lambda, X} f\|_{L^\infty(\tilde{Q}_\delta^{J_0})})_j\|_{\ell^r} \end{aligned}$$

for some  $J_0$  in  $\{1, \dots, K\}$ . Using the fact that  $\|\cdot\|_{\ell^r} \leq \|\cdot\|_{\ell^1}$ , we can write that

$$\begin{aligned} \|T_{\Lambda, X} f\|_{\dot{B}_{\infty, r}^{-1}} &\geq \|(2^{-j} \|\Delta_j T_\Lambda^{J_0} f\|_{L^\infty(\tilde{Q}_\delta^{J_0})})_j\|_{\ell^r} - \sum_{j < 0} 2^{-j} \|\Delta_j T_\Lambda^{J_0} f\|_{L^\infty(\tilde{Q}_\delta^{J_0})} \\ &\quad - \sum_{j \geq 0} 2^{-j} \|\Delta_j (T_{\Lambda, X} f - T_\Lambda^{J_0} f)\|_{L^\infty(\tilde{Q}_\delta^{J_0})}. \end{aligned}$$

Using (4.14) and (4.16), we infer that

$$\|T_{\Lambda, X} f\|_{\dot{B}_{\infty, r}^{-1}} \geq \left\| (2^{-j} \|\Delta_j T_{\Lambda, X}^{J_0} f\|_{L^\infty(\tilde{Q}_\delta^{J_0})})_j \right\|_{\ell^r} - C_{K, \delta} \Lambda^{-2} \|f\|_{\dot{B}_{\infty, \infty}^{-3}}. \tag{4.17}$$

By scaling and translation, we have

$$\left\| (2^{-j} \|\Delta_j T_{\Lambda, X}^{J_0} f\|_{L^\infty(\tilde{Q}_k^{J_0})})_j \right\|_{\ell^r} = \left\| (2^{-j} \|\Delta_j f\|_{L^\infty(Q_{\Lambda, \delta})})_j \right\|_{\ell^r}$$

where  $Q_{\Lambda, \delta}$  is the cube of size  $2\Lambda\delta$ . Using (4.11) with  $R = 2\delta\Lambda$  and (4.17), we infer that

$$\|T_{\Lambda, X} f\|_{\dot{B}_{\infty, r}^{-1}} \geq \|f\|_{\dot{B}_{\infty, r}^{-1}} - C_{K, \delta} \Lambda^{-2} \|f\|_{\dot{B}_{\infty, \infty}^{-3}}.$$

This concludes the proof of the first part of the lemma.

Now let us prove the second part of the lemma, namely estimate (4.7) on the  $\dot{H}^{-1}$  norm. Let  $f \in \mathcal{D}(Q)$  be given. Stating  $f_m \stackrel{\text{def}}{=} -\mathcal{F}^{-1}(i\xi_m |\xi|^{-2} \hat{f})$ , we can write

$$f = \sum_{m=1}^3 \partial_m f_m \quad \text{with} \quad \|f\|_{\dot{H}^{-1}} \sim \sum_{m=1}^3 \|f_m\|_{L^2}.$$

Let us recall that

$$\|T_{\Lambda, X} f\|_{\dot{H}^{-1}} = \sup_{\substack{g \in \mathcal{D}(Q) \\ \|g\|_{\dot{H}^1} \leq 1}} \int_{\mathbf{R}^3} T_{\Lambda, X} f(x) g(x) dx.$$

Let  $\chi \in \mathcal{D}(Q)$  be equal to one on the support of  $g$ . We have

$$\begin{aligned} \int_{\mathbf{R}^3} T_{\Lambda, X} f(x) g(x) dx &= \Lambda \sum_J \int_{\mathbf{R}^3} f(\Lambda(x - x_J)) g(x) dx \\ &= \Lambda^{-2} \sum_J \sum_m \int_{\mathbf{R}^3} \partial_m f_m(x) g(\Lambda^{-1}x + x_J) dx, \end{aligned}$$

so after an integration by parts and a change of variables again, we infer that

$$\begin{aligned} \int_{\mathbf{R}^3} T_{\Lambda, X} f(x) g(x) dx &= -\Lambda^{-3} \sum_J \sum_m \int_{\mathbf{R}^3} \chi(x) f_m(x) (\partial_m g)(\Lambda^{-1}x + x_J) dx \\ &= -\Lambda^{-1} \sum_m \int_{\mathbf{R}^3} T_{\Lambda, X}(\chi f_m)(x) \partial_m g(x) dx. \end{aligned}$$

