Available online at www.sciencedirect.com

ScienceDirect

Ann. I. H. Poincaré – AN 33 (2016) 397–408

ANNALES DE L'INSTITUT HENRI POINCARÉ
ANALYSE NON LINÉAIRE

www.elsevier.com/locate/anihpc

Dynamics of nematic liquid crystal flows: The quasilinear approach

Matthias Hieber [a,b,*], Manuel Nesensohn [a,1], Jan Prüss [c], Katharina Schade [a,1]

[a] *Technische Universität Darmstadt, Fachbereich Mathematik, Schlossgarten-Strasse 7, D-64289 Darmstadt, Germany*
[b] *University of Pittsburgh, 607 Benedum Engineering Hall, Pittsburgh, PA 15261, USA*
[c] *Martin-Luther-Universität Halle-Wittenberg, Institut für Mathematik, Theodor-Lieser-Strasse 5, D-06120 Halle, Germany*

Received 31 July 2013; received in revised form 1 September 2014; accepted 4 November 2014

Available online 13 November 2014

Abstract

Consider the (simplified) Leslie–Ericksen model for the flow of nematic liquid crystals in a bounded domain $\Omega \subset \mathbb{R}^n$ for $n > 1$. This article develops a complete dynamic theory for these equations, analyzing the system as a quasilinear parabolic evolution equation in an $L_p - L_q$-setting. First, the existence of a unique local strong solution is proved. This solution extends to a global strong solution, provided the initial data are close to an equilibrium or the solution is eventually bounded in the natural norm of the underlying state space. In this case the solution converges exponentially to an equilibrium. Moreover, the solution is shown to be real analytic, jointly in time and space.

Résumé

On considère le modèle de Leslie–Ericksen pour les cristaux liquides nématiques dans un domaine borné $\Omega \subset \mathbb{R}^n$. On obtient une théorie dynamique complète pour ce système, analysé comme une équation d'évolution quasi-linéare dans le cadre $L^p - L^q$. En particulier, on démontre l' existence et l'unicité locales d'une solution forte, qui s'étend en un solution forte globale si les conditions initiales sont près d'un équilibre. De plus, on montre que la solution est analytique réelle en espace et temps.

MSC: 35Q35; 76A15; 76D03; 35K59

Keywords: Nematic liquid crystals; Quasilinear parabolic evolution equations; Regularity; Global solutions; Convergence to equilibria

* Corresponding author.
E-mail addresses: hieber@mathematik.tu-darmstadt.de (M. Hieber), nesensohn@mathematik.tu-darmstadt.de (M. Nesensohn), jan.pruess@mathematik.uni-halle.de (J. Prüss), schade@mathematik.tu-darmstadt.de (K. Schade).
1 The second and fourth authors are supported by the DFG International Research Training Group 1529 on Mathematical Fluid Dynamics at TU Darmstadt.

http://dx.doi.org/10.1016/j.anihpc.2014.11.001

1. Introduction

We consider the following system modeling the flow of nematic liquid crystal materials in a bounded domain $\Omega \subset \mathbb{R}^n$

$$\begin{cases} \partial_t u + (u \cdot \nabla)u - \nu\Delta u + \nabla\pi = -\lambda \operatorname{div}\big([\nabla d]^{\mathsf{T}}\nabla d\big) & \text{in } (0,T)\times\Omega, \\ \partial_t d + (u\cdot\nabla)d = \gamma\big(\Delta d + |\nabla d|^2 d\big) & \text{in } (0,T)\times\Omega, \\ \operatorname{div} u = 0 & \text{in } (0,T)\times\Omega, \\ (u, \partial_\nu d) = (0,0) & \text{on } (0,T)\times\partial\Omega, \\ (u,d)_{|t=0} = (u_0, d_0) & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here, the function $u : (0,\infty)\times\Omega \to \mathbb{R}^n$ describes the velocity field, $\pi : (0,\infty)\times\Omega \to \mathbb{R}$ is the pressure, and $d : (0,\infty)\times\Omega \to \mathbb{R}^n$ represents the macroscopic molecular orientation of the liquid crystal. Due to the physical interpretation of d it is natural to impose the condition

$$|d| = 1 \quad \text{in } (0,T)\times\Omega. \tag{1.2}$$

We will show in the following that this condition is indeed preserved by the above system; see Proposition 4.3 below for details.

The constants $\nu > 0, \lambda > 0$, and $\gamma > 0$ represent viscosity, the competition between kinetic energy and potential energy and the microscopic elastic relaxation time for the molecular orientation field, respectively. For simplicity, we set $\nu = \lambda = \gamma = 1$, which does not change our analysis.

The continuum theory of liquid crystals was developed by Ericksen and Leslie during the 1950's and 1960's in [10,19]. The Ericksen–Leslie theory is widely used as a model for the flow of liquid crystals, see for example the survey articles by Leslie in [11] and also [4,7,15,22].

The set of Eqs. (1.1) was considered first in [23], however for the situation where in the second equation of (1.1) the term $|\nabla d|^2 d$ is replaced by $f(d) = \nabla F(d)$, i.e.

$$d_t + (u\cdot\nabla)d = \gamma\big(\Delta d - f(d)\big),$$

where $F : \mathbb{R}^3 \to \mathbb{R}$ is a smooth, bounded function. Note that in this situation, the condition (1.2) *cannot* be preserved in general. Thus, this condition was replaced in [22] and [23] by the Ginzburg–Landau energy functional, i.e. f is assumed to satisfy $f(d) = \nabla F(d) = \nabla \frac{1}{\varepsilon^2}(|d|^2 - 1)^2$. In 1995, Lin and Liu [23] proved the existence of global weak solutions to (1.1) in dimension 2 or 3 under the assumptions that $u_0 \in L_2(\Omega), d_0 \in H^1(\Omega)$, and $d_0 \in H^{3/2}(\partial\Omega)$. Existence and uniqueness of global classical solutions were also obtained by them in dimension 2 provided $u_0 \in H^1(\Omega), d_0 \in H^2(\Omega)$, and provided the viscosity ν is large in dimension 3. For regularity results of weak solutions in the spirit of Caffarelli–Kohn–Nirenberg we refer to [24].

