On the local and global well-posedness theory for the KP-I equation

Sur le caractère bien posé local et global pour l'équation KP-I

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Abstract

In this paper we obtain new local and global well-posedness results for the KP-I equation.
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Résumé

Dans cet article nous obtenons de nouveaux résultats sur le caractère bien posé local et global de l'équation KP-I.
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Introduction

In this note we prove a new local well-posedness theorem for the KP-I equation (1.2), and use the well-known conservation laws to establish a new global well-posedness result as a consequence. These, our main results, are Theorems 3.1 and 3.4.

As has been shown in [13–15], the KP-I equation is badly behaved with respect to Picard iterative methods in the standard Sobolev spaces and the natural energy space, since the flow map fails to be C^2 at the origin in these spaces. Thus, one is left with two choices: one either considers function spaces that are different from the standard Sobolev spaces, and then shows that the Picard iteration scheme does apply for them (this is the strategy

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of [4–6], where weighted spaces are used), or one decides to abandon the Picard iteration scheme and tries a different approach to well-posedness theory, which is the approach taken in the present paper. The KP-I equation arises in water wave theory, as modeling capillary gravity waves, in the presence of strong surface tension effects. Its companion equation, the KP-II equation (where the sign in front of the $\partial_x^{-1}\partial_y^2 u$ term in (1.2) is $+$ instead of $-$) is also of importance in water wave theory, and from the point of view of well-posedness theory it is much better understood, since the above mentioned difficulty with Picard iterative methods does not occur for it. In fact (see [2]) KP-II is known to be both locally and globally well-posed in $L^2(\mathbb{R}^2)$, "by Picard iteration", and even in some Sobolev spaces of negative index [18–20]. In [13], the authors use the local well-posedness theorem of [7], obtained by a version of the classical energy method, and they couple it with some of the known (formal) conserved quantities, and an ingenious use of the dispersive estimates of Strichartz type for KP-I (see [17]), to show global well-posedness for KP-I in the space $Z = \{\phi \in L^2(\mathbb{R}^2)\colon \|\phi\|_{L^2} + \|\partial_x^3\phi\|_{L^2} + \|\partial_y\phi\|_{L^2} + \|\partial_x\partial_y\phi\|_{L^2} + \|\partial_x^{-1}\partial_y\phi\|_{L^2} + \|\partial_x^{-2}\partial_y^2\phi\|_{L^\infty} < \infty\}$. Our contribution comes from improving on the local well-posedness result given by the classical energy estimate, by showing local well-posedness in the space $Y_s = \{\phi \in L^2(\mathbb{R}^2)\colon \|\phi\|_{L^2} + \|J_x^s\phi\|_{L^2} + \|\partial_x^{-1}\partial_y\phi\|_{L^2} < \infty\}$ for $s > 3/2$, where $\widehat{J_x^s f}(\xi, \eta) = (1 + |\xi|^2)^{s/2}\hat{f}(\xi, \eta)$. We first observe that, in this context, it is still enough to control $\|\partial_x u\|_{L_t^1 L_{xy}^\infty}$ (Lemma 1.3). We then adapt the version of the argument first introduced by Koch and Tzvetkov [12] in the context of the Benjamin–Ono equation, given in [10], also for the Benjamin–Ono equation. This uses the 'interpolation inequality' of Lemma 1.7, which is the main new ingredient of our proof. Another important ingredient of the proof is a 'product theory' Leibniz rule for fractional derivatives, given in Lemma 1.8(ii), which follows from [16]. Once our local well-posedness result is established, we use the results in [13], to establish global well-posedness in

$$Z_0 = \big\{\phi \in L^2(\mathbb{R}^2)\colon \|\phi\|_{L^2} + \|\partial_x^{-1}\partial_y\phi\|_{L^2} + \|\partial_x^2\phi\|_{L^2} + \|\partial_x^{-2}\partial_y^2\phi\|_{L^2} < \infty\big\}.$$

The question of the local and global well-posedness of KP-I in the energy space $Y = \{\phi \in L^2(\mathbb{R}^2)\colon \|\phi\|_{L^2} + \|\partial_x\phi\|_{L^2} + \|\partial_x^{-1}\partial_y\phi\|_{L^2} < \infty\}$ remains a challenging open problem.

1. Preliminary estimates

We introduce the space, for $s \in \mathbb{R}$,

$$H_{-1}^s(\mathbb{R}^2) = \big\{u \in S'(\mathbb{R}^2)\colon \|u\|_{H_{-1}^s(\mathbb{R}^2)} < \infty\big\},$$

where

$$\|u\|_{H_{-1}^s(\mathbb{R}^2)} = \left(\iint \big(1 + |\xi|^{-1}\big)^2\big(1 + |\xi|^2 + |\eta|^2\big)^s \cdot \big|\hat{u}(\xi, \eta)\big|^2 \, d\xi \, d\eta\right)^{1/2}$$

and $H_{-1}^\infty(\mathbb{R}^2) = \bigcap_{s \geqslant 0} H_{-1}^s(\mathbb{R}^2)$. We recall a local well-posedness result of Iorio and Nunes [7]. For $u \in H_{-1}^s(\mathbb{R}^2)$, $s \geqslant 2$ we define $\widehat{\partial_x^{-1}\partial_y^2 u}(\xi, \eta) = -\eta^2/\xi\,\hat{u}(\xi, \eta)$.

Lemma 1.1. *The Cauchy problem*

$$\begin{cases} \partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u + u\partial_x u = 0, \\ u|_{t=0} = u_0 \end{cases} \tag{1.2}$$

is locally well-posed in $H_{-1}^s(\mathbb{R}^2)$, for $s > 2$.

Also, recall that, from [14], if, in addition, $\|u_0\|_Z < \infty$, where $Z = \{\phi \in L^2(\mathbb{R}^2)\colon \|\phi\|_Z < \infty\}$, and

$$\|\phi\|_Z = \|\phi_{L^2} + \|\partial_x^3\phi\|_{L^2} + \|\partial_y\phi\|_{L^2} + \|\partial_{xy}^2\phi\|_{L^2} + \|\partial_x^{-1}\partial_y\phi\|_{L^2} + \|\partial_x^{-2}\partial_y^2\phi\|_{L^2},$$

then u in fact extends for all time and remains also in Z for all time.