In particular we get that

$$\begin{aligned} \|T_{\Lambda, X} f\|_{\dot{H}^{-1}} &\leq C \Lambda^{-1} \sum_m \|T_{\Lambda, X}(\chi f_m)\|_{L^2} \\ &\leq C \Lambda^{-1} \sum_m \|f_m\|_{L^2} \Lambda^{-\frac{1}{2}} \sqrt{K} \\ &\leq C \Lambda^{-\frac{3}{2}} \sqrt{K} \|f\|_{\dot{H}^{-1}}, \end{aligned}$$

and the result is proved.

#### 4.2.4. Proof of Lemma 4.2

First, we observe that

$$S(t)T_{\Lambda, X} f \cdot \nabla S(t)T_{\Lambda, X} g = \sum_{\ell=1}^3 \partial_\ell (S(t)T_{\Lambda, X} f^\ell S(t)T_{\Lambda, X} g), \tag{4.18}$$

so using Bernstein’s inequalities, we can write

$$\begin{aligned} \mathcal{E}_j &\stackrel{\text{def}}{=} \|\Delta_j(\mathbf{S}(t)T_{\Lambda,X}f \cdot \nabla \mathbf{S}(t)T_{\Lambda,X}g)\|_{L^1(\mathbf{R}^+;L^\infty)} + \|t^{\frac{1}{2}}\Delta_j(\mathbf{S}(t)T_{\Lambda,X}f \cdot \nabla \mathbf{S}(t)T_{\Lambda,X}g)\|_{L^2(\mathbf{R}^+;L^\infty)} \\ &\leq C2^{4j}(\|\mathbf{S}(t)T_{\Lambda,X}f\|_{L^2(\mathbf{R}^+;L^2)}\|\mathbf{S}(t)T_{\Lambda,X}g\|_{L^2(\mathbf{R}^+;L^2)} + \|t^{\frac{1}{2}}\mathbf{S}(t)T_{\Lambda,X}f\|_{L^\infty(\mathbf{R}^+;L^2)}\|\mathbf{S}(t)T_{\Lambda,X}g\|_{L^2(\mathbf{R}^+;L^2)}) \\ &\leq C2^{4j}\|T_{\Lambda,X}f\|_{\dot{H}^{-1}}\|T_{\Lambda,X}g\|_{\dot{H}^{-1}}. \end{aligned}$$

Using Lemma 4.1, we get

$$\mathcal{E}_j \leq C2^{4j}K\Lambda^{-3}\|f\|_{\dot{H}^{-1}}\|g\|_{\dot{H}^{-1}}.$$

We therefore infer a bound on the low frequencies:

$$\sum_{j \leq 0} 2^{-j}\mathcal{E}_j \leq CK\Lambda^{-3}\|f\|_{\dot{H}^{-1}}\|g\|_{\dot{H}^{-1}}. \tag{4.19}$$

The high frequencies are more delicate to estimate. Let us write that

$$\begin{aligned} \mathbf{S}(t)T_{\Lambda,X}f \cdot \nabla \mathbf{S}(t)T_{\Lambda,X}g &= H_{\Lambda,X}(t) + K_{\Lambda,X}(t) \quad \text{with} \\ H_{\Lambda,X}(t) &\stackrel{\text{def}}{=} \sum_{J \neq J'} \sum_{\ell=1}^3 \partial_\ell(\mathbf{S}(t)T_\Lambda^J f^\ell \mathbf{S}(t)T_\Lambda^{J'} g) \quad \text{and} \quad K_{\Lambda,X}(t) \stackrel{\text{def}}{=} \sum_J \sum_{\ell=1}^3 \partial_\ell(\mathbf{S}(t)T_\Lambda^J f^\ell \mathbf{S}(t)T_\Lambda^J g). \end{aligned}$$

We observe that

$$\begin{aligned} B_{\Lambda,X}^{J,J'}(f,g) &\stackrel{\text{def}}{=} \partial_\ell(\mathbf{S}(t)T_\Lambda^J f^\ell \mathbf{S}(t)T_\Lambda^{J'} g) \\ &= \frac{1}{(4\pi t)^3} \int_{\mathbf{R}^6} \partial_{x_\ell} \exp\left(-\frac{|x-y|^2 + |x-z|^2}{4t}\right) T_\Lambda^J f(y) T_\Lambda^{J'} g(z) dy dz \\ &= \frac{1}{(4\pi t)^3} \int_{\mathbf{R}^6} \frac{2x_\ell - y_\ell - z_\ell}{2t} \exp\left(-\frac{|x-y|^2 + |x-z|^2}{4t}\right) T_\Lambda^J f(y) T_\Lambda^{J'} g(z) dy dz. \end{aligned}$$