Hu and Wang [16] considered in 2010 the case of $f(d) = 0$ and proved existence and uniqueness of a global strong solution for small initial data in this case. They proved moreover that whenever a strong solutions exist, all global weak solutions as constructed in [23] must be equal to this strong solution. The idea of their approach was to consider the above system (1.1) as a semilinear equation with a forcing term $\lambda \operatorname{div}([\nabla d]^{\mathsf{T}}\nabla d)$ on the right-hand side.

The system (1.1) with $f(d) = |\nabla d|^2 d$ was revisited by Lin, Lin, and Wang in 2010. They proved in [21] interior and boundary regularity theorems under smallness condition in dimension 2 and established the existence of global weak solutions on bounded smooth domains $\Omega \subset \mathbb{R}^2$ that are smooth away from a finite set. Furthermore, Wang proved in [33] global well-posedness for this system for initial data being small in $BMO^{-1} \times BMO$ in the case of a whole space, i.e. $\Omega = \mathbb{R}^n$, by combining techniques of Koch and Tataru with methods from harmonic maps to certain Riemannian manifolds.

Let us emphasize at this point that the system (1.1) represents a simplification of the full Ericksen–Leslie system. In particular, stretching and rotational effects of the director field are not taken into account in (1.1). Coutand and Shkoller [6] considered in 2001 a modification of system (1.1) in which the second line of (1.1) is replaced by

$$\partial_t d + u\cdot\nabla d - d\cdot\nabla u = \gamma\left(\Delta d - \frac{1}{\varepsilon^2}\big(|d|^2 - 1\big)d\right) \quad \text{in } (0,T)\times\Omega. \tag{1.3}$$

They proved local wellposedness for this system and gave as well a global existence result for small data within this setting. Note, however, that the presence of the stretching term $d \cdot \nabla u$ causes the loss of the total energy balance and, moreover, condition (1.2), i.e. $|d| = 1$ in $(0, T) \times \Omega$, cannot be preserved anymore. For these reasons and since we would like to keep the interpretation of d as an orientation vector, we do not follow these lines. For further results in this direction we refer to Wu, Xu and Liu in [34]. For results on thermodynamical consistent models including the above mentioned stretching term, see [12] and references therein.

The full Ericksen–Leslie system based on the general Oseen–Frank energy density functional takes into account stretching as well as rotational effects for the director field. In the special case of isotropic elasticity the equation for d reads then as

$$\partial_t d + u \cdot \nabla d - V d + \frac{\lambda_2}{\lambda_1} D d = -\frac{1}{\lambda_1}\big(\Delta d + |\nabla d|^2 d\big) + \frac{\lambda_2}{\lambda_1}(Dd \cdot d)d \quad \text{in } (0, T) \times \Omega. \tag{1.4}$$

Here $D = \frac{1}{2}([\nabla u]^T + \nabla u)$ denotes the symmetric, $V = \frac{1}{2}([\nabla u]^T - \nabla u)$ the anti-symmetric part of the deformation tensor and $\lambda_1, \lambda_2 \in \mathbb{R}\backslash\{0\}$ are material coefficients subject to Leslie's relation. Wellposedness results concerning the full Ericksen–Leslie system will be subject of a forthcoming article.

The situation in which the fluid is modelled by a compressible fluid was treated e.g. by [17,25]. Here, local existence of strong solutions is proved. The latter turn out to be even local classical solution.

Summarizing, we observe that in particular results for local as well as global, *strong* solutions in the three dimensional setting for the system (1.1), obeying also the condition (1.2), do not seem to exist so far.

Recently, Li and Wang claimed in [20] such a result. More precisely, they claimed the existence and uniqueness of a strong solutions to (1.1) in bounded, smooth domains (however, not satisfying (1.2)). Their idea was to rewrite (1.1) as a *semilinear* equation for the Stokes equation coupled to the heat equation with a right hand side of the form

$$\widetilde{F}(u, d) := \big(-(u \cdot \nabla)u - \operatorname{div}\big([\nabla d]^{\mathsf{T}} \nabla d\big), -(u \cdot \nabla)d + |\nabla d|^2 d\big).$$

Unfortunately, their approach and their main result [20, Theorem 2.1] relies on an incorrect regularity property for the solution of the heat equation [20, Theorem 3.1]. This result would imply further regularity properties for d and hence for $\widetilde{F}(u, d)$, which however are not true. Note that the (incorrect) assertion of [20, Theorem 3.1] is crucial for their approach. Thus, the theory for local as well as for global strong solutions to (1.1), also satisfying (1.2), needs clarification.

It is the aim of this paper to present a complete theory for global strong solutions to (1.1) satisfying (1.2) as well as for their dynamical behaviour in the n-dimensional setting, where $n > 1$.

Our main idea is to consider (1.1) not as a semilinear equation as done in all of the previous approaches but as a *quasilinear* evolution equation. We thus incorporate the term $\operatorname{div}([\nabla d]^{\mathsf{T}} \nabla d)$ into the quasilinear operator A given by

$$A(d) = \begin{bmatrix} \mathcal{A} & \mathbb{P}\mathcal{B}(d) \\ 0 & \mathcal{D} \end{bmatrix},$$

where $\mathcal{A}$ denotes the Stokes operator, $\mathcal{D}$ the Neumann Laplacian, and $\mathcal{B}$ is given by

$$\big[\mathcal{B}(d)h\big]_i := \partial_i d_l \Delta h_l + \partial_k d_l \partial_k \partial_i h_l,$$

for which we employ the sum convention. Note that $\mathcal{B}(d)d = \operatorname{div}([\nabla d]^{\mathsf{T}} \nabla d)$.

We then develop a complete dynamic theory for (1.1)–(1.2). In fact, first by local existence theory for abstract quasilinear parabolic problems, we prove the existence and uniqueness of a strong solution to (1.1)–(1.2) on a maximal time interval. Thus, (1.1)–(1.2) give rise to a local semi-flow in the natural state space.

