Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 33 (2016) 329–336

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

A proof of Alexandrov's uniqueness theorem for convex surfaces in $\mathbb{R}^3$

Pengfei Guan [a,*,1], Zhizhang Wang [b], Xiangwen Zhang [c]

[a] *Department of Mathematics and Statistics, McGill University, Montreal, Canada*
[b] *Department of Mathematics, Fudan University, Shanghai, China*
[c] *Department of Mathematics, Columbia University, New York, United States*

Received 4 May 2013; accepted 2 September 2014

Available online 31 October 2014

Abstract

We give a new proof of a classical uniqueness theorem of Alexandrov [4] using the weak uniqueness continuation theorem of Bers–Nirenberg [8]. We prove a version of this theorem with the minimal regularity assumption: the spherical Hessians of the corresponding convex bodies as Radon measures are nonsingular.

MSC: 53A05; 53C24

Keywords: Uniqueness; Convex surfaces; Nonlinear elliptic equations; Unique continuation; Alexandrov theorem

We give a new proof of the following uniqueness theorem of Alexandrov, using the weak unique continuation theorem of Bers–Nirenberg [8].

Theorem 1. *(See Theorem 9 in [4].) Suppose M_1 and M_2 are two closed strictly convex C^2 surfaces in $\mathbb{R}^3$, suppose $f(y_1, y_2) \in C^1$ is a function such that $\frac{\partial f}{\partial y_1}\frac{\partial f}{\partial y_2} > 0$. Denote by $\kappa_1 \geq \kappa_2$ the principal curvatures of surfaces, and denote by ν_{M_1} and ν_{M_2} the Gauss maps of M_1 and M_2 respectively. If*

$$f\left(\kappa_1\left(\nu_{M_1}^{-1}(x), \kappa_2\left(\nu_{M_1}^{-1}(x)\right)\right)\right) = f\left(\kappa_1\left(\nu_{M_2}^{-1}(x), \kappa_2\left(\nu_{M_2}^{-1}(x)\right)\right)\right), \quad \forall x \in \mathbb{S}^2 \tag{1}$$

then M_1 is equal to M_2 up to a translation.

This classical result was first proved for analytical surfaces by Alexandrov in [3], for C^4 surfaces by Pogorelov in [20], and Hartman and Wintner [14] reduced regularity to C^3, see also [21]. Pogorelov [22,23] published certain uniqueness results for C^2 surfaces, these general results would imply Theorem 1 in C^2 case. It was pointed out

* Corresponding author.
E-mail addresses: guan@math.mcgill.ca (P. Guan), zwang@math.mcgill.ca (Z. Wang), xzhang@math.columbia.edu (X. Zhang).
1 Research of the first author was supported in part by an NSERC Discovery Grant.

http://dx.doi.org/10.1016/j.anihpc.2014.09.011

in [19] that the proof of Pogorelov is erroneous, it contains an uncorrectable mistake (see pp. 301–302 in [19]). There is a counter-example of Martinez-Maure [15] (see also [19]) to the main claims in [22,23]. The results by Han–Nadirashvili–Yuan [13] imply two proofs of Theorem 1, one for C^2 surfaces and another for $C^{2,\alpha}$ surfaces. The problem is often reduced to a uniqueness problem for linear elliptic equations in appropriate settings, either on $\mathbb{S}^2$ or in $\mathbb{R}^3$, we refer to [4,21]. Here we will concentrate on the corresponding equation on $\mathbb{S}^2$, as in [11]. The advantage in this setting is that it is globally defined.

If M is a strictly convex surface with support function u, then the principal curvatures at $\nu^{-1}(x)$ are the reciprocals of the principal radii λ_1, λ_2 of M, which are the eigenvalues of spherical Hessian $W_u(x) = (u_{ij}(x) + u(x)\delta_{ij})$ where u_{ij} are the covariant derivatives with respect to any given local orthonormal frame on $\mathbb{S}^2$. Set

$$\tilde{F}(W_u) =: f\left(\frac{1}{\lambda_1(W_u)}, \frac{1}{\lambda_2(W_u)}\right) = f(\kappa_1, \kappa_2). \tag{2}$$

In view of Lemma 1 in [5], if f satisfies the conditions in Theorem 1, then $\tilde{F}^{ij} = \frac{\partial \tilde{F}}{\partial w_{ij}} \in L^\infty$ is uniformly elliptic. In the case $n = 2$, it can be read off from the explicit formulas

$$\lambda_1 = \frac{\sigma_1(W_u) - \sqrt{\sigma_1(W_u)^2 - 4\sigma_2(W_u)}}{2}, \qquad \lambda_2 = \frac{\sigma_1(W_u) + \sqrt{\sigma_1(W_u)^2 - 4\sigma_2(W_u)}}{2}.$$

As noted by Alexandrov in [5], $\tilde{F}^{ij}$ in general is not continuous if $f(y_1, y_2)$ is not symmetric (even f is analytic).