Lemma 1.3. *Let u be a solution to* (1.2), *with* $u_0 \in H^\infty_{-1}(\mathbb{R}^2) \cap Z$. *Let* $\widehat{J^s_x f}(\xi, \eta) = (1 + |\xi|^2)^{s/2} \hat{f}(\xi, \eta)$. *Then, for* $s \geqslant 0$, *we have, for any* $T > 0$:

$$\sup_{0 < t < T} \|J^s_x u\|_{L^2_{xy}} + \|\partial^{-1}_x \partial_y u\|_{L^2_{xy}} \leqslant C_s \exp\left(C_s \int\limits_0^T \|\partial_x u\|_{L^\infty_{xy}} \right) \left[\|J^s_x u_0\|_{L^2_{xy}} + \|\partial^{-1}_x \partial_y u_0\|_{L^2_{xy}} \right].$$

Proof. We apply J^s_x to (1.2), multiply by $J^s_x u$, and integrate in (x, y). We obtain

$$\partial_t \int\limits_{\mathbb{R}^2} \frac{(J^s_x u)^2}{2} - \iint\limits_{\mathbb{R}^2} \partial^{-1}_x \partial^2_y J^s_x u J^s_x u + \iint\limits_{\mathbb{R}^2} \partial^3_x J^s_x u J^s_x u = - \iint J^s_x (u \partial_x u) J^s_x (u).$$

The last two terms in the left-hand side of the equality are 0. To bound the right-hand side, we use the Kato and Ponce [9] commutator estimate, in the form

$$\left\| J^s(fg) - f J^s(g) \right\|_{L^2(\mathbb{R})} \leqslant C_s . \left[\|\partial_x f\|_{L^\infty(\mathbb{R})} \|J^{s-1} g\|_{L^2(\mathbb{R})} + \|J^s f\|_{L^2(\mathbb{R})} \|g\|_{L^\infty(\mathbb{R})} \right]. \tag{1.4}$$

We thus see that the right-hand side above equals (with $f = u$, $g = \partial_x u$), integrating first in x, and then in y

$$= - \int u J^s_x \partial_x u . J^s_x u + O\left(\|\partial_x u\|_{L^\infty(\mathbb{R}^2)} \|J^s_x u\|^2_{L^2(\mathbb{R}^2)} \right).$$

Thus,

$$\frac{d}{dt} \int |J^s_x u|^2 \, dx \, dy \leqslant C_s \|\partial_x u\|_{L^\infty(\mathbb{R}^2)} \|J^s_x u\|^2_{L^2(\mathbb{R}^2)},$$

and the desired estimate for $\|J^s_x u\|_{L^2(\mathbb{R}^2)}$ follows. To estimate $\partial^{-1}_x \partial_y u$, we apply $\partial^{-1}_x \partial_y$ to (1.2), multiply by $(\partial^{-1}_x \partial_y u)$ and integrate. The right-hand side will be, this time,

$$- \iint \partial_y \left(\frac{u^2}{2} \right) \cdot \partial^{-1}_x \partial_y u = - \iint u \partial_y u \cdot \partial^{-1}_x \partial_y u = - \iint u \partial_x \partial^{-1}_x \partial_y u \cdot \partial^{-1}_x \partial_y u = \iint \frac{\partial_x u \partial^{-1}_x \partial_y u \partial^{-1}_x \partial_y u}{2},$$

and the desired estimate follows. $\quad\square$

We next recall the Strichartz estimates associated to the free KP-I evolution.

Remark. For future use, for $f \in S'(\mathbb{R}^2)$ we define $\widehat{D^a_x f}(\xi, \eta) = |\xi|^a \hat{f}(\xi, \eta)$, $\widehat{D^b_y f}(\xi, \eta) = |\eta|^b \hat{f}(\xi, \eta)$.

Lemma 1.5. *Let* $U(t) = \exp(-t(\partial^3_x - \partial^{-1}_x \partial^2_y))$ *be the unitary group defining the free KP-I evolution. Then, the following estimates hold*

$$\|U(t)\phi\|_{L^q_T L^r_{xy}} \lesssim \|\phi\|_{L^2},$$

$$\left\| \int\limits_0^t U(t - t') F(t') \, dt' \right\|_{L^q_T L^r_{xy}} \lesssim \|F\|_{L^{\bar{q}}_T L^{\bar{r}}_{xy}},$$

where $\frac{1}{q} + \frac{1}{r} = \frac{1}{\bar{q}} + \frac{1}{\bar{r}} - 1 = \frac{1}{2}$, $2 \leqslant r < +\infty$, $1 < \bar{r} \leqslant 2$.

Note that the constant in the two inequalities of Lemma 1.5 is independent of T. The proof of Lemma 1.5 is given in [17]. Combining Lemma 1.5 with a Sobolev-embedding, we obtain

Corollary 1.6. *For each $T > 0$, $\varepsilon > 0$, we have*

$$\|U(t)\phi\|_{L_T^2 L_{xy}^\infty} \lesssim C_{\varepsilon,T} \{ \|\phi\|_{L^2} + \|D_x^\varepsilon \phi\|_{L^2} + \|D_y^\varepsilon \phi\|_{L^2} \}.$$

Our main new linear estimate is contained in the next lemma. It is an adaptation to the present setting of Proposition 2.8 in [10], which in turn is a reformulation of the main estimate in [12]. See also [15] for another use of the [12] idea in the context of KP-I.

Lemma 1.7. *Let $\delta > 0$, $T \in (0, 1]$. Assume that $w \in C([0, T]; H_{-1}^3(\mathbb{R}^2))$ is a solution to the linear equation*

$$\partial_t w + \partial_x^3 w - \partial_x^{-1} \partial_y^2 w = F.$$

Then,

$$\|\partial_x w\|_{L_T^1 L_{xy}^\infty} \lesssim C_{\delta,T} \left[\sup_{0<t<T} \|J_x^{\frac{3}{2}+2\delta} w\|_{L^2} + \sup_{0<t<T} \|J_x^{\frac{3}{2}+\delta} D_y^\delta w\|_{L^2} \right.$$
$$\left. + \int_0^T \|J_x^{1/2+2\delta} F\|_{L^2} + \|J_x^{1/2+\delta} D_y^\delta F\|_{L^2} \right].$$