Furthermore, the equilibria $\mathcal{E}$ of (1.1)–(1.2) are determined to be

$$\mathcal{E} = \big\{(0, d_*) : d_* \in \mathbb{R}^n,\ |d_*| = 1\big\},$$

and the well-known energy functional

$$\mathsf{E} = \frac{1}{2}\int_\Omega \big[|u|^2 + |\nabla d|^2\big] dx$$

for (1.1)–(1.2) is shown to be a strict Lyapunov-functional. In addition, the equilibria are shown to be normally stable, i.e. for an initial value close to $\mathcal{E}$, the solution of (1.1)–(1.2) exists globally and the solution converges exponentially to an equilibrium. More generally, a solution, eventually bounded on its maximal interval of existence, exists globally and converges to an equilibrium exponentially fast.

Due to the polynomial character of the nonlinearities, we can even show that the solution of (1.1)–(1.2) is real analytic, jointly in time and space.

Our approach is based on the theory of quasilinear parabolic problems and relies in particular on the maximal L_p-regularity property for the heat and the Stokes equation. In particular, we refer here to [1,2,9,5,18,27,28,30].

The plan for this paper is as follows. We begin by collecting general results from the theory of quasilinear parabolic evolution equations. Then, in Section 3 we introduce our formulation of (1.1). Section 4 deals with local well-posedness and regularity of solutions to (1.1)–(1.2); in particular we see that the solution is real analytic. The generalized principle of linearized stability yields the stability of equilibria and convergence of solutions is proved in Section 5. Moreover, by means of the associated energy functional, we prove convergence of a solution to an equilibrium, whenever the solution is eventually bounded in the natural state space.

2. Quasilinear evolution equations

Let X_0 and X_1 be Banach spaces such that $X_1 \overset{d}{\hookrightarrow} X_0$, i.e. X_1 is continuously and densely embedded in X_0. Let $J = [0, a]$ for an $a > 0$. By a *quasilinear autonomous parabolic evolution equation* we understand an equation of the form

$$\dot{z}(t) + A\big(z(t)\big)z(t) = F\big(z(t)\big), \quad t \in J, \qquad z(0) = z_0, \tag{QL}$$

where A is a mapping from a real interpolation space $X_{\gamma,\mu}$ with suitable weights between X_0 and X_1 into $\mathcal{L}(X_0, X_1)$. Our approach relies on the maximal L_p-regularity of $A(v)$ for $v \in X_{\gamma,\mu}$. For details we refer e.g. to [9].

Eq. (QL) is investigated in spaces of the form $L_p(J; X_0)$ with temporal weights. More precisely, for $p \in (1, \infty)$ and $\mu \in (1/p, 1]$, the spaces $L_{p,\mu}$ and $H^1_{p,\mu}$ are defined by

$$L_{p,\mu}(J; X_1) := \big\{z: J \to X_1: t^{1-\mu}z \in L_p(J; X_1)\big\},$$
$$H^1_{p,\mu}(J; X_0) := \big\{z \in L_{p,\mu}(J; X_0) \cap W^1_1(J; X_0): \dot{z} \in L_{p,\mu}(J; X_0)\big\}.$$

It is clear, that

$$L_p(J; X) \hookrightarrow L_{p,\mu}(J; X) \quad \text{and} \quad L_p\big([0, a]; X\big) \hookrightarrow L_{p,\mu}\big([\tau, a]; X\big),$$

for all Banach spaces X and $\tau \in (0, a)$. It has been shown in [28, Theorem 2.4] that L_p-maximal regularity implies also $L_{p,\mu}$-maximal regularity, provided $p \in (1, \infty)$ and $\mu \in (1/p, 1]$. The trace space of the maximal regularity class containing temporal weights,

$$z \in H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1)$$

has been characterized in [28, Theorem 2.4] as the real interpolation space $(X_0, X_1)_{\mu-1/p,p}$, i.e.

$$X_{\gamma,\mu} = (X_0, X_1)_{\mu-1/p,p},$$

provided $p \in (1, \infty)$ and $\mu \in (1/p, 1]$; see also [26, Theorem 4.2]. The space X_γ is given by $X_\gamma := X_{\gamma,1}$.

We now impose precise assumptions on A and F.

(A) $A \in C^\omega(X_{\gamma,\mu}; \mathcal{L}(X_1, X_0))$, and $A(v)$ has maximal L_p-regularity for each $v \in X_{\gamma,\mu}$.
(F) $F \in C^\omega(X_{\gamma,\mu}; X_0)$.

Even under less restrictive Lipschitz type assumptions on A and F, local in time existence of (QL) was shown by Clément and Li [5] in the case $\mu = 1$ and by Köhne, Prüss and Wilke [18, Theorem 2.1, Corollary 2.2] for the case $\mu \in (1/p, 1]$.

Proposition 2.1. *Let $1 < p < \infty$, $\mu \in (1/p, 1]$, $z_0 \in X_{\gamma,\mu}$, and suppose that the assumptions (A) and (F) are satisfied. Then, there exists $a > 0$, such that (QL) admits a unique solution z on $J = [0, a]$ in the regularity class*

$$z \in H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1) \hookrightarrow C(J; X_{\gamma,\mu}) \cap C\big((0, a]; X_\gamma\big).$$

The solution depends continuously on z_0, and can be extended to a maximal interval of existence $J(z_0) = [0, t^+(z_0))$.

Parabolic problems allow for additional smoothing effects. In this respect, a method due to Angenent [3] is well known. We only state here a variant of it which is adapted to (QL); see [27, Theorem 5.1] for the case $\mu = 1$. By a slight adjustment of its proof to the situation of temporal weights, this result remains true also for maximal regularity classes using this type of weights.

Proposition 2.2. *Let $1 < p < \infty$, $\mu \in (1/p, 1]$, $a > 0$, and assume that (A) and (F) hold. Let $z \in H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1)$ be a solution of (QL) on $J = [0, a]$ and assume $A(z(t))$ has maximal L_p-regularity for all $t > 0$. Then*

$$t^k \left[\frac{d}{dt}\right]^k z \in H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1), \quad k \in \mathbb{N}.$$

Furthermore, z is real analytic with values in X_1 on $(0, a)$.