We want to address when Theorem 1 remains true for convex bodies in $\mathbb{R}^3$ with weakened regularity assumption. In the Brunn–Minkowski theory, the uniqueness of Alexandrov–Fenchel–Jessen [1,2,10] states that, if two bounded convex bodies in $\mathbb{R}^{n+1}$ have the same kth area measures on $\mathbb{S}^n$, then these two bodies are the same up to a rigidity motion in $\mathbb{R}^{n+1}$. Though for a general convex body, the principal curvatures of its boundary may not be defined. But one can always define the support function u, which is a function on $\mathbb{S}^2$. By the convexity, then $W_u = (u_{ij} + u\delta_{ij})$ is a Radon measure on $\mathbb{S}^2$. Also, by Alexandrov's theorem for the differentiability of convex functions, W_u is defined for almost every point $x \in \mathbb{S}^2$. Denote $\mathcal{N}$ to be the space of all positive definite 2×2 matrices, and let G be a function defined on $\mathcal{N}$. For a support function u of a bounded convex body Ω_u, $G(W_u)$ is defined for *a.e.* $x \in \mathbb{S}^2$. For fixed support functions u^l of Ω_{u^l}, $l = 1, 2$, there is $\Omega \subset \mathbb{S}^2$ with $|\mathbb{S}^2 \setminus \Omega| = 0$ such that W_{u^1}, W_{u^2} are pointwise finite in Ω. Set $P_{u^1,u^2} = \{W \in \mathcal{N} \mid \exists x \in \Omega,\ W = W_{u^1}(x), \text{ or } W = W_{u^2}(x)\}$, let $\mathcal{P}_{u^1,u^2}$ be the convex hull of P_{u^1,u^2} in $\mathcal{N}$.

We establish the following slightly more general version of Theorem 1.

Theorem 2. *Suppose Ω_1 and Ω_2 are two bounded convex bodies in $\mathbb{R}^3$. Let u^l, $l = 1, 2$ be the corresponding supporting functions respectively. Suppose the spherical Hessians $W_{u^l} = (u^l_{ij} + \delta_{ij}u^l)$ (in the weak sense) are two non-singular Radon measures. Let $G : \mathcal{N} \to \mathbb{R}$ be a $C^{0,1}$ function such that*

$$\Lambda I \geq \left(G^{ij}\right)(W) := \left(\frac{\partial G}{\partial W_{ij}}\right)(W) \geq \lambda I > 0, \quad \forall W \in \mathcal{P}_{u^1,u^2},$$

for some positive constants Λ, λ. If

$$G(W_{u^1}) = G(W_{u^2}), \tag{3}$$

at almost every parallel normal $x \in \mathbb{S}^2$, then Ω_1 is equal to Ω_2 up to a translation.

Suppose u^1, u^2 are the support functions of two convex bodies Ω_1, Ω_2 respectively, and suppose W_{u^l}, $l = 1, 2$ are defined and they satisfy Eq. (3) at some point $x \in \mathbb{S}^2$. Then, for $u = u^1 - u^2$, $W_u(x)$ satisfies equation

$$F^{ij}(x)\big(W_u(x)\big) = 0, \tag{4}$$

with $F^{ij}(x) = \int_0^1 \frac{\partial \tilde{F}}{\partial W_{ij}}(tW_{u^1}(x) + (1-t)W_{u^2}(x))dt$. By the convexity, W_{u^l}, $l = 1, 2$ exist almost everywhere on $\mathbb{S}^2$. If they satisfy Eq. (3) almost everywhere, Eq. (4) is verified almost everywhere. Note that u may *not be a solution (even in a weak sense) of partial differential equation (4)*. The classical elliptic theory (e.g., [16,18,8]) requires $u \in W^{2,2}$ in order to make sense of u as a weak solution of (4). A main step in the proof of Theorem 2 is to show that with the assumptions in the theorem, $u = u^1 - u^2$ is indeed in $W^{2,2}(\mathbb{S}^2)$. The proof will appear in the last part of the paper.

Let's now focus on $W^{2,2}$ solutions of differential equation (4), with general uniformly elliptic condition on tensor F^{ij} on $\mathbb{S}^2$:

$$\lambda|\xi|^2 \le F^{ij}(x)\xi_i\xi_j \le \Lambda|\xi|^2, \quad \forall x \in \mathbb{S}^2,\ \xi \in \mathbb{R}^2, \tag{5}$$

for some positive numbers λ, Λ. The aforementioned proofs of Theorem 1 [20,14,21,13] all reduce to the statement that any solution of (5) is a linear function, under various regularity assumptions on F^{ij} and u. Eq. (4) is also related to minimal cone equation in $\mathbb{R}^3$ [13]. The following result was proved in [13].