Proof. We use a Littlewood–Paley decomposition of w in the ξ variable. More precisely, if $\eta \in C_0^\infty(\frac{1}{2} < |\xi| < 2)$, $\chi \in C_0^\infty(|\xi| < 2)$, are such that $1 = \sum_{k=1}^\infty \eta(2^{-k}\xi) + \chi(\xi)$, and for $\lambda = 2^k$, $k \geqslant 1$, we define $w_\lambda = Q_k(w)$, where $\hat{Q}_k w(\xi, \eta) = \eta(2^{-k}\xi)\hat{w}(\xi, \eta)$, $w_0 = Q_0(w)$, and $\widehat{Q_0 w}(\xi, \eta) = \chi(\xi)\hat{w}(\xi, \eta)$. We first estimate $\|\partial_x w_\lambda\|_{L_T^1 L_{xy}^\infty}$: let us assume, for simplicity, that $T = 1$. Split $[0, 1] = \bigcup_j I_j$, where $I_j = [a_j, b_j]$ and $b_j - a_j = 1/\lambda$, $j = 1, \ldots, \lambda$. Then,

$$\|\partial_x w_\lambda\|_{L_T^1 L_{xy}^\infty} \leqslant \sum_j \|\partial_x w_\lambda\|_{L_{I_j}^1 L_{xy}^\infty} \leqslant \lambda \sum_j \|w_\lambda\|_{L_{I_j}^1 L_{xy}^\infty},$$

where we have used that $\xi\eta(2^{-k}\xi)$ has inverse Fourier transform whose L^1 norm in x is bounded by $C\lambda$. We now use Cauchy–Schwartz in I_j, and continue with

$$\lesssim \lambda^{1/2} \sum_j \|w_\lambda\|_{L_{I_j}^2 L_{xy}^\infty}.$$

We now apply Duhamel's formula, in each I_j, to obtain, for $t \in I_j$,

$$w_\lambda(t) = U(t - a_j)w_\lambda(-, a_j) + \int_{a_j}^t U(t - t')F_\lambda(t') \, dt'.$$

We use Corollary 1.6, and continue with

$$\lesssim \lambda^{1/2} \sum_j \left(\|J_x^\delta w_\lambda(-, a_j)\|_{L^2} + \|D_y^\delta w_\lambda(-, a_j)\|_{L^2} \right) + \lambda^{1/2} \sum_j \int_{I_j} \|J_x^\delta F_\lambda\|_{L^2} + \|D_y^\delta F_\lambda\|_{L^2}$$

$$\lesssim \sup_{t \in [0,1]} \|J_x^{3/2+\delta} w_\lambda(-, t)\|_{L^2} + \|J_x^{3/2} D_y^\delta w_\lambda(-, t)\|_{L^2} + \int_0^t \|J_x^{1/2+\delta} F_\lambda\|_{L^2} + \|J_x^{1/2} D_y^\delta F_\lambda\|_{L^2}.$$

Write now $w_1 = \sum_{k \geqslant 1} Q_k(w)$, so that $w = w_1 + w_0$, and $w_1 = \sum_\lambda w_\lambda$. Then,

$$\|\partial_x w_1\| \leqslant \sum_\lambda \|\partial_x w_\lambda\|_{L^1_T L^\infty_{xy}} \lesssim \sum_{k \geqslant 1} \sup_t \left(\left\| J_x^{3/2+\delta} Q_k(w) \right\|_{L^2} + \left\| J_x^{3/2} D_y^\delta Q_k(w) \right\|_{L^2} \right)$$

$$+ \int_0^T \left\| J_x^{\frac{1}{2}+\delta} Q_k(F) \right\|_{L^2} + \left\| J_x^{1/2} D_y^\delta Q_k(F) \right\|_{L^2}$$

$$\leqslant \sum_{k \geqslant 1} 2^{-k\delta} \left\{ \sup_t \left(\left\| J_x^{3/2+2\delta} Q_k(w) \right\|_{L^2} + \left\| J_x^{3/2+\delta} D_y^\delta Q_k(w) \right\|_{L^2} \right) \right.$$

$$\left. + \int_0^T \left\| J_x^{\frac{1}{2}+2\delta} Q_k(F) \right\|_{L^2} + \left\| J_x^{1/2+\delta} D_y^\delta Q_k(F) \right\|_{L^2} \right\},$$

and the desired estimate for w_1 follows. The proof of the estimate for w_0 is simpler, and Lemma 1.7 follows. $\quad \square$

Lemma 1.8 (Leibniz rule for fractional differentiation).

(i) *For $0 < \alpha < 1$,*

$$\left\| D_x^\alpha(fg) \right\|_{L^2(\mathbb{R})} \leqslant C \left\{ \|D_x^\alpha f\|_{L^{p_1}(\mathbb{R})} \|g\|_{L^{q_1}(\mathbb{R})} + \|f\|_{L^{r_1}(\mathbb{R})} \|D_x^\alpha g\|_{L^{s_1}(\mathbb{R})} \right\}$$

with $\frac{1}{2} = \frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{r_1} + \frac{1}{s_1}$, $1 < p_1, q_1, r_1, s_1 \leqslant \infty$. (See, for example, [11].)

(ii) *For $0 < \alpha, \beta < 1$, we have*

$$\left\| D_x^\alpha D_y^\beta(f.g) \right\|_{L^2_{xy}} \leqslant C \left\{ \|f\|_{L^{p_1}_{xy}} \|D_x^\alpha D_y^\beta g\|_{L^{q_1}_{xy}} + \|g\|_{L^{p_2}_{xy}} \|D_x^\alpha D_y^\beta f\|_{L^{q_2}_{xy}} + \|D_x^\alpha g\|_{L^{p_3}_{xy}} \|D_y^\beta f\|_{L^{q_3}_{xy}} \right.$$

$$\left. + \|D_y^\beta g\|_{L^{p_4}_{xy}} \|D_x\alpha f\|_{L^{q_4}_{xy}} \right\},$$

where $\frac{1}{2} = \frac{1}{p_i} + \frac{1}{q_i}$, $1 < p_i \leqslant \infty$, $1 < q_i \leqslant \infty$.

The proof of (ii) reduces, using an argument similar to the one given in [9] to a (now two parameter) version of the multilinear theorem of Coifman and Meyer [3]. The proof of (ii) is given in [16] (see also [8]).

2. A priori estimates

In this section we prove:

Lemma 2.1. *Let u be a solution to* (1.2), *with $u_0 \in H^\infty_{-1}(\mathbb{R}^2) \cap Z$, defined, for all T. Let*

$$\|u_0\|_{Y_s} = \|J_x^s u_0\|_{L^2_{xy}} + \left\| (\partial_x^{-1} \partial_y u_0) \right\|_{L^2_{xy}}.$$

Then, for any $s > 3/2$, there exists $T = T_s$, depending only on $\|u_0\|_{Y_s}$, s, and a constant C_T, depending only on $\|u_0\|_{Y_s}$, s, so that

$$\int_0^T \|\partial_x u\|_{L^\infty_{xy}} \, dt + \int_0^T \|u\|_{L^\infty_{xy}} \, dt \leqslant C_T. \tag{2.2}$$

Proof. Let $s = \frac{3}{2} + \delta_0$, and choose $\delta > 0$ so that $2\delta \ll \delta_0$. Let us define

$$f(T) = \|\partial_x u\|_{L^1_T L^\infty_{xy}} + \|u\|_{L^1_T L^\infty_{xy}} = \int\limits_0^T \left(\|\partial_x u\|_{L^\infty_{xy}} + \|u\|_{L^\infty_{xy}}\right) dt. \tag{2.3}$$

We will prove the estimate

$$f(T) \leqslant C\|u_0\|_{Y_s}\big[\exp\big(Cf(T)\big) + 1\big], \tag{2.4}$$

for a fixed, universal constant C.