Due to the distance between  $x_J$  and  $x_{J'}$ , one gets that a smooth bounded function (as well as all its derivatives)  $\chi$  on  $\mathbf{R}$  exists such that  $\chi$  vanishes identically near 0 and such that

$$\begin{aligned} B_{\Lambda,X}^{J,J'}(f,g)(t,x) &= \frac{1}{(4\pi t)^3} \int_{\mathbf{R}^6} \Theta_\delta(t,x,y,z) T_\Lambda^J f(y) T_\Lambda^{J'} g(z) dy dz \quad \text{with} \\ \Theta_\delta(t,x,y,z) &\stackrel{\text{def}}{=} \frac{1}{t^{\frac{1}{2}}} \chi\left(C\delta(|x-y|^2 + |x-z|^2)\right) \frac{2x_\ell - y_\ell - z_\ell}{2t} \exp\left(-\frac{|x-y|^2 + |x-z|^2}{4t^{\frac{1}{2}}}\right). \end{aligned}$$

As we have

$$\|a \otimes b\|_{\dot{H}^{-2}(\mathbf{R}^6)} \leq \|a\|_{\dot{H}^{-1}(\mathbf{R}^3)} \|b\|_{\dot{H}^{-1}(\mathbf{R}^3)}, \tag{4.20}$$

we infer, using the scaling, that

$$\begin{aligned} \|B_{\Lambda,X}^{J,J'}(f,g)(t,\cdot)\|_{L^\infty} &\leq \frac{1}{(4\pi t)^3} \sup_{x \in \mathbf{R}^3} \|\Theta_\delta(t,x,\cdot)\|_{\dot{H}^2(\mathbf{R}^6)} \|T_\Lambda^J f\|_{\dot{H}^{-1}(\mathbf{R}^3)} \|T_\Lambda^{J'} g\|_{\dot{H}^{-1}(\mathbf{R}^3)} \\ &\leq \frac{1}{(4\pi t)^3} \sup_{x \in \mathbf{R}^3} \|\Theta_\delta(t,x,\cdot)\|_{\dot{H}^2(\mathbf{R}^6)} \Lambda^{-3} \|f\|_{\dot{H}^{-1}(\mathbf{R}^3)} \|g\|_{\dot{H}^{-1}(\mathbf{R}^3)}. \end{aligned}$$

It is obvious that

$$|\nabla^2 \Theta_\delta(t,x,y,z)| \leq \frac{C}{t^{\frac{3}{2}}} e^{-\frac{\delta}{Ct}} \exp\left(-\frac{|x-y|^2 + |x-z|^2}{8t}\right)$$

and thus that

$$\|B_{\Lambda, X}^{J, J'}(f, g)(t, \cdot)\|_{L^\infty} \leq \frac{C}{t^3} \Lambda^{-3} e^{-\frac{\delta}{Ct}} \|f\|_{\dot{H}^{-1}(\mathbf{R}^3)} \|g\|_{\dot{H}^{-1}(\mathbf{R}^3)}.$$

We immediately infer, since  $\|\Delta_j \mathbf{P}a\|_{L^\infty} \leq C \|\Delta_j a\|_{L^\infty}$ , that

$$\sum_{j \geq 0} 2^{-j} (\|\Delta_j \mathbf{P}H_{\Lambda, X}\|_{L^1(L^\infty)} + \|t^{\frac{1}{2}} \Delta_j \mathbf{P}H_{\Lambda, X}\|_{L^2(L^\infty)}) \leq C_\delta \Lambda^{-3} \|f\|_{\dot{H}^{-1}} \|g\|_{\dot{H}^{-1}}. \tag{4.21}$$