Theorem 3. *(See Theorem 1.1 in [13].) Suppose $F^{ij}(x) \in L^\infty(\mathbb{S}^2)$ satisfies (5), suppose $u \in W^{2,2}(\mathbb{S}^2)$ is a solution of (4). Then, $u(x) = a_1x_1 + a_2x_2 + a_3x_3$ for some $a_i \in \mathbb{R}$.*

There the original statement in [13] is for 1-homogeneous $W^{2,2}_{loc}(\mathbb{R}^3)$ solution v of equation

$$\sum_{i,j=1}^{3} a^{ij}(X)v_{ij}(X) = 0. \tag{6}$$

These two statements are equivalent. To see this, set $u(x) = \frac{v(X)}{|X|}$ with $x = \frac{X}{|X|}$. By the homogeneity assumption, the radial direction corresponds to null eigenvalue of $\nabla^2 v$, the other two eigenvalues coincide the eigenvalues of the spherical Hessian of $W = (u_{ij} + u\delta_{ij})$. $v(X) \in W^{2,2}_{loc}(\mathbb{R}^3)$ is a solution to (6) if and only if $u \in W^{2,2}(\mathbb{S}^2)$ is a solution to (4) with $F^{ij}(x) = \langle e_i, Ae_j\rangle$, where $A = (a^{ij}(\frac{X}{|X|}))$ and (e_1, e_2) is any orthonormal frame on $\mathbb{S}^2$.

The proof in [13] uses gradient maps and support planes introduced by Alexandrov, as in [3,20,21]. We give a different proof of Theorem 3 using the maximum principle for smooth solutions and the unique continuation theorem of Bers–Nirenberg [8], working purely on solutions of Eq. (4) on $\mathbb{S}^2$.

Note that F in Theorem 2 (and Theorem 1) is not assumed to be symmetric. The weak assumption $F^{ij} \in L^\infty$ is needed to deal with this case. This assumption also fits well with the weak unique continuation theorem of Bers–Nirenberg. This beautiful result of Bers–Nirenberg will be used in a crucial way in our proof. If $u \in W^{2,2}(\mathbb{S}^2)$, $u \in C^\alpha(\mathbb{S}^2)$ for some $0 < \alpha < 1$ by the Sobolev embedding theorem. Eq. (4) and $C^{1,\alpha}$ estimates for 2-d linear elliptic PDE (e.g., [16,18,8]) imply that u is in $C^{1,\alpha}(\mathbb{S}^2)$ for some $\alpha > 0$ depending only on $\|u\|_{C^0}$ and the ellipticity constants of F^{ij}. This fact will be assumed in the rest of the paper.

The following lemma is elementary.

Lemma 4. *Suppose $F^{ij} \in L^\infty(\mathbb{S}^2)$ satisfies (5), suppose at some point $x \in \mathbb{S}^2$, $W_u(x) = (u_{ij}(x)+u(x)\delta_{ij})$ satisfies (4). Then,*

$$|W_u|^2(x) \le -\frac{2\Lambda}{\lambda} \det W_u(x).$$

Proof. At x, by Eq. (4),

$$\det W_u = -\frac{1}{F^{22}}\big(F^{11}W_{11}^2 + 2F^{12}W_{11}W_{12} + F^{22}W_{12}^2\big) \le -\frac{\lambda}{\Lambda}\big(W_{11}^2 + W_{12}^2\big), \tag{7}$$

and similarly, $\det W_u \le -\frac{\lambda}{\Lambda}(W_{22}^2 + W_{21}^2)$. Thus,

$$\big(W_{11}^2 + W_{12}^2 + W_{21}^2 + W_{22}^2\big) \le -\frac{2\Lambda}{\lambda} \det W_u. \qquad \square \tag{8}$$

For each $u \in C^1(\mathbb{S}^2)$, set $X_u = \sum_i u_ie_i + ue_{n+1}$. For any unit vector E in $\mathbb{R}^3$, define

$$\phi_E(x) = \big\langle E, X_u(x)\big\rangle, \quad \text{and} \quad \rho_u(x) = \big|X_u(x)\big|^2, \tag{9}$$

where $\langle\, ,\rangle$ is the standard inner product in $\mathbb{R}^3$. The function ρ was introduced by Weyl in his study of Weyl's problem [25]. It played important role in Nirenberg's solution of Weyl's problem in [17]. Our basic observation is that there is a maximum principle for ρ_u and ϕ_E.