Let us take (2.4) temporarily for granted, and let us conclude the proof of (2.2). We first note that there exists $\varepsilon_0 > 0$ such that, if $\|u_0\|_{Y_s} \leqslant \varepsilon_0$, then (2.4) implies that $f(1) \leqslant M$, for a universal constant M. This follows from a standard continuity argument. (For instance, let $F(X, \varepsilon) = X - C\varepsilon \exp(Cx) - C\varepsilon$. Note that $F(0, 0) = 0$, and that $\frac{\partial F}{\partial X}(0, 0) = 1$, $\frac{\partial F}{\partial \varepsilon}(0, 0) = -2C$. The implicit function theorem now guarantees that for $0 < \varepsilon < \varepsilon_0$, there exists a smooth function $A(\varepsilon)$, which is increasing in ε, so that $F(A(\varepsilon), \varepsilon) = 0$. If $M = A(\|u_0\|_{Y_s})$, and $\|u_0\|_{Y_s} \leqslant \varepsilon_0$, it is easy to see that $f(1) \leqslant M$.) Next, note that u is a solution of (1.2), with initial data u_0, if and only if $u_\lambda(x, y, t) = \lambda^2 u(\lambda x, \lambda^2 y, \lambda^3 t)$ is a solution of (1.2) with initial data $u_{0,\lambda}(x, y) = \lambda^2 u_0(\lambda x, \lambda^2 y)$. Since $\|u_{0,\lambda}\|_{L^2} = \lambda^{1/2}\|u_0\|_{L^2}$, $\|D_x^s u_{0,\lambda}\|_{L^2} = \lambda^{\frac{1}{2}+s}\|D_x^s u_0\|_{L^2}$ and $\|\partial_x^{-1}\partial_y u_{0,\lambda}\|_{L^2} = \lambda^{1+1/2}\|\partial_x^{-1}\partial_y u_0\|_{L^2}$, we can first choose $\lambda = \lambda(\|u_0\|_{Y^s})$ so that $\|u_{0,\lambda}\|_{Y_s} \leqslant \varepsilon_0$, and apply the above conclusion to u_λ, to obtain (2.2).

We now turn to the proof of (2.4): we will use Lemma 1.7, and write (1.2) as

$$\partial_t u + \partial_x^3 u - \partial_x^{-1}\partial_y^2 u = -u\partial_x u.$$

We then obtain:

$$\|\partial_x u\|_{L^1_T L^\infty_{xy}} \leqslant C_{\delta,T}\Big[\sup_{0<t<T} \|J_x^{3/2+2\delta} u\|_{L^2} + \sup_{0<t<T} \|J_x^{3/2+\delta} D_y^\delta u\|_{L^2}\Big]$$

$$+ C_{\delta,T}\int\limits_0^T \big\|J_x^{1/2+2\delta}(u\partial_x u)\big\|_{L^2} + C_{\delta,T}\int\limits_0^T \big\|J_x^{\frac{1}{2}+\delta} D_y^\delta(u\partial_x u)\big\|_{L^2}$$

$$= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}.$$

To bound I, we use Lemma 1.3 to see that

$$\mathrm{I} \leqslant C_s \exp\left(C_s \int\limits_0^T \|\partial_x u\|_{L^\infty_{xy}}\right)\|u_0\|_{Y_s}.$$

To bound II, we use Plancherel's theorem, and the fact that $ab \leqslant \frac{a^q}{q} + \frac{b^p}{p}$, $\frac{1}{p} + \frac{1}{q} = 1$, to see that

$$\big(1 + |\xi|\big)^{\frac{3}{2}+\delta}.|\eta|^\delta = \big(1 + |\xi|\big)^{\frac{3}{2}+\delta}.|\xi|^\delta.\left(\frac{|\eta|}{|\xi|}\right)^\delta \leqslant \big(1 + |\xi|\big)^{(\frac{3}{2}+2\delta)}\left(\frac{|\eta|}{|\xi|}\right)^\delta \lesssim \big(1 + |\xi|\big)^{(\frac{3}{2}+2\delta)/(1-\delta)} + \frac{|\eta|}{|\xi|},$$

where we have chosen $q = 1/\delta$, $p = 1/(1-\delta)$. If now δ is chosen so small that $(\frac{3}{2} + 2\delta)/(1 - \delta) \leqslant \frac{3}{2} + \delta_0$, we see that $\mathrm{II} \lesssim \sup_{0<t<T} \|u\|_{Y_s}$, and in view of Lemma 1.3,

$$\mathrm{II} \leqslant C_s \exp\left(C_s \int\limits_0^T \|\partial_x u\|_{L^\infty_{xy}}\right)\|u_0\|_{Y_s}.$$

In order to estimate III, we use that $\|J_x^{1/2+2\delta}(f)\|_{L^2} \lesssim \|f\|_{L^2} + \|D_x^{1/2+2\delta}(f)\|_{L^2}$, to bound III by $\text{III}_1 + \text{III}_2$. For III_1 we see that it is bounded by $\sup_{0 < t < T} \|u\|_{L^2} \cdot \int_0^T \|\partial_x u\|_{L^\infty}$, and hence it has the desired bound because Lemma 1.3 gives $\sup_{0 < t < T} \|u\|_{L^2} \leqslant C \|u_0\|_{Y_s} \cdot \exp C_s \int_0^T \|\partial_x u\|_{L^\infty}$.