We denote the set of equilibria of (QL) by

$$\mathcal{E} = \big\{z_* \in X_1 \colon A(z_*)z_* = F(z_*)\big\}.$$

The following result on global existence and stability was proved in [30, Theorem 2.1] assuming only that A and F are of class C^1.

Proposition 2.3. *Let $1 < p < \infty$ and assume that assumptions (A) and (F) hold. Furthermore assume, that every equilibrium of (QL) is contained in a manifold of dimension $m \in \mathbb{N}$. Let A_0 be the linearization of (QL), i.e. let*

$$A_0 w = A(z_*)w + \big(A'(z_*)w\big)z_* - F'(z_*)w, \quad w \in X_1.$$

Suppose that $z_* \in \mathcal{E}$ is normally stable equilibrium, i.e.

(i) *near z_* the set of equilibria $\mathcal{E} \subset X_1$ is a C^1-manifold in X_1 of dimension m,*
(ii) *the tangent space of $\mathcal{E}$ at z_* is given by $N(A_0)$,*
(iii) *0 is semi-simple eigenvalue of A_0, i.e. $N(A_0) \oplus R(A_0) = X_0$,*
(iv) *$\sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+ = \{x \in \mathbb{C} \colon \operatorname{Re} x > 0\}$.*

Then z_* is stable in X_γ. Further, there exists a number $\rho > 0$ such that the unique solution z of (QL) with initial value $z_0 \in B_{X_\gamma}(0, \rho)$ exists on $\mathbb{R}_+$ and converges at an exponential rate to some $u_\infty \in \mathcal{E}$ in X_γ as $t \to \infty$.

We finish the section with another result on global existence result for (QL); see [18, Theorem 3.1].

Proposition 2.4. *Let $1 < p < \infty$, $\mu \in (1/p, 1]$, $z_0 \in X_{\gamma,\mu}$ and let $J = [0, a]$ or $J = \mathbb{R}_+$. Suppose that assumptions (A) and (F) are satisfied and that the embedding $X_\gamma \overset{c}{\hookrightarrow} X_{\gamma,\mu}$ is compact. Assume furthermore that the solution z of (QL) is eventually bounded in X_γ on its maximal interval of existence, i.e. that z satisfies*

$$z \in BC\big(\big[\tau, t^+(z_0)\big); X_\gamma\big)$$

for some $\tau \in (0, t^+(z_0))$. Then the solution z exists globally and for each $\delta > 0$, the orbit $\{z(t)\}_{t \geq \delta}$ is relatively compact in X_γ. If in addition $z_0 \in X_\gamma$, then $\{z(t)\}_{t \geq 0}$ is relatively compact in X_γ.

3. Nematic liquid crystals as quasilinear evolution equations

We now reformulate (1.1) equivalently as a quasilinear parabolic evolution equation for the unknown $z = (u, d)$. To this end, for $1 < q < \infty$ define the Banach spaces X_0 by

$$X_0 := L_{q,\sigma}(\Omega) \times L_q(\Omega)^n,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain with boundary $\partial\Omega \in C^2$. The subscript σ in $L_{q,\sigma}(\Omega)$ means as usual the subspace of $L_q(\Omega)^n$ consisting of solenoidal vector fields.

The Neumann–Laplacian $\mathcal{D}_q$ in $L_q(\Omega)$ is defined by $\mathcal{D}_q = -\Delta$ with domain

$$D(\mathcal{D}_q) := \{d \in H_q^2(\Omega)^n : \partial_\nu d = 0 \text{ on } \partial\Omega\}.$$

It is well-known that $\mathcal{D}_q$ has the property of L_p-maximal regularity; see [9, Theorem 8.2].

Let $\mathbb{P} : L_q(\Omega)^n \to L_{q,\sigma}(\Omega)$ denote the Helmholtz projection. We then define the Stokes operator $\mathcal{A}_q = -\mathbb{P}\Delta$ in $L_{q,\sigma}(\Omega)$ with domain

$$D(\mathcal{A}_q) = \{u \in H_q^2(\Omega)^n : \operatorname{div} u = 0 \text{ in } \Omega,\ u = 0 \text{ on } \partial\Omega\}.$$

It is also well-known that $\mathcal{A}_q$ has the property of L_p-maximal regularity; see e.g. [31,14,13].

Next, we define the space X_1 by

$$X_1 := D(\mathcal{A}_q) \times D(\mathcal{D}_q),$$

equipped with its canonical norms. Then $X_1 \overset{d}{\hookrightarrow} X_0$ densely.

The quasilinear part $A(z)$ of (QL) is given by the tri-diagonal matrix

$$A(z) = \begin{bmatrix} \mathcal{A}_q & \mathbb{P}\mathcal{B}_q(d) \\ 0 & \mathcal{D}_q \end{bmatrix},$$

where the operator $\mathcal{B}_q$ is given by

$$[\mathcal{B}_q(d)h]_i := \partial_i d_l \Delta h_l + \partial_k d_l \partial_k \partial_i h_l,$$

for which we employed the sum convention. Note that

$$\mathcal{B}_q(d)d = \operatorname{div}([\nabla d]^\mathsf{T} \nabla d).$$

Obviously, $\mathcal{B}_q(d) : X_1 \to X_0$ is bounded for each $d \in C^1(\overline{\Omega})^n$ and the map $d \mapsto \mathbb{P}\mathcal{B}_q(d)$ is polynomial, hence real analytic. By the tri-diagonal structure of $A(z)$ and by the regularity of $\mathcal{B}_q$ one can easily see that $A(z)$ also has the property of L_p-maximal regularity, for each $z \in C^1(\overline{\Omega})^{2n}$. Indeed, for a fixed right-hand side $(f_u, f_d) \in L_p(0, a; X_{\gamma,\mu})$ and initial values $(u_0, d_0) \in X_{\gamma,\mu}$, we may use the maximal regularity of $\mathcal{D}_q$ to obtain a solution $\tilde{d}$ of the heat equation with Neumann boundary condition in the right maximal regularity class. By setting

$$\widetilde{f_u} := f_u - \mathbb{P}\mathcal{B}_q(d)\tilde{d}$$

as right-hand side for the Stokes equation, we obtain a solution $\tilde{u}$ in the right maximal regularity class due to the fact that $\mathcal{B}_q(d)$ is linear and bounded.