Lemma 5. *Suppose $U \subset \mathbb{S}^2$ is an open set, $F^{ij} \in C^1(U)$ is a tensor in U and $u \in C^3(U)$ satisfies Eq. (4), then there are two constants C_1, C_2 depending only on the C^1-norm of F^{ij} such that*

$$F^{ij}(\rho_u)_{ij} \ge -C_1|\nabla\rho_u|, \qquad F^{ij}(\phi_E)_{ij} \ge -C_2|\nabla\phi_E| \quad \text{in } U. \tag{10}$$

Proof. Picking any orthonormal frame e_1, e_2, we have

$$(X_u)_i = W_{ij}e_j, \qquad (X_u)_{ij} = W_{ijk}e_k - W_{ij}\vec{x}. \tag{11}$$

By Codazzi property of W and (4),

$$\frac{1}{2}F^{ij}(\rho_u)_{ij} = \langle X_u, F^{ij}W_{ijk}e_k\rangle + F^{ij}W_{ik}W_{kj} = -u_kF^{ij}_{,k}W_{ij} + F^{ij}W_{ik}W_{kj}.$$

On the other hand, $\nabla\rho_u = 2W\cdot(\nabla u)$. At the non-degenerate points (i.e., $\det W \neq 0$), $\nabla u = \frac{1}{2}W^{-1}\cdot\nabla\rho_u$, where W^{-1} denotes the inverse matrix of W. Now,

$$2u_kF^{ij}_{,k}W_{ij} = W^{kl}(\rho_u)_lF^{ij}_{,k}W_{ij} = (\rho_u)_lF^{ij}_{,k}\frac{A^{kl}W_{ij}}{\det W}, \tag{12}$$

where A^{kl} denotes the co-factor of W_{kl}.

The first inequality in (10) follows from (8) and (12).

The proof for ϕ_E follows the same argument and the following facts:

$$F^{ij}(\phi_E)_{ij} = -\langle E, e_k\rangle F^{ij}_{,k}W_{ij}, \qquad \nabla\phi_E = W\cdot\langle E, e_k\rangle. \qquad \square$$

Lemma 5 yields immediately Theorem 1 in C^3 case, which corresponds to the Hartman–Wintner theorem [14].

Corollary 6. *Suppose $f \in C^2$ and is symmetric, M_1, M_2 are two closed convex C^3 surfaces satisfy conditions in Theorem 1, then the surfaces are the same up to a translation.*

Proof. Since $f \in C^2$ is symmetric, F^{ij} in (4) is in $C^1(\mathbb{S}^2)$ and $u \in C^3(\mathbb{S}^2)$. By Lemma 5 and the strong maximum principle, X_u is a constant vector. $\square$

To precede further, set

$$\mathcal{M} = \left\{p \in \mathbb{S}^2 : \rho_u(p) = \max_{q\in\mathbb{S}^2}\rho_u(q)\right\},$$

for each unit vector $E \in \mathbb{R}^3$,

$$\mathcal{M}_E = \left\{p \in \mathbb{S}^2 : \phi_E(p) = \max_{q\in\mathbb{S}^2}\phi_E(q)\right\}.$$

Lemma 7. *$\mathcal{M}$ and $\mathcal{M}_E$ have no isolated points.*

Proof. We prove the lemma for $\mathcal{M}$, the proof for $\mathcal{M}_E$ is the same. If point $p_0 \in \mathcal{M}$ is an isolated point, we may assume $p_0 = (0, 0, 1)$. Pick $\bar{U}$ a small open geodesic ball centered at p_0 such that $\bar{U}$ is properly contained in $\mathbb{S}^2_+$, and pick a sequence of smooth 2-tensor $(F^{ij}_\epsilon) > 0$ which is convergent to (F^{ij}) in L^∞-norm in $\bar{U}$. Consider

$$\begin{cases} F^{ij}_\epsilon\left(u^\epsilon_{ij} + u^\epsilon\delta_{ij}\right) = 0 & \text{in } \bar{U} \\ u^\epsilon = u & \text{on } \partial\bar{U}. \end{cases} \tag{13}$$

Since $x_3 > 0$ in $\mathbb{S}^2_+$, one may write $u^\epsilon = x_3v^\epsilon$ in $\bar{U}$. As $(x_3)_{ij} = -x_3\delta_{ij}$, it easy to check that v^ϵ satisfies

$$F^{ij}_\epsilon v^\epsilon_{ij} + b_kv^\epsilon_k = 0 \quad \text{in } \bar{U}.$$

Therefore, (13) is uniquely solvable.