For III_2, we use Lemma 1.8(i), to obtain:

$$\text{III}_2 \lesssim \int_0^T \|D_x^{\frac{1}{2}+2\delta}\partial_x u\|_{L^2}\|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty}\|D_x^{\frac{1}{2}+2\delta}u\|_{L^2}$$

$$\lesssim (\text{in view of Lemma 1.3}) \; C_s \|u_0\|_{Y_s} \exp\left(C_s \int_0^T \|\partial_x u\|_{L^\infty}\right)$$

$$\times \int_0^T \|u\|_{L^\infty} + C_s\|u_0\|_{Y_s} \int_0^T \|\partial_x u\|_{L^\infty} \exp\left(C_s \int_0^T \|\partial_x u\|_{L^\infty}\right)$$

$$\leqslant C_s\|u_0\|_{Y_s} \exp\bigl(C_s f(T)\bigr).$$

Finally, for IV, we bound it first by $\int_0^T \|D_y^\delta(u\partial_x u)\|_{L^2} + \int_0^T \|D_x^{\frac{1}{2}+\delta} D_y^\delta(u\partial_x u)\|_{L^2} = \text{IV}_1 + \text{IV}_2$. To bound IV_1, we use Lemma 1.8(i), in the y variable, to see that

$$\text{IV}_1 \lesssim \int_0^T \bigl\|D_y^\delta(u)\bigr\|_{L^2}\|\partial_x u\|_{L^\infty} + \|u\|_{L^\infty}\|D_y^\delta\partial_x u\|_{L^2} = \text{IV}_{1,1} + \text{IV}_{1,2}.$$

To bound $\text{IV}_{1,1}$, we will use Plancherel's theorem for the first factor, and the inequalities

$$|\eta|^\delta = |\xi|^\delta\left(\frac{|\eta|}{|\xi|}\right)^\delta \leqslant \bigl(1+|\xi|\bigr)^\delta\left(\frac{|\eta|}{|\xi|}\right)^\delta \lesssim \bigl(1+|\xi|\bigr)^{\delta/1-\delta} + \frac{|\eta|}{|\xi|},$$

where δ is chosen so that $\delta/(1-\delta) \leqslant 3/2 + \delta_0$. This shows that (using Lemma 1.3)

$$\text{IV}_{1,1} \leqslant C_s\|u_0\|_{Y_s} \exp C_s \int_0^T \|\partial_x u\|_{L^\infty}.$$

For $\text{IV}_{1,2}$ we proceed similarly, using that

$$|\eta|^\delta|\xi| = |\xi|^{1+\delta}\left(\frac{|\eta|}{|\xi|}\right)^\delta \leqslant \bigl(1+|\xi|\bigr)^{1+\delta}\left(\frac{|\eta|}{|\xi|}\right)^\delta \leqslant \bigl(1+|\xi|\bigr)^{(1+\delta)/(1-\delta)} + \frac{|\eta|}{|\xi|},$$

and choosing δ so that $(1+\delta)/(1-\delta) \leqslant 3/2 + \delta_0$. This gives the bound $\text{IV}_{1,2} \leqslant C_s\|u_0\|_{Y_s} \exp(C_s f(T))$. We next turn to IV_2, and use Lemma 1.8(ii) to bound it by

$$\int_0^T \|\partial_x u\|_{L^\infty}\|D_x^{\frac{1}{2}+\delta} D_y^\delta u\|_{L^2} + \int_0^T \|\partial_x D_x^{\frac{1}{2}+\delta} D_y^\delta u\|_{L^2}\|u\|_{L^\infty}$$

$$+ \int_0^T \|D_x^{\frac{1}{2}+\delta}u\|_{L^\infty}\|D_y^\delta\partial_x u\|_{L^2} + \int_0^T \|D_y^\delta u\|_{L^{p_1}}\|D_x^{1/2+\delta}\partial_x u\|_{L^{q_1}},$$

where $\frac{1}{p_1} + \frac{1}{q_1} = \frac{1}{2}$ are to be determined. We call each one of these four terms $IV_{2,i}$. To bound $IV_{2,1}$, we again use Plancherel, and

$$|\xi|^{\frac{1}{2}+\delta}|\eta|^\delta \leqslant (1+|\xi|)^{\frac{1}{2}+2\delta}\left(\frac{|\eta|}{|\xi|}\right)^\delta \lesssim (1+|\xi|)^{(\frac{1}{2}+2\delta)/(1-\delta)} + \frac{|\eta|}{|\xi|},$$

and if $(\frac{1}{2} + 2\delta)/(1 - \delta) \leqslant \frac{3}{2} + \delta_0$, Lemma 1.3 shows that

$$IV_{2,1} \leqslant C_s \|u_0\|_{Y_s} \exp\left(C_s \int_0^T \|\partial_x u\|_{L^\infty}\right).$$

For $IV_{2,2}$ we proceed similarly, using that

$$|\xi| \, |\xi|^{\frac{1}{2}+2\delta}\left(\frac{|\eta|}{|\xi|}\right)^\delta \lesssim (1+|\xi|)^{(\frac{3}{2}+2\delta)/(1-\delta)} + \frac{|\eta|}{|\xi|},$$

and the fact that $(\frac{3}{2} + 2\delta)/(1 - \delta) \leqslant \frac{3}{2} + \delta_0$. We obtain, using Lemma 1.3, that

$$IV_{2,2} \leqslant C_s \|u_0\|_{Y_s} \exp(C_s f(T)).$$

In order to bound $IV_{2,3}$, we will use the following inequality ($\delta < \frac{1}{2}$)

$$\|D_x^{\frac{1}{2}+\delta} u\|_{L_{xy}^\infty} \lesssim \|u\|_{L_{xy}^\infty} + \|\partial_x u\|_{L_{xy}^\infty}. \tag{2.5}$$

To establish (2.5), let us use a Littlewood–Paley decomposition, as in the proof of Lemma 1.7, and write $u = u_0 + u_1$, as in that proof. In order to estimate $D_x^{\frac{1}{2}+\delta} u_0$, note that, if $\hat{f}(\xi) = |\xi|^{1/2+\delta}\chi(\xi)$, then $f \in L^1(\mathbb{R})$. Hence, $\|D_x^{\frac{1}{2}+\delta} u_0\|_{L_{xy}^\infty} \leqslant \|u\|_{L_{xy}^\infty}$. Moreover, in order to estimate $D_x^{\frac{1}{2}+\delta} u_1 = \sum_{k \geqslant 1} D_x^{\frac{1}{2}+\delta} Q_k u$, note that

$$\begin{aligned}
D_x^{\frac{1}{2}+\delta} \widehat{Q_k u}(\xi, \eta) &= |\xi|^{\frac{1}{2}+\delta}\eta(2^{-k}\xi)\hat{u}(\xi, \eta) \\
&= \frac{|\xi|^{\frac{1}{2}+\delta}}{\xi}\xi\,\eta(2^{-k}\xi)\hat{u}(\xi, \eta) = 2^{-k(\frac{1}{2}-\delta)}\frac{|2^{-k}\xi|^{\frac{1}{2}+\delta}}{(2^{-k}\xi)}\eta(2^{-k}\xi)\xi\hat{u}(\xi, \eta),
\end{aligned}$$

and that if $\hat{g}(\xi) = (|\xi|^{\frac{1}{2}+\delta}/\xi)\eta(\xi)$, then $g \in L^1(\mathbb{R})$. Thus, $\|D_x^{\frac{1}{2}+\delta} Q_k u\|_{L_{xy}^\infty} \leqslant 2^{-k(\frac{1}{2}-\delta)}\|\partial_x u\|_{L_{xy}^\infty}$, and (2.5) follows.