The right-hand side $F(z)$ of (QL) is defined by

$$F(z) = \left(-\mathbb{P}u \cdot \nabla u, -u \cdot \nabla d + |\nabla d|^2 d\right),$$

which is also polynomial, hence a real analytic mapping from $C^1(\overline{\Omega})^{2n}$ into X_0.

Note that (A) and (F) hold, as soon as we have the embedding

$$X_{\gamma,\mu} \hookrightarrow C^1(\overline{\Omega})^{2n}.$$

The space X_γ is given by

$$X_\gamma = (X_0, X_1)_{1-1/p,p} = D_{\mathcal{A}_q}(1 - 1/p, p) \times D_{\mathcal{D}_q}(1 - 1/p, p);$$

see [1,8]. As explained in Section 2, we consider L_p-spaces with temporal weights. The trace space of the class

$$z \in H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1)$$

now reads

$$X_{\gamma,\mu} = (X_0, X_1)_{\mu-1/p,p} = D_{\mathcal{A}_q}(\mu - 1/p, p) \times D_{\mathcal{D}_q}(\mu - 1/p, p),$$

provided $p \in (1,\infty)$ and $\mu \in (1/p, 1]$; see [26, Theorem 4.12].

In order to obtain the embeddings $X_\gamma \hookrightarrow C^1(\overline{\Omega})^{2n}$ and more generally $X_{\gamma,\mu} \hookrightarrow C^1(\overline{\Omega})^{2n}$ we impose on $p, q \in (1,\infty)$ now the conditions

$$\frac{2}{p} + \frac{n}{q} < 1, \qquad \frac{1}{2} + \frac{1}{p} + \frac{n}{2q} < \mu \le 1. \tag{3.1}$$

Standard Sobolev embedding theorems can then be applied.

Further, we recall from [32, Theorem 4.3.3] and [2, Theorem 3.4], respectively, the following characterizations of the interpolation spaces involved,

$$d \in D_{\mathcal{D}_q}(\mu - 1/p, p) \quad \Leftrightarrow \quad d \in B^{2\mu-2/p}_{qp}(\Omega)^n, \qquad \partial_\nu d = 0 \text{ on } \partial\Omega,$$

and

$$u \in D_{\mathcal{A}_q}(\mu - 1/p, p) \quad \Leftrightarrow \quad u \in B^{2\mu-2/p}_{qp}(\Omega)^n \cap L_{q,\sigma}(\Omega), \qquad u = 0 \text{ on } \partial\Omega.$$

Observe that both of these characterizations make sense, since the condition (3.1) guarantees the existence of the trace.

4. Existence, uniqueness, and regularity of solutions

We start this section by applying Proposition 2.1 to obtain the following result on local well-posedness of (1.1).

Theorem 4.1. *Let p, q, μ be subject to (3.1), and assume $z_0 = (u_0, d_0) \in X_{\gamma,\mu}$, which means that $u_0, d_0 \in B^{2\mu-2/p}_{qp}(\Omega)^n$ satisfy the compatibility conditions*

$$\operatorname{div} u_0 = 0 \quad \text{in } \Omega, \qquad u_0 = \partial_\nu d_0 = 0 \quad \text{on } \partial\Omega.$$

Then for some $a = a(z_0) > 0$, there is a unique solution

$$z \in H^1_{p,\mu}(J, X_0) \cap L_{p,\mu}(J; X_1), \quad J = [0, a],$$

of (1.1) on J. Moreover,

$$z \in C\big([0, a]; X_{\gamma,\mu}\big) \cap C\big((0, a]; X_\gamma\big),$$

i.e. the solution regularizes instantly in time. It depends continuously on z_0 and exists on a maximal time interval $J(z_0) = [0, t^+(z_0))$. Therefore problem (1.1), i.e. (QL), generates a local semi-flow in its natural state space $X_{\gamma,\mu}$.

Remark 4.2. Assuming that $2/p + n/q < 1$, for $\varepsilon > 0$ we may choose μ subject to (3.1) such that

$$H_q^{1+\frac{n}{q}+\varepsilon}(\Omega)^n \hookrightarrow B^{2\mu-2/p}_{qp}(\Omega)^n \hookrightarrow H_q^{1+\frac{n}{q}-\varepsilon}(\Omega)^n$$

due to Sobolev embeddings [32, Theorem 4.6.1]. Furthermore, we can choose p, q large with

$$C^{1+\varepsilon}(\Omega)^n \hookrightarrow B^{2\mu-2/p}_{qp}(\Omega)^n.$$

Employing different time weights for u and d, an inspection of the above proofs shows that the assertion of the above theorem remains true provided $u_0 \in C^\alpha(\Omega)$.

The following result tells that the condition (1.2) is preserved by (1.1).

Proposition 4.3. *Suppose that μ, p, q are satisfying (3.1) and let $z_0 = (u_0, d_0) \in X_{\gamma,\mu}$ with $|d_0| \equiv 1$, $a > 0$. Let*

$$z \in H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1)$$

be a solution of (1.1) on the interval $J = [0, a]$. Then $|d(t)| \equiv 1$ holds for all $t \in [0, a]$.

Proof. Setting $\varphi = |d|^2 - 1$ the elementary identities,

$$\partial_t |d|^2 = 2d \cdot \partial_t d, \qquad \Delta |d|^2 = 2\Delta d \cdot d + 2|\nabla d|^2, \qquad \nabla |d|^2 = 2d \cdot \nabla d,$$

and multiplication with d of the second line in (1.1) yields the problem

$$\begin{cases} \partial_t \varphi + u \cdot \nabla \varphi = \Delta \varphi + 2|\nabla d|^2 \varphi & \text{in } \Omega, \\ \partial_\nu \varphi = 0 & \text{on } \partial\Omega, \\ \varphi(0) = 0 & \text{in } \Omega, \end{cases}$$

provided $|d_0| \equiv 1$. Uniqueness of this parabolic convection–reaction–diffusion equations yields $\varphi \equiv 0$, i.e. $|d| \equiv 1$. □

As the nonlinearities A and F are real analytic we may employ Angenent's method (Proposition 2.2) to obtain further regularity of the solutions of (1.1).