Now let us consider the term  $K_{\Lambda, X}$ . To start with, let us write

$$\begin{aligned} \|\Delta_j \mathbf{P}K_{\Lambda, X}(t, \cdot)\|_{L^\infty(\mathbf{R}^3)} &\leq \|\Delta_j \mathbf{P}K_{\Lambda, X}(t, \cdot)\|_{L^\infty(\tilde{Q}_\delta)} + \|\Delta_j \mathbf{P}K_{\Lambda, X}(t, \cdot)\|_{L^\infty({}^c\tilde{Q}_\delta)} \\ &\leq \sup_J \|\Delta_j \mathbf{P}K_{\Lambda, X}(t, \cdot)\|_{L^\infty(\tilde{Q}_\delta^J)} + \|\Delta_j \mathbf{P}K_{\Lambda, X}(t, \cdot)\|_{L^\infty({}^c\tilde{Q}_\delta)}. \end{aligned}$$

By definition of  $K_{\Lambda, X}$ , and denoting  $\tilde{\Delta}_j = \Delta_j \mathbf{P}$ , we get

$$\begin{aligned} \|\Delta_j \mathbf{P}K_{\Lambda, X}(t, \cdot)\|_{L^\infty(\mathbf{R}^3)} &\leq \sup_J \|\Delta_j \mathbf{P}(S(t)T_\Lambda^J f \nabla S(t)T_\Lambda^J g)\|_{L^\infty(\mathbf{R}^3)} \\ &\quad + \sup_J \left\| \tilde{\Delta}_j \sum_{J' \neq J} \partial_\ell (S(t)T_\Lambda^{J'} f^\ell S(t)T_{\Lambda, X}^{J'} g) \right\|_{L^\infty(\tilde{Q}_\delta^J)} \\ &\quad + \sum_{J'} \|\tilde{\Delta}_j \partial_\ell (S(t)T_\Lambda^{J'} f^\ell S(t)T_\Lambda^{J'} g)\|_{L^\infty({}^c\tilde{Q}_\delta)} \\ &\leq \sup_J \|\Delta_j \mathbf{P}(S(t)T_\Lambda^J f \nabla S(t)T_\Lambda^J g)\|_{L^\infty(\mathbf{R}^3)} \\ &\quad + \sup_J \sum_{J' \neq J} \|\tilde{\Delta}_j \partial_\ell (S(t)T_\Lambda^{J'} f^\ell S(t)T_{\Lambda, X}^{J'} g)\|_{L^\infty(\tilde{Q}_\delta^J)} \\ &\quad + \sum_{J'} \|\tilde{\Delta}_j \partial_\ell (S(t)T_\Lambda^{J'} f^\ell S(t)T_{\Lambda, X}^{J'} g)\|_{L^\infty({}^c\tilde{Q}_\delta)} \\ &\leq \sup_J \|\Delta_j \mathbf{P}(S(t)T_\Lambda^J f \nabla S(t)T_\Lambda^J g)\|_{L^\infty(\mathbf{R}^3)} \\ &\quad + \sum_{J'} \|\tilde{\Delta}_j \partial_\ell (S(t)T_\Lambda^{J'} f^\ell S(t)T_{\Lambda, X}^{J'} g)\|_{L^\infty({}^c\tilde{Q}_\delta^J)}. \end{aligned}$$

By translation and scaling we infer that

$$\begin{aligned} \|\Delta_j \mathbf{P}K_{\Lambda, X}(t, \cdot)\|_{L^\infty(\mathbf{R}^3)} &\leq \Lambda \|\Delta_{j - \log_2 \Lambda} \mathbf{P}(S(t) f \nabla S(t) g)\|_{L^\infty(\mathbf{R}^3)} \\ &\quad + \sum_{J'} \|\tilde{\Delta}_j \partial_\ell (S(t)T_\Lambda^{J'} f^\ell S(t)T_{\Lambda, X}^{J'} g)\|_{L^\infty({}^c\tilde{Q}_\delta^J)}. \end{aligned}$$

By definition of  $S(t)$ , we have, for some  $\tilde{h} \in \mathcal{S}(\mathbf{R}^3)$ ,

$$\begin{aligned} B_{\Lambda, j}^{J'}(f, g)(t, x) &\stackrel{\text{def}}{=} \tilde{\Delta}_j \partial_\ell (S(t)T_\Lambda^{J'} f^\ell S(t)T_{\Lambda, X}^{J'} g)(t, x) \\ &= \frac{2^{3j}}{(4\pi t)^3} \int_{\mathbf{R}^9} \tilde{h}(2^j(x - x')) \partial_{x'_\ell} \exp\left(-\frac{|x' - y|^2 + |x' - z|^2}{4t}\right) T_\Lambda^{J'} f(y) T_\Lambda^{J'} g(z) dx' dy dz. \end{aligned}$$