Because of (2.5), $IV_{2,3}$ is bounded by

$$\sup_{0<t<T} \|D_y^\delta \partial_x u\|_{L^2}\left\{\int_0^T \|u\|_{L^\infty} + \|\partial_x u\|_{L^\infty}\right\}.$$

Using Plancherel, and

$$|\eta|^\delta|\xi| \leqslant \left(\frac{|\eta|}{|\xi|}\right)^\delta (1+|\xi|)^{1+\delta} \lesssim \frac{|\eta|}{|\xi|} + (1+|\xi|)^{1+\delta/(1-\delta)},$$

choosing δ so that $(1 + \delta)/(1 - \delta) \leqslant \frac{3}{2} + \delta_0$, and using Lemma 1.3, we see that

$$IV_{2,3} \leqslant C_s \|u_0\|_{Y_s} \exp(C_s f(T)).$$

To bound $IV_{2,4}$, we need to prove first a couple of auxiliary estimates:

$$\|D_x^{1/2+\delta}\partial_x u\|_{L_T^{s_1} L_{xy}^{q_1}} \lesssim \|\partial_x u\|_{L_T^1 L_{xy}^\infty}^\theta \cdot \|J_x^{3/2+\delta_0} u\|_{L_T^\infty L_{xy}^2}^{1-\theta}, \tag{2.6}$$

$$\|D_y^\delta u\|_{L_T^{r_1} L_{xy}^{p_1}} \lesssim (\|u\|_{L_T^1 L_{xy}^\infty})^\eta (\|D_y^{1/2} u\|_{L_T^\infty L_{xy}^2} + \|u\|_{L_T^\infty L_{xy}^2})^{1-\eta}, \tag{2.7}$$

where $q_1 > 2$, $s_1 < \infty$, $\frac{1}{p_1} = \frac{1}{2} - \frac{1}{q_1}$, $\frac{1}{r_1} = 1 - \frac{1}{s_1}$ are suitably chosen, and θ, η are also suitably chosen. In order to establish (2.6), we use a Littlewood–Paley decomposition in x, as in the proof of Lemma 1.7. We then have $\partial_x u = \sum_{k \geqslant 0} Q_k(\partial_x u)$, and $D_x^{\frac{1}{2}+\delta} \partial_x u = \sum_{k \geqslant 0} D_x^{\frac{1}{2}+\delta} Q_k(\partial_x u)$. Now, for $k \geqslant 1$, $D_x^{\frac{1}{2}+\delta} Q_k = 2^{k(\frac{1}{2}+\delta)} \tilde{Q}_k$, where $\widehat{\tilde{Q}_k f}(\xi) = \tilde{\eta}(2^{-k}\xi) \hat{f}(\xi)$, and $\tilde{\eta}$ has similar properties to η, and $D_x^{\frac{1}{2}+\delta} Q_0 = \tilde{Q}_0$, where $\widehat{\tilde{Q}_0 f}(\xi) = \tilde{\chi}(\xi) \hat{f}(\xi)$, and if $\hat{g} = \tilde{\chi}$, $g \in L^1(\mathbb{R})$. We then have:

$$\|D_x^{1/2+\delta} \partial_x u\|_{L_T^{s_1} L_{xy}^{q_1}} \leqslant \sum_{k \geqslant 0} 2^{k(\frac{1}{2}+\delta)} \|\tilde{Q}_k(\partial_x u)\|_{L_T^{s_1} L_{xy}^{q_1}}.$$

We first bound the term $k = 0$ in the right-hand side. Let $\frac{1}{s_1} = \theta$, $\frac{1}{q_1} = \frac{(1-\theta)}{2}$, where $0 \leqslant \theta \leqslant 1$. Then,

$$\|\tilde{Q}_0 \partial_x u\|_{L_T^{s_1} L_{xy}^{q_1}} \leqslant \|\tilde{Q}_0 \partial_x u\|_{L_T^1 L_{xy}^\infty}^\theta \cdot \|\tilde{Q}_0 \partial_x u\|_{L_T^\infty L_{xy}^2}^{1-\theta},$$

and this term has the correct control.

For $k \geqslant 1$, we again use the same choices of s_1, q_1, and use

$$\begin{aligned}
\|\tilde{Q}_k \partial_x u\|_{L_T^{s_1} L_{xy}^{q_1}} &\leqslant \|\tilde{Q}_k \partial_x u\|_{L_T^1 L_{xy}^\infty}^\theta \cdot \|\tilde{Q}_k \partial_x u\|_{L_T^\infty L_{xy}^2}^{1-\theta} \\
&\lesssim \|\tilde{Q}_k \partial_x u\|_{L_T^1 L_{xy}^\infty}^\theta \cdot 2^{-k(\frac{1}{2}+\delta_0)(1-\theta)} \|\tilde{\tilde{Q}}_k D_x^{(\frac{1}{2}+\delta_0)} \partial_x u\|_{L_t^\infty L_{xy}^2}^{1-\theta} \\
&\lesssim \|\partial_x u\|_{L_T^1 L_{xy}^\infty}^\theta \cdot 2^{-k(\frac{1}{2}+\delta_0)(1-\theta)} \|D_x^{(\frac{1}{2}+\delta_0)} \partial_x u\|_{L_T^\infty L_{xy}^2}^{1-\theta},
\end{aligned}$$

where $\tilde{\tilde{Q}}_k$ is associated to $\tilde{\tilde{\eta}}(2^{-k}\xi)$, where $\tilde{\tilde{\eta}}$ has similar properties to $\tilde{\eta}$. From this, we obtain (2.6), as long as $(\frac{1}{2}+\delta) < (\frac{1}{2}+\delta_0)(1-\theta)$, or, choosing δ small enough, as long as $\frac{1}{2} < (\frac{1}{2}+\delta_0)(1-\theta)$, or $0 < \theta < 1 - [\frac{1}{2}/(\frac{1}{2}+\delta_0)] = \eta_0$, and $\frac{1}{s_1} = \theta$, $\frac{1}{q_1} = (1-\theta)/2$. We next turn to (2.7). This time we perform a similar Littlewood–Paley decomposition, but in the y variable. We need $\frac{1}{p_1} = \frac{\theta}{2}$, $\frac{1}{r_1} = 1 - \theta$. We proceed in a similar manner, and are led to controlling

$$\sum_{k \geqslant 0} 2^{\delta k} \|\tilde{Q}_k u\|_{L_T^{r_1} L_{xy}^{p_1}}, \tag{2.8}$$

where $\tilde{Q}_k$ acts on the y variable. But, for $0 \leqslant \eta \leqslant 1$, $\frac{1}{r_1} = \eta$, $\frac{1}{p_1} = \frac{(1-\eta)}{2}$, we have

$$\|\tilde{Q}_k u\|_{L_t^{r_1} L_{xy}^{p_1}} \leqslant \|\tilde{Q}_k u\|_{L_t^1 L_{xy}^\infty}^\eta \|\tilde{Q}_k u\|_{L_T^\infty L_{xy}^2}^{1-\eta},$$

and we see that, for (2.7) to hold, we need $\delta < \frac{1}{2}(1 - \eta)$. Since $\eta = (1 - \theta)$, we need $\delta < \frac{1}{2}\theta$, and $0 < \theta < 1 - [\frac{1}{2}/(\frac{1}{2}+\delta_0)] = \eta_0$, which we can clearly achieve.