Since $p_0 \in \mathcal{M}$ is an isolated point, there are open geodesic balls $\bar{U}' \subset \bar{U}$ centered at p_0 and a small $\delta > 0$ such that

$$\rho_u(p_0) - \rho_u(p) \geq \delta \quad \text{for } \forall p \in \partial \bar{U}'. \tag{14}$$

By the $C^{1,\alpha}$ estimates for linear elliptic equation in dimension two and the uniqueness of the Dirichlet problem [16,8,18], $\exists \epsilon_k$ such that

$$\|u - u^{\epsilon_k}\|_{C^{1,\alpha}(\bar{U}')} \to 0, \qquad \|\rho_u - \rho_{u^{\epsilon_k}}\|_{C^{\alpha}(\bar{U}')} \to 0.$$

Together with (14), if ϵ_k is small enough, there is a local maximal point of $\rho_{u^{\epsilon_k}}$ in $\bar{U}' \subset \bar{U}$. Since $u^{\epsilon_k}, F_\epsilon^{ij} \in C^\infty(\bar{U}')$ satisfy (13), it follows from Lemma 5 and the strong maximum principle that $\rho_{u^{\epsilon_k}}$ must be constant in $\bar{U}'$, when ϵ_k is small enough. This implies that ρ is constant in $\bar{U}'$. A contradiction. □

We now prove Theorem 3.

Proof of Theorem 3. For any $p_0 \in \mathcal{M}$, if $\rho_u(p_0) = 0$, then $u \equiv 0$. We may assume $\rho_u(p_0) > 0$. Set $E := \frac{X_u(p_0)}{|X_u(p_0)|}$. Choose another two unit constant vectors β_1, β_2 with $\langle \beta_i, \beta_j \rangle = \delta_{ij}, \beta_i \perp E$ for $i, j = 1, 2$. Under these orthogonal coordinates in $\mathbb{R}^3$,

$$X_u(p) = a(p)E + b_1(p)\beta_1 + b_2(p)\beta_2, \quad \forall p \in \mathcal{M}_E. \tag{15}$$

On the other hand, $\phi_E(p) = \rho_u^{1/2}(p_0), \forall p \in \mathcal{M}_E$. Thus,

$$a(p) = \rho_u^{1/2}(p_0), \qquad b_1(p) = b_2(p) = 0, \quad \forall p \in \mathcal{M}_E. \tag{16}$$

Consider the function $\tilde{u}(x) = u(x) - \rho_u^{1/2}(p_0)E \cdot x$. (15) and (16) yield, $\forall p \in \mathcal{M}_E$,

$$\nabla_{e_i}\tilde{u}(p) = \nabla_{e_i}u(p) - \rho_u^{1/2}(p_0)\langle E, e_i\rangle = \big\langle X_u(p), e_i\big\rangle - \rho_u^{1/2}(p_0)\langle E, e_i\rangle = 0. \tag{17}$$

Moreover, $\tilde{u}(x)$ also satisfies Eq. (4). As pointed out in [8], if $\tilde{u}$ satisfies an elliptic equation, $\nabla\tilde{u}$ satisfies an elliptic system of equations. Lemma 7, (17) and the unique continuation theorem of Bers–Nirenberg (p. 113 in [7]) imply $\nabla\tilde{u} \equiv 0$. Thus, $\tilde{u}(x) \equiv \tilde{u}(p_0) = 0$ and $u(x)$ is a linear function on $\mathbb{S}^2$. □

Theorem 1 is a direct consequence of Theorem 3. We now prove Theorem 2.

Proof of Theorem 2. The main step is to show $u = u^1 - u^2 \in W^{2,2}(\mathbb{S}^2)$, using the assumption that $W_{u^l}, l = 1, 2$ are non-singular Radon measures. It follows from the convexity, the spherical Hessians $W_{u^l}, l = 1, 2$ and W_u are defined almost everywhere on $\mathbb{S}^2$ (Alexandrov's theorem). So, we can define $G(W_{u^l}), l = 1, 2$ almost everywhere in $\mathbb{S}^2$. As $W_u^l, l = 1, 2$ are nonsingular Radon measures, $W_{u^l} \in L^1(\mathbb{S}^2)$ (see [9]), we also have $W_u \in L^1(\mathbb{S}^2)$. Since u^1, u^2 satisfy $G(W_{u^1}) = G(W_{u^2})$ for almost every parallel normal $x \in \mathbb{S}^2$, there is $\Omega \subset \mathbb{S}^2$ with $|\mathbb{S}^2 \setminus \Omega| = 0$, such that W_u satisfies the following equation *pointwise* in Ω,

$$G^{ij}(x)\big(u_{ij}(x) + u(x)\delta_{ij}\big) = 0, \quad x \in \Omega,$$

where $G^{ij} = \int_0^1 \frac{\partial G}{\partial w_{ij}}(tW_u^1 + (1-t)W_u^2)dt$. By Lemma 4, we can obtain that

$$|W_u|^2 = W_{11}^2 + W_{12}^2 + W_{21}^2 + W_{22}^2 \leq -\frac{2\Lambda}{\lambda}\det W_u, \quad x \in \Omega.$$

On the other hand,

$$\det W_u \leq \det W_{\tilde{u}},$$

where $\tilde{u} = u^1 + u^2$. Thus, to prove $u \in W^{2,2}(\mathbb{S}^2)$, it suffices to get an upper bound for $\int_{\mathbb{S}^2}\det W_{\tilde{u}}$.