Proposition 4.4. *Suppose that μ, p, q satisfy (3.1), $z_0 \in X_{\gamma,\mu}$, and $a > 0$ and let*

$$z \in H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1)$$

be a solution of (1.1) on the interval $J = [0, a]$. Then for each $k \in \mathbb{N}$,

$$t^k \left[\frac{d}{dt} \right]^k z \in H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1).$$

Moreover, $z \in C^\omega((0, a); X_1)$.

We will employ Proposition 4.4 in the following to justify the regularity of time derivatives of the energy functional.

Remark 4.5. Employing scaling techniques jointly in time and space, it is possible to show via maximal regularity and the implicit function theorem that u, π, d are real analytic in $(0, t^+(z_0)) \times \Omega$; see [27, Section 5] for parabolic problems, and specifically for a Navier–Stokes problem [29]. As we will not use this result below we omit the details, here.

5. Stability and convergence to equilibria

We consider the set $\mathcal{E}_0 = \{0\} \times \mathbb{R}^n$, which is obviously equilibria of (1.1). This set forms an n-dimensional subspace of X_1, hence a C^1-manifold with tangent space $\{0\} \times \mathbb{R}^n$ at each point $(0, d_*) \in \mathcal{E}_0$. The linearization of (1.1) at $z_* \in \mathcal{E}_0$ is given by the linear evolution equation

$$\dot z + A_* z = f, \qquad z(0) = z_0,$$

in X_0, where

$$A_* = \operatorname{diag}(\mathcal{A}_q, \mathcal{D}_q), \qquad D(A_*) = X_1.$$

As Ω is bounded, the spectrum $\sigma(\mathcal{A}_q)$ consists only of positive eigenvalues and $0 \notin \sigma(\mathcal{A}_q)$. On the other hand, $\mathcal{D}_q$ has 0 as an eigenvalue, which is semi-simple and the remaining part of $\sigma(\mathcal{D}_q)$ consists only of positive eigenvalues. Thus $\sigma(A_*) \setminus \{0\} \subset [\delta, \infty)$ for some $\delta > 0$ and the kernel of A_* is given by

$$N(A_*) = \{0\} \times \mathbb{R}^n,$$

which equals the tangent space. In Remark 2.2 of [30] it is shown that all equilibria close to z_* are contained in a manifold $\mathcal{M}$ of dimension $n = \dim(N(A_*))$. Since the dimension of $\mathcal{E}_0$ is also n, there exists an open set $V \subset X_1$ with $\mathcal{M} \cap V = \mathcal{E} \cap V = \mathcal{E}_0 \cap V$; i.e. $\mathcal{E} \cap V$ contains no other equilibrium. As a result we see that the equilibrium is normally stable.

Now we are in position to apply Proposition 2.3 to conclude the following stability result for the equilibria of (1.1).

Theorem 5.1. *Let p, q satisfy the first inequality in (3.1). Then each equilibrium $z_* \in \{0\} \times \mathbb{R}^n$ is stable in X_γ, i.e. there exists $\epsilon > 0$ such that a solution $z(t)$ of (1.1) with initial value $z_0 \in X_\gamma$, $|z_0 - z_*|_{X_\gamma} \le \epsilon$, exists globally and converges exponentially to some $z_\infty \in \{0\} \times \mathbb{R}^n$ in X_γ, as $t \to \infty$.*

We next consider the energy of the system given by

$$\mathsf{E} = \frac{1}{2}\int_\Omega \big[|u|^2 + |\nabla d|^2\big]dx = \mathsf{E}_{kin} + \mathsf{E}_{pot}. \tag{5.1}$$

Let $z = H^1_{p,\mu}(J; X_0) \cap L_{p,\mu}(J; X_1)$ be a solution of (1.1)–(1.2) on $J = [0, a]$; according to Proposition 4.4 it belongs to $C^1((0, a); X_1)$. Using sum convention we have by an integration by parts

$$\begin{aligned}
\frac{d}{dt}\mathsf{E}_{kin}(t) &= \int_\Omega \partial_t u \cdot u dx \\
&= \int_\Omega \big[-(u \cdot \nabla)u - \nabla\pi + \Delta u - \operatorname{div}\big([\nabla d]^{\mathsf{T}}\nabla d\big]\big) \cdot u dx \\
&= -\int_\Omega |\nabla u|^2 dx + \int_\Omega \partial_k d_j \partial_i d_j \partial_k u_i dx,
\end{aligned}$$

as $\operatorname{div} u = 0$ in Ω and $u = 0$ on $\partial\Omega$. On the other hand, we have by another integration by parts

$$\begin{aligned}
\int_\Omega \big|\Delta d + |\nabla d|^2 d\big|^2 dx &= \int_\Omega \big[\Delta d + |\nabla d|^2 d\big]\big[\partial_t d + (u \cdot \nabla)d\big]dx \\
&= -\int_\Omega \big[\partial_t \nabla d : \nabla d - |\nabla d|^2 \partial_t |d|^2/2\big]dx \\
&\quad + \int_\Omega \big[(u \cdot \nabla)d \cdot \Delta d + |\nabla d|^2 (u \cdot \nabla)|d|^2/2\big]dx \\
&= -\frac{d}{dt}\mathsf{E}_{pot}(t) - \int_\Omega \partial_k(u_i \partial_i d_j)\partial_k d_j dx \\
&= -\frac{d}{dt}\mathsf{E}_{pot}(t) - \int_\Omega \partial_k u_i \partial_i d_j \partial_k d_j dx,
\end{aligned}$$

by $|d| \equiv 1$ and the Neumann boundary condition for d. Combining these equations, we obtain the energy identity

$$\frac{d}{dt}\mathsf{E}(t) = -\int_\Omega \big[|\nabla u|^2 + |\Delta d + |\nabla d|^2 d|^2\big]dx. \tag{5.2}$$