Now, in order to bound $\mathrm{IV}_{2,4}$, we choose p_1 as in (2.7), q_1 as in (2.6). Observe also that, since

$$|\eta|^{1/2} \leqslant \left(\frac{|\eta|^{1/2}}{|\xi|}\right)(1 + |\xi|)^{1/2} \leqslant \frac{|\eta|}{|\xi|} + (1 + |\xi|) \leqslant \frac{|\eta|}{|\xi|} + (1 + |\xi|)^{\frac{3}{2}+\delta_0},$$

we have $\|D_y^{1/2} u\|_{L_T^\infty L_{xy}^2} + \|u\|_{L_T^\infty L_{xy}^2} \leqslant \sup_{0<t<T} \|u\|_{Y_s}$. We then have (recalling that $\eta = 1 - \theta$):

$$\begin{aligned}
\mathrm{IV}_{2,4} &\leqslant \|D_x^{\frac{1}{2}+\delta} \partial_x u\|_{L_T^{s_1} L_{xy}^{q_1}} \|D_y^\delta u\|_{L_T^{r_1} L_{xy}^{p_1}} \lesssim \|\partial_x u\|_{L_T^1 L_{xy}^\infty}^\theta \|u\|_{L_T^1 L_{xy}^\infty}^{1-\theta} \cdot \sup_{0<t<T} \|u\|_{Y_s} \\
&\lesssim C_s \|u_0\|_{Y_s} \|\partial_x u\|_{L_T^1 L_{xy}^\infty}^\theta \|u\|_{L_T^1 L_{xy}^\infty}^{1-\theta} \exp(C_s \|\partial_x u\|_{L_t^1 L_{xy}^\infty}) \lesssim C_s \|u_0\|_{Y_s} \exp(C_s f(T)),
\end{aligned}$$

where in the next to last inequality we have used Lemma 1.3.

All in all, we have shown that

$$\|\partial_x u\|_{L_T^1 L_{xy}^\infty} \leqslant C_s \|u_0\|_{Y_s} \exp\big(C_s f(T)\big). \tag{2.9}$$

We now turn to the bound for $\|u\|_{L_T^1 L_{xy}^\infty}$. We use Duhamel's formula, and write

$$u(t) = U(t)u_0 + \int_0^t U(t - t')u\partial_x u \, dt'.$$

We apply Corollary 1.6, and Minkowski's integral inequality, to see that

$$\|u\|_{L_T^1 L_{xy}^\infty} \leqslant C_T \left(\|u_0\|_{Y_s} + \int_0^T \big[\|u\partial_x u\|_{L^2} + \big\|D_x^\varepsilon(u\partial_x u)\big\|_{L^2} + \big\|D_y^\varepsilon(u\partial_x u)\big\|_{L^2}\big] \right).$$

We proceed to estimate the three terms in the integral, which we call $A + B + C$.

$$A \leqslant \|u\|_{L_T^1 L_{xy}^\infty} \sup_{0 < t < T} \|\partial_x u\|_{L_{xy}^2} \leqslant C_s \|u_0\|_{Y_s} \exp\big(C_s \|\partial_x u\|_{L_{xy}^\infty}\big) \cdot \|u\|_{L_T^1 L_{xy}^\infty}$$

$$\leqslant C_s \|u_0\|_{Y_s} \exp\big(C_s f(T)\big),$$

where in the next to last inequality we used Lemma 1.3. For B, we use first Lemma 1.8(i) and obtain:

$$B \leqslant \int_0^T \|u\|_{L_{xy}^\infty} \|D_x^\varepsilon \partial_x u\|_{L_{xy}^2} + \|D_x^\varepsilon u\|_{L_{xy}^2} \|\partial_x u\|_{L_{xy}^\infty}$$

$$\lesssim \sup_{0 < t < T} \|u\|_{Y_s} \cdot \big(\|u\|_{L_T^1 L_{xy}^\infty} + \|\partial_x u\|_{L_T^1 L_{xy}^\infty}\big)$$

$$\leqslant C_s \|u_0\|_{Y_s} \exp\big(C_s \|\partial_x u\|_{L_T^1 L_{xy}^\infty}\big)\big(\|u\|_{L_T^1 L_{xy}^\infty} + \|\partial_x u\|_{L_T^1 L_{xy}^\infty}\big)$$

$$\leqslant C_s \|u_0\|_{Y_s} \exp\big(C_s f(T)\big),$$

where we have used Lemma 1.3 to bound $\sup_{0 < t < T} \|u\|_{Y_s}$.

For C, we use Lemma 1.8(i) in the y-variable, and obtain:

$$C \lesssim \int_0^T \|u\|_{L_{xy}^\infty} \|D_y^\varepsilon \partial_x u\|_{L_{xy}^2} + \|D_y^\varepsilon u\|_{L_{xy}^2} \|\partial_x u\|_{L_{xy}^\infty},$$

and since

$$|\eta|^\varepsilon |\xi| = \left(\frac{|\eta|}{|\xi|}\right)^\varepsilon \big(|\xi|^{1+\varepsilon}\big) \leqslant \left(\frac{|\eta|}{|\xi|}\right)^\varepsilon (1 + |\xi|)^{1+\varepsilon} \leqslant \frac{|\eta|}{|\xi|} + \big(1 + |\xi|\big)^{(1+\varepsilon)/(1-\varepsilon)},$$

and we can choose ε so that $(1 + \varepsilon)/(1 - \varepsilon) \leqslant \frac{3}{2} + \delta_0$, a similar argument gives $C \lesssim C_s \|u_0\|_{Y_s} \exp(C_s f(T))$, and so

$$\|u\|_{L_T^1 L_{xy}^\infty} \lesssim C_T \|u_0\|_{Y_s} + C_s \|u_0\|_{Y_s} \cdot \exp\big(C_s f(T)\big), \tag{2.10}$$

which, together with (2.9) gives (2.4), and hence Lemma 2.1.