Recall that $W_{u^l} \in L^1(\mathbb{S}^2)$, so $u^l \in W^{2,1}(\mathbb{S}^2), l = 1, 2$ and the same for $\tilde{u}$. This allows us to choose two sequences of smooth convex bodies Ω_ϵ^l with supporting functions u_ϵ^l such that $\|\tilde{u}_\epsilon - \tilde{u}\|_{W^{2,1}(\mathbb{S}^2)} \to 0$ as $\epsilon \to 0$. By Fatou's Lemma and continuity of the area measures,

$$\int_{\mathbb{S}^2} \det W_{\tilde{u}} = \int_{\Omega} \det W_{\tilde{u}} \le \liminf_{\epsilon\to 0} \int_{\mathbb{S}^2} \det W_{\tilde{u}_\epsilon} \le V(\Omega^1) + V(\Omega^2) + 2V(\Omega^1, \Omega^2),$$

where $V(\Omega^1)$, $V(\Omega^2)$ denote the volumes of the convex bodies Ω^1 and Ω^2 respectively and $V(\Omega^1, \Omega^2)$ is the mixed volume.

It follows that $W_u \in L^2(\mathbb{S}^2)$ and thus, $u \in W^{2,2}(\mathbb{S}^2)$. This implies that u is a $W^{2,2}$ weak solution of the differential equation

$$G^{ij}(x)\big(u_{ij}(x) + u(x)\delta_{ij}\big) = 0, \quad \forall x \in \mathbb{S}^2.$$

Finally, the theorem follows directly from Theorem 3. □

Remark 8. Alexandrov proved in [3] that, if u is a homogeneous degree 1 analytic function in $\mathbb{R}^3$ with $\nabla^2 u$ definite nowhere, then u is a linear function. As a consequence, Alexandrov proved in [6] that if an analytic closed convex surface in $\mathbb{R}^3$ satisfies the condition $(\kappa_1 - c)(\kappa_2 - c) \le 0$ at every point for some constant c, then it is a sphere. Martinez-Maure gave a C^2 counter-example in [15] to this statement, see also [19]. The counter-examples in [15,19] indicate that Theorem 3 is not true if F^{ij} is merely assumed to be degenerate elliptic. It is an interesting question that under what degeneracy condition on F^{ij} so that Theorem 3 is still true, even in smooth case. This question is related to similar questions in this nature posted by Alexandrov [4] and Pogorelov [21].

We shall wrap up this paper by mention a stability type result related with uniqueness. Indeed, by using the uniqueness property proved in Theorem 3, we can prove the following stability theorem via compactness argument.

Proposition 9. *Suppose $F^{ij}(x) \in L^\infty(\mathbb{S}^2)$ satisfies (5), and $u(x) \in W^{2,2}(\mathbb{S}^2)$ is a solution of the following equation*

$$F^{ij}(x)(W_u)_{ij} = f(x), \quad \forall x \in \mathbb{S}^2. \tag{18}$$

Assume that $f(x) \in L^\infty(\mathbb{S}^2)$ and there exists a point $x_0 \in \mathbb{S}^2$ such that $\rho_u(x_0) = 0$ (see (9) for the definition of ρ_u). Then,

$$\|u\|_{L^\infty(\mathbb{S}^2)} \le C_3 \|f\|_{L^\infty(\mathbb{S}^2)} \tag{19}$$

holds for some positive constant C_3 only depending on the ellipticity constants λ, Λ.

Proof. As mentioned above, we will prove this proposition by a compactness argument. Suppose the desired estimate (19) does not hold, then there exists a sequence of functions $\{f_n(x)\}_{n=1}^\infty$ on $\mathbb{S}^2$ with $\|f\|_{L^\infty(\mathbb{S}^2)} \le C_4$ and a sequence of points $\{x_n\}_{n=1}^\infty \subset \mathbb{S}^2$ such that $\rho_{u_n}(x_n) = 0$ and $K_n := \frac{\|u\|_{L^\infty(\mathbb{S}^2)}}{\|f\|_{L^\infty(\mathbb{S}^2)}} \to +\infty$, where $u_n(x)$ is the solution of Eq. (18) with right hand side replaced by $f_n(x)$.