Therefore $\mathsf{E}(t)$ is non-increasing along solutions. But E is also a strict Lyapunov functional, i.e. strictly decreasing along non-constant solutions. In fact, if $d\mathsf{E}(t)/dt = 0$ at some time instant, then by the energy equality we have $\nabla u = 0$ and $\Delta d + |\nabla d|^2 d = 0$ in Ω. Therefore $u = 0$ by the no-slip condition on $\partial\Omega$, and d satisfies the nonlinear eigenvalue problem

$$\begin{cases} \Delta d + |\nabla d|^2 d = 0 & \text{in } \Omega, \\ |d|^2 = 1 & \text{in } \Omega, \\ \partial_\nu d = 0 & \text{on } \partial\Omega. \end{cases} \tag{5.3}$$

But, as the lemma below shows, this implies $\nabla d = 0$ in Ω, hence $d = d_*$ is constant and $z_* := (0, d_*)$, $|d_*| = 1$ is an equilibrium of the problem.

Lemma 5.2. *Suppose that $d \in H_2^2(\Omega; \mathbb{R}^n)$ satisfies (5.3). Then d is constant in Ω.*

Proof. The idea is to reduce inductively the dimension $N = n$ of the vector d. This can be achieved by introducing polar coordinates according to

$$d_1 = c_1 \cos\theta, \qquad d_2 = c_1 \sin\theta, \qquad d_j = c_{j-1}, \quad j \geq 3.$$

Simple computations yield

$$1 = |d|^2 = |c|^2, \qquad |\nabla d|^2 = |\nabla c|^2 + c_1^2 |\nabla\theta|^2,$$

and

$$\Delta c_j + \big[|\nabla c|^2 + c_1^2|\nabla\theta|^2\big] c_j = 0 \quad \text{in } \Omega,$$

as well as $\partial_\nu c_j = 0$ on $\partial\Omega$ for $j = 2, \dots, n-1$. Moreover, by an easy calculations we further obtain

$$-\Delta c_1 + c_1 |\nabla\theta|^2 = \big[|\nabla c|^2 + c_1^2|\nabla\theta|^2\big] c_1 \quad \text{in } \Omega,$$

and

$$c_1 \Delta\theta + 2\nabla c_1 \cdot \nabla\theta = 0 \quad \text{in } \Omega,$$

as well as

$$\partial_\nu c_1 = c_1 \partial_\nu \theta = 0 \quad \text{on } \partial\Omega.$$

Multiplying the former equation by $c_1\theta$ and integrating over Ω we deduce

$$0 = \int_\Omega \big[c_1 \Delta\theta + 2\nabla c_1 \cdot \nabla\theta\big] c_1 \theta \, dx = \int_\Omega \operatorname{div}\big[c_1^2 \nabla\theta\big]\theta \, dx = -\int_\Omega c_1^2 |\nabla\theta|^2 dx,$$

hence $c_1 \nabla\theta = 0$. This implies that c satisfies the problem (5.3) where the vector c has dimension $N - 1$. Inductively, we arrive at dimension $N = 1$ and if d is a solution of (5.3) with dimension 1, then $d = 1$ or $d = -1$ by connectedness of Ω. □

Note that the side condition $|d| \equiv 1$ is important at this point. Summarizing we proved the following result.

Proposition 5.3. *The energy functional* E *defined on X_γ is a strict Lyapunov function for system (1.1)–(1.2). The equilibria of this system are given by the set*

$$\mathcal{E} = \big\{z_* = (u_*, d_*) : u_* = 0,\ d_* \in \mathbb{R}^n,\ |d_*| = 1\big\},$$

which forms a manifold of dimension $n - 1$. The corresponding pressures p_* are constant as well.

Suppose finally that z is a solution of (1.1)–(1.2) which is eventually bounded in X_γ on its maximal interval of existence. Then, by Proposition 2.3 this solution is global and $z([\delta, \infty)) \subset X_\gamma$ is relatively compact. Therefore its limit set

$$\omega(z_0) = \big\{v \in X_\gamma : \exists t_n \uparrow \infty \text{ s.t. } z(t_n; z_0) \to v \text{ in } X_\gamma\big\}$$

is nonempty. As E is a strict Lyapunov functional for (1.1)–(1.2), we obtain $\operatorname{dist}(z(t, z_0), \omega(z_0)) \to 0$ in X_γ for $t \to \infty$ and $\omega(z_0) \subset \mathcal{E} \subset X_1$. Now Theorem 5.1 applies and we may conclude that $z(t) \to z_\infty \in \mathcal{E}$ in X_γ as $t \to \infty$. In summary we proved the following result.

Theorem 5.4. *Let μ, p, q satisfy (3.1). Let $z_0 = (u_0, d_0) \in X_{\gamma,\mu}$ with $|d_0| \equiv 1$ and suppose that the solution $z(t)$ of (1.1) is eventually bounded in X_γ on its maximal interval of existence, i.e.*

$$z \in BC\big(\big[\tau, t^+(z_0)\big); X_\gamma\big)$$

for some $\tau \in (0, t^+(z_0))$. Then $z(t)$ exists globally and converges to an equilibrium $z_\infty \in \mathcal{E}$ in X_γ, as $t \to \infty$. The converse is also true.

Conflict of interest statement

There is no conflict of interest.