Now if  $x$  is in  ${}^c\tilde{Q}_\delta^{J'}$  and  $y$  in  $\tilde{Q}_\delta^{J'}$ , one gets that a smooth bounded function (as well as all its derivatives)  $\chi$  on  $\mathbf{R}$  exists such that  $\chi$  vanishes identically near 0 and has value 1 outside a ball centered at the origin, and such that

$$\begin{aligned}
 B_{\Lambda,j}^{J',1}(f, g) &= B_{\Lambda,j}^{J',1}(f, g) + B_{\Lambda,j}^{J',2}(f, g) \quad \text{with} \\
 B_{\Lambda,j}^{J',1}(f, g)(t, x) &\stackrel{\text{def}}{=} \frac{2^{3j}}{(4\pi t)^3} \int_{\mathbf{R}^9} \tilde{h}(2^j(x-x')) \chi(C\delta|x-x'|^2) \\
 &\quad \times \partial_{x'_\ell} \exp\left(-\frac{|x'-y|^2 + |x'-z|^2}{4t}\right) T_{\Lambda}^{J'} f(y) T_{\Lambda}^{J'} g(z) dx' dy dz \quad \text{and} \\
 B_{\Lambda,j}^{J',2}(f, g)(t, x) &\stackrel{\text{def}}{=} \frac{2^{3j}}{(4\pi t)^3} \int_{\mathbf{R}^9} \tilde{h}(2^j(x-x')) \chi(C\delta|x'-y|^2) \\
 &\quad \times \partial_{x'_\ell} \exp\left(-\frac{|x'-y|^2 + |x'-z|^2}{4t}\right) T_{\Lambda}^{J'} f(y) T_{\Lambda}^{J'} g(z) dx' dy dz.
 \end{aligned}$$

By integration by parts, we get that

$$B_{\Lambda,j}^{J',1}(f, g)(t, x) = 2^{3j} \int_{\mathbf{R}^9} \partial_{x'_\ell} (\tilde{h}(2^j(x-x')) \chi(C\delta|x-x'|^2)) (S(t)T_{\Lambda}^{J'} f)(x') (S(t)T_{\Lambda}^{J'} g)(x') dx'.$$

By the Leibnitz formula we have,

$$\begin{aligned}
 2^{3j} \partial_{x'_\ell} (\tilde{h}(2^j(x-x')) \chi(C\delta|x-x'|^2)) &= 2^{4j} (\partial_{x'_\ell} \tilde{h})(2^j(x-x')) \chi(C\delta|x-x'|^2) \\
 &\quad + 2^{3j} \tilde{h}(2^j(x-x')) 2C\delta(x_\ell - x'_\ell) \chi'(C\delta|x-x'|^2).
 \end{aligned}$$

Using the properties of the function  $\chi$ , we infer

$$|2^{3j} \partial_{x'_\ell} (\tilde{h}(2^j(x-x')) \chi(C\delta|x-x'|^2))| \leq C_\delta \underline{h}(2^j(x-x'))$$

for some bounded function  $\underline{h}$ . Thus, by integration, we infer that

$$\|B_{\Lambda,j}^{J',1}(f, g)(t, \cdot)\|_{L^\infty} \leq C_\delta \|S(t)T_{\Lambda}^{J'} f\|_{L^2} \|S(t)T_{\Lambda}^{J'} g\|_{L^2}.$$

By definition of Besov spaces, we deduce that

$$\|B_{\Lambda,j}^{J',1}(f, g)\|_{L^1(L^\infty)} + \|t^{\frac{1}{2}} B_{\Lambda,j}^{J',1}(f, g)\|_{L^2(L^\infty)} \leq C_\delta \|T_{\Lambda}^{J'} f\|_{\dot{H}^{-1}} \|T_{\Lambda}^{J'} g\|_{\dot{H}^{-1}}.$$

By scaling, we infer that

$$\|B_{\Lambda,j}^{J',1}(f, g)\|_{L^1(L^\infty)} + \|t^{\frac{1}{2}} B_{\Lambda,j}^{J',1}(f, g)\|_{L^2(L^\infty)} \leq C_\delta \Lambda^{-3} \|f\|_{\dot{H}^{-1}} \|g\|_{\dot{H}^{-1}}. \tag{4.22}$$