Let $v_n(x) = \frac{u_n(x)}{K_n \|f\|_{L^\infty(\mathbb{S}^2)}}$, then $\|v_n\|_{L^\infty(\mathbb{S}^2)} = 1$ and $v_n(x)$ satisfies

$$F^{ij}(x)(W_{v_n})_{ij} = \tilde{f}_n := \frac{f_n(x)}{K_n \|f_n\|_{L^\infty(\mathbb{S}^2)}}. \tag{20}$$

By the interior $C^{1,\alpha}$ estimates for linear elliptic equation in dimension two [16,8,18], we have

$$\|v_n\|_{C^{1,\alpha}(\mathbb{S}^2)} \le C_5\big(\|v_n\|_{L^\infty(\mathbb{S}^2)} + \|\tilde{f}_n\|_{L^\infty(\mathbb{S}^2)}\big) \le 2C_5$$

for some positive constant $C_5 = C_5(\lambda, \Lambda)$. In particular, this gives that $\|\nabla v_n\|_{L^\infty(\mathbb{S}^2)} \le C_6$. Now, apply the *a priori* $W^{2,2}$ estimate for linear elliptic equation in dimension two [16,8,18,12], we see that $\|v_n\|_{W^{2,2}(\mathbb{S}^2)} \le C_7$ for some constant $C_7 = C_7(\lambda, \Lambda, C_6)$. It follows from this uniform estimate that, up to a subsequence, $\{v_n(x)\}_{n=1}^\infty$ converges to some function $v(x) \in W^{2,2}(\mathbb{S}^2)$ and $v(x)$ satisfies

$$F^{ij}(x)(W_v)_{ij} = 0, \quad a.e.\ x \in \mathbb{S}^2.$$

Then, the previous uniqueness result Theorem 3 tells that $v(x)$ must be a linear function, i.e., there exists a constant vector $\vec{a} = (a_1, a_2, a_3) \in \mathbb{R}^3$ such that $v(x) = a_1x_1 + a_2x_2 + a_3x_3$.

On the other hand, recall that, by the assumption at the beginning, there exists $x_n \in \mathbb{S}^2$ such that $\rho_{v_n}(x_n) = 0$. Then, up to a subsequence, $x_n \to x_\infty \in \mathbb{S}^2$ and $\rho_v(x_\infty) = 0$. This together with the linear property of $v(x)$ imply that $v(x) \equiv 0$. However, this contradicts with the fact that $\|v\|_{L^\infty(\mathbb{S}^2)} = 1$ as $\|v_n\|_{L^\infty(\mathbb{S}^2)} = 1$. □

As a direct corollary, we have the following stability property for convex surfaces.

Theorem 10. *Suppose M_1, M_2 and f satisfy the same assumptions as in Theorem 3. Define $\mu_1(x) := f(\kappa_1(\nu_{M_1}^{-1}(x), \kappa_2(\nu_{M_1}^{-1}(x)))$ and $\mu_2(x) := f(\kappa_1(\nu_{M_2}^{-1}(x), \kappa_2(\nu_{M_2}^{-1}(x)))$ for $\forall x \in \mathbb{S}^2$. If $\|\mu_1 - \mu_2\|_{L^\infty(\mathbb{S}^2)} < \epsilon$, then, module a linear translation, M_1 is very close to M_2. More precisely, suppose u_1, u_2 are the supporting functions of M_1 and M_2 after module the linear translation, then there exists a constant C such that*

$$\|u_1 - u_2\|_{L^\infty(\mathbb{S}^2)} \le C\|\mu_1 - \mu_2\|_{L^\infty(\mathbb{S}^2)}. \tag{21}$$

Finally, it is worth to remark that there are many stability type results for convex surfaces proved in the literature (see [24]). However, almost all the proofs need to use the assumption that $f(\kappa_1, \kappa_2, \cdots, \kappa_n)$ satisfies divergence property. Here, we do not make such kind assumption in this dimension two case. There is one drawback in the above stability result: one could not get the sharp constant via the compactness argument. It would be an interesting question to derive a sharp estimate for (21).

Conflict of interest statement

There is no conflict of interest.

Acknowledgements

The first author would like to thank Professor Louis Nirenberg for stimulation conversations. Our initial proof was the global maximum principle for C^3 surfaces Lemma 5 and Corollary 6 (we only realized the connection of the result of [13] to Theorem 1 afterward). It was Professor Louis Nirenberg who brought our attention to the paper of [15] and suggested using the unique continuation theorem of [8]. That leads to Theorem 2. We want to thank him for his encouragement and generosity.