3. Local and global well-posedness

Our first result, a local well-posedness one in the space Y_s, $s > 3/2$, follows readily from Lemmas 1.3 and 2.1.

Theorem 3.1. *The (IVP) (2.1) is locally well-posed in Y_s, $s > 3/2$. More precisely, given $u_0 \in Y_s$, $s > 3/2$, there exists $T = T(\|u_0\|_{Y_s})$, and a unique solution u to (1.2), such that $u \in C([0, T] : Y_s)$, $u, \partial_x u \in L_T^1 L_{xy}^\infty$. Moreover, the mapping $u_0 \mapsto u \in C([0, T]; Y_s)$ is continuous.*

Proof. Let $u_0 \in Y_s$, find $u_{0,\varepsilon} \in Y_s \cap H_{-1}^\infty(\mathbb{R}^2) \cap Z$, such that $\|u_0 - u_{0,\varepsilon}\|_{Y_s} \to 0$, and $\|u_{0,\varepsilon}\|_{Y_s} \leqslant 2\|u_0\|_{Y_s}$. Let now u_ε be the solutions associated to $u_{0,\varepsilon}$, guaranteed by Lemma 1.1. Note that Lemma 2.1 gives us $T = T(\|u_0\|_{Y_s})$ so that

$$\|\partial_x u_\varepsilon\|_{L_T^1 L_{xy}^\infty} + \|u_\varepsilon\|_{L_T^1 L_{xy}^\infty} \leqslant C_T. \tag{3.2}$$

Lemma 1.3 now guarantees that

$$\sup_{0 < t < T} \|u_\varepsilon\|_{Y_s} \leqslant C_T. \tag{3.3}$$

Next, a use of Gronwall's inequality, combined with (3.2), shows that, as $\varepsilon, \varepsilon' \to 0 \sup_{0 < t < T} \|u_\varepsilon - u_{\varepsilon'}\|_{L^2} \to 0$. hence, we can find $u \in C([0, T]; Y_{s'}) \cap L^\infty([0, T]; Y_s)$, $(s' < s)$ such that $u_\varepsilon \to u$ in $C([0, T], Y_{s'})$. The fact that u is a solution to (1.2) is now clear. The uniqueness of u follows from the same use of Gronwall's inequality as above. Finally, the continuity of $u(t)$ in Y_s, and the continuity of the flow map in Y_s are consequences of the well-known Bona and Smith [1] argument. $\quad\square$

We now introduce the space Z_0, related to the conservation laws associated with KP-I.

$$Z_0 = \left\{ \phi \in L^2(\mathbb{R}^2) : \|\phi\|_{Z_0} < +\infty \right\}, \quad \text{where}$$
$$\|\phi\|_{Z_0} = \|\phi\|_{L^2} + \|\partial_x^2 \phi\|_{L^2} + \|\partial_x \phi\|_{L^2} + \|\partial_x^{-1} \partial_y \phi\|_{L^2} + \|\partial_x^{-2} \partial_y^2 \phi\|_{L^2}.$$

Note that, since $\partial_y \phi = \partial_x \partial_x^{-1} \partial_y \phi$, we have that $\|\partial_y \phi\|_{L^2} \lesssim \|\phi\|_{Z_0}$.

Theorem 3.4. *Let $u_0 \in Z_0$. then there exists a unique global solution u of (1.2) with initial data u_0, such that $u \in L^\infty(\mathbb{R}_+; Z_0)$.*

Proof. Find $u_{0,\varepsilon}$ as in the proof of Theorem 3.1, which converge to u_0 in Z_0. Recall from [14] the conserved quantities

$$M(\phi) = \int |\phi|^2,$$
$$E(\phi) = \frac{1}{2} \int (\partial_x \phi)^2 + \frac{1}{2} \int (\partial_x^{-1} \partial_y \phi)^2 - \frac{1}{6} \int u^3,$$
$$F(\phi) = \frac{3}{2} \int (\partial_x^2 \phi)^2 + 5 \int (\partial_y \phi)^2 + \frac{5}{6} \int (\partial_x^{-2} \partial_y^2 \phi)^2 - \frac{5}{6} \int \phi^2 \partial_x^{-2} \partial_y^2 \phi$$
$$\quad - \frac{5}{6} \int \phi (\partial_x^{-1} \partial_y \phi)^2 + \frac{5}{4} \int \phi^2 \partial_{xx} \phi + \frac{5}{24} \int \phi^4.$$

Let u_ε be the global solutions to (1.2) given by Lemma 1.1. Recall from Proposition 4 of [14] that $M(u_\varepsilon(t)) = M(u_{0,\varepsilon})$, $E(u_\varepsilon(t)) = E(u_{0,\varepsilon})$, $F(u_\varepsilon(t)) = F(u_{0,\varepsilon})$. As in [14], Proposition 5, we conclude that

$$\|\partial_x u_\varepsilon(t)\|_{L^2} + \|\partial_x^{-1} \partial_y u_\varepsilon(t)\|_{L^2} \lesssim E(u_{0,\varepsilon}) + \|u_{0,\varepsilon}\|_{L^2}^6,$$

and $E(u_{0,\varepsilon}) \lesssim C(\|u_0\|_{Z_0})$, and that $\|u_\varepsilon(t)\|_{L^q(\mathbb{R}^2)} \leqslant C(\|u_{0,\varepsilon}\|_{L^2}, E(u_{0,\varepsilon}))$, $q \in [2, 6]$. Moreover, we also see that

$$\left\|\partial_x^2 u_\varepsilon(t)\right\|_{L^2}^2 + \left\|\partial_x^{-2}\partial_y^2 u_\varepsilon(t)\right\|_{L^2}^2 + \left\|\partial_y u_\varepsilon(t)\right\|_{L^2}^2 \leqslant C(\|u_0\|_{Z_0}).$$

This shows that

$$\left\|u_\varepsilon(t)\right\|_{Z_0} \leqslant C(\|u_0\|_{Z_0}) \quad \text{for all } t.$$

Note also that, for $\frac{3}{2} < s \leqslant 2$, we have $\|\phi\|_{Y_s} \leqslant \|\phi\|_{Z_0}$. As in the proof of Theorem 3.1, we can now construct a unique global solution $u \in C(\mathbb{R}^+; Y_s)$, $\frac{3}{2} < s \leqslant 2$, with $u \in L^\infty(\mathbb{R}^+; Z_0)$. $\square$

Remark 3.5. One can give an improvement of Theorem 3.1 in the spirit of [10], by also establishing a 'local smoothing' estimate of Kato-type for KP-I. The resulting result would be the analog of Theorem 3.1, for $s > 5/4$, but, since it would be quite intricate technically, and still falls short of the desired local well-posedness for Y_1, we have decided not to include this improvement.

Acknowledgements

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