References

[1] H. Amann, Linear and Quasilinear Parabolic Problems, vol. I: Abstract Linear Theory, Monogr. Math., vol. 89, Birkhäuser Boston Inc., Boston, MA, 1995.
[2] H. Amann, On the strong solvability of the Navier–Stokes equations, J. Math. Fluid Mech. 2 (1) (2000) 16–98.
[3] S.B. Angenent, Nonlinear analytic semiflows, Proc. R. Soc. Edinb. Sect. A 115 (1–2) (1990) 91–107.
[4] S. Chandrasekhar, Liquid Crystals, Cambridge University press, 1992.
[5] P. Clément, S. Li, Abstract parabolic quasilinear equations and application to a groundwater flow problem, Adv. Math. Sci. Appl. 3 (Special Issue) (1993/1994) 17–32.
[6] Daniel Coutand, Steve Shkoller, Well-posedness of the full Ericksen–Leslie model of nematic liquid crystals, C. R. Acad. Sci. Paris Sér. I Math. 333 (10) (2001) 919–924. MR 1873808 (2002j:82118).
[7] P.G. DeGennes, J. Prost, The Physics of Liquid Crystals, Oxford, 1974.
[8] R. Denk, M. Hieber, J. Prüss, Optimal L^p-L^q-estimates for parabolic boundary value problems with inhomogeneous data, Math. Z. 257 (1) (2007) 193–224.
[9] Robert Denk, Matthias Hieber, Jan Prüss, $\mathcal{R}$-boundedness, Fourier multipliers and problems of elliptic and parabolic type, Mem. Am. Math. Soc. 166 (788) (2003), viii+114 pp. MR 2006641 (2004i:35002).
[10] J.L. Ericksen, Hydrostatic theory of liquid crystals, Arch. Ration. Mech. Anal. 9 (1962) 371–378.
[11] J.L. Ericksen, D. Kinderlehrer (Eds.), Theory and Applications of Liquid Crystals, Papers from the IMA workshop held in Minneapolis, Minn., January 21–25, 1985, IMA Vol. Math. Appl., vol. 5, Springer-Verlag, New York, 1987.
[12] Eduard Feireisl, Michel Frémond, Elisabetta Rocca, Giulio Schimperna, A new approach to non-isothermal models for nematic liquid crystals, Arch. Ration. Mech. Anal. 205 (2) (2012) 651–672. MR 2947543.
[13] M. Geissert, M. Hess, M. Hieber, C. Schwarz, K. Stavrakidis, Maximal L^p-L^q-estimates for the Stokes equation: a short proof of Solonnikov's theorem, J. Math. Fluid Mech. 12 (1) (2010) 47–60.
[14] Y. Giga, Domains of fractional powers of the Stokes operator in L_r spaces, Arch. Ration. Mech. Anal. 89 (3) (1985) 251–265.
[15] R. Hardt, D. Kinderlehrer, F.-H. Lin, Existence and partial regularity of static liquid crystal configurations, Commun. Math. Phys. 105 (4) (1986) 547–570.
[16] X. Hu, D. Wang, Global solution to the three-dimensional incompressible flow of liquid crystals, Commun. Math. Phys. 296 (3) (2010) 861–880.
[17] T. Huang, C. Wang, H. Wen, Strong solutions of the compressible nematic liquid crystal flow, J. Differ. Equ. 252 (3) (2012) 2222–2265.
[18] M. Köhne, J. Prüss, M. Wilke, On quasilinear parabolic evolution equations in weighted L_p-spaces, J. Evol. Equ. 10 (2) (2010) 443–463.
[19] F.M. Leslie, Some constitutive equations for liquid crystals, Arch. Ration. Mech. Anal. 28 (4) (1968) 265–283.
[20] X. Li, D. Wang, Global solution to the incompressible flow of liquid crystals, J. Differ. Equ. 252 (1) (2012) 745–767.
[21] F. Lin, J. Lin, C. Wang, Liquid crystal flows in two dimensions, Arch. Ration. Mech. Anal. 197 (1) (2010) 297–336.
[22] F.-H. Lin, Nonlinear theory of defects in nematic liquid crystals; phase transition and flow phenomena, Commun. Pure Appl. Math. 42 (6) (1989) 789–814.
[23] F.-H. Lin, C. Liu, Nonparabolic dissipative systems modeling the flow of liquid crystals, Commun. Pure Appl. Math. 48 (5) (1995) 501–537.
[24] F.-H. Lin, C. Liu, Partial regularity of the dynamic system modeling the flow of liquid crystals, Discrete Contin. Dyn. Syst. 2 (1) (1996) 1–22.
[25] S. Ma, Classical solutions for the compressible liquid crystal flows with nonnegative initial densities, J. Math. Anal. Appl. 397 (2) (2013) 595–618.
[26] M. Meyries, R. Schnaubelt, Interpolation, embeddings and traces of anisotropic fractional Sobolev spaces with temporal weights, J. Funct. Anal. 262 (3) (2012) 1200–1229.
[27] J. Prüss, Maximal regularity for evolution equations in L_p-spaces, Conf. Semin. Mat. Univ. Bari (285) (2002) 1–39, 2003.
[28] J. Prüss, G. Simonett, Maximal regularity for evolution equations in weighted L_p-spaces, Arch. Math. (Basel) 82 (5) (2004) 415–431.
[29] J. Prüss, G. Simonett, Analytic solutions for the two-phase Navier–Stokes equations with surface tension and gravity, in: Parabolic Problems, in: Prog. Nonlinear Differ. Equ. Appl., vol. 80, Springer, Basel, 2011, pp. 507–540.
[30] J. Prüss, G. Simonett, R. Zacher, On convergence of solutions to equilibria for quasilinear parabolic problems, J. Differ. Equ. 246 (10) (2009) 3902–3931.

[31] V.A. Solonnikov, Estimates for solutions of nonstationary Navier–Stokes equations, J. Math. Sci. 8 (4) (1977) 467–529.
[32] H. Triebel, Interpolation Theory, Function Spaces, Differential Operators, North-Holl. Math. Libr., vol. 18, North-Holland Publishing Co., Amsterdam, 1978.
[33] C. Wang, Well-posedness for the heat flow of harmonic maps and the liquid crystal flow with rough initial data, Arch. Ration. Mech. Anal. 200 (1) (2011) 1–19.
[34] Hao Wu, Xiang Xu, Chun Liu, On the general ericksen-leslie system: Parodi's relation, well-posedness and stability, Arch. Ration. Mech. Anal. 208 (2013) 59–107.