In order to estimate  $B_{\Lambda,j}^{J',2}(f, g)$ , let us write

$$\begin{aligned}
 B_{\Lambda,j}^{J',2}(f, g)(t, x) &= \frac{1}{(4\pi t)^3} \int_{\mathbf{R}^6} \Theta_{\delta,j}(t, x, y, z) T_{\Lambda}^{J'} f(y) T_{\Lambda}^{J'} g(z) dy dz \quad \text{with} \\
 \Theta_{\delta,j}(t, x, y, z) &\stackrel{\text{def}}{=} \frac{2^{3j}}{t^{\frac{1}{2}}} \int_{\mathbf{R}^3} \tilde{h}(2^j(x-x')) \chi(C\delta|x'-y|^2) \times \frac{2x'_\ell - y_\ell - z_\ell}{2t^{\frac{1}{2}}} \exp\left(-\frac{|x'-y|^2 + |x'-z|^2}{4t}\right) dx'.
 \end{aligned}$$

Using (4.20), the definition of the Besov norm and the scaling property, we deduce that

$$\begin{aligned}
 \|B_{\Lambda,j}^{J',2}(f, g)(t, \cdot)\|_{L^\infty} &\leq \sup_{x \in \mathbf{R}^3} \|\nabla_{y,z}^2 \Theta_{\delta,j}(t, x, \cdot, \cdot)\|_{L^2(\mathbf{R}^6)} \|T_{\Lambda}^{J'} f\|_{\dot{H}^{-1}} \|T_{\Lambda}^{J'} g\|_{\dot{H}^{-1}} \\
 &\leq \sup_{x \in \mathbf{R}^3} \|\nabla_{y,z}^2 \Theta_{\delta,j}(t, x, \cdot, \cdot)\|_{L^2(\mathbf{R}^6)} \Lambda^{-3} \|f\|_{\dot{H}^{-1}} \|g\|_{\dot{H}^{-1}}.
 \end{aligned}$$

A straightforward computation shows that

$$\sup_{x \in \mathbf{R}^3} \|\nabla_{y,z}^2 \Theta_{\delta,j}(t, x, \cdot, \cdot)\|_{L^2(\mathbf{R}^6)} \leq \frac{C_\delta}{t^{\frac{3}{2}}} e^{-\frac{\delta}{Ct}}.$$



Thus, we get that

$$\|B_{\Lambda,j}^{J',2}(f, g)\|_{L^1(L^\infty)} + \|t^{\frac{1}{2}} B_{\Lambda,j}^{J',2}(f, g)\|_{L^2(L^\infty)} \leq C_\delta \Lambda^{-3} \|f\|_{\dot{H}^{-1}} \|g\|_{\dot{H}^{-1}}.$$

Using (4.19), (4.21) and (4.22) we infer that

$$\begin{aligned} \|\mathbf{P}(S(t)T_{\Lambda,X}f \nabla S(t)T_{\Lambda,X}g)\|_E &\leq \sum_j 2^{-j} \Lambda (\|\Delta_{j-\log_2 \Lambda} \mathbf{P}(S(t)f \nabla S(t)g)\|_{L^1(L^\infty)} \\ &\quad + \|t^{\frac{1}{2}} \Delta_{j-\log_2 \Lambda} \mathbf{P}(S(t)f \nabla S(t)g)\|_{L^2(L^\infty)}) + C_{K,\delta} \Lambda^{-3} \|f\|_{\dot{H}^{-1}} \|g\|_{\dot{H}^{-1}} \\ &\leq \|\mathbf{P}(S(t)f \nabla S(t)g)\|_E + C_{K,\delta} \Lambda^{-3} \|f\|_{\dot{H}^{-1}} \|g\|_{\dot{H}^{-1}}. \end{aligned}$$

That ends the proof of Lemma 4.2.

#### 4.2.5. Proof of Theorem 4

The proof is straightforward: in order to apply Theorem 3, we define  $\eta > 0$  and we need to find  $\Lambda_0$  uniform in  $\varepsilon$  so that, according to Lemmas 4.1 and 4.2, the following two conditions are satisfied:

$$\Lambda_0^{-3} C_{\delta,K} \|u_{0,\varepsilon}\|_{\dot{H}^{-1}}^2 = \eta \quad \text{and} \quad \Lambda_0^{-2} C_{\delta,K} \|u_{0,\varepsilon}\|_{\dot{B}_{\infty,\infty}^{-3}} = \eta.$$

Due to Corollary 3.1 this is trivially possible as soon as  $\alpha > 0$ .

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