References

[1] A.D. Alexandrov, Zur Theorie der gemischten Volumina von konvexen Körpern, II. Neue Ungleichungen zwischen den gemischten Volumina und ihre Anwendungen, Mat. Sb. (N.S.) 2 (1937) 1205–1238 (in Russian).
[2] A.D. Alexandrov, Zur Theorie der gemischten Volumina von konvexen Körpern, III. Die Erweiterung zweeier Lehrsätze Minkowskis über die konvexen Polyeder auf beliebige konvexe Flächen, Mat. Sb. (N.S.) 3 (1938) 27–46 (in Russian).
[3] A.D. Alexandrov, Sur les théorèmes d'unicité pour les surfaces fermeès, C. R. (Dokl.) Acad. Sci. URSS, N.S. 22 (1939) 99–102; translation in: Selected Works. Part I. Selected Scientific Papers, in: Class. Sov. Math., vol. 4, Gordon and Breach, Amsterdam, 1996, pp. 149–153.
[4] A.D. Alexandrov, Uniqueness theorems for surfaces in the large. I, Vestn. Leningr. Univ. 11 (19) (1956) 5–17 (in Russian).
[5] A.D. Alexandrov, Uniqueness theorems for surfaces in the large. II, Vestn. Leningr. Univ. 12 (7) (1957) 15–44 (in Russian).
[6] A.D. Alexandrov, On the curvature of surfaces, Vestn. Leningr. Univ. 21 (19) (1966) 5–11 (in Russian).
[7] L. Bers, L. Nirenberg, On a Representation Theorem for Linear Elliptic System with Discontinuous Coefficients and Its Application, Edizioni cremonese dells S.A. editrice perrella, Roma, 1954, pp. 111–140.
[8] L. Bers, L. Nirenberg, On linear and non-linear elliptic boundary value problems in the plane, in: Convegno Internazionale sulle Equazioni Lineari alle Derivate Parziali, Trieste, 1954, pp. 141–167.
[9] L.C. Evans, R.F. Gariepy, Measure Theory and Fine Properties of Functions, CRC Press Inc., 1992.
[10] W. Fenchel, B. Jessen, Mengenfunktionen und konvexe Körper, Det. Kgl. Danske Videnskab. Selskab, Math.-fys. Medd. 16 (3) (1938) 1–31.
[11] B. Guan, P. Guan, Convex hypersurfaces of prescribed curvatures, Ann. Math. 256 (2) (2002) 655–673.
[12] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, Springer, 1998.
[13] Q. Han, N. Nadirashvili, Y. Yuan, Linearity of homogeneous order-one solutions to elliptic equations in dimension three, Commun. Pure Appl. Math. 56 (2003) 425–432.
[14] P. Hartman, A. Wintner, On the third fundamental form of a surface, Am. J. Math. 75 (1953) 298–334.
[15] Y. Martinez-Maure, Contre-example à une caracteérisation conjecturée de; as sphére, C. R. Acad. Sci. Paris Sér. I Math. 332 (2001) 41–44.
[16] C. Morrey, On the solutions of quasi-linear elliptic partial differential equations, Trans. Am. Math. Soc. 43 (1) (1938) 126–166.
[17] L. Nirenberg, The Weyl and Minkowski problems in differential geometry in the large, Commun. Pure Appl. Math. 6 (1953) 337–394.

[18] L. Nirenberg, On nonlinear elliptic partial differential equations and Hölder continuity, Commun. Pure Appl. Math. 6 (1953) 103–156.
[19] G. Panina, New counterexamples to A. D. Alexandrov's hypothesis, Adv. Geom. 5 (2005) 301–317.
[20] A.V. Pogorelov, Extension of a general uniqueness theorem of A. D. Aleksandrov to the case of nonanalytic surfaces, Dokl. Akad. Nauk SSSR (N.S.) 62 (1948) 297–299 (in Russian).
[21] A.V. Pogorelov, Extrinsic Geometry of Convex Surfaces, translated from Russian by Israel Program for Scientific Translations, Transl. Math. Monogr., vol. 35, American Mathematical Society, Providence, RI, 1973.
[22] A.V. Pogorelov, Solution of a problem of A. D. Aleksandrov, Dokl. Akad. Nauk 360 (1998) 317–319 (in Russian).
[23] A.V. Pogorelov, Uniqueness theorems for closed convex surfaces, Dokl. Akad. Nauk 366 (1999) 602–604 (in Russian).
[24] R. Schneider, Convex Bodies: The Brunn–Minkowski Theory, Encycl. Math. Appl., vol. 44, Cambridge Univ. Press, Cambridge, 1993.
[25] H. Weyl, Über die Bestimmung einer geschlossenen konvexen Fläche durch ihr Linienelement, Vierteljahrschrift Naturforsch. Gesell. Zurich 3 (2) (1916) 40–72.